Three-way crossed classification

Introduction

- Say we have 3 factors A, B, and C with levels a, b, and c, respectively.
- The observations at treatment combination (i, j, k) are labeled y_{ijkl}.
- Consider balanced case: We have fixed number of observations n for each treatment combination.
- $N = abcn$

model: A, B, C fixed

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

where ϵ_{ijkl} are i.i.d. $N(0, \sigma_e^2)$

$$
\begin{aligned}
\underbrace{abcn-1}_{dftotal} = \underbrace{a-1}_{dfA} + \underbrace{b-1}_{dfB} + \underbrace{c-1}_{dfC} + \\
\underbrace{(a-1)(b-1)}_{dfAB} + \underbrace{(a-1)(c-1)}_{dfAC} + \underbrace{(b-1)(c-1)}_{dfBC} + \\
\underbrace{(a-1)(b-1)(c-1)}_{dfABC} + \underbrace{abc(n-1)}_{dferror}
\end{aligned}
$$

$$\sum_{i=1}^{a} \alpha_i = 0$$

$$\sum_{i=1}^{a} (\alpha\beta)_{ij} = 0 \text{ for all } j$$

$$\sum_{i=1}^{a} (\alpha\beta\gamma)_{ijk} = 0 \text{ for all } j, k$$

Example

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

$i = 1, 2, 3; j = 1, 2, 3; k = 1, 2; l = 1, 2, 3, 4$

Minitab - 3way_eg.MPJ - [Worksheet 1 ***]

File Edit Data Calc Stat Graph Editor Tools Window Help

↓	C1	C2	C3	C4	C5	C6
	y	A	B	C		
1	17.9	1	1	1		
2	18.0	1	1	1		
3	18.7	1	1	1		
4	19.0	1	1	1		
5	18.1	1	1	2		
6	18.9	1	1	2		
7	18.6	1	1	2		
8	19.1	1	1	2		
9	19.2	1	2	1		
10	19.0	1	2	1		
11	20.4	1	2	1		
12	19.2	1	2	1		
13	19.2	1	2	2		
14	19.3	1	2	2		
15	20.7	1	2	2		
16	20.4	1	2	2		
17	19.9	1	3	1		
18	20.1	1	3	1		
19	20.0	1	3	1		
20	20.8	1	3	1		

- ANOVA decomposition:

$$
\begin{aligned}
SStot = & SSA + SSB + SSC + \\
& SSAB + SSAC + SSBC + \\
& SSABC + SSE
\end{aligned}
$$

- The effect estimates are

$$\hat{\alpha}_i = \bar{y}_{i...} - \bar{y}_{....}$$

$$SSA = \sum_{i,j,k,l} \hat{\alpha}_i^2 = \sum_{i,j,k,l} (\bar{y}_{i...} - \bar{y}_{....})^2$$

expected mean squares

$$y_{i\ldots} = \mu + \alpha_i + \bar{\beta}_{.} + \bar{\gamma}_{.} + (\bar{\alpha\beta})_{i.} + (\bar{\alpha\gamma})_{i.} + (\bar{\beta\gamma})_{..} + (\alpha\bar{\beta}\gamma)_{i..} + \bar{\epsilon}_{i\ldots}$$

$$y_{\ldots.} = \mu + \bar{\alpha}_{.} + \bar{\beta}_{.} + \bar{\gamma}_{.} + (\bar{\alpha\beta})_{..} + (\bar{\alpha\gamma})_{..} + (\bar{\beta\gamma})_{..} + (\alpha\bar{\beta}\gamma)_{\ldots} + \bar{\epsilon}_{\ldots.}$$

Therefore,

$$SSA = bcn \sum_{i=1}^{a} (\alpha_i + \bar{\epsilon}_{i\ldots} - \bar{\epsilon}_{\ldots.})^2$$

It can be shown that

$$E(SSA) = bcn \sum_{i=1}^{a} \alpha_i^2 + (a-1)\sigma_\epsilon^2$$

Therefore,

$$E(MSA) = \frac{1}{a-1} E(SSA) = \frac{bcn}{a-1} \sum_{i=1}^{a} \alpha_i^2 + \sigma_\epsilon^2$$

model: A, B, C random

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

where all parameters (except the overall mean μ) are mutually independent and distributed as

$$\alpha_i \sim N(0, \sigma^2_\alpha), \beta_j \sim N(0, \sigma^2_\beta), \gamma_k \sim N(0, \sigma^2_\gamma),$$
$$(\alpha\beta)_{ij} \sim N(0, \sigma^2_{\alpha\beta}), (\alpha\gamma)_{ik} \sim N(0, \sigma^2_{\alpha\gamma}), (\beta\gamma)_{jk} \sim N(0, \sigma^2_{\beta\gamma}),$$
$$(\alpha\beta\gamma)_{ijk} \sim N(0, \sigma^2_{\alpha\beta\gamma})$$

and $\epsilon_{ijkl} \sim N(0, \sigma^2_e)$

Mean Square	Expected Mean Square: A, B, and C Fixed	A, B, and C Random	A Fixed, B and C Random	A Random, B and C Fixed
MS_A	$\sigma_e^2 + \frac{bcn}{a-1}\sum_{i=1}^{a}\alpha_i^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + bcn\sigma_{\alpha}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + \frac{bcn}{a-1}\sum_{i=1}^{a}\alpha_i^2$	$\sigma_e^2 + bcn\sigma_{\alpha}^2$
MS_B	$\sigma_e^2 + \frac{acn}{b-1}\sum_{j=1}^{b}\beta_j^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + an\sigma_{\beta\gamma}^2 + acn\sigma_{\beta}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2 + acn\sigma_{\beta}^2$	$\sigma_e^2 + cn\sigma_{\alpha\beta}^2 + \frac{acn}{b-1}\sum_{j=1}^{b}\beta_j^2$
MS_C	$\sigma_e^2 + \frac{abn}{c-1}\sum_{k=1}^{c}\gamma_k^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2 + an\sigma_{\beta\gamma}^2 + abn\sigma_{\gamma}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2 + abn\sigma_{\gamma}^2$	$\sigma_e^2 + bn\sigma_{\alpha\gamma}^2 + \frac{abn}{c-1}\sum_{k=1}^{c}\gamma_k^2$
MS_{AB}	$\sigma_e^2 + \frac{cn}{(a-1)(b-1)} \times \sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2$	$\sigma_e^2 + cn\sigma_{\alpha\beta}^2$
MS_{AC}	$\sigma_e^2 + \frac{bn}{(a-1)(c-1)} \times \sum_{i=1}^{a}\sum_{k=1}^{c}(\alpha\gamma)_{ik}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2$	$\sigma_e^2 + bn\sigma_{\alpha\gamma}^2$
MS_{BC}	$\sigma_e^2 + \frac{an}{(b-1)(c-1)} \times \sum_{j=1}^{b}\sum_{k=1}^{c}(\beta\gamma)_{jk}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + an\sigma_{\beta\gamma}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + \frac{an}{(b-1)(c-1)} \times \sum_{j=1}^{b}\sum_{k=1}^{c}(\beta\gamma)_{jk}^2$
MS_{ABC}	$\sigma_e^2 + \frac{n}{(a-1)(b-1)(c-1)} \times \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\alpha\beta\gamma)_{ijk}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$
MS_E	σ_e^2	σ_e^2	σ_e^2	σ_e^2

Hypothesis	Test Statistic	Percentile
H_0^A : all $\alpha_i = 0$ versus H_1^A : not all $\alpha_i = 0$	$F_A = \frac{\text{MS}_A}{\text{MS}_E}$	$F[a-1, abc(n-1); 1-\alpha]$
H_0^B : all $\beta_j = 0$ versus H_1^B : not all $\beta_j = 0$	$F_B = \frac{\text{MS}_B}{\text{MS}_E}$	$F[b-1, abc(n-1); 1-\alpha]$
H_0^C : all $\gamma_k = 0$ versus H_1^C : not all $\gamma_k = 0$	$F_C = \frac{\text{MS}_C}{\text{MS}_E}$	$F[c-1, abc(n-1); 1-\alpha]$
H_0^{AB} : all $(\alpha\beta)_{ij} = 0$ versus H_1^{AB} : not all $(\alpha\beta)_{ij} = 0$	$F_{AB} = \frac{\text{MS}_{AB}}{\text{MS}_E}$	$F[(a-1)(b-1), abc(n-1); 1-\alpha]$
H_0^{AC} : all $(\alpha\gamma)_{ik} = 0$ versus H_1^{AC} : not all $(\alpha\gamma)_{ik} = 0$	$F_{AC} = \frac{\text{MS}_{AC}}{\text{MS}_E}$	$F[(a-1)(c-1), abc(n-1); 1-\alpha]$
H_0^{BC} : all $(\beta\gamma)_{jk} = 0$ versus H_1^{BC} : not all $(\beta\gamma)_{jk} = 0$	$F_{BC} = \frac{\text{MS}_{BC}}{\text{MS}_E}$	$F[(b-1)(c-1), abc(n-1); 1-\alpha]$
H_0^{ABC} : all $(\alpha\beta\gamma)_{ijk} = 0$ versus H_1^{ABC} : not all $(\alpha\beta\gamma)_{ijk} = 0$	$F_{ABC} = \frac{\text{MS}_{ABC}}{\text{MS}_E}$	$F[(a-1)(b-1)(c-1), abc(n-1); 1-\alpha]$

A, B, C fixed

A, B, C random

Hypothesis	Test Statistic*	Percentile
$H_0^A : \sigma_\alpha^2 = 0$ versus $H_1^A : \sigma_\alpha^2 > 0$	$F_A = \dfrac{\mathrm{MS}_A}{\mathrm{MS}_{AB} + \mathrm{MS}_{AC} - \mathrm{MS}_{ABC}}$	$F[a-1, \nu_a; 1-\alpha]$
$H_0^B : \sigma_\beta^2 = 0$ versus $H_1^B : \sigma_\beta^2 > 0$	$F_B = \dfrac{\mathrm{MS}_B}{\mathrm{MS}_{AB} + \mathrm{MS}_{BC} - \mathrm{MS}_{ABC}}$	$F[b-1, \nu_b; 1-\alpha]$
$H_0^C : \sigma_\gamma^2 = 0$ versus $H_1^C : \sigma_\gamma^2 > 0$	$F_C = \dfrac{\mathrm{MS}_C}{\mathrm{MS}_{AC} + \mathrm{MS}_{BC} - \mathrm{MS}_{ABC}}$	$F[c-1, \nu_c; 1-\alpha]$
$H_0^{AB} : \sigma_{\alpha\beta}^2 = 0$ versus $H_1^{AB} : \sigma_{\alpha\beta}^2 > 0$	$F_{AB} = \dfrac{\mathrm{MS}_{AB}}{\mathrm{MS}_{ABC}}$	$F[(a-1)(b-1), (a-1)(b-1)(c-1); 1-\alpha]$
$H_0^{AC} : \sigma_{\alpha\gamma}^2 = 0$ versus $H_1^{AC} : \sigma_{\alpha\gamma}^2 > 0$	$F_{AC} = \dfrac{\mathrm{MS}_{AC}}{\mathrm{MS}_{ABC}}$	$F[(a-1)(c-1), (a-1)(b-1)(c-1); 1-\alpha]$
$H_0^{BC} : \sigma_{\beta\gamma}^2 = 0$ versus $H_1^{BC} : \sigma_{\beta\gamma}^2 > 0$	$F_{BC} = \dfrac{\mathrm{MS}_{BC}}{\mathrm{MS}_{ABC}}$	$F[(b-1)(c-1), (a-1)(b-1)(c-1); 1-\alpha]$
$H_0^{ABC} : \sigma_{\alpha\beta\gamma}^2 = 0$ versus $H_1^{ABC} : \sigma_{\alpha\beta\gamma}^2 > 0$	$F_{ABC} = \dfrac{\mathrm{MS}_{ABC}}{\mathrm{MS}_E}$	$F[(a-1)(b-1)(c-1), abc(n-1); 1-\alpha]$

Factor	Type	Levels	Values
A	Fixed	3	1, 2, 3
B	Fixed	3	1, 2, 3
C	Fixed	2	1, 2

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value
A	2	410.692	205.346	699.60	0.000
B	2	80.541	40.270	137.20	0.000
C	1	0.467	0.467	1.59	0.212
A*B	4	14.814	3.704	12.62	0.000
A*C	2	0.314	0.157	0.53	0.589
B*C	2	0.702	0.351	1.20	0.310
A*B*C	4	0.875	0.219	0.75	0.566
Error	54	15.850	0.294		
Total	71	524.255			

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
0.541773	96.98%	96.02%	94.63%

Factor	Type	Levels	Values
A	Random	3	1, 2, 3
B	Random	3	1, 2, 3
C	Random	2	1, 2

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value	
A	2	410.692	205.346	56.39	0.001	x
B	2	80.541	40.270	10.50	0.023	x
C	1	0.467	0.467	1.62	0.429	x
A*B	4	14.814	3.704	16.94	0.009	
A*C	2	0.314	0.157	0.72	0.542	
B*C	2	0.702	0.351	1.60	0.308	
A*B*C	4	0.875	0.219	0.75	0.566	
Error	54	15.850	0.294			
Total	71	524.255				

x Not an exact F-test.

Model Summary

S	R-sq	R-sq(adj)	R-sq(pred)
0.541773	96.98%	96.02%	94.63%

Expected Mean Squares, using Adjusted SS

	Source	Expected Mean Square for Each Term
1	A	(8) + 4.0000 (7) + 12.0000 (5) + 8.0000 (4) + 24.0000 (1)
2	B	(8) + 4.0000 (7) + 12.0000 (6) + 8.0000 (4) + 24.0000 (2)
3	C	(8) + 4.0000 (7) + 12.0000 (6) + 12.0000 (5) + 36.0000 (3)
4	A*B	(8) + 4.0000 (7) + 8.0000 (4)
5	A*C	(8) + 4.0000 (7) + 12.0000 (5)
6	B*C	(8) + 4.0000 (7) + 12.0000 (6)
7	A*B*C	(8) + 4.0000 (7)
8	Error	(8)

Error Terms for Tests, using Adjusted SS

	Source	Error DF	Error MS	Synthesis of Error MS
1	A	3.84	3.6417	(4) + (5) - (7)
2	B	4.20	3.8358	(4) + (6) - (7)
3	C	0.97	0.2891	(5) + (6) - (7)
4	A*B	4.00	0.2187	(7)
5	A*C	4.00	0.2187	(7)
6	B*C	4.00	0.2187	(7)
7	A*B*C	54.00	0.2935	(8)

model: A fixed, B and C random

$$y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \epsilon_{ijkl}$$

where the parameters have the following (mutually independent) distributions

$$\begin{aligned}
&\beta_j \sim N(0, \sigma^2_\beta), \gamma_k \sim N(0, \sigma^2_\gamma),\\
&(\alpha\beta)_{ij} \sim N(0, \sigma^2_{\alpha\beta}), (\alpha\gamma)_{ik} \sim N(0, \sigma^2_{\alpha\gamma}), (\beta\gamma)_{jk} \sim N(0, \sigma^2_{\beta\gamma}),\\
&(\alpha\beta\gamma)_{ijk} \sim N(0, \sigma^2_{\alpha\beta\gamma})\\
&\text{ and } \epsilon_{ijkl} \sim N(0, \sigma^2_e)
\end{aligned}$$

α_i are fixed unknown quantities subject to the constraint that $\sum_{i=1}^{a} \alpha_i = 0$.

Any parameters involving random factors are also random, eg, $(\alpha\beta)_{ij}$ and $(\alpha\beta\gamma)_{ijk}$.

Restricted or unrestricted

Unrestricted

No further assumptions beyond previous slide.

Restricted

We impose sum-to-zero constraints on the interactions which have one fixed factor. For example,

$$\sum_{i=1}^{a}(\alpha\beta)_{ij} = 0 \text{ for all } j$$

$$\sum_{i=1}^{a}(\alpha\beta\gamma)_{ijk} = 0 \text{ for all } j, k$$

Mean Square	Expected Mean Square: A, B, and C Fixed	A, B, and C Random	A Fixed, B and C Random	A Random, B and C Fixed
MS_A	$\sigma_e^2 + \frac{bcn}{a-1}\sum_{i=1}^{a}\alpha_i^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + bcn\sigma_{\alpha}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + bn\sigma_{\alpha\gamma}^2 + \frac{bcn}{a-1}\sum_{i=1}^{a}\alpha_i^2$	$\sigma_e^2 + bcn\sigma_{\alpha}^2$
MS_B	$\sigma_e^2 + \frac{acn}{b-1}\sum_{j=1}^{b}\beta_j^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2 + an\sigma_{\beta\gamma}^2 + acn\sigma_{\beta}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2 + acn\sigma_{\beta}^2$	$\sigma_e^2 + cn\sigma_{\alpha\beta}^2 + \frac{acn}{b-1}\sum_{j=1}^{b}\beta_j^2$
MS_C	$\sigma_e^2 + \frac{abn}{c-1}\sum_{k=1}^{c}\gamma_k^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2 + an\sigma_{\beta\gamma}^2 + abn\sigma_{\gamma}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2 + abn\sigma_{\gamma}^2$	$\sigma_e^2 + bn\sigma_{\alpha\gamma}^2 + \frac{abn}{c-1}\sum_{k=1}^{c}\gamma_k^2$
MS_{AB}	$\sigma_e^2 + \frac{cn}{(a-1)(b-1)} \times \sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + cn\sigma_{\alpha\beta}^2$	$\sigma_e^2 + cn\sigma_{\alpha\beta}^2$
MS_{AC}	$\sigma_e^2 + \frac{bn}{(a-1)(c-1)} \times \sum_{i=1}^{a}\sum_{k=1}^{c}(\alpha\gamma)_{ik}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + bn\sigma_{\alpha\gamma}^2$	$\sigma_e^2 + bn\sigma_{\alpha\gamma}^2$
MS_{BC}	$\sigma_e^2 + \frac{an}{(b-1)(c-1)} \times \sum_{j=1}^{b}\sum_{k=1}^{c}(\beta\gamma)_{jk}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + an\sigma_{\beta\gamma}^2$	$\sigma_e^2 + an\sigma_{\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2 + \frac{an}{(b-1)(c-1)} \times \sum_{j=1}^{b}\sum_{k=1}^{c}(\beta\gamma)_{jk}^2$
MS_{ABC}	$\sigma_e^2 + \frac{n}{(a-1)(b-1)(c-1)} \times \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\alpha\beta\gamma)_{ijk}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$	$\sigma_e^2 + n\sigma_{\alpha\beta\gamma}^2$
MS_E	σ_e^2	σ_e^2	σ_e^2	σ_e^2

Tests of Hypotheses for (*A* Fixed, *B* and *C* Random)

Hypothesis	Test Statistic*	Percentile
H_0^A : all $\alpha_i = 0$ versus H_1^A : not all $\alpha_i = 0$	$F_A = \dfrac{\text{MS}_A}{\text{MS}_{AB} + \text{MS}_{AC} - \text{MS}_{ABC}}$	$F[a-1, \nu_a; 1-\alpha]$
$H_0^B : \sigma_\beta^2 = 0$ versus $H_1^B : \sigma_\beta^2 > 0$	$F_B = \dfrac{\text{MS}_B}{\text{MS}_{BC}}$	$F[b-1, (b-1)(c-1); 1-\alpha]$
$H_0^C : \sigma_\gamma^2 = 0$ versus $H_1^C : \sigma_\gamma^2 > 0$	$F_C = \dfrac{\text{MS}_C}{\text{MS}_{BC}}$	$F[(c-1), (b-1)(c-1); 1-\alpha]$
$H_0^{AB} : \sigma_{\alpha\beta}^2 = 0$ versus $H_1^{AB} : \sigma_{\alpha\beta}^2 > 0$	$F_{AB} = \dfrac{\text{MS}_{AB}}{\text{MS}_{ABC}}$	$F[(a-1)(b-1), (a-1)(b-1)(c-1); 1-\alpha]$
$H_0^{AC} : \sigma_{\alpha\gamma}^2 = 0$ versus $H_1^{AC} : \sigma_{\alpha\gamma}^2 > 0$	$F_{AC} = \dfrac{\text{MS}_{AC}}{\text{MS}_{ABC}}$	$F[(a-1)(c-1), (a-1)(b-1)(c-1); 1-\alpha]$
$H_0^{BC} : \sigma_{\beta\gamma}^2 = 0$ versus $H_1^{BC} : \sigma_{\beta\gamma}^2 > 0$	$F_{BC} = \dfrac{\text{MS}_{BC}}{\text{MS}_E}$	$F[(b-1)(c-1), abc(n-1); 1-\alpha]$
$H_0^{ABC} : \sigma_{\alpha\beta\gamma}^2 = 0$ versus $H_1^{ABC} : \sigma_{\alpha\beta\gamma}^2 > 0$	$F_{ABC} = \dfrac{\text{MS}_{ABC}}{\text{MS}_E}$	$F[(a-1)(b-1)(c-1), abc(n-1); 1-\alpha]$

Factor	Type	Levels	Values
A	Fixed	3	1, 2, 3
B	Random	3	1, 2, 3
C	Random	2	1, 2

Analysis of Variance

Source	DF	Adj SS	Adj MS	F-Value	P-Value	
A	2	410.692	205.346	56.39	0.001	x
B	2	80.541	40.270	10.50	0.023	x
C	1	0.467	0.467	1.62	0.429	x
A*B	4	14.814	3.704	16.94	0.009	
A*C	2	0.314	0.157	0.72	0.542	
B*C	2	0.702	0.351	1.60	0.308	
A*B*C	4	0.875	0.219	0.75	0.566	
Error	54	15.850	0.294			
Total	71	524.255				

x Not an exact F-test.

Expected Mean Squares, using Adjusted SS

	Source	Expected Mean Square for Each Term
1	A	(8) + 4.0000 (7) + 12.0000 (5) + 8.0000 (4) + Q[1]
2	B	(8) + 4.0000 (7) + 12.0000 (6) + 8.0000 (4) + 24.0000 (2)
3	C	(8) + 4.0000 (7) + 12.0000 (6) + 12.0000 (5) + 36.0000 (3)
4	A*B	(8) + 4.0000 (7) + 8.0000 (4)
5	A*C	(8) + 4.0000 (7) + 12.0000 (5)
6	B*C	(8) + 4.0000 (7) + 12.0000 (6)
7	A*B*C	(8) + 4.0000 (7)
8	Error	(8)

Error Terms for Tests, using Adjusted SS

	Source	Error DF	Error MS	Synthesis of Error MS
1	A	3.84	3.6417	(4) + (5) - (7)
2	B	4.20	3.8358	(4) + (6) - (7)
3	C	0.97	0.2891	(5) + (6) - (7)
4	A*B	4.00	0.2187	(7)
5	A*C	4.00	0.2187	(7)
6	B*C	4.00	0.2187	(7)
7	A*B*C	54.00	0.2935	(8)

Type 3 Analysis of Variance								
Source	DF	Sum of Squares	Mean Square	Expected Mean Square	Error Term	Error DF	F Value	Pr > F
A	2	410.692500	205.346250	Var(Residual) + 4 Var(A*B*C) + 12 Var(A*C) + 8 Var(A*B) + Q(A)	MS(A*B) + MS(A*C) - MS(A*B*C)	3.8403	56.39	0.0014
B	2	80.540833	40.270417	Var(Residual) + 4 Var(A*B*C) + 8 Var(A*B) + 12 Var(B*C) + 24 Var(B)	MS(B*C) + MS(A*B) - MS(A*B*C)	4.2008	10.50	0.0232
C	1	0.467222	0.467222	Var(Residual) + 4 Var(A*B*C) + 12 Var(A*C) + 12 Var(B*C) + 36 Var(C)	MS(B*C) + MS(A*C) - MS(A*B*C)	0.9736	1.62	0.4286
B*C	2	0.701944	0.350972	Var(Residual) + 4 Var(A*B*C) + 12 Var(B*C)	MS(A*B*C)	4	1.60	0.3078
A*B	4	14.814167	3.703542	Var(Residual) + 4 Var(A*B*C) + 8 Var(A*B)	MS(A*B*C)	4	16.94	0.0090
A*C	2	0.313611	0.156806	Var(Residual) + 4 Var(A*B*C) + 12 Var(A*C)	MS(A*B*C)	4	0.72	0.5418
A*B*C	4	0.874722	0.218681	Var(Residual) + 4 Var(A*B*C)	MS(Residual)	54	0.75	0.5656
Residual	54	15.850000	0.293519	Var(Residual)	.	.	.	.