Finite Dimensional Normed Vector Spaces are Banach Spaces

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1 Introduction

We will show that all finite dimensional vector spaces equipped with a norm are complete with respect to the metric induced by the norm, that is, all Cauchy sequences in a finite dimensional vector space converge to a vector in the space. We will first prove a lemma stating all norms on a finite dimensional vector space are equivalent. We then will show any Cauchy sequence in a finite dimensional vector space converges.

2 All norms on a finite dimensional vector space are equivalent

Let V be a finite dimensional vector space over $\mathbb{R}$ such that $dim(V) = k$ and $||\cdot|| : V \to \mathbb{R}$ be an arbitrary norm on V. Two norms on V, $||\cdot||_a, ||\cdot||_b : V \to \mathbb{R}$, are equivalent if there exists real numbers $C, D > 0$ such that $C||x||_b \leq ||x||_a \leq D||x||_b$ for all $x \in V$. Let $x \in V$ and $e_1, \ldots, e_k$ be a basis of V, then x can be written $x = x_1e_1 + \ldots + x_ke_k$. Note that $|x_i| \leq ||x||_\infty = \max_i |x_i|$ for each $1 \leq i \leq k$. Using the triangle inequality $k-1$ times we see that,

$$||x|| = ||\sum_{i=1}^{k} x_ie_i|| \leq \sum_{i=1}^{k} ||x_ie_i|| = \sum_{i=1}^{k} |x_i| \cdot ||e_i|| \leq ||x||_\infty \sum_{i=1}^{k} ||e_i||$$

Setting $D = \sum_{i=1}^{k} ||e_i||$, we see that $\exists D > 0$ such that $||x|| \leq D||x||_\infty$ for all $x \in V$.
Now consider set $S = \{x \in X : ||x||_\infty = 1\}$ and $f : S \to \mathbb{R}$ defined by $f(x) = ||x||$. We proceed using a couple theorems in Analysis by first proving f is continuous with respect to $||\cdot||_\infty$, S is compact by the Heine-Borel Theorem, then finding our value for C with the Extreme Value Theorem. Let $\varepsilon > 0$ and choose $\delta = \varepsilon/D$. Then if $||x-y||_\infty < \delta$, we have for all $x, y \in S$

$$|f(x) - f(y)| = |\ ||x|| - ||y||\ | = |\ ||x-y+y|| - ||y||\ | \leq ||x-y|| \leq D||x-y||_\infty < \varepsilon$$

Thus f is continuous on S with respect to $|\cdot|_\infty$ by definition. Note that S, the unit ball with respect to $|\cdot|_\infty$, is a closed and bounded set, so S is compact by the Heine-Borel Theorem. Since f is a continuous function on a compact set, it attains a minimum value by the Extreme Value Theorem. That is, there exists real number $C > 0$ such that $C \leq ||x||\ \forall x \in S$. Then for $v = \frac{x}{||x||_\infty} \in V$, $||v|| = ||\frac{x}{||x||_\infty}|| = \frac{1}{||x||_\infty} \cdot \frac{||x||}{1} = ||x|| \geq C, \forall x \in V$ implying $C||x||_\infty \leq ||x||$ for some $C > 0$ and all $x \in V$. Thus all norms on V are equivalent to the ℓ_∞ norm: $\exists C, D > 0$ such that $C||x||_\infty \leq ||x|| \leq D||x||_\infty\ \forall x \in V$.
Let $||\cdot||_a$ and $||\cdot||_b$ be two arbitrary norms on V. Since they're both equivalent to $||\cdot||_\infty$, it is clear that there exists real numbers $C_1, C_2, D_1, D_2 > 0$ such that $C_1||x||_\infty \leq ||x||_a \leq D_1||x||_\infty$ and $C_2||x||_\infty \leq ||x||_b \leq D_2||x||_\infty$ $\forall x \in V$. Thus $\frac{C_1}{D_2}||x||_b \leq ||x||_a \leq \frac{C_2}{D_1}||x||_b\ \forall x \in V$, hence it is clear $||\cdot||_a$ and $||\cdot||_b$ are equivalent so all norms on a finite dimensional vector space are equivalent.

3 All Cauchy sequences in a normed vector space converge

Let V be a finite dimensional vector space over $\mathbb{R}$ such that $dim(V) = k$, equipped with a norm $||\cdot|| : V \to \mathbb{R}$, let $d(x, y) = ||x - y||$ be the metric induced by the norm and let (x_n) be a Cauchy sequence in V. Using the definition of a Cauchy sequence, $\forall \varepsilon > 0, \exists M \in \mathbb{N}$ such that positive integers $n, m > M$ implies $d(x_n, x_m) < C\varepsilon$. Thus $\forall \varepsilon > 0$ and $n, m > M$, using equivalence of norms with the ℓ_∞ norm,

$$C\varepsilon > d(x_n, x_m) = ||x_n - x_m|| \geq C||x_n - x_m||_\infty = C \max_i |x_{n,i} - x_{m,i}| \geq C|x_{n,i} - x_{m,i}|$$

for all $1 \leq i \leq k$ and some real $C > 0$. Thus each $(x_{n,i})$ is a Cauchy sequence in $\mathbb{R}$ for all $1 \leq i \leq k$. Since $\mathbb{R}$ is complete, there exists $x_i \in \mathbb{R}$ such that $(x_{n,i})$ converges to x_i for each $1 \leq i \leq k$. Let $y = (x_1, \ldots, x_k)$ and let us show (x_n) converges to y in V. Let $\varepsilon' > 0$ and using equivalence of norms with the ℓ_1 norm, $||x_n - y|| \leq D||x_n - y||_1$ for some $D > 0$. Since $(x_{n,i})$ converges to x_i, there exists $N_i \in \mathbb{N}$ such that $n > N_i$ implies $|x_{n,i} - x_i| < \frac{\varepsilon'}{kD}$ for each $1 \leq i \leq k$. Set $N = max\{N_1, \ldots, N_k\}$, then we have $\forall \varepsilon' > 0, \exists N \in \mathbb{N}$ such that $n > N$ implies

$$d(x_n, y) = ||x_n - y|| \leq D||x_n - y||_1 = D\sum_{i=1}^{k} |x_{n,i} - x_i| < D\sum_{i=1}^{k} \frac{\varepsilon'}{kD} = kD\frac{\varepsilon'}{kD} = \varepsilon'$$

Thus, (x_n) converges to y in V and it follows V is complete. $\blacksquare$

4 Reference

Johnson, S. G. (2020, October 28). Notes on the equivalence of norms - Massachusetts Institute of Technology. Notes on the equivalence of norms. Retrieved April 4, 2023, from https://math.mit.edu/ stevenj/18.335/norm-equivalence.pdf