Name : Dalvi Mangesh Ramdas

Roll No : 112001010

Least - square function approximation.

Interpolation vs curve fitting:

In function interpolation our goal was to find a function f that fits some given points $\{ (x_i, y_i) = 1, 2, \dots, m \}$

Assume we have $n+1$ points $(x_1, y_1), \dots, (x_{n+1}, y_{n+1})$. (given $x_i \neq x_j$ if $i \neq j$). A polynomial of degree n is of the form $P_n(x) = a_n x^n + \dots + a_1 x + a_0$. To study the existence and uniqueness of such polynomial consider the system of linear equations,

$$y_1 = a_0 + a_1 x_1 + \dots + a_n x_1^n$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$y_{n+1} = a_0 + a_1 x_{n+1} + \dots + a_n x_{n+1}^n$$

We ~~would~~ write system as

$$\begin{pmatrix} x_1^n & x_1^{n-1} & \dots & 1 \\ \vdots & & & \vdots \\ x_{n+1}^n & x_{n+1}^{n-1} & \dots & 1 \end{pmatrix} \begin{pmatrix} a_n \\ \vdots \\ a_0 \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n+1} \end{pmatrix}$$

since matrix of coefficients of the system is non-singular (by fact vandermonde matrix) the system have unique solution, that ~~e~~ there exist one polynomial of degree n through the $(n+1)$ given points and it is unique.

Discarding the points that is most different from other points is good idea.

Paramaterized Functions :- and Measure of fitness :-

Let f_θ be a function paramaterized ~~function~~ θ, which can be scalar, vector fitness OR countable sequence,

$$f_\theta(x) = \sin(\theta x) \quad \text{or} \quad f_\theta(x) = \sum_{i=0}^{k} \theta_i x^i$$

Natural measure of fit,

$$\sum_{i=1}^{M} \left(y_i - f_\theta(x_i)\right)^2 \quad \ldots \quad (\text{LSMA})$$

We can find best fit ~~line~~ curve as follows,

$$\min_{\theta} \sum_{i=1}^{M} \left(y_i - f_\theta(x_i)\right)^2$$

for best fit ~~curve~~ line,

$$\min_{a_0, a_1} \sum_{i=1}^{m} \left[y_i - (a_0 + a_1 x)\right]^2$$

We want to minimize,

$$E = \sqrt{\frac{1}{n} \sum_{i=1}^{n} \left[f(x_i) - y_i\right]^2}$$

when considering all possible functions,

$f(x) = mx + b$,

Note that minimizing E is equivalent to ~~mini~~ minimizing sum, ~~althoug~~ i.e. to minimize

$$G(b, m) = \sum_{i=1}^{n} \left[m x_i + b - y_i\right]^2$$

as b and m are allowed vary arbitarily

$$\therefore \frac{\partial G(b,m)}{\partial b} = 0 \quad , \quad \frac{\partial G(b,m)}{\partial m} = 0$$

use

$$\frac{\partial G}{\partial b} = \sum_{i=1}^{n} 2 [mx_i + b - y_i]$$

$$\frac{\partial G}{\partial b} = \sum_{i=1}^{n} 2 [mx_i + b - y_i] x_i = \sum_{i=1}^{n} 2 [mx_i^2 + bx_i - x_i y_i]$$

This leads to Linear equation,

$$nb + \left(\sum_{i=1}^{n} x_i\right) m = \sum_{i=1}^{n} y_i$$

in ~~other words~~:

$$\therefore \left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right) m = \sum_{i=1}^{n} x_i y_i$$

∴ In other words, $a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i$

⇒ equation for best fit polynomial will be of form,

$$\min_{a_0, a_1 \ldots a_n} \sum_{i=1}^{m} \left[y_i - \sum_{j=0}^{n} a_j x^j \right]^2$$

can be proved using similar methode above mentioned.

* Least-square approximations ~~using~~ of a function using monomial polynomials:-

Given a function $f(x)$ continous $[a,b]$ find a polynomial $P_n(x)$ of degree at most n.

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

such that integral of square is minimized,

$$\min_{a_0 \ldots a_n} \int_a^b (f(x) - P_n(x))^2 dx$$

We would need to solve following equations

$$\sum_{k=0}^{n} a_k \cdot \int_a^b x^{j+k} dx = \int_a^b x^j f(x) dx \quad \forall j \{0, 1, \ldots n\}$$

we have,

ILL Conditioned Matrix:-

The Normal equations have following form,

$$Sa = b$$

Every matrix can be written in SVD form,

$$M = U\Sigma V^T$$ where Σ ~~for~~ is diagonal entries having singular values.

condition number is matrix ratio to max. Σ to min Σ.

A matrix is said to be ill-conditioned if condition number is large.

We can use such matrix for finding error when solving system of Linear equations.

orthogonal functions:-

A set of functions $\{\phi_1, \phi_2, \dots, \phi_n\}$ in $[a,b]$ are called orthogonal functions wrt $w(x)$ if,

$$\int_a^b w(x)\, \phi_i(x)\, \phi_j(x)\, dx = \begin{cases} 0 & \text{if } i \neq j \\ c_j & \text{if } i = j \end{cases}$$

where c_j is +ve number. if $c_j = 1$ $\forall j$ then set is called orthonormal set.

using orthogonal functions:-

we are interested in ~~LSA~~ LSEA of $f(x)$ on $[a,b]$ by means of polynomial form,

$$\cancel{Q_n(x) = \sum_{i=0}^{b} a_i \phi_i} \quad Q_n(x) = \sum_{i=0}^{n} a_i \phi_i(x)$$

where $\{\phi_i\}_{i=0}^{n}$ is a set of orthogonal polynomial $[a,b]$ s.t

$$\min_{a_0, a_1, \dots a_n} \int_a^b w(x) \cdot (f(x) - Q_n(x))^2\, dx.$$

Using orthogonal functions,

$$\int_a^b w(x)\, \phi_j(x)\, f(x)\, dx = \int_a^b w(x)\, \phi_j(x) \left(\sum_{i=0}^{n} \phi(x)\right) dx = c_j a_j$$

or we have,

$$a_j = \frac{1}{c_j} \int_a^b w(x)\, \phi_j(x)\, f(x)\, dx \quad \forall j$$

Where, $c_j = \int_a^b w(x)\, \phi_j^2\, dx$.

Lengendree Polynomial

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

The above polynomial is orthogonal in $[-1, 1]$ wrt weight function $w(x) = 1$.

$$L_1(x) = \frac{1}{2^1\, 1!} \frac{d^1}{dx^1} (x^2-1)^1 = x$$

$$L_2(x) = \frac{1}{8}(12x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$L_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

consider, $L_1(x) \cdot L_2(x)$ we get $\left(\frac{3}{2}x^3 - \frac{1}{2}x\right)$ ~~we~~ ve by itegrating this function in range $[-1, 1]$ ~~even function~~. will get 0.
~~gete as it will ge~~

Chebyshev Polynomial:-

$$T_n(x) = \cos(n \cos^{-1}(x))$$

is $T_n(x)$ is really polynomial, in fact we have, $T_0(x) = 1$ and $T_1(x) = x$,

further we have, $T_{n+1}(x) = 2x\, T_n(x) - T_{n-1}(x)$

Recall trigonometric addition formulas,

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

Let $n \geq 1$ apply these identities to get,

i) $T_{n+1}(x) = \cos[(n+1)\theta] = \cos(n\theta + \theta)$
$= \cos(n\theta)\cos\theta - \sin(n\theta)\sin(\theta)$

ii) $T_{n-1}(x) = \cos[(n-1)\theta] = \cos(n\theta - \theta)$
$= \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin\theta$

Add, above two equations, we will get

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\theta)\cos(\theta) = 2x\,T_n(x)$$

Hence, $T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)$, $n \geq 1$

Fourier Series:-

For any positive integer n, the set of functions $\{\cos(0), \cos(x), \dots \cos(nx), \sin(0), \dots \sin(nx)\}$ is orthogonal in interval $[-\pi, \pi]$ wrt weight function $w(x) = 1$.

for any two $\cos(ax) \cdot \sin(bx)$ $\forall a, b$ we have,
We have, $\cos(ax)\sin(bx) = \frac{1}{2}(\sin(ax+bx) - \sin(ax-bx))$

On Integration, $= \int_0^{2\pi} \cos(ax)\sin(bx)\,dx$

$$= \int_0^{2\pi} \frac{1}{2}(\sin(ax+bx) - \sin(ax-bx))$$

$$= \left[\frac{\cos(a-b)x}{2(a-b)} - \frac{\cos(a+b)x}{2(a+b)}\right]_0^{2\pi} = 0$$

Let, $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx)$

Such that least square is minimized i.e.

$$\min \int^{\pi} (f(x) - S_n(x))^2\,dx$$

Equating partial derivatives to zero,
due to orthogonality we get,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad , \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

Suppose we have $2m$ data points x_k, y_k ~~we have,~~
where,

$$x_k = -\pi + \frac{k\pi}{m} \quad \text{and} \quad y_k = f(x_k),\ k = \{0,1,\dots,2m-1\}$$

The discrete least square trigonometric polynomial does the following,

$$\min \sum_{k=0}^{2m-1} \left(S_n(x_k) - y_k\right)^2$$

<u>DFT</u> if r is not multiple of $2m$

$$\sum_{k=0}^{2m-1} \cos(r x_k) = \sum_{k=0}^{2m-1} \sin(r x_k) = 0$$

if $r \neq 0$ is not multiple of m,

$$\sum_{k=0}^{2m-1} [\cos(r x_k)]^2 = \sum_{k=0}^{2m-1} [\sin(r x_k)]^2 = m$$

if $r \neq l$ and $r+l$ is not a multiple of $2m$,

$$\sum_{k=0}^{2m-1} \cos(r x_k)\cos(l x_k) = \sum_{k=0}^{2m-1} \sin(r x_k)\sin(l x_k) = 0$$

$$\sum_{k=0}^{2m-1} \cos(r x_k)\sin(l x_k) = \sum_{k=0}^{2m-1} \sin(r x_k)\cos(l x_k)$$

Then for any $n < m$ the best approximation is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kx) + \sum_{k=1}^{n} b_k\sin(kx)$$

due to previous 3 lemma, we have

$$a_K = \frac{1}{m} \sum_{j=0}^{2m-1} y_i \cos(K x_j) \text{ and } b_K = \frac{1}{m} \sum_{j=0}^{2m-1} y_i - \sin(K x_j)$$

let $n = m-1$ Then

$$\{y_i\}_{j=0}^{2m-1} \xrightarrow{DFT} \{(a_K, b_K)\}_{K=0}^{m-1}$$

if $n \geq m$ then ? we may not have enough equations to solve for all different values of a_K and b_K

Since every value of m gave one pair of values a_K and b_K.