Non Parametric Finite Time Identification of Closed Loop Systems

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***Abstract*—In this paper, we develop a method to estimate the parameters of an unknown closed loop discrete time LTI system with feedback. This is accomplished using regularized least squares and a subspace algorithm for system identification. Under several assumptions on the underlying system dynamics we obtain non-asymptotic bounds on estimation error and demonstrate that our algorithm converges to the true parameters of the system over time. We show that even when the dimension of the underlying system is unknown, accurate reduced order approximations can be recovered. We also present extensions of these results to unstable open loop plants and more general multiple input multiple output systems. Simulations demonstrate the efficacy of this method for single input single output systems and its potential for use in adaptive control.**

I. Introduction

Linear time invariant (LTI) models are ubiquitous in engineered systems. Not only are they easily analyzed and manipulated but they can also be used to model and control a vast array of real world systems from circuits to cruise controllers and even complex astrodynamical systems [1]. In general, the first step in controller design is to obtain a model for a system of interest. This can be derived from first principals or identified through data. In the latter case a *system identification* algorithm is employed with collected measurements to estimate parameters that best represent the behavior of the system.

Unlike some learning problems, correlations appear in data collected from systems with internal dynamics leading to difficulties guaranteeing the convergence of estimation techniques, especially in finite time. Finite time system identification can be seen as an extension of traditional statistical learning formulations [2] and related to the asymptotic convergence of system identification methods studied in [3]. The first consideration of this problem was in [4] where the estimation of transfer functions and parameters of autoregressive models was discussed.

In the statistics literature, problems with correlated data can be dealt with by using mixing time arguments [5]. However, in our case, these only apply to a small class of LTI systems. More recent work [6] analyzes self-normalization to demonstrate the convergence of a least squares estimator of LTI system parameters. The authors in [7] show similar results using a mixing time argument and Mendelson's small-ball method.

The estimation of impulse responses in order to generate a realization of the system parameters using subspace identification techniques was considered in [8]–[10]. This is similar to [11] in which the authors attempt to both estimate and control using state feedback a LTI system with estimation error and regret guarantees. The problem of estimating the parameters of a system with feedback is approached in [12] where the innovation form of the linear dynamics are used to decouple the outputs from the noise. Our work is also related to the adaptive control algorithms designed in [13], [14].

This paper derives an algorithm that effectively estimates the parameters of an SISO LTI system in closed loop. In it, we formulate the system identification problem using regularized least squares and obtain bounds on the resulting estimators convergence. We then demonstrate how our estimates can be used to generate a realization of the parameters of the original system as in [10]. Additionally we consider the case of unstable open loop plants with stabilizing static output feedback and demonstrate that with observed white external excitation, the parameters of such a system can also be recovered and show that our results naturally extend to the MIMO case.

This problem is especially challenging since, in closed loop, inputs and noise become correlated and standard identification algorithms as in [8], [9] yield biased estimates. In this paper, careful analysis of statistical convergence shows that this bias can be overcome. Additionally, in contrast [12], [15] who estimate the Markov parameters of the so called "innovations form" of an LTI system, we attempt to explore the benefits and limitations of simply regressing outputs upon themselves without any external excitation. Finally, different from prior work we show that even without prior knowledge of the state dimension, accurate reduced order estimates can be generated in the presence of feedback, a novel non parametric identification result.

The rest of the paper is organized as follows. In section II we formulate a the least squares problem to estimate system parameters. In section III we show that the error in the estimator proposed in the previous section converges to zero as the number of samples increase and that the resulting estimated system parameters also approach their true values at the same rate. We demonstrate a method to estimate an unstable open loop system with stabilizing control. Section IV presents several simulation results demonstrating the efficacy of the algorithm on several systems. We conclude in section V.

II. Notation and Preliminaries

The standard discrete time LTI system is parametrized by a set of matrices A, B, C where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. Unless otherwise specified in this paper we assume that $p = m = 1$, a SISO system. We show that our analysis may be extended to the MIMO case in a straightforward way. The system dynamics are given as

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the latent state of the system, $y \in \mathbb{R}^p$ is the output and $w_t \in \mathbb{R}^n$, $v_t \in \mathbb{R}^p$ are the process and measurement noise. Let $\rho(A)$ be the spectral radius of a matrix A. The matrix A is Schur stable if $\rho(A) < 1$ in which case the associated system is stable. The Markov parameters of the system in Equation 1 are given by:

$$G = \begin{bmatrix} CB & CAB & CA^2B & CA^3B & \dots \end{bmatrix}. \tag{2}$$

In the absence of noise, the output of the open loop system can be directly calculated by convolving G with the sequence of inputs: $y_t = \sum_{i=1}^{t} G[i]u_{t-i}$. A system is said to be in closed loop with static output feedback if there exists a matrix $K \in \mathbb{R}^{m \times p}$ such that $u_t = Ky_t$.

Assumption II.1. *w is a vector of i.i.d. mean zero Gaussian random variables with covariance matrix $\sigma_w^2 I$ and v is a scalar mean zero Gaussian random variable with variance σ_v^2.*

When measurements of states x_i of the system are available, the least squares solution of

$$\min \sum_{t=0}^{T-1} ||x_{t+1} - Ax_t + Bu_t||_2^2$$
$$\text{subject to } A \in \mathbb{R}^{n\times n}, \;\; B \in \mathbb{R}^{n \times m}$$

given by $\left(X_-^\top X_-\right)^{-1} X_-^\top X_+$ where $X_-[i,:] = [x_i, u_i]$ and $X_+[i,:] = x_{i+1}$ has error given as $\left(X_-^\top X_-\right)^{-1} X_-^\top E$ where $E = [w_1, w_2, ..., w_T]^\top$. This term can be shown [6] to converge by showing that with high probability $||\left(X_-^\top X_-\right)^{-1/2} X_-^\top E||$ is constant and $||\left(X_-^\top X_-\right)^{-1/2}||$ converges to zero with the number of samples. This problem becomes more difficult without access to state measurements.

Consider an LTI system running in closed loop with static output feedback as in equation 1 where only the measurements $\{y_t\}$ are observed and K is known. Note that y_t can be expressed as a linear combination of past measurements y_j plus some noise term η_t, e.g.:

$$y_t = \eta_t + CBKy_{t-1} + CABKy_{t-2} + ... + CA^{t-1}BKy_0. \tag{3}$$

One might consider trying to solve a least squares problem for the Markov parameters G but since the terms y_t depend on increasingly many outputs y_j, this is ill defined. If the system A is stable, for some $r > 0$ we see that $y_t = \eta_t + \left(\sum_{j=0}^{r} CA^jBKy_{t-j}\right) + \gamma_{t,r}$ where the term $\gamma_{t,r}$ can be made arbitrarily small. This suggests that the solution $\hat{G}_r$ of

$$\min_{G \in \mathbb{R}^r} \sum_{t=0}^{T-1} ||y_t - GK\tilde{y}_{t-1,r}||_2^2 + ||\lambda G||_2^2 \tag{4}$$

converges to G_r as T grows for the proper selection of r and the regularization parameter λ. In equation 4 $\tilde{y}_{t-1,r} = [y_{t-1}, y_{t-2}, ..., y_{t-r}]^\top$ and G_r is a vector of the first r Markov parameters of the system. The regularized least squares solution is then given by $(Z_T + \lambda I)^{-1} S_T$ where

$$\begin{aligned} Z_T &= Y_T^\top Y_T \\ S_T &= Y_T^\top \begin{bmatrix} y_{r+1} & y_{r+2} & \dots & y_{r+T} \end{bmatrix}^\top \\ Y_T &= \begin{bmatrix} \tilde{y}_{r,r} & \tilde{y}_{r+1,r} & \dots & \tilde{y}_{T+r,r} \end{bmatrix}^\top. \end{aligned}$$

Unfortunately the error in the solution of the least squares problem above is no longer conditionally sub-Gaussian (since the conditional expectation is no longer zero) and the mechanics of [6], [16] can not be directly applied. However, through careful analysis of the error, convergence of the estimator can be guaranteed at a rate of $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$. The estimates $\hat{G}_r$ can then be used to construct a Hankel matrix which, by using subspace identification techniques [10] one obtains estimates $\hat{A}, \hat{B}, \hat{C}$. We show in this paper that these parameters can be estimated also at the rate $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$.

III. Closed Loop Parameter Estimation

In this section we consider our proposed estimator to show that with high probability it converges at a rate of $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$. We begin with several technical results that will be used to bound the error of the estimator given by optimization problem 4.

Proposition III.1. *Define $\tilde{y}_{t,r} = \begin{bmatrix} y_t & y_{t-1} & \dots & y_{t+1-r} \end{bmatrix}^\top$, and assume the system given by Equation 1 with feedback K is stable, $x_0 = 0$. Then for any $\epsilon > 0$, there exists a $r \in \mathbb{N}$ the dynamics can be written as the following vector autoregressive system:*

$$\tilde{y}_{t+1,r} = \begin{pmatrix} G[:r-1]K & G[r]K \\ I_{r-1} & \mathbb{0} \end{pmatrix} \tilde{y}_{t,r} + \begin{pmatrix} \eta_t + \gamma_{t,r} \\ \mathbb{0} \end{pmatrix} \tag{5}$$

where G are the Markov parameters of the open loop system, I_{r-1} is the $r-1^{th}$ dimensional identity matrix, η_t is a normal random variable, and $\gamma_{t,r}$ is a zero mean Gaussian random variable. Furthermore, the covariance $\mathbb{E}(\eta_j\gamma_{k,r}) \leq C\zeta^{|j-k|}$ for some constants $\zeta \in (0,1)$ and $C > 0$ dependent on A,B,C,K, σ_w^2, and σ_v^2.

Proof. Under feedback $u_t = Ky_t$ observe that $y_t = \sum_{i=1}^{t-1} CA^iBKy_{t-i} + Cw_{t-1} + v_t$. Define $\eta_t = Cw_{t-1} + v_t$ and note that η_t is a Gaussian variable with variance $||C||_2^2\sigma_w^2 + \sigma_v^2$. Hence we see that

$$y_t = \sum_{i=1}^{t-1} CA^iBKy_{t-i} + \eta_t. \tag{6}$$

we can then express the error term as the tail of the above sum such that

$$\gamma_{t,r} = \sum_{j=r}^{t-1} CA^j BKy_{t-j-1} = CA^r x_{t-r}$$
$$= CA^r \sum_{i=0}^{t-r} C(A+BCK)^i (w_{t-r-i} + BKv_{t-r-i})$$

where the first equality comes from the fact that $y_t = \sum_{j=0}^{r} CA^j BKy_{t-1} + CA^r x_{t-r}$ and the second from the fact that in closed loop the state x evolves as $x_m = \sum_{j=0}^{m}(A + BCK)^i(w_{m-i} + BKv_{m-i})$.

By construction of x and γ and since $A,\ A+BKC$ are stable, let $\zeta = \rho(A+BKC)$, there exists a constant $M > 0$ depending on $A,\ B,\ C,\ K,\ \sigma_w^2,\ \sigma_v^2$ such that

$$\mathbb{E}(\gamma_{t,r}^2) \le M\rho(A)^{2r} \sum \zeta^{2j} < M\rho(A)^{2r} \frac{1}{1-\zeta^2}$$

since we are taking the variance of the sum of Gaussian random variables with coefficients that decrease at a rate $\sim \zeta^k$. Furthermore, noting the structure of $\gamma_{t,r}$ it is clear that, selecting r from the preceding equation, that $\mathbb{E}(\eta_j \gamma_{k,r}) \lesssim \zeta^{|k-j|}\sigma_\eta^2$ and that $\mathbb{E}(\gamma_{j,r}\gamma_{k,r}) \lesssim \zeta^{|k-j|}\sigma_\eta^2$. This shows that $\mathrm{Var}(\gamma_{t,r}) \lesssim M\rho(A)^{2r}\frac{1}{1-\zeta^2}$ and that the off diagonal covariance terms scale at a rate of $\phi\zeta^{|k-j|}$ for some constant $\phi > 0$. Supposing that A is stable and that we redefine $\zeta = \max\{\rho(A), \rho(A+BKC)\}$, then noting that $\zeta < 1$, as long as $r > \log\left(M/(\epsilon(1-\zeta^2))\right)/2\log(1/\zeta)$, we see that $\mathbb{E}(\gamma_{t,r}^2) < \epsilon$.

□

In essence, this suggests that the evolution of the closed loop system with stable A can be well approximated by an autoregressive system with correlated noise γ that can be forced to be arbitrarily small. This formulation is useful to upper bound the matrix $\sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top$ which is important to bound the finite time convergence rate of the estimator. We would like to show for large enough T, with high probability:

$$c_{\min} TI \preceq \sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top \preceq c_{\max}(T+r)I. \tag{7}$$

We will prove both bounds below.

Definition III.2. *A system running in closed loop is said to be persistently exciting if there exists a $c_{min} \in \mathbb{R}_{\ge 0}$ such that for $T > T_0$ and any r, with probability at least $1-\delta$,*

$$c_{min} TI \preceq \sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top.$$

To demonstrate the lower bound in Equation 7 we first present a technical result from [17].

Theorem III.3. *(Theorem 2.1 in [17].) Given a linear system as in Equation 1 the transfer function satisfies $C(zI - A)^{-1}B = \frac{N(z)}{d(z)}$ for some polynomials $N(z)$, $d(z)$ where N has degree ν. Let N be the vector of coefficients of the polynomial N. Then the output vector is persistently exciting over an interval of time $[t_0-\nu, t_0+l]$ if $N^\top\tilde{u}_i$ is persistently exciting over $[t_0, t_0+l]$ where $\tilde{u}_i = [u_i, u_{i-1}, \dots, u_{i-\nu}]^\top$.*

Theorem III.4. *The closed loop system given in Equation 1 with feedback K has persistently exciting outputs $c_{min}TI \preceq \sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top$ for some c_{min}, $T > T_0$ with probability at least $1-\delta$.*

Proof. We can use the autoregressive state space description of the closed loop system given in III.1 for the dynamics of $\tilde{y}$ such that $\tilde{y}_{t+1,r} = A_{\text{autoreg}}\tilde{y}_{t,r} + \begin{bmatrix}1 & \mathbb{0}\end{bmatrix}^\top(\eta_t + \gamma_{t,r})$ where both η and γ are Gaussian random variables and

$$A_{\text{autoreg}} = \begin{pmatrix} G[:r-1]K & G[r]K \\ I_{r-1} & \mathbb{0} \end{pmatrix}.$$

But then we see that the coefficients N of the numerator of the transfer function of the above system is simply $N = \begin{bmatrix}1 & 0 & 0 & \dots & 0\end{bmatrix}$ in which case we see that $\tilde{y}$ are persistently exciting iff $N^\top\tilde{\eta} + \tilde{\gamma}$ are. However, $N'(\tilde{\eta}+\tilde{\gamma})$ are persistently exciting if $\sum(\eta_j + \gamma_{j,r})^2 > c_{\min}T$. Since $\eta_j + \gamma_{j,r}$ has strictly greater variance than η_j, this happens with higher probability than $\mathbb{P}(\sum \eta_t^2 > c_{\min}T)$. Since each η_t is a Gaussian random variable, $\eta_t^2 = \sigma_\eta^2 + e_t$ where e_t is a subexponential random variable. Using Bernstein's inequality applied to

$$\mathbb{P}\left(\left(\sum \eta_t^2\right)/T > c_{\min}\right) = \mathbb{P}\left(\sum e_t > Tc_{\min} - T\sigma_\eta^2\right)$$

we can see that with probability $1-\delta$, there exists an $c_{\min} > 0$ such that $\sum \eta_t^2 > c_{\min}T$. Bernstein's inequality also shows that $c_{\min}$ is a monotone increasing function of T, and hence that setting $c_{\min}$ corresponding to T_0, we obtain the result. □

We may now derive the upper bound in Equation 7. First we state a useful result from [18].

Lemma III.5. *Suppose $A, B \succeq 0$, then $tr(AB) \le \lambda_{max}(A)tr(B)$, $tr(AB) \le \lambda_{max}(B)tr(A)$.*

Theorem III.6. *Consider the LTI system given by Equation 1 such that $A+BKC$ is Schur stable. The matrix*

$$Z_T = \sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top \preceq c_{max}(T+r)I$$

with probability at least $1-\delta$ for some constant $c_{max} \in \mathbb{R}_{\ge 0}$.

Proof. We have that $\sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top \preceq \mathrm{tr}\left(\sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top\right) I$ and as such we will focus on bounding the trace term

$$\mathrm{tr}\left(\sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top\right) = \mathrm{tr}\left(\sum_{i=r}^{T+r} \tilde{y}_{i,r}^\top\tilde{y}_{i,r}\right) = \mathrm{tr}\left(\tilde{Y}^\top\tilde{Y}\right)$$

where $\tilde{Y} = \begin{bmatrix}\tilde{y}_{r,r} & \tilde{y}_{r+1,r} & \dots & \tilde{y}_{T+r,r}\end{bmatrix}^\top$. We aim to control the magnitude of $\mathrm{tr}\left(\tilde{Y}^\top\tilde{Y}\right)$.

Note that the states of the system $x_1, ..., x_{T+r}$ can be written as a function of noise using the stable feedback matrix $M = (A + BKC)$ and the expression

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{T+r} \end{bmatrix} = \begin{bmatrix} I & 0 & \dots & 0 \\ M & I & \dots & 0 \\ \vdots & \vdots & \ddots & \\ M^{t+r-1} & M^{t+r-2} & \dots & I \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_2 \\ \vdots \\ \xi_{T+r-1} \end{bmatrix}$$

where each ξ_i are Gaussian vectors given by $\xi_i = w_i + BKv_i$ with w_i and v_i mean zero Gaussian process and measurement noise respectively. For ease of notation, define write the concatenate the above equation in the form $\tilde{x} = \tilde{M}\tilde{\xi}$.

Then we have that, using $\tilde{x}$ we can calculate $\tilde{Y}$ by the linear map $\tilde{Y} = \tilde{C}\tilde{x}$ where $\tilde{C}$ is a $Tr \times Tn$ real valued matrix such that for $j \in \{1, ..., T\}$

$$\tilde{C}[jr : j(r+1), jn : nr + jn] = \begin{bmatrix} C & 0 & \dots & 0 \\ 0 & C & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C \end{bmatrix}$$

for any $j \in \{0, ..., T-1\}$ and 0 elsewhere.

Thus $\tilde{Y} = \tilde{C}\tilde{M}\tilde{\xi}$ and hence we would like to control the value

$$\begin{aligned} \text{tr}\left(\tilde{Y}^\top \tilde{Y}\right) &= \text{tr}\left(\tilde{\xi}^\top \tilde{M}^\top \tilde{C}^\top \tilde{C}\tilde{M}\tilde{\xi}\right) \\ &= \text{tr}\left(\tilde{C}\tilde{M}\tilde{\xi}\tilde{\xi}^\top \tilde{M}^\top \tilde{C}^\top\right) \end{aligned}$$

Where we used the cyclicity of the trace to arrive at the final equality. We will now use Markov's inequality as well as Lemma III.5 to demonstrate that with high probability

$$\text{tr}\left(\tilde{Y}^\top \tilde{Y}\right) \preceq c_{\max} T I.$$

Note that $\mathbb{E}\left(\tilde{\xi}\tilde{\xi}^\top\right)$ is a block diagonal matrix with blocks $\sigma_w^2 I_n + \sigma_v^2 BKK^\top B^\top$. In particular this means that

$$\lambda_{\max}\left(\mathbb{E}\left(\tilde{\xi}\tilde{\xi}^\top\right)\right) = \sigma_w^2 + \sigma_v^2 ||B||_2^2 ||K||_2^2.$$

Furthermore, we also have that $\tilde{C}^\top \tilde{C}$ is a block diagonal matrix with $\lambda_{\max}\left(\tilde{C}^\top \tilde{C}\right)$ bounded above by $r||C||_2^2$.

With all of this in place, taking the expectation of the expression above and using Lemma III.5 we get:

$$\begin{aligned} &\mathbb{E}\left(\text{tr}\left(\tilde{C}\tilde{M}\tilde{\xi}\tilde{\xi}^\top \tilde{M}^\top \tilde{C}^\top\right)\right) \\ &= \text{tr}\left(\tilde{C}\tilde{M}\mathbb{E}\left(\tilde{\xi}\tilde{\xi}^\top\right)\tilde{M}^\top \tilde{C}^\top\right) \\ &\leq \text{tr}\left(\tilde{M}\tilde{M}^\top\right)\left(r||C||_2^2\right)\left(\sigma_w^2 + \sigma_v^2 ||B||_2^2 ||K||_2^2\right). \end{aligned}$$

But due to the structure of $\tilde{M}$ we have that

$$\begin{aligned} \text{tr}\left(\tilde{M}\tilde{M}^\top\right) &= \text{tr}\left(\sum_{j=0}^{T+r} \Gamma_j(M)\right) \\ &\leq (T+r)\text{tr}\left(\Gamma(M)\right) \end{aligned}$$

where $\Gamma_T(A) = \sum_{j=0}^{T} (A^j)^\top A^j$ and $\Gamma(A) = \sum_{j=0}^{\infty} (A^j)^\top A^j$. Note that since $M = A + BKC$ is closed loop stable, $\text{tr}\left(\Gamma(M)\right)$ is a finite quantity and hence we have that

$$\begin{aligned} &\mathbb{E}\left(\text{tr}\left(\tilde{C}\tilde{M}\tilde{\xi}\tilde{\xi}^\top \tilde{M}^\top \tilde{C}^\top\right)\right) \\ &\leq (T+r)\text{tr}\left(\Gamma(M)\right)\left(r||C||_2^2\right)\left(\sigma_w^2 + \sigma_v^2 ||B||_2^2 ||K||_2^2\right). \end{aligned}$$

Finally, using Markov's inequality, we see that with probability at least $1 - \delta$,

$$\begin{aligned} &\sigma_{\max}\left(\sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top\right) \leq \\ &(T+r)\text{tr}\left(\Gamma(M)\right)\left(r||C||_2^2\right)\left(\sigma_w^2 + \sigma_v^2 ||B||_2^2 ||K||_2^2\right)/\delta \end{aligned}$$

proving the result.

□

We are now in a position to derive the convergence rate of the regularized least squares estimator given by Equation 4.

Theorem III.7. *Let $\hat{G}_T K$ be the solution to the regularized least squares problem given in Equation 4 for a stable open and closed loop system at time $T + r$. Suppose that the delay r satisfies $\mathbb{E}(\gamma_{t,r}^2) < \epsilon = \sqrt{r}/c_h T^2$ where $c_h = n\sqrt{2\left(\sigma_w^2 + ||B||_2^2 ||C||_2^2 |K|^2 \sigma_v^2\right)\log(1/\delta)}$. Then defining $Z_T = \lambda I + \sum_{i=r}^{T+r} \tilde{y}_{i,r}\tilde{y}_{i,r}^\top$ and noting that (by stability of A) the Markov parameters satisfy $||G||_F \leq S$ we have with probability at least $1 - \delta$ that the true G_T satisfies $\text{tr}\left((\hat{G}_T - G_T)Z_T(\hat{G}_T - G_T)\right) \leq \beta_T$*

$$\begin{aligned} &\beta_T = \\ &\frac{1}{K^2}\left(\sqrt{c_{var}\log\left(\frac{\sqrt{\lambda + c_{max}(T+r)}}{\delta\sqrt{\lambda}}\right)} + S\sqrt{\lambda} + \frac{\sqrt{r}}{T}\right)^2. \end{aligned}$$

where $c_{var} = ||C||_2^2(\sigma_w^2 + ||B||_2^2 |K|^2 \sigma_v^2)$

The proof of the above result is a straightforward extension of Theorem 3 in [15] applied to the term $\left(\hat{G}_r - G_r\right)K$. We then have

Theorem III.8. *The error in the estimate $\hat{G}_T$ scales as*

$$\begin{aligned} &||\hat{G}_T - G_T||_F \leq \\ &\frac{\left(\sqrt{c_{var}\log\left(\frac{\sqrt{\lambda + c_{max}(T+r)}}{\delta\sqrt{\lambda}}\right)} + S\sqrt{\lambda} + \frac{\sqrt{r}}{T}\right)}{|K|\sqrt{c_{min} T}} \end{aligned}$$

Proof. Using the result in Theorem III.7 and the fact that the closed loop system is persistently exciting we have that

$$\begin{aligned} &c_{\min} T ||\hat{G}_T - G_T||_F^2 \leq \\ &\frac{1}{K^2}\left(\sqrt{c_{\text{var}}\log\left(\frac{\sqrt{\lambda + c_{\max}(T+r)}}{\delta\sqrt{\lambda}}\right)} + S\sqrt{\lambda} + \frac{\sqrt{r}}{T}\right)^2. \end{aligned}$$

Taking square roots and dividing by $\sqrt{c_{\min}T}$ gives the desired result. $\square$

Note that as T grows, r stays fixed resulting in an error rate of approximately $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$. We then use the estimate $\hat{G}_r$ to construct a Hankel matrix and perform the Ho-Kalman algorithm to obtain estimates of the original system parameters. The following two results describe the convergence of these estimates. In the first, the dimension of the system n is unknown, in the second it is assumed known.

Corollary III.8.1. *Suppose that (A_d, B_d, C_d) are the parameters of the order d balanced truncation of the original system obtained through the Ho-Kalman algorithm and that $(\hat{A}_d, \hat{B}_d, \hat{C}_d)$ is the approximation achieved through the Ho-Kalman algorithm and the estimate of $\hat{G}$ using regularized least squares. Then for $T > T_0$ and some unitary matrix $U \in \mathbb{R}^{d\times d}$ we have*

$$||C_dU - \hat{C}_d||_2 + ||U^\top A_dU - \hat{A}_d||_2 + ||U^\top B_d - \hat{B}_d||_2$$
$$\lesssim d\frac{1}{|K|}\sqrt{\frac{1}{T}\left(d+\log\left(\frac{T}{\delta}\right)\right)}$$

with probability at least $1-\delta$.

This result follows from theorem 13.8 and theorem 5.3 in [8] where the Frobenius norm of the Hankel matrix used in the Ho-Kalman algorithm can be bounded by a term of order $\mathcal{O}(1/\sqrt{T})$. The constant T_0 depends on the structure of the original system.

Corollary III.8.2. *Let $(\hat{A}, \hat{B}, \hat{C})$ be the estimates of the dimension n LTI system derived from the Ho-Kalman algorithm using the estimate $\hat{G}_T$. Then for $T > T_0$ there exists a unitary matrix $U \in \mathbb{R}^{n\times n}$ such that*

$$||\hat{A} - U^\top AU||,\ \ ||\hat{B} - U^\top B||,\ \ ||\hat{C} - CU|| \lesssim ||\hat{G}_T - G_T||$$

where $||\hat{G}_T - G_T|| \sim \frac{\log(T)}{\sqrt{T}}$.

The proof of this result is a corollary to the results in [9].

Before moving on, it should be noted that in the above algorithm, adding independent subgaussian inputs $\{u'_t\}$ in addition to the closed loop input $u_t = Ky_t$ does not effect the rate of estimation. This is because it we can calculate a similar lower and upper bound, $c_{\min}$ and $c_{\max}$ since the empirical covariance matrix can then be calculated as $\sum(K\tilde{y}_i+\tilde{u}_i)(K\tilde{y}_i+\tilde{u}_i)^\top$. Not only does external noise not impact our previous convergence results, it can also enhance the effectiveness of estimation.

These novel results require only measured outputs to estimate the parameters of a LTI system in closed loop. However, they rely on the fact that the underlying open loop system is stable. In practice most systems employ feedback for its stabilizing properties and similar results for unstable open loop systems with stabilizing feedback are desirable.

Theorem III.9. *Consider the system given by Equation 1 with multiple inputs and outputs and with stabilizing static output feedback K and a random sequence of inputs $\{u_j\}_{j=0}^\infty$ where each u_j is an independent Gaussian with mean zero and variance σ_u^2. Using measurements $\{u_j\}$ and outputs $\{y_j\}$, the parameters of the system A, B, C can be estimated (up to similarity transformation) with error of order $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$.*

Proof. The system response from the inputs $\{u_j\}$ to outputs $\{y_t\}$ can be described by the linear system

$$\begin{aligned} x_{t+1} &= (A + BKC)x_t + Bu_t + w_t \\ y_t &= Cx_t + v_t \end{aligned} \tag{8}$$

with corresponding Markov parameters

$$\begin{bmatrix} CB & C(A+BKC)B & C(A+BKC)^2B & \dots \end{bmatrix}^\top.$$

Define $A_{\text{cl}} = A + BKC$. Then with independent Gaussian inputs $\{u_j\}_{j=0}^\infty$ we may use the algorithm in [8], using least squares to estimate a Hankel made up of the above Markov parameters and applying the Ho-Kalman subspace realization method, to show that we can generate estimates: $\hat{A_{\text{cl}}}, \hat{B}, \hat{C}$ of A_{cl}, B, C respectively. For $T > T_0$ samples, where T_0 depends on the underlying system, the estimates satisfy $\hat{A_{\text{cl}}} - U^{\text{cl}}U = E_{A_{\text{cl}}},\ \hat{B} - U^\top B = E_B,\ \hat{C} - CU = E_C$

for some unitary matrix U and where each error term $E_{A_{\text{cl}}}$, E_B, $E_C \lesssim \sqrt{\frac{\log(T/\delta)}{T}}$ with probability at least $1-\delta$.

Since we know the value of K, we can then estimate A by simply taking $\hat{A} = \hat{A_{\text{cl}}} - \hat{B}K\hat{C}$. We can easily see that $\hat{A}$ again has error that scales as $\sqrt{\frac{\log(T/\delta)}{T}}$ by noting that

$$\begin{aligned} &||U^\top AU - \hat{A}||_2 = ||U^\top AU - \hat{A_{\text{cl}}} + \hat{B}\hat{C}K||_2 \\ &= ||U^\top AU - U^\top(A_{\text{cl}})U + U^\top E_{A_{\text{cl}}}U \\ &+ U^\top(B + E_B)(C + E_C)KU||_2 \\ &\le ||E_{A_{\text{cl}}}||_2 + ||B||_2||E_C||_2||K||_2 \\ &+ ||C||_2||E_B||_2||K||_2 + ||E_B||_2||E_C||_2||K||_2 \end{aligned}$$

where the inequality follows from the submultiplicativity of of the induced 2-norm. But note that each of the terms in the above sum scale as $\lesssim \sqrt{\frac{\log(T/\delta)}{T}}$ with probability at least $1-\delta$ for $T > T_0$ and thus we must have that

$$||U^\top AU - \hat{A}||_2 \lesssim \sqrt{\log(T/\delta)/T}.$$

$\square$

This gives us another way to estimate a system in closed loop with static output feedback even when the original open loop system is not stable. Note that this result assumes that the system dimension n is known a priori. If not, it is unclear whether the reduced order estimate $\hat{A}_{\text{cl}\ell}$ corresponds to the matrix $A_\ell + B_\ell C_\ell K$ where A_ℓ, B_ℓ, C_ℓ are optimal reduced order parameters of dimension $\ell < n$. This and the fact that random inputs are required are limitations of this otherwise very powerful method for isolating and estimating a LTI system in feedback. This result also follows for MIMO systems. Additionally, we have with no external forcing the following corollary.

Corollary III.9.1. *The error in the estimate $\hat{G}_T$ obtained through regularized least squares on the MIMO closed loop system with static output feedback scales as*

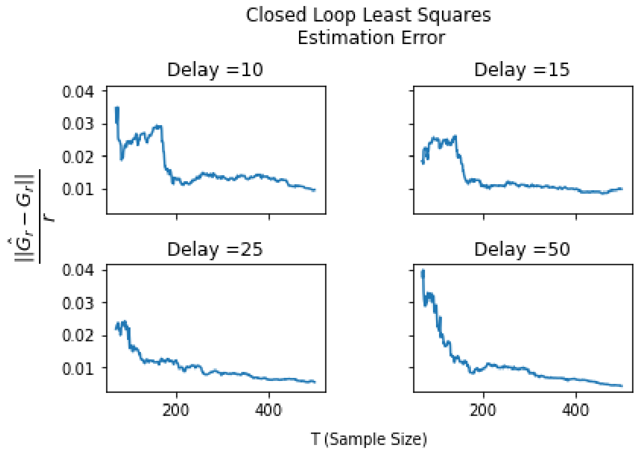

Fig. 1. Normalized estimation error for a given stable LTI system using different delays r. Initially, estimation for longer delays produces more error, but as more data becomes available, more Markov parameters allow for increasingly accurate estimation as $\gamma_{t,r}$ add bias into the estimate for smaller values of r. All error rates decay at a rate on the order of $\sim \mathcal{O}\left(\log(T)/\sqrt{T}\right)$.

$$||\hat{G}_T - G_T||_F \lesssim \left(\sqrt{m \log\left(\frac{\sqrt{\lambda + c_{max}(T+r)}}{\delta\sqrt{\lambda}}\right)}\right) / \left(\sigma_{min}(K)\sqrt{c_{min}T}\right)$$

such that m is the input dimension such that $B \in \mathbb{R}^{n\times m}$. The proof of this again follows from [15] and the fact that, since K and y are known, the input U is also known, and since $||\hat{G}_T K - G_T K||_F \geq ||\hat{G}_T - G_T||_F \sigma_{\min}(K)$.

In this section, we show that a regularized least squares estimator of system parameters converges in finite time. This estimate can then be used as an input to the Ho-Kalman algorithm to arrive at estimates of the original parameters of the system.

IV. Simulation Results

In order to test the effectiveness of the system identification algorithm, random stable systems were generated which then ran in closed loop with different delays and different levels of noise. Data was collected from these systems over time with different levels of delay r and over differing amounts of time T. Over time it is clearly seen in Figure 1, while different values of delay can alter the convergence rate, this technique, given a stable A and stable closed loop system can effectively estimate the Markov parameters of the system over time.

Our experiments suggest that the selection of r and T should depend on the convergence rate calculation derived in the previous section. It should also be noted that the bound from theorem III.8 is also generally not tight for small values of T. This suggest that a tighter bound could be derived, in particular for systems in which there is good understanding of $c_{\min}$ and $c_{\max}$ in Equation 7.

V. Conclusion

In this paper, we analyzed a method to estimate the parameters of a closed loop LTI system using output measurements. Our method does not require multiple independent trajectories as many system identification algorithms do and can thus use data efficiently. Moreover, through a novel analysis regularized least squares problem in this domain, we conclude that the parameters of the system can be estimated at a rate proportional to $\mathcal{O}\left(\log(T)/\sqrt{T}\right)$. All of this analysis is done without estimating an intermediate observer as in [12]. In our case, the original system description can be directly used in the least squares problem. Our results suggest that $\rho(A) < 1$ is necessary to show the convergence of the least squares problem, however it follows that an unstable open loop system with stabilizing feedback should also have decaying error. Future work will explore the limitations of learning a system from only collected outputs. Other extensions include estimating the parameters of a system under stabilizing dynamic linear feedback and combining our results to generate confidence sets around parameters along with with robust control synthesis methods as in [13].

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