CS5016: Computational Methods and Applications

Partial Differential Equations and Finding Roots

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What is an PDE?

An equation involving one or more derivatives of an unknown function.

If all derivatives are taken with respect to a several independent variable we get an **partial differential equation**.

The well-known 1-D heat equation

$$\frac{\partial u(x,t)}{\partial t} - \mu \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in (a,b), t > 0$$

where $\mu > 0$ is the coefficient representing thermal diffusivity.

Boundary value problem

Differential equations in an open multidimensional region $\Omega \subset \mathbb{R}^d$ for which the value of the unknown solution (or its derivatives) is prescribed on the boundary $\partial\Omega$ of the multidimensional region.

$$\frac{\partial u(x,t)}{\partial t} - \mu \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t), \quad x \in (a,b), t > 0$$

with the initial condition

$$u(x,0) = g(x) \quad \forall x \in [a,b]$$

and boundary condition

$$u(a,t) = u(b,t) = 0 \quad \forall t > 0$$

Approximation by finite differences

Consider the following approximation for $h > 0$

$$\frac{\partial u(x,t)}{\partial x} \approx g(x,t) = \frac{u(x+h/2,t) - u(x-h/2,t)}{h}$$

Then, we have

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial x^2} &\approx \frac{g(x+h/2,t) - g(x-h/2,t)}{h} \\
&\approx \frac{u(x+h,t) - u(x,t)}{h^2} - \frac{u(x,t) - u(x-h,t)}{h^2} \\
&= \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}
\end{aligned}$$

Approximation by finite differences

Let us partition the interval $[a, b]$ into interval $I_j = [x_j, x_{j+1}]$ of length h for $j = 0, 1, \ldots, N$ with $x_0 = a$ and $x_N = b$.

Let $u_j(t)$ denote an approximation of $u(x_j, t)$, $j \in \{0, \ldots N\}$. Then, for all $t > 0$, we should have $\forall j \in \{1, \ldots, N-1\}$

$$\frac{du_j(t)}{dt} - \frac{\mu}{h^2}(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)) = f_j(t)$$

with initial condition $u_0(t) = 0$ and $u_N(t) = 0$, $f_j(t) = f(x_j, t)$, and $u_j(0) = g(x_j)$

Giving us the following system of ODE

$$\frac{d\boldsymbol{u}(t)}{dt} = \frac{\mu}{h^2}\boldsymbol{A}\boldsymbol{u}(t) + \boldsymbol{f}(t)$$

with $\boldsymbol{u}(0) = \boldsymbol{g}$

Handling problem with 2 spatial dimensions

We have $u(\boldsymbol{x}, t)$, where $\boldsymbol{x} \in \mathbb{R}^2$. The heat equation is given as

$$\frac{\partial u(\boldsymbol{x}, t)}{\partial t} - \mu \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_1^2} - \mu \frac{\partial^2 u(\boldsymbol{x}, t)}{\partial x_2^2} = f(\boldsymbol{x}, t) \quad \forall x \in \Omega$$

with initial condition $u(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ and boundary condition $u(\boldsymbol{x}, t) = 0 \, \forall \boldsymbol{x} \in \partial\Omega, t \geq 0$

Approximate Ω as a grid of points such that $x_{1,i} = x_{1,0} + ih$ and $x_{2,j} = x_{2,0} + jh$

Figure out approximations for $\frac{\partial^2 u(\boldsymbol{x},t)}{\partial x_1^2}$ and $\frac{\partial^2 u(\boldsymbol{x},t)}{\partial x_2^2}$, and solve the resultant system of ODE.

Root-finding algorithms

In mathematics and computing, a root-finding algorithm is an algorithm for finding **zeroes**, also called "**roots**", of continuous functions[1].

A zero of a function $f : \mathbb{R} \to \mathbb{R}$, is a number x such that $f(x) = 0$. As, generally, the zeroes of a function cannot be computed exactly nor expressed in closed form, root-finding algorithms provide approximations to zeroes.

Most root-finding algorithms do not guarantee that they will find all the roots; in particular, if such an algorithm does not find any root, that does not mean that no root exists.

[1] https://en.wikipedia.org/wiki/Root-finding_algorithms

The bisection method

Consider a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an interval $[a, b]$. If $f(a) \cdot f(b) <= 0$, then function f has at least one zero/root in the interval $[a, b]$, i.e., there exists a point $x^* \in [a, b]$ such that $f(x^*) = 0$.

Algorithm 1 Pesudocode

```
while |a − b| > ϵ do
    let c = (a + b)/2
    if sgn(f(c)) == sgn(f(a)) then
        a = c
    else
        b = c
    end if
end while
return (a + b)/2
```

What is the run-time complexity of the above algorithm?

Newton-Raphson method

Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$. If the f satisfies sufficient assumptions and the initial guess x_0 is close, a root can be found using the following iterative method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

What are the conditions that f should satisfy?

If $F : \mathbb{R}^k \to \mathbb{R}^k$ multivariate vector-valued function, then the iteration is given by

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{J}(\boldsymbol{x}_k)^{-1}\boldsymbol{F}(x_k)$$

where $\boldsymbol{J}(\boldsymbol{x}_k)$ is the **Jacobian matrix** of $\boldsymbol{F}$.

More root-finding methods

The SciPy module `scipy.optimize` offers methods to find zeros/roots of function. To know more visit `https://docs.scipy.org/doc/scipy/reference/tutorial/optimize.html#root-finding`

Thank You