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Partial Differencial Equations and finding roots.

* PDE is an equation involving one or more derivative of unknown function if all derivative are taken with respect to several independent Variable.

Heat equation,

$$\frac{\partial u(x,t)}{\partial t} - \mu \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t),\ x \in (a,b),\ t > 0.$$

Here, μ is coefficient of termal diffusivity.

* Boundary value problem:-

Differencial equations is an open multidiamentional region $\Omega \in \mathbb{R}^d$ for which the value of unknown solⁿ is prescribed on boundary $\partial\Omega$ of multidiamentional region,

example, $\frac{\partial u(x,t)}{\partial t} - \mu \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$

with initial conditions. $u(x,0) = g(x)\ \forall x \in [a,b]$ and boundary equation. $u(a,t) = u(b,t) = 0,\ \forall t > 0$

* Approximation by Finite differences:-

let, $\frac{\partial u(x,t)}{\partial x} \approx g(x,t) = \frac{u(x+h/2,t) - u(x-h/2,t)}{h}$

Then, we have, $\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{g(x+\frac{h}{2},t) - g(x-\frac{h}{2},t)}{h}$

$$\approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

let, partition interval $[a,b]$ into $I_j = [x_j, x_{j+1}]$ of length h for $j = 0,1,\dots,N$ with $x_0 = a$, $x_N = b$

let, $u_j(t)$ denotes approximation of $u(x_j,t)$, $j \in \{0,\dots,N\}$. Then $\forall t > 0$, we should have

$\forall j \in \{1, 2, \dots, N-1\}$

$$\frac{du_j(t)}{dt} - \frac{u}{h^2}\left(u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)\right) = f_j(t)$$

with initial condition $u_0(t) = 0$ and $u_N(t) = 0$,
$f_j(t) = f(x_j, t)$ and $u_j(0) = g(x_j)$

giving us following ode,

$$\frac{du(t)}{dt} = \frac{u}{h^2} A\, u(t) + f(t) \text{ with}$$

$u(0) = g$.

Q. * Figure out approximations for $\frac{\partial^2 u(x,t)}{\partial x_1^2}$ and $\frac{\partial^2 u(x,t)}{\partial x_2^2}$, and solve ODE.

$$\cancel{\frac{\partial^2 u}{\partial x_1^2} = \frac{u(i+1,j) - 2u(i,j) + u(i-1,j)}{\Delta x_1^2}}$$

$$\cancel{\frac{\partial^2 u}{\partial x_1^2} = u_{xx} = \frac{u(x+h) - 2f(x,y) + f(x-h,y)}{h^2}}$$

$$\frac{\partial^2 u}{\partial x_1^2} = \frac{u(x+h) - 2f(x,t) + f(x-h,t)}{h^2}$$

$$\frac{\partial^2 u}{\partial x_2^2} = \frac{u(x+k) - 2f(x,t) + f(x-k,t)}{k^2}$$

$$h \approx k$$

$$\therefore \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = \frac{2}{h^2}\left[u(x+h) + f(x-h,t)\right]$$

Assumming we have system of ODE,

$$\frac{\partial^2 u(x,t)}{\partial x_1^2} = F_1(x_1, x_2, t)$$

$$\frac{\partial^2 u(x,t)}{\partial x_2^2} = F_2(x_1, x_2, t).$$

we need more info like form of F_1, F_2 and boundry condⁿ to solve.

* Root Finding Algorithms :-

In mathematics and computing, root finding algorithm is an algorithm for finding zeros also called roots of continous function.

zero ~~function~~ of function $f: \mathbb{R} \to \mathbb{R}$ is number x s.t. $f(x) = 0$.

⇒ Most algorithms (root finding) do not garrantee ~~they~~ ~~thee~~ that they will find all the roots; in perticular if such an algorithm does not find any root, that does not mean root not exist.

* Bisection Method :-

An approximation method to finds roots of given equation repeatedly dividing the intervals. This method will divide interval until resulting interval is found which is extremely small.

input : f, a, b, TOL, NMAX.

condition : $a < b$

~~either $f(a) < f(b)$ or~~

either $f(a) < 0$ and $f(b) > 0$ or $f(a) > 0$ and $f(b) < 0$

output : value which differs from a root of $F(x) = 0$ by less than TOL.

$N \leftarrow 1$

While $N \leq$ NMAX do

$c \leftarrow (a+b)/2$

if $f(c) = 0$ or $(b-a)/2 <$ TOL Then

output (c); stop

~~stop~~

end if
N ← N+1
if sgn(f(c)) = sgn(f(a)) then
a ← c else b ← c.
end while;
Output "Method failed".

* The ~~no.~~ number n of iterations needed to achieve a required ~~tolerance~~ tolerance ε is bounded by

$$n \le n_{1/2} = \left\lceil \log_2\left(\frac{\varepsilon_0}{\varepsilon}\right) \right\rceil$$

where $\varepsilon_0 = |b-a|$

* Newton - Raphson method,
Consider a differencial ~~equation~~ function, $f: \mathbb{R} \to \mathbb{R}$ if f satisfies sufficient assumptions and initial guess x_0 is close, root can be found using following iterative method,

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

f → must be continous in [a,b]
→ must be differencial in [a,b], derivative $f'(x)$ must exist and should be continous
→ $f'(x) \neq 0$
→ initial guess x_0 must be close to root of function.

* if $f: \mathbb{R}^k \to \mathbb{R}^k$, multivariate vector-value function, iteration is given by $x_{k+1} = x_k - J(x_k)^{-1} F(x_k)$
Where $J(x_k)$ is Jacobian Matrix of F