Signals and Systems | (2nd Edition)

Problem

Compute the Fourier transform of each of the following signals:

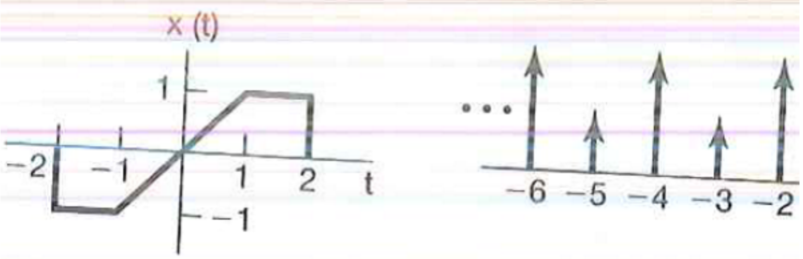
(a) $[e^{-\alpha t}\cos\omega_0 t]u(t),\ \alpha > 0$ (b

(c) $x(t) = \begin{cases} 1+\cos\pi t, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$ (d

(e) $[te^{-2t}\sin 4t]u(t)$ (f)

(g) $x(t)$ as shown in Figure P4.21(a) (h

(i) $x(t) = \begin{cases} 1-t^2, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$ (j)



(a)

Figure P4.2

Determine the

Step-by-step solution

Step 1 of 15

(a)

The signal, $x(t) = \left[e^{-\alpha t}\cos\omega_0 t\right]u(t), \alpha > 0$

Compute the Fourier transform of this signal.

$$= \int_0^{\infty} e^{-\alpha t}\left(\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right)e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{(-\alpha+j(-\omega+\omega_0))t}\,dt + \int_0^{\infty} e^{(-\alpha+j(-\omega-\omega_0))t}\right]dt$$

Simplify further.

$$X(j\omega) = \frac{1}{2}\left[\frac{-e^{-(\alpha+j(\omega-\omega_0))t}}{\alpha+j(\omega-\omega_0)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{-e^{-(\alpha+j(\omega+\omega_0))t}}{\alpha+j(\omega+\omega_0)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[-0-\left(\frac{-1}{\alpha+j(\omega-\omega_0)}\right)\right] + \frac{1}{2}\left[-0-\left(\frac{-1}{\alpha+j(\omega_0+\omega)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{\alpha+j(\omega-\omega_0)} + \frac{1}{\alpha+j(\omega_0+\omega)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\alpha-j(\omega_0-\omega)} + \frac{1}{\alpha+j(\omega_0+\omega)}\right]$$

Comment

Step 2 of 15

Simplify further.

$$X(j\omega) = \frac{1}{2}\left[\frac{1}{\alpha-j(\omega_o-\omega)} + \frac{1}{\alpha+j(\omega_o+\omega)}\right]$$

$$= \frac{0.5}{\alpha-j(\omega_o-\omega)} + \frac{0.5}{\alpha+j(\omega_o+\omega)}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{0.5}{\alpha-j(\omega_o-\omega)} + \frac{0.5}{\alpha+j(\omega_o+\omega)}}$.

Comment

Step 3 of 15

(b)

The signal, $x(t) = e^{-3|t|}\sin 2t$

Compute the Fourier transform of this signal.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty}\left(e^{-3|t|}\sin 2t\right)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{0} e^{3t}\left(\frac{e^{2jt}-e^{-2jt}}{2j}\right)e^{-j\omega t}\,dt + \int_0^{\infty} e^{-3t}\left(\frac{e^{j2t}-e^{-j2t}}{2j}\right)e^{-j\omega t}\,dt$$

$$= \frac{1}{2j}\left[\int_{-\infty}^{0}\left(e^{(3+2j-j\omega)t}-e^{(3-2j-j\omega)t}\right)dt\right] + \frac{1}{2j}\left[\int_0^{\infty}\left(e^{-(3-2j+j\omega)t}-e^{-(3+2j+j\omega)t}\right)dt\right]$$

Comment

Step 4 of 15

Simplify further.

$$=\frac{1}{2j}\left\{\left.\frac{e^{-(3-2j+j\omega)t}}{-(3-2j+j\omega)}\right|_0^\infty-\left.\frac{e^{-(3+2j+j\omega)t}}{-(3+2j+j\omega)}\right|_0^\infty\right\}$$

$$=\frac{1}{2j}\left\{\begin{array}{l}\left(\frac{1}{3+j(2-\omega)}\right)-\left(\frac{1}{3-j(2+\omega)}\right)+\left(\frac{1}{(3-2j+j\omega)}\right)\\-\left(\frac{1}{(3+2j+j\omega)}\right)\end{array}\right\}$$

Simplify further.

$$X(j\omega)=\frac{1}{2j}\left\{\begin{array}{l}\frac{1}{3+j(2-\omega)}-\frac{1}{3-j(2+\omega)}+\frac{1}{(3-j(2-\omega))}-\\\frac{1}{(3+j(2+\omega))}\end{array}\right\}$$

$$=\frac{1}{2j}\left[\begin{array}{l}\left\{\frac{1}{3+j(2-\omega)}+\frac{1}{(3-j(2-\omega))}\right\}-\\\left\{\frac{1}{3-j(2+\omega)}+\frac{1}{(3+j(2+\omega))}\right\}\end{array}\right]$$

$$=\frac{1}{2j}\left\{\frac{6}{9+(2-\omega)^2}\right\}+\frac{1}{2j}\left\{\frac{-6}{9+(2+\omega)^2}\right\}$$

$$=\frac{-3j}{9+(2-\omega)^2}+\frac{3j}{9+(2+\omega)^2}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{3j}{9+(2+\omega)^2}-\frac{3j}{9+(2-\omega)^2}}$.

Comment

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(c)

The signal, $x(t)=\begin{cases}1+\cos\pi t, & |t|\le 1\\ 0, & |t|>1\end{cases}$

Compute the Fourier transform of this signal.

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

$$=\int_{-1}^{1}(1+\cos\pi t)e^{-j\omega t}dt$$

$$=\int_{-1}^{1}e^{-j\omega t}dt+\int_{-1}^{1}\left(\frac{e^{j\pi t}+e^{-j\pi t}}{2}\right)e^{-j\omega t}dt$$

$$=\int_{-1}^{1}e^{-j\omega t}dt+\frac{1}{2}\left\{\int_{-1}^{1}e^{jt(\pi-\omega)}dt+\int_{-1}^{1}e^{-jt(\pi+\omega)}dt\right\}$$

Simplify further.

$$X(j\omega)=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-1}^{1}+\frac{1}{2}\left\{\left.\frac{e^{jt(\pi-\omega)}}{j(\pi-\omega)}\right|_{-1}^{1}+\left.\frac{e^{-jt(\pi+\omega)}}{-j(\pi+\omega)}\right|_{-1}^{1}\right\}$$

$$=\frac{e^{-j\omega}-e^{j\omega}}{-j\omega}+\frac{1}{2}\left\{\frac{e^{j(\pi-\omega)}-e^{-j(\pi-\omega)}}{j(\pi-\omega)}-\frac{e^{-j(\pi+\omega)}-e^{j(\pi+\omega)}}{j(\pi+\omega)}\right\}$$

$$=\frac{-2j\sin\omega}{-j\omega}+\frac{2j\sin(\pi-\omega)}{2j(\pi-\omega)}+\frac{2j\sin(\pi+\omega)}{2j(\pi+\omega)}$$

$$=\frac{2\sin\omega}{\omega}+\frac{\sin(\pi-\omega)}{\pi-\omega}+\frac{\sin(\pi+\omega)}{\pi+\omega}$$

Simplify further.

$$X(j\omega)=\frac{2\sin\omega}{\omega}+\frac{\sin(\pi-\omega)}{\pi-\omega}+\frac{\sin(\pi+\omega)}{\pi+\omega}$$

Note: $\sin(\pi-\omega)=\sin\omega$ and $\sin(\pi+\omega)=-\sin\omega$

Hence,

$$X(j\omega)=\frac{2\sin\omega}{\omega}+\frac{\sin\omega}{\pi-\omega}-\frac{\sin\omega}{\pi+\omega}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{2\sin\omega}{\omega}+\frac{\sin\omega}{\pi-\omega}-\frac{\sin\omega}{\pi+\omega}}$.

Comment

Step 7 of 15

(d)

The signal, $x(t)=\sum_{k=0}^{\infty}\alpha^k\delta(t-kT),|\alpha|<1$

Compute the Fourier transform of this signal.

$$\begin{aligned}X(j\omega)&=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\\&=\int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\alpha^k\delta(t-kT)e^{-j\omega t}dt\\&=\sum_{k=0}^{\infty}\alpha^k\int_{-\infty}^{\infty}\delta(t-kT)e^{-j\omega t}dt\\&=\sum_{k=0}^{\infty}\alpha^k\left(e^{-j\omega t}\Big|_{t=kT}\right)\end{aligned}$$

Simplify further.

$$\begin{aligned}X(j\omega)&=\sum_{k=0}^{\infty}\alpha^k e^{-j\omega kT}\\&=\sum_{k=0}^{\infty}\left(\alpha e^{-j\omega T}\right)^k\\&=\frac{1}{1-\alpha e^{-j\omega T}}\end{aligned}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{1}{1-\alpha e^{-j\omega T}}}$.

(e)

The signal, $x(t)=\left[te^{-2t}\sin 4t\right]u(t)$

Compute the Fourier transform of this signal.

$$\begin{aligned}X(j\omega)&=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\\&=\int_{-\infty}^{\infty}\left[te^{-2t}\sin 4t\right]u(t)e^{-j\omega t}dt\\&=\int_{0}^{\infty}te^{-2t}\left(\frac{e^{4jt}-e^{-4jt}}{2j}\right)e^{-j\omega t}dt\\&=\frac{1}{2j}\left[\int_{0}^{\infty}\left\{te^{-(2-4j+j\omega)t}-te^{-(2+4j+j\omega)t}\right\}dt\right]\end{aligned}$$

Simplify further.

$$= \frac{1}{2j}\left\{(0-0)+\int_0^\infty \frac{e^{-(2-4j+j\omega)t}}{(2-4j+j\omega)}dt-(0-0)-\int_0^\infty \frac{e^{-(2+4j+j\omega)t}}{(2+4j+j\omega)}dt\right\}$$
$$= \frac{1}{2j}\left\{\int_0^\infty \frac{e^{-(2-4j+j\omega)t}}{(2-4j+j\omega)}dt-\int_0^\infty \frac{e^{-(2+4j+j\omega)t}}{(2+4j+j\omega)}dt\right\}$$
$$= \frac{1}{2j}\left\{\begin{array}{l}\left(\frac{1}{(2-4j+j\omega)}\right)\left(\frac{-e^{-(2-4j+j\omega)t}}{(2-4j+j\omega)}\right)\Big|_0^\infty - \\ \left(\frac{1}{(2+4j+j\omega)}\right)\left(\frac{-e^{-(2+4j+j\omega)t}}{(2+4j+j\omega)}\right)\Big|_0^\infty\end{array}\right\}$$

Comment

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Simplify further.

$$X(j\omega) = \frac{1}{2j}\left\{\begin{array}{l}\left(\frac{1}{(2-4j+j\omega)}\right)\left(\frac{1}{(2-4j+j\omega)}\right) - \\ \left(\frac{1}{(2+4j+j\omega)}\right)\left(\frac{1}{(2+4j+j\omega)}\right)\end{array}\right\}$$
$$= \frac{1}{2j}\left\{\frac{1}{(2-4j+j\omega)^2}-\frac{1}{(2+4j+j\omega)^2}\right\}$$

Thus, the Fourier transform of the signal $x(t)$ is

$$\boxed{\frac{1}{2j}\left\{\frac{1}{(2-4j+j\omega)^2}-\frac{1}{(2+4j+j\omega)^2}\right\}}.$$

Comment

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(f)

The signal, $x(t)=\left[\frac{\sin \pi t}{\pi \tau}\right]\left[\frac{\sin 2\pi(t-1)}{\pi(t-1)}\right]$

Compute the Fourier transform of this signal.

Assume,

$$x_1(t)=\frac{\sin \pi t}{\pi t}$$
$$x_2(t)=\frac{\sin 2\pi(t-1)}{\pi(t-1)}$$

Write the Fourier transform of $x_1(t)=\frac{\sin \pi t}{\pi t}$

$$X_1(j\omega)=\begin{cases}1, & |\omega|<\pi \\ 0, & \text{otherwise}\end{cases}$$

Write the Fourier transform of $x_2(t)=\frac{\sin 2\pi(t-1)}{\pi(t-1)}$

$$X_2(j\omega)=\begin{cases}e^{-2\omega} & |\omega|<2\pi \\ 0 & \text{otherwise}\end{cases}$$

Write the multiplication property in Fourier domain.

$$x(t) = x_1(t)x_2(t) \xleftrightarrow{F\cdot T} X(j\omega) = \frac{1}{2\pi}\{X_1(j\omega) * X_2(j\omega)\}$$

Thus, the Fourier transform of the signal $x(t)$ is

$$\boxed{X(j\omega) = \begin{cases} e^{-j\omega} & |\omega| < \pi \\ \left(\frac{1}{2\pi}\right)(3\pi + \omega)e^{-j\omega} & -3\pi < \omega < -\pi \\ \left(\frac{1}{2\pi}\right)(3\pi - \omega)e^{-j\omega} & \pi < \omega < 3\pi \\ 0 & \text{otherwise} \end{cases}}.$$

(g)

Consider the following figure:

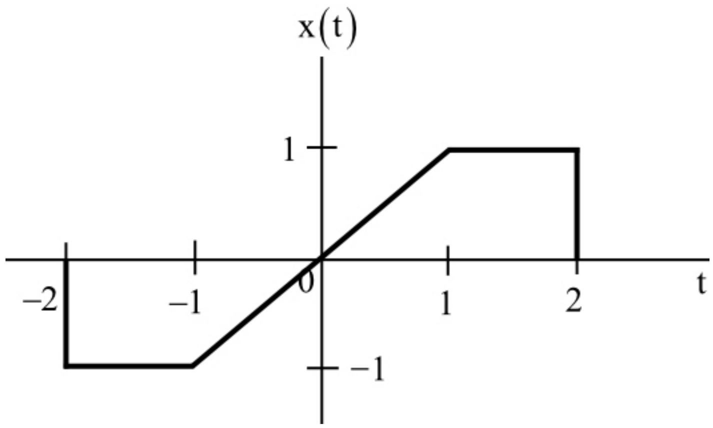

Figure 1

Write the mathematical representation of the signal in Figure 1.

$$x(t) = \begin{cases} -1, & -2 < t < -1 \\ t, & -1 < t < 1 \\ 1, & 1 < t < 2 \end{cases}$$

Compute the Fourier transform of this signal.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-2}^{-1}(-1)e^{-j\omega t}dt + \int_{-1}^{1} te^{-j\omega t}dt + \int_{1}^{2}(1)e^{-j\omega t}dt$$

$$= \frac{-e^{-j\omega t}}{-j\omega}\bigg|_{-2}^{-1} + t\frac{e^{-j\omega t}}{-j\omega}\bigg|_{-1}^{1} - \int_{-1}^{1}\frac{e^{-j\omega t}}{-j\omega}dt + \frac{e^{-j\omega t}}{-j\omega}\bigg|_{1}^{2}$$

$$= \left[\frac{e^{j\omega} - e^{2j\omega}}{j\omega}\right] - \left[\frac{e^{-j\omega} - (-1)e^{j\omega}}{j\omega}\right] - \left[\frac{e^{-j\omega t}}{(-j\omega)^2}\bigg|_{-1}^{1}\right] - \left[\frac{e^{-2j\omega} - e^{-j\omega}}{j\omega}\right]$$

Comments (1)

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Simplify further.

$$= -\left(\frac{\cdots}{j\omega}\right) - \left(\frac{\cdots}{\omega^2}\right)$$

$$= \frac{-2\cos 2\omega}{j\omega} - \frac{2j\sin\omega}{\omega^2}$$

Simplify further.

$$\begin{aligned} X(j\omega) &= \frac{-2j\cos 2\omega}{j \cdot j\omega} - \frac{2j\sin\omega}{\omega^2} \\ &= \frac{2j\cos 2\omega}{\omega} - \frac{2j\sin\omega}{\omega^2} \end{aligned}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{2j\cos 2\omega}{\omega} - \frac{2j\sin\omega}{\omega^2}}$.

Comment

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(h)

Consider the following figure:

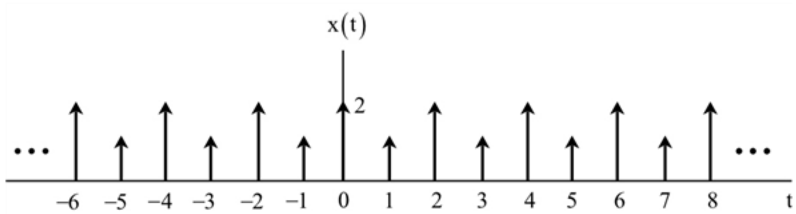

Figure 2

Write the mathematical representation of the signal in Figure 2.

Assume,

$$x_1(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

Write the Fourier transform of this signal.

$$X_1(j\omega) = \frac{2\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{2}\right)$$

Clearly from Figure 2, the input signal is,

$$x(t) = 2x_1(t) + x_1(t-1)$$

Compute the Fourier transform of this signal.

$$\begin{aligned} X(j\omega) &= 2X_1(j\omega) + e^{-j\omega} X_1(j\omega) \\ &= X_1(j\omega)\left[2 + e^{-j\omega}\right] \\ &= \frac{2\pi}{2} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{2}\right)\left[2 + e^{-j\omega}\right] \\ &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)\left(2 + e^{-j\omega}\right) \end{aligned}$$

Simplify further.

$$\begin{aligned} X(j\omega) &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)\left(2 + e^{-j\omega}\right) \\ &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)\left[2 + e^{-j\pi k}\right] \\ &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)\left[2 + \left(e^{-j\pi}\right)^k\right] \\ &= \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi k)\left[2 + (-1)^k\right] \end{aligned}$$

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(i)

The signal, $x(t)=\begin{cases}1-t^2 & 0<t<1\\ 0 & \text{otherwise}\end{cases}$

Compute the Fourier transform of this signal.

$$\begin{aligned}X(j\omega)&=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\\&=\int_0^1\left(1-t^2\right)e^{-j\omega t}dt\\&=\int_0^1 e^{-j\omega t}dt-\int_0^1 t^2e^{-j\omega t}dt\\&=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^1-t^2\left.\frac{e^{-j\omega t}}{-j\omega}\right|_0^1+\int_0^1 2t\frac{e^{-j\omega t}}{-j\omega}dt\end{aligned}$$

Simplify further.

$$\begin{aligned}X(j\omega)&=\frac{e^{-j\omega}-1}{-j\omega}+\frac{e^{-j\omega}}{j\omega}+2\left[\left.\frac{te^{-j\omega t}}{(-j\omega)^2}\right|_0^1-\int_0^1\frac{e^{-j\omega t}}{(-j\omega)^2}dt\right]\\&=\frac{1-e^{-j\omega}+e^{-j\omega}}{j\omega}+\frac{2e^{-j\omega}}{(-j\omega)^2}-\left.\frac{2e^{-j\omega t}}{(-j\omega)^3}\right|_0^1\\&=\frac{1}{j\omega}-\frac{2e^{-j\omega}}{-\omega^2}-\frac{2e^{-j\omega}}{(-j\omega)^3}+\frac{2}{(-j\omega)^3}\\&=\frac{1}{j\omega}-\frac{2e^{-j\omega}}{-\omega^2}+\frac{2-2e^{-j\omega}}{(-j\omega)^3}\end{aligned}$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{1}{j\omega}-\frac{2e^{-j\omega}}{-\omega^2}+\frac{2-2e^{-j\omega}}{(-j\omega)^3}}$.

Comment

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(j)

The signal, $x(t)=\sum_{n=-\infty}^{\infty}e^{-|t-2n|}$

Compute the Fourier transform of this signal.

$$\begin{aligned}X(j\omega)&=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\\&=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}e^{-|t-2n|}e^{-j\omega t}dt\\&=\int_{-\infty}^{0}\sum_{n=-\infty}^{0}e^{t-2n}e^{-j\omega t}dt+\int_0^{\infty}\sum_{n=0}^{\infty}e^{-t+2n}e^{-j\omega t}dt\\&=\sum_{n=-\infty}^{0}e^{-2n}\int_{-\infty}^{0}e^{t}e^{-j\omega t}dt+\sum_{n=0}^{\infty}e^{2n}\int_0^{\infty}e^{-t}e^{-j\omega t}dt\end{aligned}$$

Simplify further.

$$= \left(\frac{1}{1-e^2}\right)\left[\frac{e^{(1-j\omega)t}}{(1-j\omega)}\Bigg|_{-\infty}^{0} - \frac{e^{-(1+j\omega)t}}{(1+j\omega)}\Bigg|_{0}^{\infty}\right]$$

$$= \left(\frac{1}{1-e^2}\right)\left[\frac{1}{1-j\omega} + \frac{1}{1+j\omega}\right]$$

Comments (1)

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Simplify further.

$$X(j\omega) = \left(\frac{1}{1-e^2}\right)\left[\frac{1}{1-j\omega} + \frac{1}{1+j\omega}\right]$$
$$= \left(\frac{1}{1-e^2}\right)\left(\frac{2}{1+\omega^2}\right)$$

Thus, the Fourier transform of the signal $x(t)$ is $\boxed{\frac{2}{(1-e^2)(1+\omega^2)}}$.

Comment