1. Consider the following quadratic optimization problem

$$\min_{x\in\mathbb{R}^n}\{f(x) = 5x_1^2 + 6x_1x_2 + 10x_1x_3 - 39x_1 + 4x_2^2 + 42x_2 - 4x_3^2 - 101x_3\},$$

subject to

$$h_1(x) = -2x_1 + 3x_2 - 9x_3 + 57 = 0,$$

$$h_2(x) = -5x_1 - 10x_2 + x_3 - 56 = 0.$$

a. Find a stationary point;(0.5)

b. Use variables substitution technique to check whether the found stationary point is a point of minimum. (0.5)

$$f(x) = x^T A x + B^T x$$

$$A = \begin{pmatrix} 5 & 3 & 5 \\ 3 & 4 & 0 \\ 5 & 0 & -4 \end{pmatrix} \qquad B = (-39, 42, -101)^T$$

$$h_1(x) = B_1^T x + d_1 \qquad B_1 = (-2, 3, -9)^T \quad d_1 = 57$$

$$h_2(x) = B_2^T x + d_2 \qquad B_2 = (-5, -10, 1)^T \quad d_2 = -56$$

$$L = x^T A x + B^T x + \lambda_1 h_1(x) + \lambda_2 h_2(x)$$

$$\nabla L = 2Ax + B + \lambda_1 B_1 + \lambda_2 B_2 = 0$$

$$x^* = -\frac{1}{2} A^{-1}(B + \lambda_1 B_1 + \lambda_2 B_2)$$

$$\begin{cases} B_1^T x^* = -\frac{1}{2} B_1^T A^{-1}(B + \lambda_1 B_1 + \lambda_2 B_2) = -d_1 \\ B_2^T x^* = -\frac{1}{2} B_2^T A^{-1}(B + \lambda_1 B_1 + \lambda_2 B_2) = -d_2 \end{cases}$$

$$A^{-1} = \begin{pmatrix} 1/9 & -1/12 & 5/36 \\ -1/12 & 0{,}3125 & -5/48 \\ 5/36 & -5/48 & -11/144 \end{pmatrix}$$

$\Rightarrow$ System for λ_1, λ_2

$$\begin{cases} b_1^T A^{-1}(b + \lambda_1 b_1 + \lambda_2 b_2) = -2\alpha_1 \\ b_2^T A^{-1}(b + \lambda_1 b_1 + \lambda_2 b_2) = -2\alpha_1 \end{cases}$$

2) $h_2(x) = 0 \Rightarrow 5x_1 + 10x_2 + 56 = x_3$

$h_1(x) = 0 \Rightarrow -2x_1 + 3x_2 - 9(5x_1 + 10x_2 + 56) + 57 = 0$

$$-47x_1 - 87x_2 - 9 \cdot 56 + 57 = 0$$

$$\Rightarrow x_2 = \frac{-47x_1 - 9 \cdot 56 + 57}{87}$$

$$f(x) = 5x_1^2 + 6x_1\left(\frac{-47x_1 - 9 \cdot 56 + 57}{87}\right) +$$

$$+ 10x_1\left(5x_1 + 10\left(\frac{-47x_1 - 9 \cdot 56 + 57}{87}\right)\right) - 39x_1 +$$

$$+ 4\left(\frac{-47x_1 - 9 \cdot 56 + 57}{87}\right) - 4\left(\quad \right)^2 - 101\left(\quad \right)$$

we are interested in only $\alpha(x_y)^2$ $\quad$ $\alpha > 0 \Rightarrow \min$

$\alpha < 0 \Rightarrow \max$

2. Nonlinear optimization problem is given in the following form

$$\min\{f(x) = 4x_1^4 + 2x_1^2x_2^2 - 4x_1^2x_3^2 - x_2^4 - 6x_2^2x_3^2 - 4x_3^4\},$$

subject to

$$h_1(x) = 14x_1^2 + 6x_1x_2 + 8x_1x_3 + 232x_1 + 14x_2^2 - 10x_2x_3 - 734x_2 + 12x_3^2 - 154x_3 - 120 = 0,$$

$$g_1(x) = -257x_1 + 377.5x_2 + 216x_3 - 213.5 \leqslant 0,$$
$$g_2(x) = 70x_1 + 317x_2 + 215x_3 - 1457 \leqslant 0.$$

Check whether $\bar{x} = (4,1,4)^\top$, $\tilde{x} = (7.641261, 4.692335, -2.629574)^\top$, $\hat{x} = (3.931046, 10.638642, -12.927354)^\top$ are stationary points. (1.0)

$$L = f(x) + \lambda h_1(x) + \mu g_1(x) + \mu_2 g_2(x)$$

1) $\begin{cases} h_1(x) = 0 \\ g_1 \leq 0 \\ g_2 \leq 0 \end{cases}$ $\quad \mu_1 g_1 = 0 \quad \mu_2 g_2 = 0 \quad \mu_1 \geqslant 0 \quad \mu_2 \geqslant 0 \quad \lambda \in \mathbb{R}$

$\nabla_x L = 0$

1. Let $f(x) = (x_1^2 + 7x_1 + 2x_2 + 2)^2 + (x_2 - 2)^2$.

a. find all stationary point of f; (0.4)

b. check whether each found point is point of minimum, maximum, saddle point or neither of them. (0.6)

$$\nabla f(x) = \begin{pmatrix} 2(x_1^2 + 7x_1 + 2x_2 + 2)(2x_1 + 7) \\ 2(x_1^2 + 7x_1 + 2x_2 + 2)\cdot 2 + 2(x_2 - 2) \end{pmatrix} = \vec{0}$$

$$\cancel{2}(x_1^2 + 7x_1 + 2x_2 + 2)(2x_1 + 7) = 0$$
$$\cancel{2}(x_1^2 + 7x_1 + 2x_2 + 2)\cdot 2 + \cancel{2}(x_2 - 2) = 0$$

1) $(x_1^2 + 7x_1 + 2x_2 + 2) = 0$

$\Rightarrow x_2 - 2 = 0 \qquad x_2 = 2$

$x_1^2 + 7x_1 + 6 = 0$

$-1 \quad -6$

$\Rightarrow \begin{cases} x_2 = 2 \\ x_1 = -1 \end{cases} \qquad \begin{cases} x_2 = 2 \\ x_1 = -6 \end{cases}$

$2x_1 + 7 = 0 \Rightarrow x_1 = -\frac{7}{2}$

$$(\underbrace{-\frac{49}{4} + \frac{49}{2}}_{49/4} + 2x_2 + 2)\cdot 2 + (x_2 - 2) = 0$$

$4x_2 + 57/2 + x_2 - 2 = 0$

$5x_2 + 53/2 = 0$

$\Rightarrow \begin{cases} x_2 = \frac{53}{10} = 5,3 \\ x_1 = -3,5 \end{cases}$

2) $$\nabla^2 f(x) = \begin{pmatrix} 2(2x_1 + 7)^2 + 2(*)\cdot 2 ; & 2(2x_1 + 7)\cdot 2 \\ 4(2x_1 + 7) ; & 4\cdot 2 + 2 \end{pmatrix}$$

$\begin{cases} x_1 = -1 \\ x_2 = 2 \end{cases} \Rightarrow \nabla^2 f = \begin{pmatrix} 2\cdot 5^2 + 10 & 2\cdot 5\cdot 2 \\ 4\cdot 5 & 10 \end{pmatrix}$

$\Delta_1 > 0 \quad \Delta_2 = 2 \cdot 5^2 \cdot 10 - 4^2 \cdot 5^2 = 4^2 \cdot 5^2 (5-1) > 0$

$\Rightarrow \min$

$$\begin{cases} x_1 = -6 \\ x_2 = 2 \end{cases} \quad \nabla^2 f = \begin{pmatrix} 2(-5)^2 + 0 & 4(-5) \\ 4 \cdot (-5 & 10 \end{pmatrix}$$

$\Delta_1 > 0 \quad \Delta_2 = 2 \cdot 5^2 \cdot 10 - 4^2 \cdot 5^2 > 0 \Rightarrow \min$

$$\begin{cases} x_1 = -3{,}5 \\ x_2 = 5{,}3 \end{cases} \quad \nabla^2 f = \begin{pmatrix} 2 \cdot 0 + 4(1-3{,}5)^2 - 7 \cdot 3{,}6 + 2 \cdot 5{,}3 + 2) & 0 \\ 0 & 10 \end{pmatrix}$$

$\Delta_1 > 0 \Rightarrow \Delta_2 > 0 \Rightarrow \min$