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Ordinary differential Equations

ODE ⇒ (i) An equation involving one or more derivative of an unknown function.

(ii) if all derivative are taken ~~under~~ wrt single independent variable we get an ordinary differencial equation.

Ex. $$\frac{d(x(t))}{dt} = -x(t) \quad \text{---(1)}$$

Maximum order of differenciation in equation is Maximum order of differencial equation. Known as order of differencial equation.

above example has order one.

q. varify that $x(t) = e^{-t}$ satisfies above ODE.

$$\frac{d(x(t))}{dt} = \frac{d(e^{-t})}{dt} = -e^{-t}$$

$$\therefore -e^{-t} = -e^{-t} \quad \text{.... (From (1))}$$

$$\therefore LHS = RHS.$$

Prey-Predator equations:-

The Lokta-Valterra model assumes that the prey consumption rate by predator is directly proportional to the prey abundance. means predator feeding is limited only by the amount of prey in enviroment.

the Lokta-volterra equations (predator-prey equation) (2)
are pair of first-order ~~equations~~ non-Linear ~~equations~~ differencial equations, frequently used to describe the dynamic of biological systems in which two species interact, one as a predator and other as prey. the population change through time according to ~~of~~ the pair of equations,

$$\frac{d x(t)}{dt} = \alpha x(t) - \beta x(t) y(t)$$

$$\frac{d y(t)}{dt} = \delta x(t) y(t) - \gamma y(t).$$

$x(t)$ and $y(t)$ denotes no. of prey and predator at time t.

* An ODE of order $p>1$ can always be reduced to a system of p equations of order 1.

9. We have, $u(t) = \frac{d x(t)}{dt}$

$$\therefore v(t) = \frac{d(u(t))}{dt} = \frac{d^2 x(t)}{dt^2}$$

we have, $u(t) + x(t) v(t) + 3 x(t)^2 \frac{d v(t)}{dt} - 4 x(t)^3 = 0$

$$\therefore \frac{d x(t)}{dt} + x(t) \frac{d^2 x(t)}{dt^2} + 3 x(t)^2 \frac{d^3 x(t)}{dt^3} = 4 x(t)^3$$

$\therefore$ which is equal to given equation,

Hence proved.

~~Cauchy problem~~

An ODE in general ~~a~~ admits infinites number of solutions.

For Ex. $\frac{d(x(t))}{dt} = -x(t)$ admits solution $x(t) = ce^{-t}$,

where c an arbitary constant.

if we impose condition $x(0) = 2$, we get unique sol^n

$x(t) = 2e^{-t}$.

Caushy problem:-

Find $x : I \to R$ such that

$x'(t) = f(t, x(t)) \quad \forall t \in I$ and $x(t_0) = x_0$

where I is an interval of R.

We say that caushy problem has unique ~~an~~ solution if it has exactly one global solution in the sense that there exists a ~~unique~~ solution $\tilde{y} : \tilde{I} \to \mathbb{R}^n$, such that for every other solution $y : I \to \mathbb{R}^n$ we have $I \subseteq \tilde{I}$ and $y = \tilde{y}$ on I.

Explicit and Implicit Solutions:-

The ODE ~~dx~~ $\frac{d(x(t))}{dt} = -x(t)$ has explicit solution $x(t) = ce^{-t}$ i.e. x can be written as function of t.

Q. consider the following ODE, $\frac{d(x(t))}{dt} = \frac{(x(t) - t)}{(x(t) + t)}$

Show that following satisfies given ode.

$$\frac{1}{2} \ln(t^2 + x(t)^2) + \tan^{-1}\frac{x(t)}{t} = c.$$

Let, $x(t) = y$.

$\therefore \frac{1}{2}\ln(t^2+y^2) + \tan^{-1}\left(\frac{y}{t}\right) = c$

$\therefore \frac{1}{y^2+t^2}\left(t + y\frac{dy}{dt}\right) + \frac{1}{1+\frac{y^2}{t^2}}\left(\frac{t\frac{dy}{dt} - y}{t^2}\right)$

$$\therefore \frac{t + y\frac{dy}{dt}}{y^2+t^2} + \frac{t\frac{dy}{dt} - y}{y^2+t^2} = 0$$

$$\therefore t + y\frac{dy}{dt} = y - t\frac{dy}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{y-t}{y+t}$$

$$\therefore \frac{d\,x(t)}{dt} = \frac{x(t) - t}{x(t) + t}$$

Hence proved.

Euler Methods :

Subdivide integration interval $I = [t_0, T]$ with $T < \infty$ into N_H intervals of length $h = \frac{(T_0 - t_0)}{N_h}$; h is called discretization step.

At each t_n, $n \in \{0, 1, \ldots, N_H - 1\}$ we seek unknown value x_n that approximate $x(t_n)$. The set of values $\{x_n\}_{n=0}^{N_h-1}$ is our numerical solution.

Forward Euler method, $x_{n+1} = x_n + h\, f(t_n, x_n)$
$\forall n \in \{0, 1, \ldots N_h - 1\}$

Backward Euler method, $x_{n+1} = x_n + h\, f(t_{n+1}, x_{n+1})$
$\forall n \in \{0, 1, \ldots, N_h - 1\}$

Consider ODE, $\frac{d(x(t))}{dt} = -x(t)^4$

FEM gives, $x_{n+1} = x_n - h\, x_n^4$ ... (Explicit Expr)

BEM gives, $x_{n+1} = x_n - h.\, x_{n+1}^4$

i.e. x_{n+1} should be real root of polynomial,

$$y^4 - \frac{y}{h} - \frac{x_n}{h} = 0 \quad \text{... (Implicit Expr)}.$$

* Implicite methods are more stable than explicit methods beacause they have no contraint on the time step size and can handle stiff and non-linear system more effectively. This is due to their use of iterative methods to solve for future state, which allows for a larger time step size and better properties.

Stability on unbounded intervals:-

$x'(t) = \lambda\, x(t) \quad \forall t \in (0, \infty)$ and $x(0) = 1$

it is easy to check that $x(t) = e^{\lambda t}$ is exact solution.

if $\lambda < 0$ then $\lim_{t \to \infty} x(t) = 0$

Forward euler method. with $x_0 = 1$ gives

$$x_{n+1} = x_n(1 + \lambda h) = (1 + \lambda h)^n \quad \forall n \geq 0$$

$\lim_{n \to \infty} x_n = 0$ only if $h \in (0, \frac{2}{|\lambda|})$

Backward euler method gives at $x_0 = 1$,

$$x_{n+1} = \frac{x_n}{1 - \lambda h} = \frac{1}{(1 - \lambda h)^n} \quad \forall n \geq 0$$

as $n \to \infty$ when $y_n \to 0$. This is true if (6)

$\left| \frac{1}{1-h\lambda} \right| < 1$. The hypothesis that $\lambda < 0$ ~~or Re(λ) < 0~~

is sufficient to show this.

<u>System of ODE's :-</u>

Consider the following system of first order ODE's with unknowns $x_1(t), \ldots, x_m(t)$

$$x'(t) = f_1(t, x_1(t), \ldots, x_m(t))$$

$$\vdots \qquad\qquad \vdots$$

$$x'_m(t) = f_m(t, x_1(t), \ldots, x_m(t))$$

where $t \in [t_0, T]$ with initial conditions, $x_1, x_2, \ldots x_m$.

~~let write~~ We can write above system of ODEs as $x'(t) = F(t, x(t))$

and we can apply any of method used to solve cauchy problem to solve this system of equations.