axiom™

The 30 Year Horizon

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New Foreword

On October 1, 2001 Axiom was withdrawn from the market and ended life as a commercial product. On September 3, 2002 Axiom was released under the Modified BSD license, including this document. On August 27, 2003 Axiom was released as free and open source software available for download from the Free Software Foundation's website, Savannah.

Work on Axiom has had the generous support of the Center for Algorithms and Interactive Scientific Computation (CAISS) at City College of New York. Special thanks go to Dr. Gilbert Baumslag for his support of the long term goal.

The online version of this documentation is roughly 1000 pages. In order to make printed versions we've broken it up into three volumes. The first volume is tutorial in nature. The second volume is for programmers. The third volume is reference material. We've also added a fourth volume for developers. All of these changes represent an experiment in print-on-demand delivery of documentation. Time will tell whether the experiment succeeded.

Axiom has been in existence for over thirty years. It is estimated to contain about three hundred man-years of research and has, as of September 3, 2003, 143 people listed in the credits. All of these people have contributed directly or indirectly to making Axiom available. Axiom is being passed to the next generation. I'm looking forward to future milestones.

With that in mind I've introduced the theme of the "30 year horizon". We must invent the tools that support the Computational Mathematician working 30 years from now. How will research be done when every bit of mathematical knowledge is online and instantly available? What happens when we scale Axiom by a factor of 100, giving us 1.1 million domains? How can we integrate theory with code? How will we integrate theorems and proofs of the mathematics with space-time complexity proofs and running code? What visualization tools are needed? How do we support the conceptual structures and semantics of mathematics in effective ways? How do we support results from the sciences? How do we teach the next generation to be effective Computational Mathematicians?

The "30 year horizon" is much nearer than it appears.

Tim Daly
CAISS, City College of New York
November 10, 2003 ((iHy))

Ultimately we would like Axiom to be able to prove that an algorithm generates correct results. There are many steps between here and that goal, including proving one Axiom algorithm correct through all of the levels from Spad code, to the Lisp code, to the C code, to the machine code; a daunting task of its own.

The proof of a single Axiom algorithm is done with an eye toward automating the process. Automated machine proofs are not possible in general but will exist for known algorithms.

This is an exercise in Omphaloskepsis
– Tim Daly

Coq is the tactical nuclear weapons of programming. You need to be quite clever to use them. And indeed, when people first start to use them you don't hear from them for two years, they sort of 'go dark' and then they emerge with all their pain receptors destroyed and they say Coq the only way to live, to do everything in Coq. "I don't write programs anymore, I just write Coq proofs"
– Simon Peyton-Jones [Peyt17]

It is an intrinsic part of the duty of everyone who professes to compose algorithms to supply a proof that his text indeed represents a proper algorithm.
– Edsgar Dijkstra [Dijk71]

When correctness concerns come as an afterthought and correctness proofs have to be given once the program is already completed, the programmer can indeed expect severe troubles. If, however, he adheres to the discipline to produce the correctness proofs as he programs along [my italics], he will produce program and proof with less effort than programming alone would have taken. – Edsgar Dijkstra [Dijk70]

If what you are working on is not important, why are you working on it?
– Richard Hamming [Hamm95]

Q: Why bother doing proofs about programming languages? They are almost always boring if the definitions are right.

A: The definitions are almost always wrong.

The Misfortunes of a Trio of Mathematicians Using Computer Algebra Systems: Can We Trust In Them?
– Durán, Pérez, and Varona [Dura14]
Certainly not. – Tim Daly
Yeah, well, that's just, like, your opinion, man
– Jeff Bridges, The Big Lebowski

If programming is understood not as the writing of instructions for this or that computing machine but as the design of methods of computation that it is the computer's duty to execute, then it no longer seems possible to distinguish the discipline of programming from constructive mathematics.
– Per Martin-Löf [Mart79]

Our basic premise is that the ability to construct and modify programs will not improve without a new and comprehensive look at the entire

programming process. Past theoretical research, say, in the logic of programs, has tended to focus on methods for reasoning about individual programs; little has been done, it seems to us, to develop a sound understanding of the process of programming – the process by which programs evolve in concept and in practice. At present, we lack the means to describe the techniques of program construction and improvement in ways that properly link verification, documentation and adaptability.

– Scherlis and Scott (1983) in [Maso86]

The intrinsically discrete nature of symbol processing makes programming such a tricky job that the application of formal techniques becomes a necessity.

– Edsger W. Dijkstra [Dijk83]

The notion of a proof serves not only to organize information, but to direct the analysis of a problem and produce the necessary insights. It is as much an analytical tool as it is a final product.

– Bates and Constable [Bate85]

By June 1949 people had begun to realize that it was not so easy to get programs right as at one time appeared. ... the realization came over me with full force that a good part of the remainder of my life was going to be spent in finding errors in my own programs.

– Maurice Wilkes [Wilk85a]

I believe the hard part of building software to be the specification, design, and testing of this conceptual construct, not the labor of representing it and testing the fidelity of the representation. If this is true, building software will always be hard. There is inherently no silver bullet.

– Fredrick P. Brooks Jr. [Broo87]

I hold the opinion that the construction of computer programs is a mathematical activity like the solution of differential equations, that programs can be derived from their specifications through mathematical insight, calculation, and proof, using algebraic laws as simple and elegant as those of elementary arithmetic.

– C. A. R. Hoare [Fetz88]

It might be said that programs are conjectures, while executions are attempted refutations.

– James Fetzer [Fetz88]

The existence of the computer is giving impetus to the discovery of algorithms that generate proofs. I can still hear the echos of the collective sigh of relief that greeted the announcement in 1970 that there is no general algorithm to test for integer solutions to polynomial Diophantine equations; Hilbert's tenth problem has no solution. Yet, as I look at my own field, I see that creating algorithms that generate proofs constitutes some of the most important mathematics being done. The all-purpose proof machine may be dead, but tightly targeted machines are thriving.

– Dave Bressoud [Bres93]

In contrast to humans, computers are good at performing formal processes. There are people working hard on the project of actually formalizing parts of mathematics by computer, with actual formally correct formal deductions. I think this is a very big but very worthwhile project, and I am confident that we will learn a lot from it. The process will help simplify and clarify mathematics. In not too many years, I expect that we will have interactive computer programs that can help people compile significant chunks of formally complete and correct mathematics (based on a few perhaps shaky but at least explicit assumptions) and that they will become part of the standard mathematician's working environment.
– William P. Thurston [Thur94]

...constructive mathematics provides a way of viewing the language of logical propositions as a *specification* **language for programs. An ongoing thrust of work in computer science has been to develop program specification languages and formalisms for systematically deriving programs from specifications. For constructive mathematics to provide such a methodology, techniques are needed for systematically extracting programs from constructive proofs. Early work in this field includes that of Bishop and Constable [Cons98]. What distinguished Martin-Löf's '82 type theory was that the method it suggested for program synthesis was exceptionally simple: a direct correspondence was set up between the constructs of mathematical logic, and the constructs of a functional programming language. Specifically, every proposition was considered to be isomorphic to a type expression, and the proof of a proposition would suggest precisely how to construct an inhabitant of the type, which would be a term in a functional programming language. The term that inhabits the type corresponding to a proposition is often referred to as the** *computational content* **of the proposition.**
– Paul Bernard Jackson [Jack95]

Writing is nature's way of letting you know how sloppy your thinking is.
– Guindon [Lamp02]

Mathematics is nature's way of letting you know how sloppy your writing is.
– Leslie Lamport [Lamp02]

Type theory is nothing short of a grand unified theory of computation unified with mathematics so ultimately there is no difference between math and the code.
– Robert Harper [Harp13]

Informal proofs are algorithms. Formal proofs are code.
– Benjamin Pierce [Pier17]

Chapter 1

Why this effort will not succeed

> When something is important enough, you do it even if the odds are not in your favor.
> – Elon Musk

> **The ancient Greeks gave (western) civilization quite a few gifts, but we should beware of Greeks bearing gifts. The gifts of theatre and democracy were definitely good ones, and perhaps even the gift of philosophy, but the 'gift' of the so-called 'axiomatic method' and the notion of 'rigorous' proof did much more harm than good. If we want to maximize Mathematical Knowledge, and its Management, we have to return to Euclid this dubious gift, and give-up our fanatical insistence on perfect rigor. Of course, we should not go to the other extreme, of demanding that everything should be non-rigorous. We should encourage diversity of proof-styles and rigor levels, and remember that nothing is absolutely sure in this world, and there does not exist an absolutely rigorous proof, nor absolute certainty, and 'truth' has many shades and levels.**
> – Doron Zeilberger [Zeil10]

> **The thing that differentiates scientist is purely an artistic ability to discern what is a good idea, what is a beautiful idea, what is worth spending time on, and most importantly, what is a problem that is sufficiently interesting, yet sufficiently difficult, that it hasn't yet been solved, but the time for solving it has come now.**
> – Savas Dimopoulos; Stanford University

> **Every morning one should wake up and reflect on the conceptual and foundational significance of one's work.**
> – Harvey Friedman

This is an effort to prove Axiom correct. That is, we wish to prove that the algorithms provide correct, trustworthy answers.

All prior attempts at combining a Computer Algebra system and a Proof system raise the issue that the CAS is untrustworthy.

Axiom tries to encode mathematical algorithms. Unlike other systems it is built on the

scaffold of group theory which provides a sound mathematical foundation. As such, it seems only reasonable that the algorithms in Axiom can be proven correct, hence the project to Prove Axiom Sane (PAS).

The PAS project will not succeed. This is perfectly obvious from the outset. But, given the law of the excluded middle (that is, $A \vee \ A$) is not applicable in this case, the fact that the project "does not succeed" does not imply failure. Learning will occur.

That said, we can list quite a few reasons why PAC will not succeed, most of which are explained in more detail in [Cyph17]. We provide useful names for the likely criticisms presented there and paraphrase those criticisms applied to our context.

1. **Leap of Faith** All of these efforts either require making a leap of faith to go from verified code to a real-world problem, or required the use of an artificially restricted system in order to function (the NewSpeak approach, create a language in which it is impossible to think bad thoughts), indicating that formal verification down to the binary code level is unlikely to be practical in any generally accepted formal methods sense.

2. **Tools for Toy Problems** The tools used to support formal methods arose from an academic research envvironment characterised by a small number of highly skilled users. The tools were tested on small problems (usually referred to somewhat disparagingly as "toy problems") which were targeted more at exercising the tools than exercising the problem.

3. **Opaque Specifications** Another factor which complicates the use of formal methods is that the mathematical methods available to software engineers are often very difficult to use and plagued by notation which is cumbersome and hard to read and understand.

4. **Too Large a Task** The national technology base for this level of task is essentially non-existent. There do not appear to be even 20 people in the world that have undertaken the essential steps. "In order to get a system with excellent system integrity, you must ensure that it is designed and built by geniuses. Geniuses are in short supply" [Blak96].

5. **False Axioms** Problems occur when a proof has to be manually augmented with "self-vidient" truths which sometimes turn out to be false and end up misguiding the prover.

6. **Intellectual Cost** Requiring a proof of an algorithm adds to the already outsized cost of creating the algorithm in Axiom, given the steep intellectual hill such a system already presents.

7. **Ripple Cost** Multiple layers of abstraction are required to go from a proof to its implementation and a change in any of the layers can result in a ripple effect as changes propagate. If a change manages to propagate its way into a formally proven section, portions of, or possibly all of, the proof might need to be redone.

8. **Informal Implementation Details** Current techniques rarely reach down any further than the high-level specificaiton resulting in large amounts of time and energy being poured into producing a design specification which is void of any implementation details.

9. **Specification Mismatch** Formal proofs that a specification is correct don't show that the assumptions made in the proof are correct in the actual physical system. If the code which is supposed to implement the formal specification doesn't quite conform to it or the compiler can't produce an executable which quite matches what was intended in the code then no amount of formal proving will be able to guarantee the execution

behaviour of the code.

10. **Natural / Formal Mismatch** Natural language descriptions of what is intended may mismatch the formal language descriptions. In many cases "common sense" assumptions come into play which are never formally stated.

11. **Specification Flaws** The code may implement a specification exactly as stated but the specification can be flawed.

12. **Mis-implemented Specifications** The specification may be correct but the implementation does not match the specification.

13. **Specification Narrowing** The specification language may not be sufficient to state all of the required concepts so a "narrowing" of the specification is made to approximate the intent.

14. **Specification Widening** The specification language may generalize the required concepts so a "widening" of the specification is made beyond the required intent.

15. **Specification Impedence Mismatch** The specification language does not cover the concepts in the domain of interest, requiring considerable effort to "model" the domain.

16. **Specification Blindness** The specification writers can't take into account every eventuality which will arise. As a result, an implemention of a specification doesn't just implement it, it alters it in order to fit real-world constraints which weren't foreseen by the original authors of the specification.

17. **Contradictory Proofs** If the system is inconsistent, it would be possible to find two contradictory proofs. It is likely that such an inconsistency will not be detected and only one of the proofs will be accepted since the other one is never generated.

18. **Likely to be Ignored** There are thousands of proofs published, most of which are "write only" and are never checked. This is likely to be the case with Axiom's proofs.

19. **Proven Programs are Wrong** It is possible to prove a program correct and still get wrong results. Testing is still required.

20. **Proofs are a Social Process** Axiom's "social circle" is vanishingly small.

21. **Chicken and Egg** Do proofs follow from programs or programs follow from proofs?

22. **Boiling the Ocean** The task is much too large.

23. **MetaTheory Cost** There is a cost to developing theories for new data types; these are unnecessary "meta" costs.

24. **Partial Functions** Not all of the functions are total so it is difficult to prove theorems about them.

25. **Undecidable Theories** Some theories are known to be undecidable so there will be problems with their proofs.

Fenton [Fent93] states that there is no hard evidence to show that

1. formal methods have been used cost effectively on a real safety-critical system development

2. the use of formal methods can deliver reliability more cost effectively than traditional structured methods with enhanced testing

3. sufficient numbers of either developers or users can ever be trained to make proper use of formal methods

Calude, et al. [Calu07] take a quite different attack on the idea of proving and programming, specifically

1. Theorems (in mathematics) correspond to algorithms and not programs (in computer science); algorithms are subject to mathematical proofs (for example correctness)

2. The role of proof in mathematical modeling is very small: adequacy is the main issue

3. Programs (in computer science) correspond to mathematical models. They are not subject to proofs, but to an adequacy and relevance analysis; in this type of analysis, some proofs may appear. Correctness proofs in computer science (if any) are not cost-effective.

4. Rigour in programming is superior to rigour in mathematical proofs.

5. Programming gives mathematics a new form of understanding

6. Although the Hilbertian notion of proof has few chances to change, future proofs will be of various types and will play different roles, and their truth will be checked differently.

7. In general, correctness is undecidable

8. for most non-trivial cases, correctness is a monumental task which gives an added confidence at a disproportionate cost.

Still another attack comes from Hoare [Hoar96] which states that "By surveying current software engineering practice, this paper reveals that the techniques employed to achieve reliability are little different from those which have proved effective in all other branches of modern engineering: rigorous management of procedures for design inspection and review; quality assurance based on a wide range of targeted tests; continuous evolution by removal of errors from products already in widespread use; and defensive programming, among other forms of deliberate over-engineering. Formal methods and proof play a small direct role in large scale programming; but they do provide a conceptual framework and basic understanding to promote the best of current practice, and point directions for future improvement."

Fetzer [Fetz88] adds "The notion of program verification appears to trade upon an equvocation. Algorithms, as logical structures, are appropriate subjects for deductive verification. Programs, as causal models of those structures, are not. The success of program verification as a generally applicable and completely reliable method for guaranteeing program performance is not even a theoretical possibility."

An unnamed letter writer to the CACM says "It is time somebody said it – loud and clear – the formal approach to software verification does not work now and probably never will." [Glas02]

DeMillo et al. [Demi79] says "It is argued that formal verifications of programs, no matter how obtained, will not play the same key role in the development of computer science and software engineering as proofs do in mathematics. Furthermore the absence of continuity, the inevitability of change, and the complexity of specification of significantly many real programs make the formal verification process difficult to justify and manage. It is felt that ease of formal verification should not dominate program language design.",

DeMillo argues that proofs are social constructs. "Mathematiical proofs increase our confidence in the truth of mathematical statements only after they have been subjected to the social mechanisms of the mathematical community. These same mechanisms doom the so-called proofs of software, the long formal verifications that correspond, not to the working mathematical proof, but to the imaginary logical structure that the mathematician conjures up to describe his feeling of belief. Verifications are not messages, a person who ran out into

the hall to communicate his latest verification would rapidly find himself a social pariah. Verifications cannot really be read; a reader can flay himself through one of the shorter ones by dint of heroic effort, but that's not reading. Being unreadable and – literally – unspeakable, verifications cannot be internalized, transformed, generalized, used, connected to other disciplines, and eventually incorporated into a community consciousness. They cannot acquire credibility gradually, as a mathematical theorem does; one either believes them blindly, as a pure act of faith, or not at all."

And still more with "There is a fundamental logical objection to verification, an objection on its own ground of formalistic rigor. Since the requirement for a program is informal and the program is formal, there must be a transition, and the transition itself must necessarily be informal."

And "So, having for the moment suspended all rational disbelief, let us suppose that the programmer gets the message "VERIFIED." And let us suppose further that the message does not result from a failure on the part of the verifying system. What does the programmer know? He knows that his program is formally, logically, provably, certifiably correct. He does not know, however, to what extent it is reliable, dependable, trustworthy, safe; he does not know within what limits it will work; he does not know what happens when it exceeds those limits. And yet he has that mystical stamp of approval "VERIFIED.""

Manna and Waldinger [Mann78] mentions

1. We can never be sure that the specifications are correct

2. No verification system can verify every correct program

3. We can never be certain that a verification system is correct

Hall [Hall90] presents 7 myths of formal methods.

1. Formal methods can guarantee that software is perfect.

2. They work by proving that programs are correct.

3. Only highly critical systems benefit from their use.

4. They involve complex mathematics.

5. They increase the cost of development.

6. They are incomprehensible to clients.

7. Nobody uses them for real projects.

Bowen and Hinchey [Bowe95] follow up with 7 more myths.

1. Formal methods delay the development process

2. Formal methods lack tools

3. Formal methods replace traditional engineering design methods

4. Formal methods only apply to software

5. Formal methods are unnecessary

6. Formal methods are not supported

7. Formal methods people always use formal methods

Murray and Oorschot [Murr18] provide comments that, while they are indented for security-related proofs, also apply to proofs in general. We paraphrase the comments that we feel are relevant to all proofs.

Proofs have been used as a means to aid the development of systems for at least the past five decades. There has been much debate about proof's role in developing software and systems, including its practicality for mainstream software and relevance for reasoning about the real-world.

There has been a rash of large-scale software verification projects that have resulted in the construction of software systems, whose assurance is backed by (machine-checked) proofs about the software's behaviour.

As an assurance mechanism proof remains poorly understood expecially outside of formal verification practitioners and researchers.

Proof surely delivers many benefits, and examples abound of defects and vulnerabilities found during the process of proving (a property about) a system that led to improvements in the system that might have been unlikely otherwise. Proof also brings with it its own *side effects* (code changes and deployment constraints), not all of which may be positive for the system and some of which might even be negative (aside from on performance).

There is both confusion, and wide variety of interpretations by different stakeholders and communities, about the value and role of formal proofs in the construction of systems.

What is the value of a formal proof if it cannot establish properties of the real system?

The most common interpretation of the meaning of a proof is that it provides *guarantees* about a system. This perspective is perhaps strongest *outside* of the formal methods community. *Within* the formal methods community – i.e., those practising formal verification – things are more nuanced. This apparent contraiction – frequent in the literature, and not to be discounted as a source of confusion – is resolved by observing that a proof provides guarantees subject to the accuracy of the model and proof assumptions, i.e., provides guarantees about the real world when, and only when, the formal model of the system matches the system's real-world behaviour with respect tot he property being proven.

All proofs have assumptions. Yet, as noted by DeMillo, Lipton, and Perlis in their infamous critique of formal methods [Demi79], what sets formal verification of software apart from proofs in pure mathematics is their sheer volume of assumptions.

One conclusion about the value of formal proofs is that they allow an evaluator to concentrate their efforts on validating the accuracy of the proof's assumptions, enabling them to ignore large partf of the implementation. The value of a formal guarantee is that it concentrates remaining doubt on the assumptions.

An alternaitve perspective is that the value in proofs is not to provide (qualified) guarantees, but instead to force careful and rigorous understanding of a systme and its operation. There is no question that proof processes generate more robust code. Yet the general consensus seems to be that this robustness is much more an emergent consequence of rigorously understanding the problem than a result of the proof discharge. If discharging the entire proof is merely a side effect, we note that under this point of view even *partial* proofs have value. The same is less clear for the viewpoint of proofs as qualified guarantees, since an unfinished proof provides no quarantees about the formal object under study.

Interative proofs provide a critical mechanism for performing structured, interactive exploration of a system (via a formal model). Carrying out the proof forces the human to carefully examine each part of the system model, and allows human intuition to bring to light issues that might be missed by the proof itself.

Proofs have some established ideas on their benefits: providing qualified guarantees, allowing auditors to concentrate effor on validating proof assumptions and the formal model; providing

a means for structured exploration of a system to better understand and improve it.

Yet against these benefits stand proof's disadvantages, which as with proof itself, remain poorly understood expecially outside the formal methods community. One is proof *brittleness*: e.g., changing one line of code can potentially invalidate large amounts of formal reasoning. It is difficult to judge the guarantees provided by a proof when even *one* proof assumption deviates from reality. Another is a dearth of techniques for reliably predicting the cost of formal methods. Still another is the difficulty that both non-experts and experts have in discerning precisely the nature of the formal property that a proof establishes, and the difficulty of validating implicit assumptions in large-scale mormal models on which such proofs rest. This leaves plenty of room for gaps between what non-expert might *think* has been proven and the precise formal property *acutally* proven.

There are effects *on the system*, both beneficial and otherwise, for which proof is a proximate cause – i.e., concrete effects on the system brought about directly as a result of performing the proof. We refer to these as the *side effects* of a proof.

A proof might *induce* two kinds of side effects on the system. One is *changes to the code* either to fix a bug found during the proof or to modify some aspect of the design or implementation to open a path enabling a proof. A second is *deployment constraints* imposed on the system to enforce *environmental assumptions* necessary for proofs.

1.1 General problems with formal verification

In Moy and Wallenburg [Moyx10] we find:

1. Formal verification is not applied to all parts of a system. Around a formally verified core, there are usually parts of the system that cannot be verified formally, because they deal with interfacing with other systems or a user, or because it would be too costly to verify them (effort/time/money). The cost/benefit ratio of formal verification is parricularly important to take into account for those less critical parts.

2. Formal verification is only as strong as the specification is. When formally verifying the absence of run-time errors in programs, the specification is given by the semantics of the programming language. However, in general there is no way to "guess" the intended meaning of some part of a system, but the developer has to indicate it in a specificaiton. Writing specifications can be as error-prone as writing programs. In practice, these specifcations are not complete functional specifications, as this would be much too costly. Only the properties that a developer specified can be formally verified.

3. Formal verification techniques do not target all desirable properties of programs. Most formal verification frameworks verify the absence of run-time errors, and perperties expressed by developers are invariants or function contracts. In particular, formal verification techniques are not usually good at detecting dead code and "strange" code (inefficient or unidiomatic code), or covert channels.

4. The formal verification method itself may be flawed, for example if the immplementation of the underlying theorem prover or static analyzer contains problems.

Chapter 2

Progress Will Occur

At the very lowest level there have been some truly impressive steps toward formal verification of low-level implmentation details.

Reid [Reid17] has created and verified a huge specification (over 1/2 Million nodes) of the ARM processor. The specification passes almost every test in their huge (11k) test suite. Using the specification with an SMT Solver allows the ability to do things like ask what input will give the known output, for example.

Chlipala has done both machine-level [Chli17a] and higher-level [Chli17] formal proofs. He claims that within 10 years it will be normal to have deployed computer systems with top-to-bottom correctness proofs which were checked algorithmically.

Chapter 3

Here is a problem

> The world needs both logicians and verifiers, and we need one another. Logicians need verifiers to keep their work relevant, and verifiers need logicians to keep them honest. (In typical human fashion, most logicians think they know all they need to about real verification problems for developing theories, and most verifiers think they know enough about logic to verify programs. They're probably both wrong.)
> – Lamport and Owicki [Lamp81]

Axiom has a domain-specific language for mathematical computation called Spad. This is implemented in Common Lisp. Our version of Common Lisp compiles to C. The C code is then compiled to machine code. The machine code is executed by a physical machine.

There are various attacks on each of these areas. We will be addressing them all at some point, from the mathematical specification, through the Spad code all the way down to the hardware timing signals.

One useful technique to exploit is mentioned by Myreen, Slind, and Gordon [Myre09a]. They 'forward compile' the code to f, collect the result as f', and then prove the $f = f'$. This proof is sufficient to show that the compile step is correct.

3.1   Proving the Algebra

3.1.1   Defining the Spad syntax

3.1.2   Defining the Spad semantics

Gordon [Gord89] describes formal reasoning about programs that can be based directly on the semantics of the programming language, or done in a special purpose logic like Hoare logic.

O'Donnell [Odon81] raises questions about the logical soundness of rules used in Hoare logic reasoning.

Manna and Waldinger [Mann78a] introduce the notion of *sometime* as a conditional invariant that sometimes is true.

Floyd [Floy67], Back [Back81], Caldwell [Cald97], Filliatre [Fill03], Filliatre and Paskevich [Fill13]

3.1.3   Type Resolution

In Kaes [Kaes88] we find that there is a semantical difference between overloading and polymorphism: in the case of the overloaded + operator, we would expect different code to be executed for integer and real addition respectively, whereas we expect the code of *len* to be usable for lists of any type. This distinction has led to the name "adhoc polymorphism".

Parametric Polymorphism: One semantic object can have different types at each usage, all being instances of a single type expression over variables.

Overloading ("adhoc polymorphism"): A single name can be used to denote different semantic objects, the types of these objects being completely unrelated.

Parametric Overloading: A single name can be used to denote several objects, the types of these objects being instances of a single type expression over some extended set of type variables.

Cardelli and Wegner [Card85] refines Strachey by introducing a new form of polymorphism called *inclusion polymorphism* to model subtypes and inheritance. Parametric and inclusion polymorphism are classified as the two major subcategories of "universal polymorphism" which is contrasted with nonuniversal or ad-hoc polymorphism.

```
                              |- parametric
                |- universal -|
                |             |- inclusion
  polymorphism -|
                |             |- overloading
                |- ad hoc    -|
                              |- coercion
```
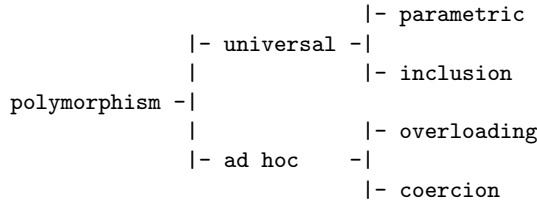
Parametric polymorphism is so called because the uniformity of type structure is normally achieved by type parameters, but uniformity can be achieved in different ways, and this more general concept is called *universal polymorphism*. Universally polymorphic functions will normally work on an infinite number of types (all the types having a given common structure), whereas an ad-hoc polymorphic function will only work on a finite set of different and potentially unrelated types.

In *parametric polymorphism*, a polymorphic function has an implicit or explicit type parameter which determines the type of the argument for each application of that function. In *inclusion polymorphism* an object can be viewed as belonging to many different classes that need not be disjoint; that is, there may be inclusion of classes.

There are at least 4 kinds of escapes from a monomorphic type.

- *Overloading*: Integer constants may have both type integer and real.

- *Coercion*: An integer value can be used where a real is expected, and vice versa.

- *Subtyping*: Elements of a subrange type also belong to superrange types.

- *Value Sharing*: `nil` in Pascal is a constant that is shared by all the pointer types.

Wadler and Blott [Wadl88] point out that it would be nice if polymorphic equality could be extended to include user-defined equality operations over abstract types. To implement this, we would need to require that every object carry with it a pointer to a *method*, a

procedure for performing the equality test. If we are to have more than one operation with this property, then each object should carry with it a pointer to a *dictionary* of appropriate methods.

In the case of polymorphic equality, this means that *both* arguments of the equality function will contain a pointer to the same dictionary (since they are both of the same type). This suggests that perhaps dictionaries should be passed around independently of objects, now polymorphic equality would be passed one dictionary and two objects (minus dictionaries). This is the intuition behind type classes and the translation method they describe.

It is natural to think of adding assertions to the class declaration, specifying properties that each instance must satisfy:

```
class EQ a where
  (==) :: a -> a -> Bool
  % (==) is an equivalence relation

class Num a where
  zero, one :: a
  (+), (*) :: a -> a -> a
  negate :: a -> a
  % (zero, one, (+), (*), negate) form a ring
```

It is valid for any proof to rely on these properties, so long as one proves that they hold for each instance declaration. Here the assertions have simply been written as comments; a more sophisticated system could perhaps verify or use such assertions.

It is reasonable to allow a class to apply to more than one type variable. For instance, we might have

```
class Coerce a b where
  coerce :: a -> b

instance Coerce Int Float where
  coerce - convertIntToFloat
```

In this case, the assertion `Coerce a b` might be taken as equivalent to the assertion that `a` is a subtype of `b`. This suggests a relation between this work and work on bounded quantification and on subtypes.

3.2 Proving the Logic

Soundness has two aspects, closed world and open world. [Bonn18]

Closed world soundness assumes all progremas are type checked by the same type system. Open world soundness not only require closed world soundness, but also that the guarantees upheld under closed world assumptions cannot be broken by foreign code. The introduction and consideration of foreign code is the essential difference between open-world and closed world soundness. See gradual-typing.

Proving programs correct involves working with a second programming language, the proof language, that is well-founded on some theory. Proofs (programs), can be reduced (compiled) in this new language to the primitive constructs (machine language).

The ideal case would be that the programming language used, such as Spad, can be isomorphic, or better yet, syntactically the same as the proof language. Unfortunately that is not (yet?) the case with Spad.

The COQ system language, Gallina, is the closest match to Spad.

Barras and Werner [Barr18] formalize the definition and the metatheory of the Calculus of Constructions (CC) using the proof assistant Coq. In particular, we prove strong normalization and decidability of type inference. From the latter proof, we extract a certified Objective Caml program which performs type inference in CC and use this code to build a small-scale certified proof-checker.

Gordon [Gord89] derives the rules of Hoare logic *mechanically* from the semantics of a simple imperative programming language.

Igarashi et al. [Igar75] defined a program to compute the subgoals from a program and its post-condition.

Katz and Manna [Katz75] look at various methods of proving termination including

- using the 'no-infinitely-descending-chain property of well-founded sets (Floyd's approach)
- bounding a counter associated with each loop (*loop* approach)
- showing that some exit of each loop must be taken (*exit* approach)
- inducting on the structure of the data domain (Burstall's approach)

Davis [Davi09], Myreen and Davis [Myre11], Myreen and Davis [Myre14], Davis and Myreen [Davi15]

3.2.1 Typing and Subtyping

Dolan's PhD Thesis [Dola16] on Algebraic Subtyping. Henri Tuhola [Tuho18] notes that solving the problem of subtyping does not directly solve the problem of coercion. Coercion extends the type and doesn't narrow it.

Dolan and Mycroft [Dola17] note that, given

```
select p v d = if (p v) then v else d
```

only by ignoring the orientation of the data flows could we conclude that d's type flows to the argument of p, even though p never mentions d. ML's type inference turns data flows into equality constraints between types so information about the *direction* of data flow is ignored. Since equality is symmetric, data flow is treated as unidirectional. But support for subtyping cares about the direction of data flow. With subtyping, a source of data must provide at least the guarantees that the destination requires, but is free to provide more.

3.2.2 Defining the Algebra specifications

Talpin and Jouvelot [Talp92]

3.2.3 Termination

Hughes [Hugh90] talks about a lazy approach to termination, while gluing programs together. Functional languages allow whole progams to be glued together. A complete functional program is just a function from its inputs to its output. If f and g are such programs, then $(g \cdot f)$ is a program that, when applied to its input, computes

$$g(f \text{ input})$$

The program f computes its output, which is used as the input to program g. This might be implemented conventionally by storing the output from f in a temporary file. The problem with this is that the temporary file might occupy so much memory that it is impractical to glue the programs together in this way. Functional languages provde a solution to this problem. The two programs f and g are run together in strict synchronization. Program f is started only when g is trying to read. Then f is suspended and g is run until it ties to read another input. As an added bonus, if g terminates without reading all of f's output, then f is aborted. Program f can even be a nonterminating program, producing an infinite amount of output, since it will be terminated forcibly as soon as g is finished. This allows termination conditions to be separated from loop bodies – a powerful modularization.

Since this method of evaluation runs f as little as possible, it is called "lazy evaluation". It makes it practical to modularize a program as a generator that constructs a large number of possible answers, and a selector that chooses the appropriate one.

IDEA: write (lazy f) for non-terminating programs to simplify proofs.

Sites [Site74] offers some thoughts on proving clean termination of programs.

Katz and Manna [Katz75] take a closer look at termination.

3.3 Proving the Lisp

Guttman, et al. [Gutt95] in the VL!SP project showed how to produce a comprehensively verified implementation for a programming language, namely Scheme.

Leroy [Lero09] describes the development and formal verification (proof of semantic preservation) of a compiler back-end from Cminor (a simple imperative intermediate language) to PowerPC assembly code, using the Coq proof assistant.

Manolios and Moore [Mano03] describe a method for introducing 'partial functions' in to ACL2. This is necessary for Common Lisp as most functions are not total.

Myreen [Myre12]

3.4 Proving the Compiler

Myreen [Myre10]

3.5 Proving to the metal

Myreen, Slind, and Gordon [Myre09a] presents a compiler which produces machine code from functions defined in the logic of a theorem prover, and at the same time proves that

the generated code executes the source functions.

Myreen and Gordon [Myre09],

Gordon, et al. [Gord06] describes a compiler that automatically translates recursive function definitions in higher order logic to clocked synchronous hardware Compilation is by mechanised proof in the HOL4 system. In particular, this models functions as circuits.

Domipheus [Domi18] provides a VHDL (hardware description language) CPU which provides the actual electrical timing of signals all the way up to a machine language which consists of

OPERATION	FUNCTION
Add	D = A + B
Substrct	D = A - B
Bitwise Or	D = A or B
Bitwise Xor	D = A xor B
Bitwise And	D = A and B
Bitwise Not	B = not A
Read	D = Memory[A]
Write	Memory[A] = B
Load	D = 8-bit immediate value
Compare	D = cmp(A,B)
Shift Left	D = A << B
Shift Right	D = A >> B
Jump/Branch	PC = A Register or Immediate Value
Jump/Brance conditionally	PC = Register if (condition) else nop

3.6   Setting up the problem

The GCD function will be our first example of a proof.

The goal is to prove that Axiom's implementation of the Euclidean GCD algorithm is correct.

We need to be clear about what is to be proven. In this case, we need to show that, given GCD(a,b),

1. **GCD** is a function from $a \times b \Rightarrow c$

2. **a** and **b** are elements of the correct type

3. **c**, the result, is the correct type

4. the meaning of **divisor**

5. the meaning of a **common divisor**

6. **GCD** terminates

We next need to set up the things we know in "the global environment", generally referred to as **E** in Coq.

Axiom's GCD is categorically defined to work over any Euclidean domain. This means that the axioms of a Euclidean domain are globally available. In fact, this is stronger than we need since

- commutative rings $\subset$ integral domains

- integral domains $\subset$ integrally closed domains

- integrally closed domains $\subset$ GCD domains

- GCD domains $\subset$ unique factorization domains

- unique factorization domains $\subset$ principal ideal domains

- principal ideal domains $\subset$ Euclidean domains

A Euclidean function on R is a function f from $\mathbb{R}$
$\{0\}$ to the non-negative integers satisfying the following fundamental division-with-remainder property [WikiED]:

$D(a, b) = $ set of common divisors of a and b.

$\gcd(a, b) = \max D(a, b)$

3.7 Axiom NNI GCD

NonNegativeInteger inherits `gcd` from `Integer` up the "add chain" since it is a subtype of `Integer`. `Integer` has `EuclideanDomain` as an ancestor [Book103]:

```
(1) -> getAncestors "Integer"

  (1)
  {AbelianGroup, AbelianMonoid, AbelianSemiGroup, Algebra, BasicType,
   BiModule, CancellationAbelianMonoid, CharacteristicZero, CoercibleTo,
   CombinatorialFunctionCategory, CommutativeRing, ConvertibleTo,
   DifferentialRing, EntireRing, EuclideanDomain, GcdDomain,
   IntegerNumberSystem, IntegralDomain, LeftModule, LeftOreRing,
   LinearlyExplicitRingOver, Module, Monoid, OpenMath, OrderedAbelianGroup,
   OrderedAbelianMonoid, OrderedAbelianSemiGroup,
   OrderedCancellationAbelianMonoid, OrderedIntegralDomain, OrderedRing,
   OrderedSet, PatternMatchable, PrincipalIdealDomain, RealConstant,
   RetractableTo, RightModule, Ring, Rng, SemiGroup, SetCategory, StepThrough,
   UniqueFactorizationDomain}
                                                    Type: Set(Symbol)
```

From category `EuclideanDomain` (EUCDOM) we find the implementation of the Euclidean GCD algorithm [Book102]:

```
    gcd(x,y) ==                 --Euclidean Algorithm
       x:=unitCanonical x
       y:=unitCanonical y
       while not zero? y repeat
          (x,y):= (y,x rem y)
          y:=unitCanonical y   -- this doesn't affect the
                               -- correctness of Euclid's algorithm,
                               -- but
                               -- a) may improve performance
                               -- b) ensures gcd(x,y)=gcd(y,x)
                               --    if canonicalUnitNormal
       x
```

The `unitCanonical` function comes from the category `IntegralDomain` (INTDOM) where we find:

```
    unitNormal: % -> Record(unit:%,canonical:%,associate:%)
        ++ unitNormal(x) tries to choose a canonical element
```

```
        ++ from the associate class of x.
        ++ The attribute canonicalUnitNormal, if asserted, means that
        ++ the "canonical" element is the same across all associates of x
        ++ if \spad{unitNormal(x) = [u,c,a]} then
        ++ \spad{u*c = x}, \spad{a*u = 1}.
    unitCanonical: % -> %
        ++ \spad{unitCanonical(x)} returns \spad{unitNormal(x).canonical}.
```

implemented as

```
    UCA ==> Record(unit:%,canonical:%,associate:%)
    if not (% has Field) then
      unitNormal(x) == [1$%,x,1$%]$UCA -- the non-canonical definition
    unitCanonical(x) == unitNormal(x).canonical -- always true
    recip(x) == if zero? x then "failed" else _exquo(1$%,x)
    unit?(x) == (recip x case "failed" => false; true)
    if % has canonicalUnitNormal then
      associates?(x,y) ==
        (unitNormal x).canonical = (unitNormal y).canonical
     else
      associates?(x,y) ==
        zero? x => zero? y
        zero? y => false
        x exquo y case "failed" => false
        y exquo x case "failed" => false
        true
```

Coq proves the following GCD function:

```
  Fixpoint gcd a b :=
    match a with
      | 0 => b
      | S a' => gcd (b mod (S a')) (S a')
    end.
```

This can be translated directly to working Spad code:

```
  GCD(x:NNI,y:NNI):NNI ==
    zero? x => y
    GCD(y rem x,x)
```

with the test case results of:

```
(1) -> GCD(2415,945)
  Compiling function mygcd2 with type (NonNegativeInteger,
    NonNegativeInteger) -> NonNegativeInteger

  (1)   105
                                          Type: PositiveInteger
(2) -> GCD(0,945)

  (2)   945
                                          Type: PositiveInteger
(3) -> GCD(2415,0)

  (3)   2415
                                          Type: PositiveInteger
(4) -> GCD(17,15)
```

   (4) 1

<div align="right">Type: PositiveInteger</div>

3.8 Mathematics

From Buchberger [Buch97],

Define "divides"

$$t|a \iff \exists u(t \cdot u = a)$$

Define "greatest common divisor"

$$\mathrm{GCD}(a, b) = \forall t \; max(t|a \wedge t|b)$$

Theorem:

$$(t|a \wedge t|b) \iff t|(a - b) \wedge t|b$$

Euclid's Algorithm

$$a > b \Rightarrow \mathrm{GCD}(a, b) = \mathrm{GCD}(a - b, b)$$

By the definition of GCD we need to show that

$$\forall t \; max(t|a \wedge t|b) = \forall t \; max(t|(a - b) \wedge t|b)$$

Thus we need to show that

$$(t|a \wedge t|b) \iff (t|(a - b) \wedge t|b)$$

Let t be arbitrary but fixed and assume

$$(t|a \wedge t|b) \tag{3.1}$$

We have to show

$$t|(a - b) \tag{3.2}$$

and

$$t|b \tag{3.3}$$

Equation 3.3 follows propositionally. For equation 3.2, by definition of "divides", we have to find a w such that

$$t \cdot w = a - b \tag{3.4}$$

From 3.1, by definition of "divides", we know that for certain u and v

$$t \cdot u = a$$

and

$$t \cdot v - b$$

Hence,
$$a - b = t \cdot u - t \cdot v$$

But
$$t \cdot u - t \cdot v = t \cdot (u - v)$$

So we need to find
$$w = u - v$$

and
$$\text{Find w such that } t \cdot u - t \cdot v = t \cdot w$$

3.9   Approaches

There are several systems that could be applied to approach the proof.

The plan is to initially look at Coq and ACL2. Coq seems to be applicable at the Spad level. ACL2 seems to be applicable at the Lisp level. Both levels are necessary for a proper proof.

Coq is very close to Spad in spirit so we can use it for the high-level proofs.

ACL2 is a Lisp-level proof technology which can be used to prove the Spad-to-Lisp level.

There is an LLVM to ACL2 translator which can be used to move from the GCL Lisp level to the hardware since GCL compiles to C. In particular, the "Vellvm: Verifying the LLVM" [Zdan14] project is important.

Quoting from Hardin [Hard14]

> LLVM is a register-based intermediate in Static Single Assignment (SSA) form. As such, LLVM supports any number of registers, each of which is only assigned once, statically (dynamically, of course, a given register can be assigned any number of times). Appel has observed that "SSA form is a kind of functional programming"; this observation, in turn, inspired us to build a translator from LLVM to the applicative subset of Common Lisp accepted by the ACL2 theorem prover. Our translator produces an executable ACL2 specification that is able to efficiently support validation via testing, as the generated ACL2 code features tail recursion, as well as in-place updates via ACL2's single-threaded object (stobj) mechanism. In order to ease the process of proving properties about these translated functions, we have also developed a technique for reasoning about tail-recursive ACL2 functions that execute in-place, utilizing a formally proven "bridge" to primitive-recursive versions of those functions operating on lists.
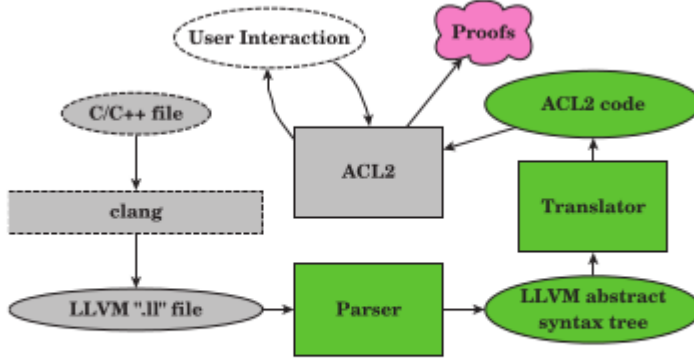
Figure 1: LLVM-to-ACL2 translation toolchain.

Hardin [Hard13] describes the toolchain thus:

> Our translation toolchain architecture is shown in Figure 1. The left side of the figure depicts a typical compiler frontend producing LLVM intermediate code. LLVM output can be produced either as a binary "bitcode" (.bc) file, or as text (.ll file). We chose to parse the text form, producing an abstract syntax tree (AST) representation of the LLVM program. Our translator then converts the AST to ACL2 source. The ACL2 source file can then be admitted into an ACL2 session, along with conjectures that one wishes to prove about the code, which ACL2 processes mostly automatically. In addition to proving theorems about the translated LLVM code, ACL2 can also be used to execute test vectors at reasonable speed.

Note that you can see the intermediate form from clang with

```
clang -O4 -S -emit-llvm foo.c
```

Both Coq and the Hardin translator use OCAML [OCAM14] so we will have to learn that language.

Chapter 4

Theory

The proof of the Euclidean algorithm has been known since Euclid. We need to study an existing proof and use it to guide our use of Coq along the same lines, if possible. Some of the "obvious" natural language statements may require Coq lemmas.

From WikiProof [Wiki14a] we quote:

Let

$$a, b \in \mathbf{Z}$$

and $a \neq 0$ or $b \neq 0$.

The steps of the algorithm are:

1. Start with (a, b) such that $|a| \geq |b|$. If $b = 0$ then the task is complete and the GCD is a.

2. if $b \neq 0$ then you take the remainder r of a/b.

3. set $a \leftarrow b$, $b \leftarrow r$ (and thus $|a| \geq |b|$ again).

4. repeat these steps until $b = 0$

Thus the GCD of a and b is the value of the variable a at the end of the algorithm.

The proof is:

Suppose

$$a, b \in \mathbf{Z}$$

and $a \, or \, b \neq 0$.

From the **division theorem**, $a = qb + r$ where $0 \leq r \leq |b|$

From **GCD with Remainder**, the GCD of a and b is also the GCD of b and r.

Therefore we may search instead for the $gcd(b, r)$.

Since $|r| \geq |b|$ and

$$b \in \mathbf{Z}$$

, we will reach $r = 0$ after finitely many steps.

At this point, $gcd(r, 0) = r$ from **GCD with Zero**.

We quote the **Division Theorem** proof [Wiki14b]:

For every pair of integers a, b where $b \neq 0$, there exist unique integers q, r such that $a = qb + r$ and $0 \leq r \leq |b|$.

4.0.1   Hoare's axioms and gcd proof

From Hoare [Hoar69]

A1	$x + y = y + x$	addition is commutative
A2	$x \times y = y \times x$	multiplication is commutative
A3	$(x + y) + z = x + (y + z)$	addition is associative
A4	$(x \times y) \times z = x \times (y \times z)$	multiplication is associative
A5	$x \times (y + z) = x \times y + x \times z$	multiplication distributes through addition
A6	$y \leq x \rightarrow (x - y) + y = x$	addition cancels subtraction
A7	$x + 0 = x$	
A8	$x \times 0 = 0$	
A9	$x \times 1 = x$	

D0 Axiom of Assignment

$$\vdash P_0\{x := f\}P$$

where

- x is a variable identifier

- f is an expression

- P_0 is obtained from P by substituting f for all occurrences of x

4.1   The Division Algorithm

From Judson [Juds15],

An Application of the Principle of Well-Ordering that we will use often is the division algorithm.

Theorem 2.9 Division Algorithm Let a and b be integers, with $b > 0$. Then there exists unique integers q and r such that

$$a = bq + r$$

where $0 \leq r < b$.

Proof

Let a and b be integers. If $b = ak$ for some integer k, we write $a|b$. An integer d is called a *common divisor* of a and b if $d|a$ and $d|b$. The *greatest common divisor* of integers a and b is a positive integer d such that d is a common divisor of a and b and if d' is any other common divisor of a and b, then $d'|d$. We write $d = gcd(a, b)$; for example, $gcd(24, 36) = 12$ and $gcd(120, 102) = 6$. We say that two integers a and b are *relatively prime* if $gcd(a, b) = 1$.

Theorem 2.10 Let a and b be nonzero integers. Then there exist integers r and s such that

$$gcd(a, b) = ar + bs$$

Furthermore, the greatest common divisor of a and b is unique.

Proof

Corollary 2.11 Let a and b be two integers that are relatively prime. Then there exist integers r and s such that

$$ar + bs = 1$$

The Euclidean Algorithm

Among other things, Theorem 2.10 allows us to compute the greatest common divisor of two integers.

Example 2.1.2 Let us compute the greatest common divisor of 945 and 2415. First observe that

$$
\begin{aligned}
2415 &= 945 \cdot 2 + 525 \\
945 &= 525 \cdot 1 + 420 \\
525 &= 420 \cdot 1 + 105 \\
420 &= 105 \cdot 4 + 0
\end{aligned}
$$

Reversing our steps, 105 divides 420, 105 divides 525, 105 divides 945, and 105 divides 2415. Hence, 105 divides both 945 and 2415. If d were another common divisor of 945 and 2415, then d would also have to divide 105. Therefore, $gcd(945, 2415) = 105$.

If we work backward through the above sequence of equations, we can also obtain numbers r and s such that

$$945r + 2415s = 105$$

$$
\begin{aligned}
105 &= 525 + (-1) \cdot 420 \\
105 &= 525 + (-1) \cdot [945 + (-1) \cdot 525] \\
105 &= 2 \cdot 525 + (-1) \cdot 945 \\
105 &= 2 \cdot [2415 + (-2) \cdot 945] + (-1) \cdot 945 \\
105 &= 2 * 2415 + (-5) \cdot 945
\end{aligned}
$$

So $r = -5$ and $s - 2$. Notice the r and s are not unique, since $r = 41$ and $s = -16$ would also work.

To compute $gcd(a, b) = d$, we are using repeated divisions to obtain a decreasing sequence of positive integers $r_1 > r_2 > \ldots > r_n = d$; that is

$$
\begin{aligned}
b &= aq_1 + r_1 \\
a &= r_1 q_2 + r_2 \\
r_1 &= r_2 q_3 + r_3 \\
&\vdots \\
r_{n-2} &= r_{n-1} q_n + r_n \\
r_{n-1} &= r_n q_{n+1}
\end{aligned}
$$

To find r and s such that $ar + bs = d$, we begin with the last equation and substitute results obtained from the previous equations:

$$
\begin{aligned}
d &= r_n \\
d &= r_{n-2} - r_{n-1} q_n \\
d &= r_{n-2} - q_n(r_{n-3} - q_{n-1} r_{n-2} \\
d &= -q_n r_{n-3} + (1 + q_n q_{n-1}) r_{n-2} \\
&\vdots \\
d &= ra + sb
\end{aligned}
$$

Chapter 5

GCD in Miranda by Broda, et al.

In Broda, et al. [Brod94, p. 55] they consider the greatest common divisor, `gcd`, of two natural numbers:

```
gcd :: num -> num -> num
||pre:   nat(x) & nat(y)
||post:  nat(z) & z|x & z|y (ie z is a common divisor)
||       &(A)n:nat(n|x & n|y -> n|z)
||            (ie any other common divisor divides it)
||       where z = (gcd x y)
```

We have introduced some notation in the pre and post-conditions:

- (A) just means $\forall$, that is 'for all', written in standard keyboard characters. $\exists$ would be (E).

- | means 'divides', or 'is a factor of'. (Note that it is not the same symbol as the division sign '/'.)
$$z \mid x \Leftrightarrow \exists y : nat.(x = z \times y)$$

- When we write '$y : nat$', we are using the predicate nat as though it were a Miranda type, though it is not. You can think of '$nat(y)$' and '$y : nat$' as meaning exactly the same, namely that y is a natural number. But the type-style notation is particularly useful with quantifiers:

$$\exists y : nat, P \qquad \text{means } \exists y.(nat(y) \wedge P)$$
$$('\text{there is a natural number } y \text{ for which } P \text{ holds}')$$
$$\forall y : nat, P \qquad \text{means } \forall y.(nat(y) \implies P)$$
$$('\text{for all natural numbers } y, P \text{ holds}')$$

Be sure to understnad these, and in particular why it is that $\exists$ goes naturally with $\wedge$, and $\forall$ with $\implies$. They are patterns that arise very frequently when you are translating from English into logic.

There is a small unexpected feature. You might expect the post-condition to say that any other common divisor is *less* than z, rather than dividing it: in other words that z is indeed the *greatest* common divisor. There is just a single case where this makes a difference,

namely when x and y are both 0. All numbers divide 0, so amongst the common divisors of x and y there is no greatest one. The specification as given has the effect of specifying

```
gcd 0 0 = 0
```

Proposition 1: For any two natural numbers x and y, there is at most one z satifying the specification for $(gcdxy)$.

Proof: Lex $z1$ and $z2$ be two values satisfying the specification for $(gcdxy)$; we must show that they are equal. All common divisors of x and y divide $z2$, so, in particular, $z1$ does. Similiarly, $z2$ divides $z1$. Hence for some positive natural numbers p and q, we have $z1 = z2 \times p$, $z2 = z1 \times q$, so $z1 = z1 \times p \times q$. It follow that either $z1 = 0$, in which case also $z2 = 0$, or $p \times q = 1$, in which case $p = q = 1$. In either case, $z1 = z2$.

Note that we have not actually proved that there is *any* value z satisfying the specification; only that there cannot be more than one. But we shall soon have an implementation showing how to find a suitable z, so then we shall know that there is exactly one possible result.

Euclid's algorithm relies on the following fact.

Proposition 2: Let x and y be natural numbers, $y \neq 0$. Then the common divisors of x and y are the same as those of y and $(x \bmod y)$.

Proof: For natural numbers x and y there are two fundamental properties of integer division, which in fact are enough to specify it uniquely: if $y \neq 0$ (pre-condition), the (post-condition)

$$x = u \times (x \text{ div } y) + (x \bmod y)$$

$$0 \leq (x \bmod y) < y$$

Suppose n is a common divisor of y and $(x \bmod yy)$. That is, there is a p such that $y = n \times p$ and a q such that $(x \bmod y) = n \times q$. Then

$$x = y \times (x \text{ div } y) + (x \bmod y) = n \times (p \times (x \text{ div } y) + q)$$

so n also divides z. Hence every common divisor of y and $(x \bmod y)$ is also a common divisor of x and y. The converse is also true, by a similar proof.   ∎

It follows that, provided $y \neq 0$, $(gcdxy)$ must equal $(gcdy(x \bmod y))$. On the other hand, $(gcdx0)$ must be x. This is because $x \mid x$ and $x \mid 0$, and any common divisor of x and 0 obviously divides x, so x satisfies the specification for $(gcdx0)$. We can therefore write the following definition:

$$
\begin{aligned}
gcdxy \quad &= x & \text{if } y = 0 \\
&= gcdy(x \bmod y) & \text{otherwise}
\end{aligned}
$$

`Question`: Does this definition satisfy the specification?

Let x and y be natural numbers, and let $z = (gcdxy)$. We must show that z has the properties given by the post-condition, and there are two cases corresponding to the two clauses in the definition:

$y = 0 : z = x$ We have already noted that this satisfies the specification.

$y \neq 0 : z = (gcdy(x \bmod y))$ What we have seen shows that *provided that* z satisfies the specification for $(gcdy(x \bmod y))$, then it also satisfies the specification for $(gcdxy)$ as required.   ∎

But how do we know that the recursive call gives the right answer? How do we know that it gives any answer at all? (conceivably, the recursion might never bottom out.) Apparently,

we have having to *assume* that `gcd` satisfies its specification in order to *prove* that it satisfies its specification.

The answer is that we *are* allowed to assume it! But there is a catch. This apparently miraculous circular reasoning must be justified, and the key is to notice that the recursive call uses simpler arguments: the pair of arguments y with x mod y) is 'simpler' than the pair x and y, in the sense that the second argument is smaller: x mod $y < y$.

As we go down the recursion, the second argument, always a natural number, becomes smaller and smaller, but never negative. This cannot go on for ever, so the recursion must eventually terminate. This at least proves termination, but it also justifies the *circular reasoning*. For suppose that `gcd` does not always work correctly. What might be the smallest bad y for which $gcdxy$ may go wrong (for some x)? Not 0 – gcd x 0 always works correctly. Suppose Y is the smallest bad y, and $gcdXY$ goes wrong. Then $Y > 0$, so

    gcd X Y = gcd Y (X mod Y)

But X mod Y is good (since X mod $Y < Y$), so the recursive call works correctly, so (we have already reasoned) $gcdXY$ does also – a contradiction.

We call the value y in $gcdxy$ a *recursion variant* for our definition of `gcd`. It is a rough measure of the depth of recursion needed, and always decreases in the recursive calls.

Let us now state this as a reasoning principle:

    In proving that a recursive function satisfies its specificaiton,
    you are allowed to assume that the recursive calls work correclty
    -- provided that you can define a recursion variant for the function.

A recursion variant for a function must obey the following rules:

- It is calculated from the arguments of the function

- It is a natural number (at least when the pre-conditions of the function hold). For instances, in `gcd` the recursion variant of y.

- It is calculated (trivially) from the function's arguments x and y)

- It always decreases in the recursive calls. For the recursive call $gcdy(x$ mod $y)$, the recursion variant x mod y is less than y, the variant for $gcdxy$.

Chapter 6

GCD in COQ

```
Fixpoint Pgcdn (n: nat) (a b : positive) : positive :=
  match n with
    | O => 1
    | S n =>
      match a,b with
        | xH, _ => 1
        | _, xH => 1
        | xO a, xO b => xO (Pgcdn n a b)
        | a, xO b => Pgcdn n a b
        | xO a, b => Pgcdn n a b
        | xI a', xI b' =>
          match Pcompare a' b' Eq with
            | Eq => a
            | Lt => Pgcdn n (b'-a') a
            | Gt => Pgcdn n (a'-b') b
          end
      end
  end.

Definition Zgcd (a b : Z) : Z :=
  match a,b with
    | Z0,_ => Zabs b
    | _,Z0 => Zabs a
    | Zpos a, Zpos b => Zpos (Pgcd a b)
    | Zpos a, Zneg b => Zpos (Pgcd a b)
    | Zneg a, Zpos b => Zpos (Pgcd a b)
    | Zneg a, Zneg b => Zpos (Pgcd a b)
  end.

Lemma Zgcd_is_gcd : forall a b, Zis_gcd a b (Zgcd a b).
```

Chapter 7

GCD in Nuprl by Anne Trostle

Quoted from [Tros13]:

Here we show how to use the Nuprl proof assistant to develop an existence proof for the greatest common divisor of two natural numbers. We then take the proof a step further and show that the greatest common divisor, or GCD, can be calculated as a linear combination of the two numbers. For each proof, we also show that Nuprl can extract a *program* from the proof that can be used to perform calculations.

The greatest common divisor is defined in Nuprl as follows:

Defintion 1: gcd_p

$GCD(m : n : g) == (g|m) \land (g|n) \land (\forall z : Z.(((z|m) \land (z|m)) \rightarrow (z|g)))$

Defintion 2: divides

$b|a == \exists c : Z.(a = (b * c))$

In words, Definition 1 means that g is the greatest common divisor of m and n when g divides both m and n, and any other common divisor of m and n divides g.

To prove that the GCD exists, we are going to use Euclid's algorithm, whicc is based on the property that for two integers m and n, the GCD of m and n is equivalent to the GCD of n and the remainder from $m \div n$:

Lemma 1: div_rem_gcd_anne

$\forall m : Z. \quad \forall n : \mathbb{Z}^{-0}, \quad \forall g : \mathbb{Z}.(GCD(m; n; g) \iff GCD(n; m \text{ rem } n; g))$

Another useful fact about the GCD is that the GCD of an integer z and 0 is z. A proof of this property can be done by showing that each part of Definition 1 is satisfied.

Lemma 2: gcd_p_zero

$\forall z : \mathbb{Z}. \quad GCD(z; 0; z)$

From these properties we can see a method for calculating the greatest common divisor of two numbers: continue finding remainders until you reach 0 and then use the fact that the GCD of an integer z and 0 is z. Since the GCD stays the same as you reduce the terms, z is also the GCD of the original pair of numbers. This is Euclid's algorithm. Here is an example of how it works, using 18 and 12:

$$
\begin{array}{rcl}
\text{GCD}(18;12;g) & = & \text{GCD}(12;18 \text{ rem } 12;g) \\
& = & \text{GCD}(12;6;g \\
& = & \text{GCD}(6;12 \text{ rem } 6;g) \\
& = & \text{GCD}(6;0;g) \\
& \rightarrow & g = 6
\end{array}
$$

Using this idea we can not only prove that the GCD exists but we can also construct a method for actually computing the GCD. A great feature of Nuprl is that when we run a constructive existence proof, we can extract a program from it and use the program to perform calculations. In the next section we show in detail how to develop a constructive existence proof for the GCD using induction. Induction proofs often to hand-in-hand with recursive programs, and sure enough, a very clean recursive program can be extracted from the proof, and this program follows exactly the method we just came up with:

$\lambda n.letrcgcd(n) =$
$\lambda m.if n = 0 then m$
$else (gcd(m rem n)n)$
$in gcd(n)$

The program here is an example of *currying*: a function of n that results in another function which then uses m. This isn't necessarily intuitive, since when we think of the GCD we think of a function of a pair (or more) of numbers, so we might expect the program to start with something like "$\gcd(m, n) = \ldots$". But the proof that follows uses natural induction on a single variable and flows very nicely, giving reason to prefer the curried function here. To develop a proof that produces a function of the pair (m, n) would require induction on the pair itself which isn't as intuitive or easy to understand as natural induction on a single variable.

Chapter 8

Software Details

8.1 Installed Software

Install CLANG, LLVM

`http://llvm.org/releases/download.html`

Install OCAML

`sudo apt-get install ocaml`

An OCAML version of gcd would be written

```
let rec gcd a b = if b = 0 then a else gcd b (a mod b)
val gcd : int -> int -> int = <fun>
```

Chapter 9

Temporal Logic of Actions (TLA)

Sloppiness is easier than precision and rigor – Leslie Lamport [Lamp14a]

Leslie Lamport [Lamp14] [Lamp16] on 21st Century Proofs.

A method of writing proofs is described that makes it harder to prove things that are not true. The method, based on hierarchical structuring, is simple and practical. The author's twenty years of experience writing such proofs is discussed.

Lamport points out that proofs need rigor and precision. Structure and Naming are important. Every step of the proof names the facts it uses.

Quoting from [Lamp16]:

Broadly speaking, a TLA+ proof is a collection of *claims*, arranged in a hierarchical structure which we describe below, where each claim has an *assertion* that is either *unjustified* or justified by a collection of *cited facts*. The purpose of TLAPS is to check the user-provided proofs of theorems, that is, to check that the hierarchy of claims indeed establishes the truth of the theorem if the claims were true, and then to check that the assertion of every justified claim indeed is implied *by* its cited facts. If a TLA+ theorem has a proof with no unjustified claims, then, as a result of checking the proof, TLAPS verifies the truth of the theorem.

9.1   The algorithm

The well-known Euclidean algorithm can be written in the PlusCal language as follows:

```
--algorithm Euclid {
  variables x \in 1..M, y \in 1..N, x0 = x, y0 = y;
  {
    while (x # y) {
      if (x < y) { y := y - x; }
      else { x := x-y; }
    };
    assert x = GCD(x0, y0) /\ y = GCD(x0, y0)
  }
```

The PlusCal translator translates this algorithm into a TLA+ specification that we could prove correct. However, in this tutorial, we shall write a somewhat simpler specification of Euclid's algorithm directly in TLA+.

9.1.1   Creating a new TLA+ module

In order to get the definitions of arithmetic operators (+, −, etc.), we shall make this specification *extend* the `Integers` standard module.

```
-------------------- Module Euclid ---------------------
EXTENDS Integers
```

9.1.2   Definitions

We shall then define the GCD of two integers. For that purpose, let us define the predicate "p divides q" as follows: p divides q iff there exists some integer d in the interval 1..q such that q is equal to p times d.

```
p | q == \E d \in 1..q : q = p * d
```

We then define the set of divisors of an integer q as the sets of integers which both belong to the interval 1..q and divide q:

```
Divisors(q) == {d \in 1..q : d | q}
```

We define the maximum of a set S as one of the elements of this set which is greater than or equal to all the other elements:

```
Maximum(S) == CHOOSE x \in S : \A y \in S : x >= y
```

And finally, we define the GCD of two integers p and q to be the maximum of the intersection of Divisors(p) and Divisors(a):

```
GCD(p,q) == Maximum(Divisors(p) \cap Divisors(q))
```

For convenience, we shall also define the set of all positive integers as:

```
Number = Nat \ {0}
```

9.1.3   Constants and variables

We then define the two constants and two variables needed to describe the Euclidean algorithm, where M and N are the values whose GCD is to be computed:

```
CONSTANTS M, N
VARIABLES x, y
```

9.1.4   The specification

We define the initial state of the Euclidean algorithm as follows:

```
Init == (x = M) /\ (y = N)
```

In the Euclidean algorithm, two actions can be performed:

- set the value of y to y - x if x < y

- set the value of x to x - y if x > y

These actions are again written as a definition of `Next`, which specifies the next-state relation. In TLA+, a primed variable refers to its value at the next state of the algorithm.

```
Next == \/ /\ x < y
           /\ y' = y - x
           /\ x' = x
        \/ /\ y < x
           /\ x' = x-y
           /\ y' = y
```

The specification of the algorithm asserts that the variables have the correct initial values and, in each execution step, either a `Next` action is performed or x and y keep the same values:

```
Spec == Init /\ [][Next]_<<x,y>>
```

(For reasons that are irrelevant to this algorithm, TLA specifications always allow *stuttering steps* that leave all the variables unchanged.)

We want to prove that the algorithm always satisfies the following property:

```
ResultCorrect == (x = y) => x = GCD(M, N)
```

Hence we want to prove the following theorem named Correctness:

```
THEOREM Correctness == Spec => []ResultCorrect
```

9.1.5   Summary

```
-------------------- Module Euclid ---------------------
EXTENDS Integers

p | q == \E d \in 1..q : q = p * d
Divisors(q) == {d \in 1..q : d | q}
Maximum(S) == CHOOSE x \in S : \A y \in S : x >= y
GCD(p,q) == Maximum(Divisors(p) \cap Divisors(q))
Number == Nat \ {0}

CONSTANTS M, N
VARIABLES x, y

Init == (x = M) /\ (y = N)

Next == \/ /\ x < y
           /\ y' = y - x
           /\ x' = x
        \/ /\ y < x
           /\ x' = x-y
           /\ y' = y

Spec == Init /\ [][Next]_<<x,y>>

ResultCorrect == (x = y) => x = GCD(M,N)

THEOREM Correctness == Spec => []ResultCorrect
```

9.2   A simple proof

9.2.1   The invariant

Intuitively, the theorem Correctness holds because the implementation guarantees the following *invariant*

```
InductiveInvariant == /\ x \in Number
                      /\ y \in Number
                      /\ GCD(x, y) = GCD(M, N)
```

That is, `InductiveInvariant` holds for the initial state (i.e., the state specified by `Init`) and is preserved by the next-state relation `[Next]`$\_<< x, y >>$

9.2.2   Checking proofs

First we need to assume that constants M and N are not equal to zero

```
ASSUME NumberAssumption == M \in Number /\ N \in Number
```

Let us then prove that `InductiveInvariant` holds for the initial state.

```
THEOREM InitProperty == Init => InductiveInvariant
```

To check whether TLAPS can prove that theorem by itself, we declare its proof obvious.

```
THEOREM InitProperty == Init => InductiveInvariant
  OBVIOUS
```

We now ask TLAPS to prove that theorem. But TLAPS does not know how to prove the proof obligation corresponding to that proof. It prints that obligation and reports failures to three backends, Zenon, Isabelle, and SMT. The default behavior of TLAPS is to send obligations first to an SMT solver (by default CVC3), then if that fails to the automatic prover Zenon, then if Zenon fails to Isabelle (with the tactic "auto").

9.2.3   Using facts and definitions

The obligation cannot be proved because TLAPS treats the symbols `Init` and `InductiveInvariant` as opaque identifiers unless it is explicitly instructed to expand their definitions using the directive `DEF`. The main purpose of this treatment of definitions is to make proof-checking tractable, because expanding definitions can arbitrarily increase the size of expressions. Explicit use of definitions is also a good hint to the (human) reader to look only at the listed definitions to understand a proof step. In that precise case, we can ask TLAPS to expand definitions of `Init` and `InductiveInvariant`, by replacing the proof `OBVIOUS` by the proof `BY DEF Init, InductiveInvariant`. In the obligations sent to the backends, the definitions of `Init` and `InductiveInvariant` have been expanded.

Unfortunately, none of the back-ends could prove that obligation. As with `definitions`, we have to specify which facts are *usable*. In this case, we have to make the fact `NumberAssumption` usable by changing the proof to

```
THEOREM InitProperty == Init => InductiveInvariant
  BY NumberAssumption DEF Init, InductiveInvariant
```

The general form of a `BY` proof is:

$$\text{BY } e_1, \ldots, e_m \text{ DEF } d_1, \ldots, d_n$$

which claims that the assertion follows by assuming $e_1, \ldots, e_m$ and expanding the definitions $d_1, \ldots, d_n$. It is the job of TLAPS to then check this claim, and also to check that the cited facts $e_1, \ldots, e_m$ are indeed true.

Finally, SMT succeeds in proving that obligation.

```
-------------------- Module Euclid ---------------------
EXTENDS Integers


p | q == \E d \in 1..q : q = p * d
Divisors(q) == {d \in 1..q : d | q}
Maximum(S) == CHOOSE x \in S : \A y \in S : x >= y
GCD(p,q) == Maximum(Divisors(p) \cap Divisors(q))
Number == Nat \ {0}


CONSTANTS M, N
VARIABLES x, y


Init == (x = M) /\ (y = N)


Next == \/ /\ x < y
           /\ y' = y - x
           /\ x' = x
        \/ /\ y < x
           /\ x' = x-y
           /\ y' = y


Spec == Init /\ [][Next]_<<x,y>>


ResultCorrect == (x = y) => x = GCD(M,N)


InductiveInvariant == /\ x \in Number
                      /\ y \in Number
                      /\ GCD(x, y) = GCD(M, N)


ASSUME NumberAssumption == M \in Number /\ N \in Number


THEOREM InitProperty == Init => InductiveInvariant
  BY NumberAssumption DEF Init, InductiveInvariant


THEOREM Correctness == Spec => []ResultCorrect
```

9.3 Divisibility Definition

In Shoup [Sho08] we find the divisibility definition.

Given the integers, a and b

$$a \text{ divides } b \implies az = b \text{ for some } z$$

so or all a,b, and c

$$a|a, \quad 1|a, \quad and \quad a|0$$

because $a \cdot 1 = a$, $1 \cdot a = a$, and $a \cdot 0 = 0$

$$0|a \iff a = 0$$

$$a|b \iff -a|b \iff a|-b$$

$$a|b \text{ and } a|c \implies a|(b+c)$$

$$a|b \text{ and } b|c \implies a|c$$

$$a|b \text{ and } b \neq 0 \implies 1 \leq |a| \leq |b|$$

$$az = b \neq 0 \text{ and } a \neq 0 \text{ and } z \neq 0 \implies |a| \geq 1 \text{ and } |z| \geq 1$$

$$a|b \text{ and } b|a \implies a = \pm b$$

proof:

$$a|b \implies |a| \leq |b|; b|a \implies |b| \leq |a|; \text{ therefore } |a| = |b| \implies a = \pm b$$

$$a|1 \iff a = \pm 1$$

Chapter 10

COQ proof of GCD

10.1 Basics of the Calculus of Constructions

Coquand [Coqu86] [Wiki17] defines the Calculus of Constructions which can be considered an extension of the Curry-Howard Isomorphism. The components are

10.1.1 Terms

A *term* in the calculus of constructions is constructed using the following rules:

- **T** is a term (also called *Type*)
- **P** is a term (also called *Prop*, the type of all propositions)
- Variables $(x, y, \ldots)$ are terms
- if **A** and **B** are terms, then so are
  - (A, B)
  - $(\lambda x : A, B)$
  - $(\forall x : A, B)$

The calculus of constructions has five kinds of objects:

1. *proofs*, which are terms whose types are *propositions*
2. *propositions*, which are also known as *small types*
3. *predicates*, which are functions that return propositions
4. *large types*, which are the types of predicates. **P** is an example of a large type)
5. **T** itself, which is the type of large types.

10.1.2 Judgements

The calculus of constructions allows proving **typing judgements**

$$x_1 : A_1, x_2 : A_2, \ldots \vdash t : B$$

which can be read as the implication

> if variables $x_1, x_2, \ldots,$ have types $A_1, A_2, \ldots,$ then term t has type B

The valid judgements for the calculus of constructions are derivable from a set of inference rules. In the following, we use Γ to mean a sequence of type assignments $x_1 : A_1, x_2 : A_2, \ldots,$ and we use $\mathbf{K}$ to mean either $\mathbf{P}$ or $\mathbf{T}$. We shall write $A : B : C$ to mean "A has type B, and B has type C". We shall write $B(x := N)$ to mean the result of substituting the term N for the variable x in the term B.

An inference rule is written in the form

$$\frac{\mathbf{\Gamma} \vdash \mathbf{A} : \mathbf{B}}{\mathbf{\Gamma'} \vdash \mathbf{C} : \mathbf{D}}$$

which means

> if $\mathbf{\Gamma} \vdash \mathbf{A} : \mathbf{B}$ is a valid judgement, then so is $\mathbf{\Gamma'} \vdash \mathbf{C} : \mathbf{D}$

10.1.3   Inference Rules

In Frade [Frad08] we find:

(axiom)	$() \vdash s_1 : s_2$	if $(s_1, s_2) \in A$
(start)	$\dfrac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A}$	if $x \notin \mathrm{dom}(\Gamma)$
(weakening)	$\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma, x : B \vdash M : A}$	if $x \notin \mathrm{dom}(\Gamma)$
(product)	$\dfrac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\prod x : A.B) : s_3}$	if $(s_1, s_2, s_3) \in \mathbb{R}$
(application)	$\dfrac{\Gamma \vdash M : (\prod x : A.B) \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[x := N]}$	
(abstraction)	$\dfrac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash (\prod x : A.B) : s}{\Gamma \vdash \lambda x : A.M : (\prod x : A.B)}$	
(conversion)	$\dfrac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s}{\Gamma \vdash M : B}$	if $A =_\beta B$

10.1.4   Defining Logical Operators

$$
\begin{aligned}
A \Rightarrow B &\equiv & \forall x : A.B & \qquad (x \notin B) \\
A \wedge B &\equiv & \forall C : P.(A \Rightarrow B \Rightarrow C) \Rightarrow C & \\
A \vee B &\equiv & \forall C : P.(A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C & \\
\neg A &\equiv & \forall C : P.(A \Rightarrow C) & \\
\exists x : A.B &\equiv & \forall C : P.(\forall x : A.(B \Rightarrow C)) \Rightarrow C &
\end{aligned}
$$

10.1.5 Defining Types

The basic data types used in computer science can be defined within the Calculus of Constructions:

Booleans

$$\forall A : P.A \Rightarrow A \Rightarrow A$$

Naturals

$$\forall A : P.(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$$

Product $A \times B$

$$A \wedge B$$

Disjoint Union $A + B$

$$A \vee B$$

Note that Booleans and Naturals are defined in the same way as in Church encoding. However additional problems raise from propositional extensionality and proof irrelevance.

10.2 Why does COQ have Prop?

From a stackexchange post [Stac17] we find the question:

"Coq has a type `Prop` of proof irrelevant propositions which are discarded during extraction. What are the reasons for having this if we use Coq only for proofs? Prop is impredicative, however, Coq automatically infers universe indexes and we can use `Type(i)` instead everywhere. It seems `Prop` complicates everything a lot."

Prop is very useful for program extraction because it allows us to delete parts of code that are useless. For example, to extract a sorting algorithm we would prove the statement "for every list l there is a list k such that k is ordered and k is a permutation of l". If we write this down in Coq and extract without using Prop, we will get:

1. "for all l there is a k" which gives us a map **sort** which takes lists to lists,

2. "such that k is ordered" will give a function **verify** which runs through k and checks that it is sorted, and

3. "k is a permutation of l" will give a permutation **p1** which takes l to k. Note that **p1** is not just a mapping, but also the inverse mapping together with programs verifying that the two maps really are inverses.

While the extra stuff is not totally useless, in many applications we want to get ride of it and keep just **sort**. This can be accomplished if we use `Prop` to state "k is ordered" and "k is a permutation of l", but *not* "for all l there is k".

In general, a common way to extract code is to consider a statement of the form

$$\forall x : A.\exists y : B.\phi(x, y)$$

where x is input, y is output, and $\phi(x, y)$ explains what it means for y to be a correct output. (In the above example A and B are the types of lists and $\phi(l, k)$ is "k is ordered and k is a permutation of l.") if ϕ is in `Prop` then extraction gives a map $f : A \Rightarrow B$ such that $\phi(x, f(x))$ holds for all $x \in A$. If ϕ is in `Set` then we also get a function g such that $g(x)$ is the proof that $\phi(x, f(x))$ holds, for all $x \in A$. Often the proof is computationally useless and we prefer to get rid of it, especially when it is nested deeply inside some other statement. `Prop` gives use the possibility to do so.

There is a question whether we could avoid `Prop` altogether by automatically optimizing away "useless extracted code". To some extent we can do that, for instance all code extracted from the negative fragment of logic (stuff build from the empty type, unit type, products) is useless as it just shuffles around the unit. But there are genuine design decisions one has to make when using `Prop`. Here is a simple example, where $\sum$ means that we are in `Type` and $\exists$ means we are in `Prop`. If we extract from

$$\prod_{n:N} \sum_{b:[0,1]} \sum_{k:N} \quad n = 2 \cdot k + b$$

we will get an inductivea program which decomposes n into its lowest bit b and the remaining bits k, i.e., it computes everything. If we extract from

$$\prod_{n:N} \sum_{b:[0,1]} \underset{k:N}{\exists} \quad n = 2 \cdot k + b$$

then the program will only compute the lowest bit b. The machine cannot tell which is the correct one, the user has to tell it what he wants.

10.3   Source code of COQ GCD Proof

This is the proof of GCD [Coqu16a] in the COQ [Coqu16] sources:

```
Library Coq.ZArith.Znumtheory

Require Import ZArith_base.
Require Import ZArithRing.
Require Import Zcomplements.
Require Import Zdiv.
Require Import Wf_nat.

For compatibility reasons, this Open Scope isn't local as it should

Open Scope Z_scope.

This file contains some notions of number theory upon Z numbers:

    a divisibility predicate Z.divide
    a gcd predicate gcd
    Euclid algorithm euclid
    a relatively prime predicate rel_prime
    a prime predicate prime
    properties of the efficient Z.gcd function
```

```
Notation Zgcd := Z.gcd (compat "8.3").
Notation Zggcd := Z.ggcd (compat "8.3").
Notation Zggcd_gcd := Z.ggcd_gcd (compat "8.3").
Notation Zggcd_correct_divisors := Z.ggcd_correct_divisors (compat "8.3").
Notation Zgcd_divide_l := Z.gcd_divide_l (compat "8.3").
Notation Zgcd_divide_r := Z.gcd_divide_r (compat "8.3").
Notation Zgcd_greatest := Z.gcd_greatest (compat "8.3").
Notation Zgcd_nonneg := Z.gcd_nonneg (compat "8.3").
Notation Zggcd_opp := Z.ggcd_opp (compat "8.3").
```

The former specialized inductive predicate Z.divide is now a generic existential predicate.

```
Notation Zdivide := Z.divide (compat "8.3").
```

Its former constructor is now a pseudo-constructor.

```
Definition Zdivide_intro a b q (H:b=q*a) : Z.divide a b := ex_intro _ q H.
```

Results concerning divisibility

```
Notation Zdivide_refl := Z.divide_refl (compat "8.3").
Notation Zone_divide := Z.divide_1_l (compat "8.3").
Notation Zdivide_0 := Z.divide_0_r (compat "8.3").
Notation Zmult_divide_compat_l := Z.mul_divide_mono_l (compat "8.3").
Notation Zmult_divide_compat_r := Z.mul_divide_mono_r (compat "8.3").
Notation Zdivide_plus_r := Z.divide_add_r (compat "8.3").
Notation Zdivide_minus_l := Z.divide_sub_r (compat "8.3").
Notation Zdivide_mult_l := Z.divide_mul_l (compat "8.3").
Notation Zdivide_mult_r := Z.divide_mul_r (compat "8.3").
Notation Zdivide_factor_r := Z.divide_factor_l (compat "8.3").
Notation Zdivide_factor_l := Z.divide_factor_r (compat "8.3").
```

```
Lemma Zdivide_opp_r a b : (a | b) -> (a | - b).
```

```
Lemma Zdivide_opp_r_rev a b : (a | - b) -> (a | b).
```

```
Lemma Zdivide_opp_l a b : (a | b) -> (- a | b).
```

```
Lemma Zdivide_opp_l_rev a b : (- a | b) -> (a | b).
```

```
Theorem Zdivide_Zabs_l a b : (Z.abs a | b) -> (a | b).
```

```
Theorem Zdivide_Zabs_inv_l a b : (a | b) -> (Z.abs a | b).
```

```
Hint Resolve Z.divide_refl Z.divide_1_l Z.divide_0_r: zarith.
Hint Resolve Z.mul_divide_mono_l Z.mul_divide_mono_r: zarith.
Hint Resolve Z.divide_add_r Zdivide_opp_r Zdivide_opp_r_rev Zdivide_opp_l
  Zdivide_opp_l_rev Z.divide_sub_r Z.divide_mul_l Z.divide_mul_r
  Z.divide_factor_l Z.divide_factor_r: zarith.
```

Auxiliary result.

```
Lemma Zmult_one x y : x >= 0 -> x * y = 1 -> x = 1.
```

Only 1 and -1 divide 1.

Notation Zdivide_1 := Z.divide_1_r (compat "8.3").

If a divides b and b divides a then a is b or -b.

Notation Zdivide_antisym := Z.divide_antisym (compat "8.3").
Notation Zdivide_trans := Z.divide_trans (compat "8.3").

If a divides b and b<>0 then |a| <= |b|.

Lemma Zdivide_bounds a b : (a | b) -> b <> 0 -> Z.abs a <= Z.abs b.

Z.divide can be expressed using Z.modulo.

Lemma Zmod_divide : forall a b, b<>0 -> a mod b = 0 -> (b | a).

Lemma Zdivide_mod : forall a b, (b | a) -> a mod b = 0.

Z.divide is hence decidable

Lemma Zdivide_dec a b : {(a | b)} + {~ (a | b)}.

Theorem Zdivide_Zdiv_eq a b : 0 < a -> (a | b) -> b = a * (b / a).

Theorem Zdivide_Zdiv_eq_2 a b c :
 0 < a -> (a | b) -> (c * b) / a = c * (b / a).

Theorem Zdivide_le: forall a b : Z,
 0 <= a -> 0 < b -> (a | b) -> a <= b.

Theorem Zdivide_Zdiv_lt_pos a b :
 1 < a -> 0 < b -> (a | b) -> 0 < b / a < b .

Lemma Zmod_div_mod n m a:
 0 < n -> 0 < m -> (n | m) -> a mod n = (a mod m) mod n.

Lemma Zmod_divide_minus a b c:
 0 < b -> a mod b = c -> (b | a - c).

Lemma Zdivide_mod_minus a b c:
 0 <= c < b -> (b | a - c) -> a mod b = c.

Greatest common divisor (gcd).
There is no unicity of the gcd; hence we define the predicate Zis_gcd a b g expressing that g is a gcd of a and

Inductive Zis_gcd (a b g:Z) : Prop :=
 Zis_gcd_intro :
  (g | a) ->
  (g | b) ->
  (forall x, (x | a) -> (x | b) -> (x | g)) ->
  Zis_gcd a b g.

Trivial properties of gcd

```
Lemma Zis_gcd_sym : forall a b d, Zis_gcd a b d -> Zis_gcd b a d.
```

```
Lemma Zis_gcd_0 : forall a, Zis_gcd a 0 a.
```

```
Lemma Zis_gcd_1 : forall a, Zis_gcd a 1 1.
```

```
Lemma Zis_gcd_refl : forall a, Zis_gcd a a a.
```

```
Lemma Zis_gcd_minus : forall a b d, Zis_gcd a (- b) d -> Zis_gcd b a d.
```

```
Lemma Zis_gcd_opp : forall a b d, Zis_gcd a b d -> Zis_gcd b a (- d).
```

```
Lemma Zis_gcd_0_abs a : Zis_gcd 0 a (Z.abs a).
```

```
Hint Resolve Zis_gcd_sym Zis_gcd_0 Zis_gcd_minus Zis_gcd_opp: zarith.
```

```
Theorem Zis_gcd_unique: forall a b c d : Z,
 Zis_gcd a b c -> Zis_gcd a b d -> c = d \/ c = (- d).
```

Extended Euclid algorithm.
Euclid's algorithm to compute the gcd mainly relies on the following property.

```
Lemma Zis_gcd_for_euclid :
  forall a b d q:Z, Zis_gcd b (a - q * b) d -> Zis_gcd a b d.
```

```
Lemma Zis_gcd_for_euclid2 :
  forall b d q r:Z, Zis_gcd r b d -> Zis_gcd b (b * q + r) d.
```

We implement the extended version of Euclid's algorithm, i.e. the one computing Bezout's coefficients as it comp

Section extended_euclid_algorithm.

  Variables a b : Z.

The specification of Euclid's algorithm is the existence of u, v and d such that ua+vb=d and (gcd a b d).

```
  Inductive Euclid : Set :=
    Euclid_intro :
    forall u v d:Z, u * a + v * b = d -> Zis_gcd a b d -> Euclid.
```

The recursive part of Euclid's algorithm uses well-founded recursion of non-negative integers. It maintains 6 in

```
  Lemma euclid_rec :
    forall v3:Z,
      0 <= v3 ->
      forall u1 u2 u3 v1 v2:Z,
        u1 * a + u2 * b = u3 ->
        v1 * a + v2 * b = v3 ->
        (forall d:Z, Zis_gcd u3 v3 d -> Zis_gcd a b d) -> Euclid.
```

We get Euclid's algorithm by applying euclid_rec on 1,0,a,0,1,b when b>=0 and 1,0,a,0,-1,-b when b<0.

```
  Lemma euclid : Euclid.

End extended_euclid_algorithm.

Theorem Zis_gcd_uniqueness_apart_sign :
  forall a b d d':Z, Zis_gcd a b d -> Zis_gcd a b d' -> d = d' \/ d = - d'.

Bezout's coefficients

Inductive Bezout (a b d:Z) : Prop :=
  Bezout_intro : forall u v:Z, u * a + v * b = d -> Bezout a b d.

Existence of Bezout's coefficients for the gcd of a and b

Lemma Zis_gcd_bezout : forall a b d:Z, Zis_gcd a b d -> Bezout a b d.

gcd of ca and cb is c gcd(a,b).

Lemma Zis_gcd_mult :
  forall a b c d:Z, Zis_gcd a b d -> Zis_gcd (c * a) (c * b) (c * d).

Relative primality

Definition rel_prime (a b:Z) : Prop := Zis_gcd a b 1.

Bezout's theorem: a and b are relatively prime if and only if there exist u and v such that ua+vb = 1.

Lemma rel_prime_bezout : forall a b:Z, rel_prime a b -> Bezout a b 1.

Lemma bezout_rel_prime : forall a b:Z, Bezout a b 1 -> rel_prime a b.

Gauss's theorem: if a divides bc and if a and b are relatively prime, then a divides c.

Theorem Gauss : forall a b c:Z, (a | b * c) -> rel_prime a b -> (a | c).

If a is relatively prime to b and c, then it is to bc

Lemma rel_prime_mult :
  forall a b c:Z, rel_prime a b -> rel_prime a c -> rel_prime a (b * c).

Lemma rel_prime_cross_prod :
  forall a b c d:Z,
    rel_prime a b ->
    rel_prime c d -> b > 0 -> d > 0 -> a * d = b * c -> a = c /\ b = d.

After factorization by a gcd, the original numbers are relatively prime.

Lemma Zis_gcd_rel_prime :
  forall a b g:Z,
    b > 0 -> g >= 0 -> Zis_gcd a b g -> rel_prime (a / g) (b / g).

Theorem rel_prime_sym: forall a b, rel_prime a b -> rel_prime b a.

Theorem rel_prime_div: forall p q r,
```

```
  rel_prime p q -> (r | p) -> rel_prime r q.

Theorem rel_prime_1: forall n, rel_prime 1 n.

Theorem not_rel_prime_0: forall n, 1 < n -> ~ rel_prime 0 n.

Theorem rel_prime_mod: forall p q, 0 < q ->
 rel_prime p q -> rel_prime (p mod q) q.

Theorem rel_prime_mod_rev: forall p q, 0 < q ->
 rel_prime (p mod q) q -> rel_prime p q.

Theorem Zrel_prime_neq_mod_0: forall a b, 1 < b -> rel_prime a b -> a mod b <> 0.

Primality

Inductive prime (p:Z) : Prop :=
  prime_intro :
    1 < p -> (forall n:Z, 1 <= n < p -> rel_prime n p) -> prime p.

The sole divisors of a prime number p are -1, 1, p and -p.

Lemma prime_divisors :
  forall p:Z,
    prime p -> forall a:Z, (a | p) -> a = -1 \/ a = 1 \/ a = p \/ a = - p.

A prime number is relatively prime with any number it does not divide

Lemma prime_rel_prime :
  forall p:Z, prime p -> forall a:Z, ~ (p | a) -> rel_prime p a.

Hint Resolve prime_rel_prime: zarith.

As a consequence, a prime number is relatively prime with smaller numbers

Theorem rel_prime_le_prime:
 forall a p, prime p -> 1 <= a < p -> rel_prime a p.

If a prime p divides ab then it divides either a or b

Lemma prime_mult :
  forall p:Z, prime p -> forall a b:Z, (p | a * b) -> (p | a) \/ (p | b).

Lemma not_prime_0: ~ prime 0.

Lemma not_prime_1: ~ prime 1.

Lemma prime_2: prime 2.

Theorem prime_3: prime 3.

Theorem prime_ge_2 p : prime p -> 2 <= p.

Definition prime' p := 1<p /\ (forall n, 1<n<p -> ~ (n|p)).
```

```
Lemma Z_0_1_more x : 0<=x -> x=0 \/ x=1 \/ 1<x.

Theorem prime_alt p : prime' p <-> prime p.

Theorem square_not_prime: forall a, ~ prime (a * a).

Theorem prime_div_prime: forall p q,
 prime p -> prime q -> (p | q) -> p = q.

we now prove that Z.gcd is indeed a gcd in the sense of Zis_gcd.

Notation Zgcd_is_pos := Z.gcd_nonneg (compat "8.3").

Lemma Zgcd_is_gcd : forall a b, Zis_gcd a b (Z.gcd a b).

Theorem Zgcd_spec : forall x y : Z, {z : Z | Zis_gcd x y z /\ 0 <= z}.

Theorem Zdivide_Zgcd: forall p q r : Z,
 (p | q) -> (p | r) -> (p | Z.gcd q r).

Theorem Zis_gcd_gcd: forall a b c : Z,
 0 <= c -> Zis_gcd a b c -> Z.gcd a b = c.

Notation Zgcd_inv_0_l := Z.gcd_eq_0_l (compat "8.3").
Notation Zgcd_inv_0_r := Z.gcd_eq_0_r (compat "8.3").

Theorem Zgcd_div_swap0 : forall a b : Z,
 0 < Z.gcd a b ->
 0 < b ->
 (a / Z.gcd a b) * b = a * (b/Z.gcd a b).

Theorem Zgcd_div_swap : forall a b c : Z,
 0 < Z.gcd a b ->
 0 < b ->
 (c * a) / Z.gcd a b * b = c * a * (b/Z.gcd a b).

Notation Zgcd_comm := Z.gcd_comm (compat "8.3").

Lemma Zgcd_ass a b c : Z.gcd (Z.gcd a b) c = Z.gcd a (Z.gcd b c).

Notation Zgcd_Zabs := Z.gcd_abs_l (compat "8.3").
Notation Zgcd_0 := Z.gcd_0_r (compat "8.3").
Notation Zgcd_1 := Z.gcd_1_r (compat "8.3").

Hint Resolve Z.gcd_0_r Z.gcd_1_r : zarith.

Theorem Zgcd_1_rel_prime : forall a b,
 Z.gcd a b = 1 <-> rel_prime a b.

Definition rel_prime_dec: forall a b,
 { rel_prime a b }+{ ~ rel_prime a b }.

Definition prime_dec_aux:
```

```
forall p m,
  { forall n, 1 < n < m -> rel_prime n p } +
  { exists n, 1 < n < m /\ ~ rel_prime n p }.

Definition prime_dec: forall p, { prime p }+{ ~ prime p }.

Theorem not_prime_divide:
 forall p, 1 < p -> ~ prime p -> exists n, 1 < n < p /\ (n | p).
```

Chapter 11

LEAN proof of GCD

This is the proof of GCD [Avig14] in the LEAN [Avig16] sources:

```
/-
Copyright (c) 2014 Jeremy Avigad. All rights reserved.
Released under Apache 2.0 license as described in the file LICENSE.
Authors: Jeremy Avigad, Leonardo de Moura

Definitions and properties of gcd, lcm, and coprime.
-/
import .div
open eq.ops well_founded decidable prod

namespace nat

/- gcd -/

private definition pair_nat.lt : nat  nat  nat  nat  Prop := measure pr
private definition pair_nat.lt.wf : well_founded pair_nat.lt :=
intro_k (measure.wf pr) 20  -- we use intro_k to be able to execute gcd efficiently in the kernel

local attribute pair_nat.lt.wf [instance]        -- instance will not be saved in .olean
local infixl ‘  ‘:50 := pair_nat.lt

private definition gcd.lt.dec (x y : nat) : (succ y, x % succ y)  (x, succ y) :=
!mod_lt (succ_pos y)

definition gcd.F :  (p : nat  nat), ( p : nat  nat, p  p  nat)  nat
| (x, 0)      f := x
| (x, succ y) f := f (succ y, x % succ y) !gcd.lt.dec

definition gcd (x y : nat) := fix gcd.F (x, y)

theorem gcd_zero_right [simp] (x : nat) : gcd x 0 = x := rfl

theorem gcd_succ [simp] (x y : nat) : gcd x (succ y) = gcd (succ y) (x % succ y) :=
well_founded.fix_eq gcd.F (x, succ y)
```

```
theorem gcd_one_right (n : ) : gcd n 1 = 1 :=
calc gcd n 1 = gcd 1 (n % 1)  : gcd_succ
          ... = gcd 1 0         : mod_one

theorem gcd_def (x : ) :  (y : ), gcd x y = if y = 0 then x else gcd y (x % y)
| 0        := !gcd_zero_right
| (succ y) := !gcd_succ  (if_neg !succ_ne_zero)


theorem gcd_self :  (n : ), gcd n n = n
| 0        := rfl
| (succ n) := calc
    gcd (succ n) (succ n) = gcd (succ n) (succ n % succ n) : gcd_succ
                    ...   = gcd (succ n) 0                   : mod_self

theorem gcd_zero_left :  (n : ), gcd 0 n = n
| 0        := rfl
| (succ n) := calc
    gcd 0 (succ n) = gcd (succ n) (0 % succ n) : gcd_succ
              ... = gcd (succ n) 0              : zero_mod

theorem gcd_of_pos (m : ) {n : } (H : n > 0) : gcd m n = gcd n (m % n) :=
gcd_def m n  if_neg (ne_zero_of_pos H)

theorem gcd_rec (m n : ) : gcd m n = gcd n (m % n) :=
by_cases_zero_pos n
  (calc
        m = gcd 0 m       : gcd_zero_left
      ... = gcd 0 (m % 0) : mod_zero)
  (take n, assume H : 0 < n, gcd_of_pos m H)

theorem gcd.induction {P :      Prop}
                  (m n : )
                  (H0 : m, P m 0)
                  (H1 : m n, 0 < n  P n (m % n)  P m n) :
               P m n :=
induction (m, n) (prod.rec (m, nat.rec ( IH, H0 m)
   ( n v (IH : p, p  (m, succ n)  P (pr p) (pr p)),
     H1 m (succ n) !succ_pos (IH _ !gcd.lt.dec))))

theorem gcd_dvd (m n : ) : (gcd m n  m)  (gcd m n  n) :=
gcd.induction m n
  (take m, and.intro (!one_mul  !dvd_mul_left) !dvd_zero)
  (take m n (npos : 0 < n), and.rec
     (assume (IH : gcd n (m % n)  n) (IH : gcd n (m % n)  (m % n)),
    have H : (gcd n (m % n)  (m / n * n + m % n)), from
      dvd_add (dvd.trans IH !dvd_mul_left) IH,
    have H1 : (gcd n (m % n)  m), from !eq_div_mul_add_mod  H,
    show (gcd m n  m)  (gcd m n  n), from !gcd_rec  (and.intro H1 IH)))

theorem gcd_dvd_left (m n : ) : gcd m n  m := and.left !gcd_dvd

theorem gcd_dvd_right (m n : ) : gcd m n  n := and.right !gcd_dvd
```

```
theorem dvd_gcd {m n k : } : k  m  k  n  k  gcd m n :=
gcd.induction m n (take m, imp.intro)
  (take m n (npos : n > 0)
    (IH : k  n  k  m % n  k  gcd n (m % n))
    (H1 : k  m) (H2 : k  n),
    have H3 : k  m / n * n + m % n, from !eq_div_mul_add_mod  H1,
    have H4 : k  m % n, from nat.dvd_of_dvd_add_left H3 (dvd.trans H2 !dvd_mul_left),
    !gcd_rec  IH H2 H4)

theorem gcd.comm (m n : ) : gcd m n = gcd n m :=
dvd.antisymm
  (dvd_gcd !gcd_dvd_right !gcd_dvd_left)
  (dvd_gcd !gcd_dvd_right !gcd_dvd_left)

theorem gcd.assoc (m n k : ) : gcd (gcd m n) k = gcd m (gcd n k) :=
dvd.antisymm
  (dvd_gcd
    (dvd.trans !gcd_dvd_left !gcd_dvd_left)
    (dvd_gcd (dvd.trans !gcd_dvd_left !gcd_dvd_right) !gcd_dvd_right))
  (dvd_gcd
    (dvd_gcd !gcd_dvd_left (dvd.trans !gcd_dvd_right !gcd_dvd_left))
    (dvd.trans !gcd_dvd_right !gcd_dvd_right))

theorem gcd_one_left (m : ) : gcd 1 m = 1 :=
!gcd.comm  !gcd_one_right

theorem gcd_mul_left (m n k : ) : gcd (m * n) (m * k) = m * gcd n k :=
gcd.induction n k
  (take n, calc gcd (m * n) (m * 0) = gcd (m * n) 0 : mul_zero)
  (take n k,
    assume H : 0 < k,
    assume IH : gcd (m * k) (m * (n % k)) = m * gcd k (n % k),
    calc
      gcd (m * n) (m * k) = gcd (m * k) (m * n % (m * k)) : !gcd_rec
                      ... = gcd (m * k) (m * (n % k))     : mul_mod_mul_left
                      ... = m * gcd k (n % k)             : IH
                      ... = m * gcd n k                   : !gcd_rec)

theorem gcd_mul_right (m n k : ) : gcd (m * n) (k * n) = gcd m k * n :=
calc
  gcd (m * n) (k * n) = gcd (n * m) (k * n) : mul.comm
                  ... = gcd (n * m) (n * k) : mul.comm
                  ... = n * gcd m k         : gcd_mul_left
                  ... = gcd m k * n         : mul.comm

theorem gcd_pos_of_pos_left {m : } (n : ) (mpos : m > 0) : gcd m n > 0 :=
pos_of_dvd_of_pos !gcd_dvd_left mpos

theorem gcd_pos_of_pos_right (m : ) {n : } (npos : n > 0) : gcd m n > 0 :=
pos_of_dvd_of_pos !gcd_dvd_right npos

theorem eq_zero_of_gcd_eq_zero_left {m n : } (H : gcd m n = 0) : m = 0 :=
or.elim (eq_zero_or_pos m)
  (assume H1, H1)
```

```
    (assume H1 : m > 0, absurd H (ne_of_lt (!gcd_pos_of_pos_left H1)))

theorem eq_zero_of_gcd_eq_zero_right {m n : } (H : gcd m n = 0) : n = 0 :=
eq_zero_of_gcd_eq_zero_left (!gcd.comm  H)

theorem gcd_div {m n k : } (H1 : k  m) (H2 : k  n) :
  gcd (m / k) (n / k) = gcd m n / k :=
or.elim (eq_zero_or_pos k)
  (assume H3 : k = 0, by subst k; rewrite *nat.div_zero)
  (assume H3 : k > 0, (nat.div_eq_of_eq_mul_left H3 (calc
        gcd m n = gcd m (n / k * k)              : nat.div_mul_cancel H2
            ... = gcd (m / k * k) (n / k * k) : nat.div_mul_cancel H1
            ... = gcd (m / k) (n / k) * k      : gcd_mul_right)))

theorem gcd_dvd_gcd_mul_left (m n k : ) : gcd m n  gcd (k * m) n :=
dvd_gcd (dvd.trans !gcd_dvd_left !dvd_mul_left) !gcd_dvd_right

theorem gcd_dvd_gcd_mul_right (m n k : ) : gcd m n  gcd (m * k) n :=
!mul.comm  !gcd_dvd_gcd_mul_left

theorem gcd_dvd_gcd_mul_left_right (m n k : ) : gcd m n  gcd m (k * n) :=
dvd_gcd  !gcd_dvd_left (dvd.trans !gcd_dvd_right !dvd_mul_left)

theorem gcd_dvd_gcd_mul_right_right (m n k : ) : gcd m n  gcd m (n * k) :=
!mul.comm  !gcd_dvd_gcd_mul_left_right

/- lcm -/

definition lcm (m n : ) :  := m * n / (gcd m n)

theorem lcm.comm (m n : ) : lcm m n = lcm n m :=
calc
  lcm m n = m * n / gcd m n : rfl
      ... = n * m / gcd m n : mul.comm
      ... = n * m / gcd n m : gcd.comm
      ... = lcm n m            : rfl

theorem lcm_zero_left (m : ) : lcm 0 m = 0 :=
calc
  lcm 0 m = 0 * m / gcd 0 m : rfl
      ... = 0 / gcd 0 m      : zero_mul
      ... = 0                   : nat.zero_div

theorem lcm_zero_right (m : ) : lcm m 0 = 0 := !lcm.comm  !lcm_zero_left

theorem lcm_one_left (m : ) : lcm 1 m = m :=
calc
  lcm 1 m = 1 * m / gcd 1 m : rfl
      ... = m / gcd 1 m     : one_mul
      ... = m / 1             : gcd_one_left
      ... = m                 : nat.div_one

theorem lcm_one_right (m : ) : lcm m 1 = m := !lcm.comm  !lcm_one_left
```

```
theorem lcm_self (m : ) : lcm m m = m :=
have H : m * m / m = m, from
  by_cases_zero_pos m !nat.div_zero (take m, assume H1 : m > 0, !nat.mul_div_cancel H1),
calc
  lcm m m = m * m / gcd m m : rfl
      ... = m * m / m        : gcd_self
      ... = m                : H

theorem dvd_lcm_left (m n : ) : m  lcm m n :=
have H : lcm m n = m * (n / gcd m n), from nat.mul_div_assoc _ !gcd_dvd_right,
dvd.intro H

theorem dvd_lcm_right (m n : ) : n  lcm m n :=
!lcm.comm  !dvd_lcm_left

theorem gcd_mul_lcm (m n : ) : gcd m n * lcm m n = m * n :=
eq.symm (nat.eq_mul_of_div_eq_right (dvd.trans !gcd_dvd_left !dvd_mul_right) rfl)

theorem lcm_dvd {m n k : } (H1 : m  k) (H2 : n  k) : lcm m n  k :=
or.elim (eq_zero_or_pos k)
  (assume kzero : k = 0, !kzero  !dvd_zero)
  (assume kpos : k > 0,
    have mpos : m > 0, from pos_of_dvd_of_pos H1 kpos,
    have npos : n > 0, from pos_of_dvd_of_pos H2 kpos,
    have gcd_pos : gcd m n > 0, from !gcd_pos_of_pos_left mpos,
    obtain p (km : k = m * p), from exists_eq_mul_right_of_dvd H1,
    obtain q (kn : k = n * q), from exists_eq_mul_right_of_dvd H2,
    have ppos : p > 0, from pos_of_mul_pos_left (km  kpos),
    have qpos : q > 0, from pos_of_mul_pos_left (kn  kpos),
    have H3 : p * q * (m * n * gcd p q) = p * q * (gcd m n * k), from
    calc
      p * q * (m * n * gcd p q)
          = m * p * (n * q * gcd p q)       : by rewrite [*mul.assoc, *mul.left_comm q,
                                                         mul.left_comm p]
      ... = k * (k * gcd p q)               : by rewrite [-kn, -km]
      ... = k * gcd (k * p) (k * q)         : by rewrite gcd_mul_left
      ... = k * gcd (n * q * p) (m * p * q) : by rewrite [-kn, -km]
      ... = k * (gcd n m * (p * q))         : by rewrite [*mul.assoc, mul.comm q, gcd_mul_right]
      ... = p * q * (gcd m n * k)           : by rewrite [mul.comm, mul.comm (gcd n m), gcd.comm,
                                                         *mul.assoc],
    have H4 : m * n * gcd p q = gcd m n * k,
      from !eq_of_mul_eq_mul_left (mul_pos ppos qpos) H3,
    have H5 : gcd m n * (lcm m n * gcd p q) = gcd m n * k,
      from !mul.assoc  !gcd_mul_lcm  H4,
    have H6 : lcm m n * gcd p q = k,
      from !eq_of_mul_eq_mul_left gcd_pos H5,
    dvd.intro H6)

theorem lcm.assoc (m n k : ) : lcm (lcm m n) k = lcm m (lcm n k) :=
dvd.antisymm
  (lcm_dvd
    (lcm_dvd !dvd_lcm_left (dvd.trans !dvd_lcm_left !dvd_lcm_right))
    (dvd.trans !dvd_lcm_right !dvd_lcm_right))
  (lcm_dvd
```

```
    (dvd.trans !dvd_lcm_left !dvd_lcm_left)
    (lcm_dvd (dvd.trans !dvd_lcm_right !dvd_lcm_left) !dvd_lcm_right))


/- coprime -/

definition coprime [reducible] (m n : ) : Prop := gcd m n = 1

lemma gcd_eq_one_of_coprime {m n : } : coprime m n  gcd m n = 1 :=
 h, h

theorem coprime_swap {m n : } (H : coprime n m) : coprime m n :=
!gcd.comm  H

theorem dvd_of_coprime_of_dvd_mul_right {m n k : } (H1 : coprime k n) (H2 : k  m * n) : k  m :=
have H3 : gcd (m * k) (m * n) = m, from
  calc
    gcd (m * k) (m * n) = m * gcd k n : gcd_mul_left
                    ... = m * 1       : H1
                    ... = m           : mul_one,
have H4 : (k  gcd (m * k) (m * n)), from dvd_gcd !dvd_mul_left H2,
H3  H4

theorem dvd_of_coprime_of_dvd_mul_left {m n k : } (H1 : coprime k m) (H2 : k  m * n) : k  n :=
dvd_of_coprime_of_dvd_mul_right H1 (!mul.comm  H2)

theorem gcd_mul_left_cancel_of_coprime {k : } (m : ) {n : } (H : coprime k n) :
  gcd (k * m) n = gcd m n :=
have H1 : coprime (gcd (k * m) n) k, from
  calc
    gcd (gcd (k * m) n) k
        = gcd (k * gcd 1 m) n : by rewrite [-gcd_mul_left, mul_one, gcd.comm, gcd.assoc]
    ... = 1                   : by rewrite [gcd_one_left, mul_one, coprime at H, H],
dvd.antisymm
  (dvd_gcd (dvd_of_coprime_of_dvd_mul_left H1 !gcd_dvd_left) !gcd_dvd_right)
  (dvd_gcd (dvd.trans !gcd_dvd_left !dvd_mul_left) !gcd_dvd_right)

theorem gcd_mul_right_cancel_of_coprime (m : ) {k n : } (H : coprime k n) :
  gcd (m * k) n = gcd m n :=
!mul.comm  !gcd_mul_left_cancel_of_coprime H

theorem gcd_mul_left_cancel_of_coprime_right {k m : } (n : ) (H : coprime k m) :
  gcd m (k * n) = gcd m n :=
!gcd.comm  !gcd.comm  !gcd_mul_left_cancel_of_coprime H

theorem gcd_mul_right_cancel_of_coprime_right {k m : } (n : ) (H : coprime k m) :
  gcd m (n * k) = gcd m n :=
!gcd.comm  !gcd.comm  !gcd_mul_right_cancel_of_coprime H

theorem coprime_div_gcd_div_gcd {m n : } (H : gcd m n > 0) :
  coprime (m / gcd m n) (n / gcd m n) :=
calc
  gcd (m / gcd m n) (n / gcd m n) = gcd m n / gcd m n : gcd_div !gcd_dvd_left !gcd_dvd_right
      ... = 1 : nat.div_self H
```

```
theorem not_coprime_of_dvd_of_dvd {m n d : } (dgt1 : d > 1) (Hm : d  m) (Hn : d  n) :
   coprime m n :=
assume co : coprime m n,
have d  gcd m n, from dvd_gcd Hm Hn,
have d  1, by rewrite [coprime at co, co at this]; apply this,
have d  1, from le_of_dvd dec_trivial this,
show false, from not_lt_of_ge ‘d  1‘ ‘d > 1‘

theorem exists_coprime {m n : } (H : gcd m n > 0) :
  exists m’ n’, coprime m’ n’  m = m’ * gcd m n  n = n’ * gcd m n :=
have H1 : m = (m / gcd m n) * gcd m n, from (nat.div_mul_cancel !gcd_dvd_left),
have H2 : n = (n / gcd m n) * gcd m n, from (nat.div_mul_cancel !gcd_dvd_right),
exists.intro _ (exists.intro _ (and.intro (coprime_div_gcd_div_gcd H) (and.intro H1 H2)))

theorem coprime_mul {m n k : } (H1 : coprime m k) (H2 : coprime n k) : coprime (m * n) k :=
calc
  gcd (m * n) k = gcd n k : !gcd_mul_left_cancel_of_coprime H1
            ... = 1       : H2

theorem coprime_mul_right {k m n : } (H1 : coprime k m) (H2 : coprime k n) : coprime k (m * n) :=
coprime_swap (coprime_mul (coprime_swap H1) (coprime_swap H2))

theorem coprime_of_coprime_mul_left {k m n : } (H : coprime (k * m) n) : coprime m n :=
have H1 : (gcd m n  gcd (k * m) n), from !gcd_dvd_gcd_mul_left,
eq_one_of_dvd_one (H  H1)

theorem coprime_of_coprime_mul_right {k m n : } (H : coprime (m * k) n) : coprime m n :=
coprime_of_coprime_mul_left (!mul.comm  H)

theorem coprime_of_coprime_mul_left_right {k m n : } (H : coprime m (k * n)) : coprime m n :=
coprime_swap (coprime_of_coprime_mul_left (coprime_swap H))

theorem coprime_of_coprime_mul_right_right {k m n : } (H : coprime m (n * k)) : coprime m n :=
coprime_of_coprime_mul_left_right (!mul.comm  H)

theorem comprime_one_left :  n, coprime 1 n :=
 n, !gcd_one_left

theorem comprime_one_right :  n, coprime n 1 :=
 n, !gcd_one_right

theorem exists_eq_prod_and_dvd_and_dvd {m n k : nat} (H : k  m * n) :
   m’ n’, k = m’ * n’  m’  m  n’  n :=
or.elim (eq_zero_or_pos (gcd k m))
 (assume H1 : gcd k m = 0,
    have H2 : k = 0, from eq_zero_of_gcd_eq_zero_left H1,
    have H3 : m = 0, from eq_zero_of_gcd_eq_zero_right H1,
    have H4 : k = 0 * n, from H2  !zero_mul,
    have H5 : 0  m, from H3  !dvd.refl,
    have H6 : n  n, from !dvd.refl,
    exists.intro _ (exists.intro _ (and.intro H4 (and.intro H5 H6)))))
  (assume H1 : gcd k m > 0,
    have H2 : gcd k m  k, from !gcd_dvd_left,
    have H3 : k / gcd k m  (m * n) / gcd k m, from nat.div_dvd_div H2 H,
```

```
have H4 : (m * n) / gcd k m = (m / gcd k m) * n, from
  calc
    m * n / gcd k m = n * m / gcd k m    : mul.comm
                ... = n * (m / gcd k m) : !nat.mul_div_assoc !gcd_dvd_right
                ... = m / gcd k m * n    : mul.comm,
have H5 : k / gcd k m  (m / gcd k m) * n, from H4  H3,
have H6 : coprime (k / gcd k m) (m / gcd k m), from coprime_div_gcd_div_gcd H1,
have H7 : k / gcd k m  n, from dvd_of_coprime_of_dvd_mul_left H6 H5,
have H8 : k = gcd k m * (k / gcd k m), from (nat.mul_div_cancel' H2),
exists.intro _ (exists.intro _ (and.intro H8 (and.intro !gcd_dvd_right H7))))

end nat
```

Chapter 12

Formal Pre- and Post-conditions

In Boldo [Bold11] we find an effort to verify floating point software using preconditions, postconditions, and assertions. Quoting:

"These conjectures can be described formally by annotations as follows.

```
/*@ requires \abs(x) <= 0x1p-5;
  @ ensures \abs(\result - \cos(x)) <= 0x1p-23;
  @*/
float my_cosine(float x) {
  //@ assert \abs(1.0 - x*x*0.5 - \cos(x)) < 0x1p-24;
  return 1.0f - x * x * 0.5f;
}
```

The *precondition*, introduced by **requires**, states that we expect argument x in the interval $[-1/32; 1/32]$. The *postcondition*, introduced by **ensures**, states that the distance between the value returned by the function, denoted by the keyword `\result`, and the model of the program, which is here the true mathematical cosine function denoted by `\cos` in ACSL, is not greater than 2^{-23}. It is important to notice that in annotations the operators like $+$ or $*$ denote operations on real numbers and not on floating-point numbers. In particular, there is no rounding error and no overflow in annotations, unlike in the early Leavens' proposal. The C variables of type `float`, like `x` and `\result` in this example, are interpreted as the real number they represent. Thus, the last annotation, given as an assertion inside the code, is a way to make explicit the reasoning we made above, making the total error the sum of the method error and the rounding error: it states that the method error is less than 2^{-24}. Again, it is thanks to the choice of having exact operations in the annotations that we are able to state a property of the method error."

In Boldo [Bold07, Bold07a] we find 'search in an array' annotated:

```
/*@ requires \valid_range(t,0,n-1)
  @ ensures
  @  (0 <= \result < n => t[\result] == v) &&
  @  (\result == n =>
  @    \forall int i; 0 <= i < n => t[i] != v) */
int index(int t[], int n, int v) {
  int i = 0;
  /*@ invariant 0 <= i &&
    @  \forall int k; 0 <= k <i => t[k] != v
```

```
   @ variant n - i */
 while (i < n) {
   if (t[i] == v) break;
   i++;
 }
 return i;
}
```

Chapter 13

Types and Signatures

We need to start from a base of the existing types in Common Lisp, eventually providing Axiom combinations or specializations. Common Lisp has these standard type specifier symbols.

Common Lisp Type Hierarchy [Pfei12]
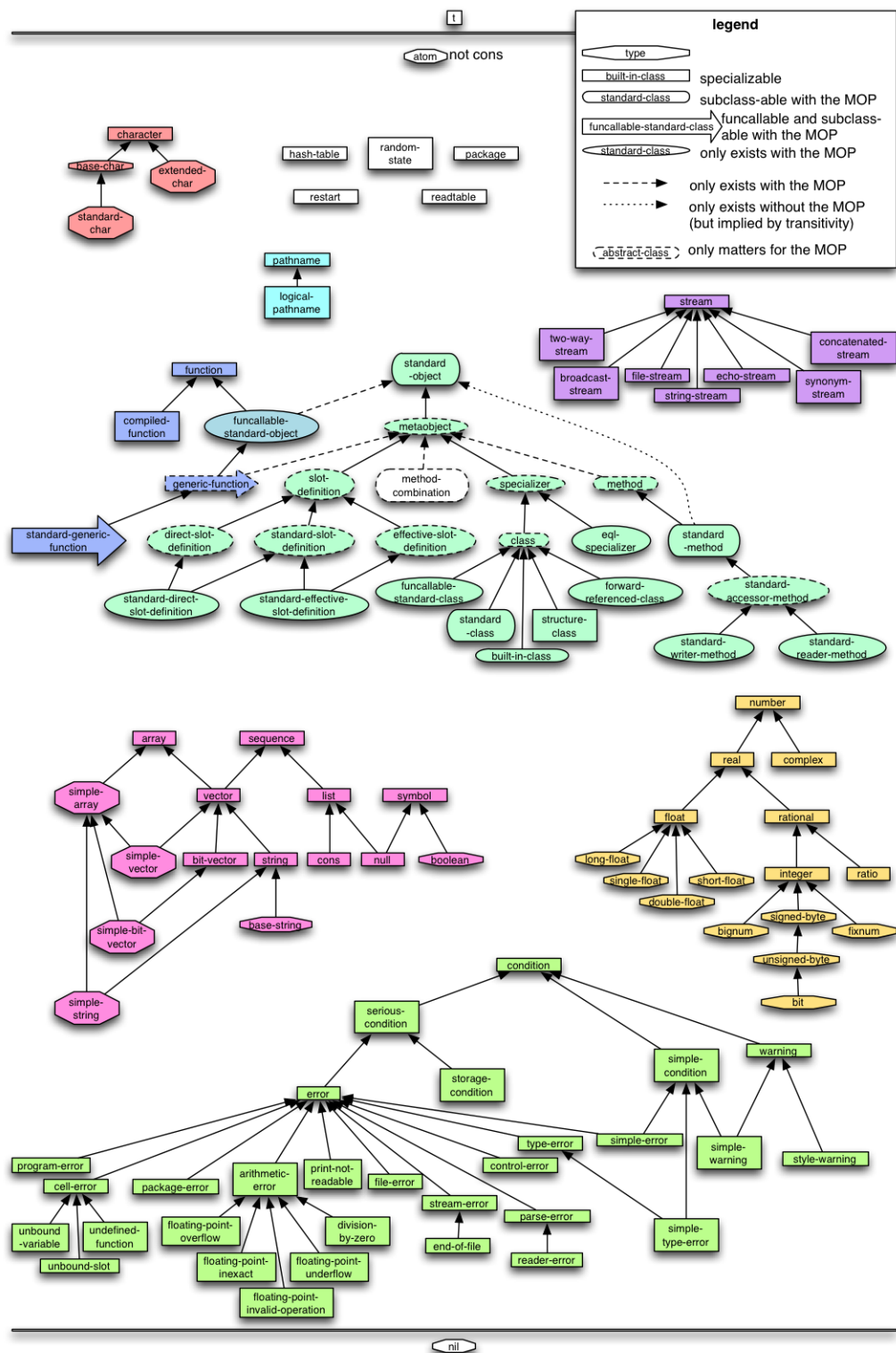
Axiom adds these types:

- Command = String

Chapter 14

COQ nat vs Axiom NNI

COQ's nat domain includes a proof of GCD.

We would like to show an isomorphism between types in Coq and types in Axiom. Having such an isomorphism will make lemmas available and simplify future proofs.

Note that Coq's `nat` domain stops at O (a symbolic 0) as does Axiom's NNI. The Axiom interpreter will promote a subtraction to Integer whereas Coq will not.

COQ's nat domain [COQnat] is

14.0.1 Library Coq.Init.Nat

```
Require Import Notations Logic Datatypes.
```

```
Local Open Scope nat_scope.
```

Peano natural numbers, definitions of operations

This file is meant to be used as a whole module, without importing it, leading to qualified definitions (e.g. Nat.pred)

```
Definition t := nat.
```

Constants

```
Definition zero := 0.
Definition one := 1.
Definition two := 2.
```

Basic operations

```
Definition succ := S.
```

```
Definition pred n :=
  match n with
    | 0 => n
    | S u => u
  end.
```

```
Fixpoint add n m :=
```

```
  match n with
  | 0 => m
  | S p => S (p + m)
  end
```

```
where "n + m" := (add n m) : nat_scope.
```

```
Definition double n := n + n.
```

```
Fixpoint mul n m :=
  match n with
  | 0 => 0
  | S p => m + p * m
  end
```

```
where "n * m" := (mul n m) : nat_scope.
```

Note that Axiom's NNI domain will be automatically promoted to Integer when the sub-traction result is negative. Coq returns O when this occurs.

```
Truncated subtraction: n-m is 0 if n<=m
```

```
Fixpoint sub n m :=
  match n, m with
  | S k, S l => k - l
  | _, _ => n
  end
```

```
where "n - m" := (sub n m) : nat_scope.
```

Comparisons

```
Fixpoint eqb n m : bool :=
  match n, m with
    | 0, 0 => true
    | 0, S _ => false
    | S _, 0 => false
    | S n', S m' => eqb n' m'
  end.
```

```
Fixpoint leb n m : bool :=
  match n, m with
    | 0, _ => true
    | _, 0 => false
    | S n', S m' => leb n' m'
  end.
```

```
Definition ltb n m := leb (S n) m.
```

```
Infix "=?" := eqb (at level 70) : nat_scope.
Infix "<=?" := leb (at level 70) : nat_scope.
Infix "<?" := ltb (at level 70) : nat_scope.
```

```
Fixpoint compare n m : comparison :=
```

```
  match n, m with
    | 0, 0 => Eq
    | 0, S _ => Lt
    | S _, 0 => Gt
    | S n', S m' => compare n' m'
  end.
```

```
Infix "?=" := compare (at level 70) : nat_scope.
```

Minimum, maximum

```
Fixpoint max n m :=
  match n, m with
    | 0, _ => m
    | S n', 0 => n
    | S n', S m' => S (max n' m')
  end.
```

```
Fixpoint min n m :=
  match n, m with
    | 0, _ => 0
    | S n', 0 => 0
    | S n', S m' => S (min n' m')
  end.
```

Parity tests

```
Fixpoint even n : bool :=
  match n with
    | 0 => true
    | 1 => false
    | S (S n') => even n'
  end.
```

```
Definition odd n := negb (even n).
```

Power

```
Fixpoint pow n m :=
  match m with
    | 0 => 1
    | S m => n * (n^m)
  end
```

```
where "n ^ m" := (pow n m) : nat_scope.
```

Euclidean division

This division is linear and tail-recursive. In divmod, y is the
predecessor of the actual divisor, and u is y minus the real remainder

```
Fixpoint divmod x y q u :=
  match x with
    | 0 => (q,u)
```

```
      | S x' => match u with
                  | 0 => divmod x' y (S q) y
                  | S u' => divmod x' y q u'
               end
  end.

Definition div x y :=
  match y with
    | 0 => y
    | S y' => fst (divmod x y' 0 y')
  end.

Definition modulo x y :=
  match y with
    | 0 => y
    | S y' => y' - snd (divmod x y' 0 y')
  end.

Infix "/" := div : nat_scope.
Infix "mod" := modulo (at level 40, no associativity) : nat_scope.
```

Greatest common divisor

We use Euclid algorithm, which is normally not structural, but Coq is
now clever enough to accept this (behind modulo there is a subtraction,
which now preserves being a subterm)

```
Fixpoint gcd a b :=
  match a with
    | 0 => b
    | S a' => gcd (b mod (S a')) (S a')
  end.
```

Square

```
Definition square n := n * n.
```

Square root

The following square root function is linear (and tail-recursive).
With Peano representation, we can't do better. For faster algorithm,
see Psqrt/Zsqrt/Nsqrt... We search the square root of
$n = k + p^2 + (q - r)$ with $q = 2p$ and $0 \le r \le q$. We start with
$p=q=r=0$, hence looking for the square root of $n = k$. Then we
progressively decrease k and r. When $k = S\ k'$ and r=0, it means we can
use (S p) as new sqrt candidate, since $(S\ k')+p^2+2p = k'+(S\ p)^2$. When k reaches 0, we have found the biggest p^2 square contained
in n, hence the square root of n is p.

```
Fixpoint sqrt_iter k p q r :=
  match k with
    | 0 => p
    | S k' => match r with
                | 0 => sqrt_iter k' (S p) (S (S q)) (S (S q))
```

```
                  | S r' => sqrt_iter k' p q r'
                end
    end.

Definition sqrt n := sqrt_iter n 0 0 0.
```

Log2

This base-2 logarithm is linear and tail-recursive. In
log2_iter, we maintain the logarithm p of the counter q, while r is
the distance between q and the next power of 2, more precisely q + S r
= 2^(S p) and r<2^p. At each recursive call, q goes up while r goes
down. When r is 0, we know that q has almost reached a power of 2, and
we increase p at the next call, while resetting r to q. Graphically
(numbers are q, stars are r) :

```
                  10
                 9
               8
             7   *
           6       *
         5           ...
       4
     3   *
   2       *
 1   *       *
0   *   *       *
```

We stop when k, the global downward counter reaches 0. At that moment,
q is the number we're considering (since k+q is invariant), and p its
logarithm.

```
Fixpoint log2_iter k p q r :=
  match k with
    | 0 => p
    | S k' => match r with
                | 0 => log2_iter k' (S p) (S q) q
                | S r' => log2_iter k' p (S q) r'
              end
  end.

Definition log2 n := log2_iter (pred n) 0 1 0.

Iterator on natural numbers

Definition iter (n:nat) {A} (f:A->A) (x:A) : A :=
 nat_rect (fun _ => A) x (fun _ => f) n.
```

Bitwise operations We provide here some bitwise operations for unary
numbers. Some might be really naive, they are just there for
fullfiling the same interface as other for natural representations. As
soon as binary representations such as NArith are available, it is
clearly better to convert to/from them and use their ops.

```
Fixpoint div2 n :=
  match n with
  | 0 => 0
  | S 0 => 0
  | S (S n') => S (div2 n')
  end.

Fixpoint testbit a n : bool :=
 match n with
   | 0 => odd a
   | S n => testbit (div2 a) n
 end.

Definition shiftl a := nat_rect _ a (fun _ => double).
Definition shiftr a := nat_rect _ a (fun _ => div2).

Fixpoint bitwise (op:bool->bool->bool) n a b :=
 match n with
  | 0 => 0
  | S n' =>
    (if op (odd a) (odd b) then 1 else 0) +
    2*(bitwise op n' (div2 a) (div2 b))
 end.

Definition land a b := bitwise andb a a b.
Definition lor a b := bitwise orb (max a b) a b.
Definition ldiff a b := bitwise (fun b b' => andb b (negb b')) a a b.
Definition lxor a b := bitwise xorb (max a b) a b.
```

Chapter 15

Binary Power in COQ by Casteran and Sozeau

From Casteran and Sozeau [Cast16]:

```
(* About integer powers (monomorphic version) *)

Set Implicit Arguments.
Require Import ZArith.
Require Import Div2.
Require Import Program.
Open Scope Z_scope.
```

Let us consider a simple arithmetic operation: raising some integer x to the n-th power, where n is a natural number. The following function definition is a direct translation of the mathematical concept:

```
Fixpoint power (a:Z)(n:nat) :=
  match n with 0%nat => 1
             | S p =>  a * power a p
  end.

Eval vm_compute in power 2 40.
 = 1099511627776 : Z
```

This definition can be considered as a very naive way of programming, since computing x^n requires n multiplications. Nevertheless, this definition is very simple to read, and everyone can admit that it is correct with respect to the mathematical definition. Thus, we can consider it as a *specification*: when we write more efficient but less readable functions for exponentiation, we should be able to prove their correctness by proving in Coq their equivalence with the naive power function.

The following function allows one to compute x^n, with a number of multiplications proportional to $\log_2(n)$:

```
Program
Fixpoint binary_power_mult (acc x:Z) (n:nat) {measure (fun i=>i) n} : Z
  (* acc * (power x n) *) :=
  match n with
```

```
    | 0%nat => acc
    | _ => if Even.even_odd_dec n
           then binary_power_mult acc (x * x) (div2 n)
           else binary_power_mult (acc * x) (x * x) (div2 n)
  end.
```

```
Solve Obligations with program_simpl; intros; apply lt_div2; auto with arith.
```

```
Definition binary_power (x:Z)(n:nat) := binary_power_mult 1 x n.
```

```
Eval vm_compute in binary_power 2 40.
 = 1099511627776 : Z
```

```
Goal binary_power 2 234 = power 2 234.
reflexivity.
Qed.
```

We want now to *prove* `binary_power`'s correctness, i.e. that this function and the naive `power` function are pointwise equivalent.

Proving this equivalence in Coq may require a lot of work. Thus it is not worth at all writing a proof dedicated only to powers of integers. In fact, the correctness of `binary_power` with respect to `power` holds in any structure composed of an associative binary operation on some domain, that admits a neutral element. For instance, we can compute powers of square matrices using the most efficient of both algorithms.

Thus, let us throw away our previous definition, and try to define them in a more generic framework.

15.1   On Monoids

Definition 2.1 *A monoid is a mathematical structure composed of*

- *a carrier A*
- *a binary, associative operation $\circ$ on A*
- *a neutral element $1 \in A$ for $\circ$*

Such a mathematical structure can be defined in Coq as a type class. [Soze08]. In the following definition, parameterized by a type A (implicit), a binary operation `dot` and a neutral element `unit`, three fields describe the properties that `dot` and `unit` must satisfy.

```
Class Monoid {A:Type}(dot : A -> A -> A)(one : A) : Prop := {
  dot_assoc : forall x y z:A, dot x (dot y z) = dot (dot x y) z;
  unit_left : forall x, dot one x = x;
  unit_right : forall x, dot x one = x }.
```

Note that other definitions could have been given for representing this mathematical structure.

From an implementational point of view, such a type class is just a record type, i.e. an inductive type with a single constructor `Build_Monoid`

```
Print Monoid.
```

```
Record Monoid (A:Type)(dot : A -> A -> A)(one : A) : Prop := Build_Monoid
```

```
  { dot_assoc : forall x y z:A, dot x (dot y z) = dot (dot x y) z;
    one_left : forall x, dot one x = x;
    one_right : forall x, dot x one = x }
```

```
For Monoid: Argument A is implicit and maximally inserted
For Build_Monoid: Argument A is implicit
For Monoid: Argument scopes are [type_scope _ _]
For Build_Monoid: Argument scopes are [type_scope _ _ _ _ _]
```

Nevertheless, implementation of type classes by M. Sozeau provides several specific tools — dedicated tactics for instance –, and we advise the reader not to replace the `Class` keyword with `Record` or `Inductive`.

With the command `About`, we can see the polymorphic type of the fields of the class `Monoid`:

```
About one_left
```

```
one_left:
forall (A : Type) (dot : A -> A -> A) (one : A),
Monoid dot one -> forall x : A, dot one x = x
```

```
Arguments A, dot, one, Monoid are implicit and maximally inserted
Argument scopes are [type_scope _ _ _ _]
one_left is transparent
```

15.1.1 Classes and Instances

Members of a given class are called *instances* of this class. Instances are defined to the Coq system through the `Instance` keyword. Our first example is a definition of the monoid structure on the set $\mathbb{Z}$ of integers, provided with integer multiplication, with 1 as the neutral element. Thus we give these parameters to the `Monoid` class (note that $\mathbb{Z}$ is implicitly given).

```
Instance ZMult : Monoid Zmult 1
```

For this instance to be created, we need to prove that the binary operation `Zmult` is associative and admits 1 as the neutral element. Applying the constructor `Build_Monoid` – for instance with the tactic `split` – generates three subgoals.

```
split.
3 subgoals
  =================================================
  forall x y z : Z, x * (y * z) = x * y * z

subgoal 2 is:
  forall x : Z, 1 * x = x
subgoal 3 is:
  forall x : Z, x * 1 = x
```

Each subgoal is easily solved by `intros; ring`.

When the proof is finished, we register our instance with a simple `Qed`. Note that we used `Qed` because we consider a class of sort `Prop`. In some cases where instances must store some information constants, ending an instance construction with `Defined` may be necessary.

```
Check Zmult.
ZMult : Monoid Zmult 1
```

We explained on the preceding page why it is better to use the `Class` keyword than `Record` or `Inductive`. For the same reason, the definition of an instance of some class should be written using `Instance` and not `Lemma`, `Theorem`, `Example`, etc. nor `Definition`.

15.1.2   A generic definition of `power`

We are now able to give a definition of the function `power` than can be applied with any instance of class `Monoid`:

A first definition could be

```
Fixpoint power {A:Type}{dot:A->A->A}{one:A}{M: Monoid dot one}
              (a:A)(n:nat) :=
  match n with 0:nat => one
           | S p => dot a (power a p)
  end.


Compute power 2 10.
 = 1024 : Z
```

Happily, we can make the declaration of the three first arguments implicit, by using the `Generalizable Variables` command:

```
Reset power.


Generalizable Variables A dot one.


Fixpoint power `{M: Monoid A dot one}(a:A)(n:nat) :=
  match n with 0%nat => one
           | S p => dot a (power a p)
  end.


Compute power 2 10.
 = 1024 : Z
```

The variables `A dot one` appearing in the binder for `M` are implicitly bound before the binder for `M` and their types are inferred from the `Monoid A dot one` type. This syntactic sugar helps abbreviate bindings for classes with parameters. The resulting internal Coq term is exactly the same as the first definition above.

15.1.3   Instance Resolution

The attentive reader has certainly noticed that in the last computation, the binary operation `Zmult` and the neutral element 1 need not to be given explicitly. The mechanism that allows Coq to infer all the arguments needed by the `power` function to be applied is called *instance resolution*.

In order to understand how it operates, let's have a look at `power`'s type:

```
About power.
power :
forall (A : Type) (dot : A -> A -> A) (one : A),
Monoid dot one -> A -> nat -> A

Arguments A, dot, one, M are implicit and maximally inserted
```

```
Compute power 2 100.
= 1267650600228229401496703205376 : Z

Set Printing Implicit.
Check power 2 100.
@power Z Zmult 1 Zmult 2 100 : Z
Unset Printing Implicit.
```

We see that the *instance* ZMult has been inferred from the type of 2. We are in the simple case where only one monoid of carrier Z has been declared as an instance of the Monoid class.

The implementation of type classes in Coq can retrieve the instance ZMult from the type Z, then filling the arguments ZMult and 1 from ZMult's definition.

15.2 More Monoids

15.2.1 Matrices over some ring

We all know that multiplication of square matrices is associative and admits identity matrices as neutral elements. For simplicity's sake let us restrict our study to 2×2 matrices over some ring.

We first load the Ring library, then open a section with some useful declarations and notations.

```
Require Import Ring.

Section matrices.
  Variables (A:Type)
            (zero one : A)
            (plus mult minus : A -> A -> A)
            (sym : A -> A).
  Notation "0" := zero.
  Notation "1" := one.
  Notation "x + y" := (plus x y).
  Notation "x * y" := (mult x y).

  Variable rt : ring_theory zero one plus mult minus sym (@eq A).

  Add Ring Aring : rt.
```

We can now define a carrier type for 2×2-matrices, as well as matrix multiplication and the identity matrix.

```
Structure M2 : Type := {c00 : A; c01 : A; c10 : A; c11 : A}.

Definition Id2 : M2 := Build_M2 1 0 0 1.

Definition M2_mult (m m':M2) : M2 :=
  Build_M2 (c00 m * c00 m' + c01 m * c10 m')
           (c00 m * c01 m' + c01 m * c11 m')
           (c10 m * c00 m' + c11 m * c10 m')
           (c10 m * c01 m' + c11 m * c11 m').
```

As for multiplication of integers, we can now define an instance of `Monoid` for the type M2.

```
Global Instance M2_Monoid : Monoid (M2_mult plus mult) (Id2 0 1).
split.
destruct x; destruct y; destruct z; simpl.
unfold M2_mult. apply M2_eq_intros; simpl; ring.
destruct x; simpl;
unfold M2_mult; apply M2_eq_intros; simpl; ring.
destruct x; simpl;
unfold M2_mult; apply M2_eq_intros; simpl; ring.
Qed.

End matrices.
```

We want now to play with 2×2 matrices over $\mathbb{Z}$. We declare an instance **M2Z** for this purpose, and can use directly the function `power`.

```
Instance M2Z : Monoid _ _ := M2_Monoid Zth.

Compute power (Build_M2 1 1 1 0) 40.
 = {|
     c00 := 165580141;
     c01 := 102334155;
     c10 := 102334155;
     c11 := 63245986 |}
   : M2 Z

Definition fibonacci (n:nat) :=
  C00 (power (Build_M2 1 1 1 0) n).

Compute fibonacci 20.
 = 10946
 :Z
```

15.3   Reasoning within a Type Class

We are now able to consider again the equivalence between two functions for computing powers. Let use define the binary algorithm for any monoid.

First, we define an auxiliary function. We use the `Program` extension to define an efficient version of exponentiation using an accumulator. The function is defined by well-founded recursion on the exponent n.

```
Function binary_power_mult (A:Type) (dot:A->A->A) (one:A)
    (M: @Monoid A dot one) (acc x:A)(n:nat){measure (fun i=>i) n} : A
  (* acc * (x ** n) *) :=
  match n with
    | 0%nat => acc
    | _ => if Even.even_odd_dec n
             then binary_power_mult _ acc (dot x x) (div2 n)
             else binary_power_mult _ (dot acc x) (dot x x) (div2 n)
  end.
intros; apply lt_div2; auto with arith.
intros; apply l2_div2; auto with arith.
Defined.
```

```
Definition binary_power '{M:Monoid} x n := binary_power_mult M one x n.
```

```
Compute binary_power 2 100.
 = 1267650600228229401496703205376 : Z
```

15.3.1   The Equivalence Proof

The proof of equivalence between `power` and `binary_power` is quite long, and can be split in several lemmas. Thus, it is useful to open a section, in which we fix some arbitrary monoid M. Such a declaration is made with the `Context` command, which can be considered as a version of `Variables` for declaring arbitrary instances of a given class.

```
Section About_power.
```

```
Require Import Arith.
 Context '(M:Monoid A dot one ).
```

It is good practice to define locally some specialized notations and tactics.

```
Ltac monoid_rw :=
    rewrite (@one_left A dot one M) ||
    rewrite (@one_right A dot one M)||
    rewrite (@dot_assoc A dot one M).
```

```
Ltac monoid_simpl := repeat monoid_rw.
```

```
Local Infix "*" := dot.
Local Infix "**" := power (at level 30, no associativity).
```

15.3.2   Some Useful Lemmas About `power`

We start by proving some well-known equalities about powers in a monoid. Some of these equalities are integrated later in simplification tactics.

```
Lemma power_x_plus : forall x n p, x ** (n + p) =  x ** n *  x ** p.
Proof.
 induction n as [| p IHp];simpl.
   intros;  monoid_simpl;trivial.
  intro q;rewrite (IHp q);  monoid_simpl;trivial.
Qed.
```

```
Ltac power_simpl := repeat (monoid_rw || rewrite <- power_x_plus).
```

```
Lemma power_commute : forall x n p,
              x ** n * x ** p = x ** p * x ** n.
Proof.
  intros x n p;power_simpl; rewrite (plus_comm n p);trivial.
Qed.
```

```
Lemma power_commute_with_x : forall x n ,
       x * x ** n = x ** n * x.
Proof.
```

```
   induction n;simpl;power_simpl;trivial.
   repeat rewrite <- (@dot_assoc A dot one M); rewrite IHn; trivial.
Qed.

Lemma power_of_power : forall x n p,  (x ** n) ** p = x ** (p * n).
Proof.
    induction p;simpl;[| rewrite power_x_plus; rewrite IHp]; trivial.
Qed.

Lemma power_S : forall x n, x *  x ** n = x ** S n.
Proof. intros;simpl;auto. Qed.

Lemma sqr : forall x, x ** 2 =  x * x.
Proof.
 simpl;intros;monoid_simpl;trivial.
Qed.

Ltac factorize := repeat (
                rewrite <- power_commute_with_x ||
                rewrite  <- power_x_plus  ||
                rewrite <- sqr ||
                rewrite power_S ||
                rewrite power_of_power).

Lemma power_of_square : forall x n, (x * x) ** n = x ** n * x ** n.
  induction n;simpl;monoid_simpl;trivial.
  repeat rewrite dot_assoc;rewrite IHn; repeat rewrite dot_assoc.
 factorize; simpl;trivial.
Qed.
```

15.3.3   Final Steps

We are now able to prove that the auxiliary function `binary_power_mult` satisfies its intuitive meaning. The proof uses well-founded induction and the lemmas proven in the previous section.

```
Lemma binary_power_mult_ok :
  forall n a x,  binary_power_mult a x n = a * x ** n.
Proof.
  intro n; pattern n;apply lt_wf_ind.
  clear n; intros n Hn;   destruct n.
   intros;simpl; monoid_simpl;  trivial.
  intros; rewrite binary_power_mult_equation.
     destruct (Even.even_odd_dec (S n)).
   rewrite Hn.   rewrite power_of_square;  factorize.
   pattern (S n) at 3;replace (S n) with (div2 (S n) + div2 (S n))%nat;auto.
   generalize (even_double _ e);simpl;auto.
   apply lt_div2;auto with arith.
   rewrite Hn.
  rewrite power_of_square ; factorize.
  pattern (S n) at 3;replace (S n) with (S (div2 (S n) + div2 (S n)))%nat;auto.

  rewrite <- dot_assoc; factorize;auto.
```

```
  generalize (odd_double _ o);intro H;auto.
  apply lt_div2;auto with arith.
Qed.
```

Then the main theorem follows immediately:

```
Lemma binary_power_ok : forall (x:A) (n:nat), binary_power x n = x ** n.
Proof.
  intros n x;unfold binary_power;rewrite binary_power_mult_ok;
  monoid_simpl;auto.
Qed.
```

15.3.4 Discharging the Context

It is time to close the section we opened for writing our proof of equivalence. The theorem `binary_power_ok` is now provided with a universal quantification over all the parameters of any monoid.

```
End About_power.

About binary_power_ok.
binary_power_ok :
forall (A : Type) (dot : A -> A -> A) (one : A) (M : Monoid dot one)
  (x : A) (n : nat), binary_power x n = power x n

Arguments A, dot, one M are implicit and maximally inserted
Argument scopes are [type_scope _ _ _ _ nat_scope]
binary_power_ok is opaque
Expands to Constant Top.binary_power_ok

Check binary_power_ok 2 20.
binary_power_ok 2 20
  : binary_power 2 20 = power 2 20

Let Mfib := Build_M2  1 1 1 0.

Check  binary_power_ok Mfib 56.
binary_power_ok Mfib 56
  : binary_power Mfib 56 = power Mfib 56
```

15.3.5 Subclasses

We could prove many useful equalities in the section `about_power`. Nevertheless, we couldn't prove the equality $(xy)^n = x^n y^n$ because it is false in general – consider for instance the free monoid of strings, or simply matrix multiplication. But this equality holds in every commutative (a.k.a Abelian) monoid.

Thus we say that Abelian monoids form a *subclass* of the class of monoids, and prove this equality in a context declaring an arbitrary instance of this subclass.

Structurally, we parameterize the new class `Abelian_Monoid` by an arbitrary instance M of `Monoid`, and add a new field stating the commutativity of `dot`. Please keep in mind that we

declared `A`, `dot`, and `one` as *generalizable variables*, hence we can use the backquote symbol here.

```
Class Abelian_Monoid '(M:Monoid A dot one) := {
  dot_comm : forall x y, dot x y = dot y x}.
```

A quick look at the representation of *Abelian_Monoid* as a record type helps us understand how this class is implemented.

```
Print Abelian_Monoid.
Record Abelian_Monoid (A : Type) (dot : A -> A -> A)
  (one : A) (M : Monoid dot one) : Prop := Build_Abelian_Monoid
  {dot_comm : forall x y : A, dot x y = dot y x }


For Abelian_Monoid: Arguments A, dot, one are implicit and maximally inserted
For Build_Abelian_Monoid: Arguments A, dot, one are implicit
For Abelian_Monoid: Arguemnt scopes are [type_scope _ _ _]
For Build_Abelian_Monoid: Argument scopes are [type_scope _ _ _ _]
```

For building an instance of `Abelian_Monoid` we can start from `ZMult`, the monoid on `Z`, adding a proof that integer multiplication is commutative.

```
Instance ZMult_Abelian : Abelian_Monoid ZMult.
split.
 exact Zmult_comm.
Qed.
```

We can now prove our equality by building an appropriate context. Note that we can specify just the parameters of the monoid here in the binder of the Abelian monoid, an instance of monoid on those same parameters is automatically generalized. Superclass parameters are automatically generalized inside quote binders. Again, this is simply syntactic sugar.

```
Section Power_of_dot.
 Context '{M: Monoid A} {AM:Abelian_Monoid M}.

Theorem power_of_mult : forall n x y,
    power (dot x y)  n =  dot (power x  n) (power y n).
Proof.
 induction n;simpl.
 rewrite one_left;auto.
 intros; rewrite IHn; repeat rewrite dot_assoc.
  rewrite <- (dot_assoc  x y (power x n)); rewrite (dot_comm y (power x n)).
 repeat rewrite dot_assoc;trivial.
Qed.

End Power_of_dot.

Check power_of_mult 3 4 5.
power_of_mult 3 4 5
     : power (4 * 5) 3 = power 4 3 * power 5 3
```

Chapter 16

Proof Tower Layer: C11 using CH$_2$O

From Krebbers [Kreb17]

Module example_gcd

Require Import String axiomatic_simple.

Section gcd.
Context '{EnvSpec K}.

Hint Extern 10 (Some Readable ⊆ _) ⇒ transitivity (Some Writable).
Hint Extern 0 (perm_locked _ = _) ⇒
    apply perm_Readable_locked; auto : typeclass_instances.

Hint Resolve ax_load' ax_var' assert_memext_l' assert_eval_int_cast_self'
    assert_memext_r' assert_and_l assert_singleton_eval assert_int_typed_eval
    assert_eval_singleton_r assert_eval_singleton_l assert_and_intro : exec.

Ltac exec :=
    repeat match goal with A := _ : assert _ ⊢ _ ⇒ progress unfold A end;
    simpl; eauto 20 with exec.

Definition gcd_stmt : stmt K :=
    "l" :; if{load (var 1)} local{uintT} (
        !(var 2 ::= (
            var 0 ::= load (var 1) @{ArithOp ModOp} load (var 2),,
            var 1 ::= load (var 2),,
            load (var 0)));;
        goto "l"
) else skip.

Lemma gcd_typed : (∅,∅,[uintT%T;uintT%T]) ⊢ gcd_stmt : (false,None).
Proof.
Lemma gcd_correct Γ δ R J T C y z μ1 γ1 μ2 γ2 :
    sep_valid γ1 → Some Writable ⊆ perm_kind γ1 →
    sep_valid γ2 → Some Writable ⊆ perm_kind γ2 →

J ”l”%string $\equiv$ {Γ,δ} ($\exists$ y' z',
 $\ulcorner$ Z.gcd y' z' = Z.gcd y z $\urcorner$%Z
 var 0 $\mapsto$ {μ1,γ1} #intV{uintT} y' : uintT
 var 1 $\mapsto$ {μ2,γ2} #intV{uintT} z' : uintT)%A $\rightarrow$
Γ δ R J T C $\models_s$
 {{ var 0 $\mapsto$ {μ1,γ1} #intV{uintT} y : uintT
  var 1 $\mapsto$ {μ2,γ2} #intV{uintT} z : uintT }}
  gcd_stmt
 {{ var 0 $\mapsto$ {μ1,γ1} #intV{uintT} (Z.gcd y z) : uintT
  var 1 $\mapsto$ {μ2,γ2} #intV{uintT} 0 : uintT }}.

Proof.
End gcd.

Chapter 17

Other Ideas to Explore

17.1   Aczel [Acze13]

Homotopy Type Theory

17.2   Chlipala [Chli15]

gives a pragmatic approach to COQ.

17.3   Dijkstra [Dijk72a]

Unlike later texts, Dijkstra does not require a descreasing value to do induction. Instead, he says

Let us consider the sequence of values

$$d_0, d_1, d_2, d_3, \ldots \tag{1}$$

given by

$$\text{for } i = 0 \, d_i = D \tag{2a}$$

$$\text{for } i > 0 \, d_i = f(d_{i-1} \tag{2b}$$

where D is a given value and f a given (computable) function. It is asked to make the value of the variable "d" equal to the first value d_k in the sequence that satisfies a given (computable) condition "prop". It is given that such a value exists for finite k. A more formal definition of the requirement is to establish the relation

$$d = d_k \tag{3}$$

where k is given by the (truth of the) expressions

$$prop(d_k) \tag{4}$$

and

$$\textbf{non } \text{prop}(d_i) \text{ for all } i \text{ satisfying } 0 \leq i < k \tag{5}$$

We now consider the following program part:

$$``d := D;$$
$$\textbf{while non } \text{prop}(d) \text{ do } d := f(d)$$

in which the first line represents the initialization and the second one the loop, controlled by the (hopefully self-explanatory) repetition **while...do**.

A more formal definition of the semantics of the repetition clause is by stating that

$$``\textbf{while } B \textbf{ do } S''$$

is semantically equivalent with

$$``\textbf{if } B \textbf{ do begin } S; \textbf{ while } B \textbf{ do } S \textbf{ end}''$$

expressing that "**non** B" is the necessary and sufficient condition for the repetition to terminate.)

17.4   Feferman [Fefe95]

Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in conputational contexts. It begins with a general logic of "partial terms", continues with partial combinatory and lambda calculi, and concludes with an expressively right theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.

In recursion theory and conputer science, questions of definedness usually have to deal with termination of an algorithm; typically, this is given recursively rather than explicitly, for example in Euclid's algorithm for $gcd(a, b)$ of natural numbers a, b:

$$gcd(a, b) = [a \text{ if } b = 0 \text{ else } gcd(b, rem(a, b))] \tag{1}$$

One proves by complete induction on b that

$$\text{forall } a, gcd(a, b) \text{ is defined}$$

Of course, justification of such reasoning presupposes a semantical account of recursive definitions. In particular, on the face of it, (1) requires definition of the functional $F(b) = \lambda a.gcd(a, b)$ by recursion of the form

$$F(b) = G(b, F \uparrow b)$$

and justification of *that* requires proof by induction on the natural numbers of the statement that for each x there is a unique function(al) F_x with domain $[0, x]$ such that for all $y \leq x$

$$F_x(y) = G(y, F_x \uparrow y)$$

then $F(b)$ is defined to be $F_b(b)$ for the unique such F_b with $\text{dom}(F_b) = [0, b]$.

17.4.1 Homann [Homa94]

algorithm schemata

Name: gcd(?a,?b)=?g
Signature: ?A × ?A → ?A
Constraints: (?A, EuclideanRing)
Definition: $(?g|?a) \land (?g|?b) \land (\forall c \in ?A : (c|?a) \land (c|?b) \Rightarrow (c|?g))$
Theorems:
gcd(u,v) = gcd(v,u)
gcd(u,v) = gcd(v, u mod v)
gcd(u,0) = u

17.5 Igarashi et al. [Igar75]

Igarashi et al. [Igar75]

The genera idea of how to go about verifying an asserted program is to reduce this problem to questions about whether certain associated logical conditions are true of theorems in various standard first-order theories.

It will become evident from the examples that a great deal of elementary simplification of verification conditions is both necessary and easy to do. The truth of many of the conditions will be established at the simplification stage.

What has become evident is that the Verification Condition Generator (VCG) is not a trivial element in this type of verification system. In order to make such a system practical, the amount of documentation the user is required to supply with his program should be restricted to what would be considered natural for human understanding of what the program and its sub-programs do.

In the logic of programs containing assertions they are of three kinds

1. assertions

2. statements of the form P{A}Q where P, Q are assertions and A is a program or asserted program. P{A}Q means "if P is true of the input state and A halts then Q is true of the output state"

3. procedure declarations of the form $p(x; y)$ PROC K where p is a procedure name, x and y are lists of variable and value parameters respectively, and K is a program or asserted program (the procedure body), in which the variables designated by y do not occur in the left-hand side of an assignment statement, nor occur as a variable parameter in a procedure statement.

If a primitive procedure name, say q, occurs in a program about which we are to prove a certain theorem, we have to either give a set of (nonlogical) axioms of the form $P\{q(x; y)\}R$ or a defining axiom for q.

Here we study the properties of the set V of axioms and rules of inference used by VCG. One of our main concerns is that the rules of inference in V should be unambiguous in the sense that only one rule is applicable to generate subgoals from any given goal. This will certainly be the case if no two rules have conclusoins which have common substitution instances, a property which is true of V.

The core rules:

Axioms

C1. assignment axioms: $P|_t^x\{x \leftarrow t\}P$

C2. frame axioms: $P\{q(x;t)\}P$ provided $\neg(x \in VAR(P))$

C3. procedure declarations: $p(x;y)$ PROC K

C4. logical theorems: P for all P s.t. $\vdash P$

Rules

C5. consequence: $$\frac{P \supset Q \quad Q\{A\}R}{P\{A\}R} \qquad \frac{P\{A\}Q \quad Q \supset R}{P\{A\}R}$$

C6. and/or: $$\frac{P\{A\}Q \quad R\{A\}S}{P \wedge R\{A\}Q \wedge S} \qquad \frac{P\{A\}Q \quad R\{A\}S}{P \vee R\{A\}Q \vee S}$$

C7. composition: $$\frac{P\{A\}Q \quad Q\{B\}R}{P\{A;B\}R}$$

C8.conditional: $$\frac{P \wedge R\{A\}Q \quad P \wedge \neg R\{B\}Q}{P\{if \ R \ then \ A \ else \ B\}Q}$$

C9. substitution: $(L) \quad \dfrac{P(x;y)\{q(x;y)\}Q(x;y)}{P(z;y)\{q(z;y)\}Q(z;y)}$

$(R) \quad \dfrac{P(x;y)\{q(x;y)\}Q(x;y)}{P(x;s)\{q(x;s)\}Q(x;s)}$

C10. procedure call: $$\frac{p(x;y) \ \text{PROC} \ K(p) \quad P\{r(x;y)\}Q \,\|\!\!-\, P\{K(r)\}Q}{P\{p(x;y)\}Q}$$

Note that in C9 (1) s does not contain members of x; (ii) members of x must be distinct and x must contain no variable occurring in $P(x;y)$ or $Q(x;y)$ except possibly members of x.

Note that in C10 p does not occur in the proof of the right hand premiss, and r does not occur in any other assumption in that proof.

The rules in the following table are stated in the form in which they are used to generate subgoals. Thus, for example in the case of the assignment rule V1, the axiom $Q(e)\{x \leftarrow e\}Q(x)$ is ommitted from the premisses since it is true and therefore not generated as a subgoal. The composition rule is not used to generate subgoals (it would be a source of ambiguity) but is included in the other rules. VCG does not require assertions at conditional statements. It 'marks' the conditional tests in the subgoals of the conditional rule, and uses them as assertions that permit a slightly different rule of consequence. The normal rule of consequence, V3 (ii) would usually lead to a verification condition of the form $Q \supset R'$ where R' is some formula involving R. Most likely the proof of R' would depend on the premiss P and in such a case $Q \supset R'$ is unlikely to be provable.

V1. *Simple assignment*

$$\frac{P\{A\}Q(e)}{P\{A; x \leftarrow e\}Q(x)}$$

V2. Array assignment

$$\frac{P\{A\}R(if\ i = j\ then\ e\ else\ B[i])}{P\{A; B[j] \leftarrow e\}R(B[i])}$$

V3. Consequence

$$(i)\frac{P \supset Q}{P\{Null\}Q'} \quad \frac{P\{A\}Q \quad Q \supset R}{P\{A; Q\}R} \quad \frac{P\{A\}Q \supset R}{P\{A; Q - if\}R}$$

V4. Iteration

$$\frac{P\{A\}R \quad R \wedge S\{B\}R \quad R \wedge \neg S \supset Q}{P\{A; R;\ while\ S\ do\ B\}Q} \quad \text{where R is an assertion}$$

V5. Conditional

$$\frac{P\{A; Q - if; B\}R \quad P\{A; \neg Q - if; C\}R}{P\{A;\ if\ Q\ then\ B\ else\ C\}R}$$

V6. Goto

$$\frac{P\{A\}\ assertion\ (L)}{P\{A; goto\ L\}Q}$$

V7. Procedure call

$$U(x; v)\{q(x; v)\}W(x; v) \Big\Vert\!\!- \frac{P\{A\}U(a; e) \wedge \forall a(W(a; e) \supset R)}{P\{A; q(a; e)\}R}$$

V8. Procedure declaration

$$\frac{P\{q(x; v)\}R \Big\Vert\!\!- P\{K\}R}{P\{procedure\ q(x; v); K\}R}$$

Notation: P,Q,R,S are Boolean Assertions. Null denotes the empty program. Q(e) denotes the substitution of e for x in Q(x). In V6, 'assertion (L)' denotes the assertion at label L.

B[i] denotes the ith element in array B. In each of the rules A can be Null. Q-if denotes a 'marked' Boolean assertion Q.

17.6 Kamareddine [Kama15]

Kamareddine [Kama15] Computerising Mathematical Text explores various ways of capturing mathematical reasoning.

17.7 Mahboubi [?]

Mathematical Components. This book contains a proof of the Euclidean algorithm using COQ.

17.8 Medina-Bulo et al. [Bulo04]

gives a formal verification of Buchberger's algorithm using ACL2 and Common Lisp.

17.9   Pierce [Pier15]

has a Software Foundations course in COQ with downloaded files in Pier15.tgz.

17.10   Santas [Sant95]

A Type System for Computer Algebra

17.11   Spitters [Spit11]

Type Classes for Mathematics in Coq. Also see `http://www.eelis.net/research/math-classes/`

17.12   Théry [Ther01]

used COQ to check an implementation of Buchberger's algorithm.

Appendix A

The Global Environment

Let S be a set. Let $\circ$ be a binary operation. Let $+$ be an additive operation. Let $*$ be a multiplicative operation.

Axiom 1 (Magma) *A* **Magma** *is the set S with a* **closed binary operation** $S \circ S \to S$ *such that*

$$\forall a, b \in S \Rightarrow a \circ b \in S$$

.

Axiom 2 (Semigroup) *A* **Semigroup** *is a* **Magma** *with the operation $\circ$ that is* **associative** *such that*

$$\forall a, b, c \in S \Rightarrow (a \circ b) \circ c = a \circ (b \circ c)$$

Axiom 3 (Abelian Semigroup) *An* **Abelian Semigroup** *is a* **Semigroup** *with the operation $\circ$ that is* **commutative** *such that*

$$\forall a, b \in S \Rightarrow a \circ b = b \circ a$$

Axiom 4 (Monoid) *A* **Monoid** *is a* **Semigroup** *with an* **identity element** $e \in S$ *such that*

$$\forall a \in S \Rightarrow e \circ a = a \circ e = a$$

Axiom 5 (Group) *A* **Group** *is a* **Monoid** *with an* **inverse element** $b \in S$ *and an identity element* $i \in S$ *such that*

$$\forall a \in S \ \exists b \in S \Rightarrow a \circ b = b \circ a = i$$

Axiom 6 (Group Unique Identity) *A* **Group** *has a* **unique identity element** $e \in S$ *such that*

$$\exists e \wedge \forall a, b \in S \wedge a \neq e \wedge b \neq e \Rightarrow a \circ b \neq e$$

Axiom 7 (Group Unique Inverse) *A* **Group** *has a* **unique inverse element** $i \in S$ *such that*

$$\exists i \wedge \forall a, b \in S \wedge a \neq i \wedge b \neq i \Rightarrow a \circ b \neq i$$

Axiom 8 (Group Right Quotient) *A* **Group** *has a* **Right Quotient** *(right division) such that*

$$x \circ a = b \Rightarrow x \circ a \circ a^{-1} = b \circ a^{-1} \Rightarrow x = b \circ a^{-1}$$

Axiom 9 (Group Left Quotient) *A* **Group** *has a* **Left Quotient** *(left division) such that*

$$a \circ x = b \Rightarrow a^{-1} \circ a \circ x = a^{-1} \circ b \Rightarrow x = a^{-1} \circ b$$

Axiom 10 (Abelian Group) *An* **Abelian Group** *is a* **Group** *with the operation $\circ$ that is* **commutative** *such that*

$$\forall a, b \in S \Rightarrow a \circ b = b \circ a$$

Axiom 11 (Abelian Group Quotient) *An* **Abelian Group** *has a* **Quotient** *(division) such that*

$$a^{-1} \circ a \circ x = a \circ a^{-1} \circ x$$

Axiom 12 (Euclidean Domain) *Let R be an integral domain. Let f be a function from $R\backslash\{0\}$ to the NonNegativeInteger domain. If a and b are in R and b is nonzero, then there are q and r in R such that $a = bq + r$ and either $r = 0$ or $f(r) < f(b)$*

Appendix B

Related work

B.1 Overview of related work

B.1.1 Adams [Adam01]

B.1.2 Ballarin [Ball95]

B.1.3 Basu, Pollack,and Roy [Basu10]

Euclidean Division and Greatest Common Divisor

We study euclidean division, compute greatest divisors, and show how to use them to decide whether or not a basic constructible set of C is empty.

In this section, C is an algebraically closed field, D a subring of C and K the quotient field of D. One can take as a typical example of this situation the field $\mathbb{C}$ of complex numbers, the ring $\mathbb{Z}$ of integers, and the field $\mathbb{Q}$ of rational numbers.

Let P be a non-zero polynomial

$$P = a_p X^p + \ldots + a_1 X + a_0 \in D[X]$$

with $a_p \neq 0$.

We denote the **degree** of P, which is p by $\deg(P)$. By convention, the degree of the zero polynomial is defined to be $-\infty$. If P is non-zero, we write $\mathrm{cof}_j(P) = a_j$ for the **coefficient** of X^j in P (which is equal to 0 if $j > \deg(P)$) and $\mathrm{lcof}(P)$ for its **leading coefficient** $a_p = \mathrm{cof}_{\deg(P)}(P)$. By convention $\mathrm{lcof}(0) = 1$.

Suppose that P and Q are two polynomials in $D[X]$. The polynomial Q is a **divisor of** P if $P = AQ$ for some $A \in K[X]$. Thus, while every P divides 0, 0 divides 0 and no other polynomial.

If $Q \neq 0$, the **remainder** in the **euclidean division of** P **by** Q, denoted $\mathrm{Rem}(P,Q)$, is the unique polynomial $R \in K[X]$ of degree smaller than the degree of Q such that $P = AQ + R$ with $A \in K[X]$. The **quotient** in the euclidean division of P by Q, denoted $\mathrm{Quo}(P,Q)$, is A.

Exercise Prove that, if $Q \neq 0$, there exists a unique pair (R, A) of polynomials in $K[X]$ such that $P = AQ + R$, $\deg(R) < \deg(Q)$.

Clearly, $\mathrm{Rem}(aP, bQ) = a\mathrm{Rem}(P, Q)$, for any $a, b \in K$ with $b \neq 0$. At a root x of Q, $\mathrm{Rem}(P, Q)(x) = P(x)$.

Exercise: Prove that x is a root of P in K if and only if $X - x$ is a divisor of P in $K[X]$.

Exercise: Prove that if C is algebraically closed, every $P \in C[X]$ can be written uniquely as
$$P = a(X - x_1)^{\mu_1} \cdots (X - x_k)^{\mu_k}$$
with $x_1, \ldots, x_k$ distinct elements of C.

A **greatest common divisor of P and Q**, denoted $gcd(P, Q)$, is a polynomial $G \in K[X]$ such that G is a divisor of both P and Q, and any divisor of both P and Q is a divisor of G. Observe that this definition implies that P is a greatest common divisor of P and 0. Clearly, any two greatest common divisors (say G_1, G_2) of P and Q must divide each other and have equal degree. Hence $G_1 = aG_2$ for some $a \in K$. Thus, any two greatest common divisors of P and Q are proportional by an element in $K \backslash \{0\}$. Two polynomials are **coprime** if their greatest common divisor is an element of $K \backslash \{0\}$.

A **least common multiple of P and Q**, $\mathrm{lcm}(P, Q)$ is a polynomial $G \in K[X]$ such that G is a multiple of both P and Q, and any multiple of both P and Q is a multiple of both G. Clearly, any two least common multiples L_1, L_2 of P and Q must divide each other and have equal degree. Hence, $L_1 = aL_2$ for some $a \in K$. Thus, any two least common multiple of P and Q are proportional by an element in $K \backslash \{0\}$.

It follows immediately from the definitions that:

Proposition: *Let $P \in K[X]$ and $Q \in K[X]$, not both zero. Then PQ/G is a least common multiple of P and Q.*

Corollary:
$$\deg(\mathrm{lcm}(P, Q)) = \deg(P) + \deg(Q) - \deg(\gcd(P, Q))$$

We now prove that greatest common divisors and least common multiple exist by using euclidean division repeatedly.

Definition [Signed remainder sequence] Given $P, Q \in K[X]$, not both 0, we define the **signed remainder sequence of P and Q**.
$$\mathrm{SRemS}(P, Q) = \mathrm{SRemS}_0(P, Q), \mathrm{SRemS}_1(P, Q), \ldots, \mathrm{SRemS}_k(P, Q)$$
by
$$
\begin{aligned}
\mathrm{SRemS}_0(P, Q) &= P \\
\mathrm{SRemS}_1(P, Q) &= Q \\
\mathrm{SRemS}_2(P, Q) &= -\mathrm{Rem}(\mathrm{SRemS}_0(P, Q), \mathrm{SRemS}_1(P, Q)) \\
&\vdots \\
\mathrm{SRemS}_k(P, Q) &= -\mathrm{Rem}(\mathrm{SRemS}_{k-2}(P, Q), \mathrm{SRemS}_{k-1}(P, Q)) \neq 0 \\
\mathrm{SRemS}_{k+1}(P, Q) &= -\mathrm{Rem}(\mathrm{SRemS}_{k-1}(P, Q), \mathrm{SRemS}_k(P, Q)) = 0
\end{aligned}
$$

The signs introduced here are unimportant in the algebraically closed case. They play an important role when we consider analogous problems over real closed fields.

In the above, each $\mathrm{SRemS}_i(P, Q)$ is the negative of the remainder in the euclidean division of $\mathrm{SRemS}_{i-2}(P, Q)$ by $\mathrm{SRemS}_{i-1}(P, Q)$ for $2 \leq i \leq k+1$, and the sequence ends with $\mathrm{SRemS}_k(P, Q)$ when $\mathrm{SRemS}_{k+1}(P, Q) = 0$, for $k \geq 0$.

Proposition: *The polynomial* $\mathrm{SRemS}_k(P, Q)$ *is a greatest common divisor of* P *and* Q

Proof: Observe that if a polynomial A divides two polynomials B, C then it also divides $UB + VC$ for arbitrary polynomials U, V. Since

$$\mathrm{SRemS}_{k+1}(P, Q) = -\mathrm{Rem}(\mathrm{SRemS}_{k-1}(P, Q), \mathrm{SRemS}_k(P, Q)) = 0$$

$\mathrm{SRemS}_k(P, Q)$ divides $\mathrm{SRemS}_{k-1}(P, Q)$ and since,

$$\mathrm{SRemS}_{k-2}(P, Q) = -\mathrm{SRemS}_k(P, Q) + A\mathrm{SRemS}_{k-1}(P, Q)$$

$\mathrm{SRemS}_k(P, Q)$ divides $\mathrm{SRemS}_{k-2}(P, Q)$ using the above observation. Continuing this process one obtains that $\mathrm{SRemS}_k(P, Q)$ divides $\mathrm{SRemS}_1(P, Q) = Q$ and $\mathrm{SRemS}_0(P, Q) = P$

Also, if any polynomial divides $\mathrm{SRemS}_0(P, Q)$, $\mathrm{SRemS}_1(P, Q)$ (that is P, Q then it divides $\mathrm{SRemS}_2(P, Q)$ and hence $\mathrm{SRemS}_3(P, Q)$ and so on. Hence, it divides $\mathrm{SRemS}_k(P, Q)$.

Note that the signed remainder sequence of P and 0 is P and when Q is not 0, the signed remainder sequence of 0 and Q is 0, Q.

Also, note that by unwinding the definitions of the $\mathrm{SRemS}_i(P, Q)$, we can express $\mathrm{SRemS}_k(P, Q) = gcd(P, Q)$ as $UP + VQ$ for some polynomials U, V in $K[X]$. We prove bounds on the degrees of U, V by elucidating the preceding remark.

Proposition: *If* G *is a greatest common divisor of* P *and* Q, *then there exist* U *and* V *with*

$$UP + VQ = G$$

Moreover, if $\deg(G) = g$, U *and* V *can be chosen so that* $\deg(U) < q - g$, $\deg(V) < p - g$.

The proof uses the extended signed remainder sequence defined as follows:

Definition: [**Extended signed remainder sequence**]

Given $P, Q \in K[X]$, not both 0, let

$$
\begin{aligned}
\mathrm{SRemU}_0(P, Q) &= 1 \\
\mathrm{SRemV}_0(P, Q) &= 0 \\
\mathrm{SRemU}_1(P, Q) &= 0 \\
\mathrm{SRemV}_1(P, Q) &= 1 \\
A_{i+1} &= \mathrm{Quo}(\mathrm{SRemS}_{i-1}(P, Q), \mathrm{SRemS}_i(P, Q)) \\
\mathrm{SRemS}_{i+1}(P, Q) &= -\mathrm{SRemS}_{i-1}(P, Q) + A_{i+1}\mathrm{SRemS}_i(P, Q) \\
\mathrm{SRemU}_{i+1}(P, Q) &= -\mathrm{SRemU}_{i-1}(P, Q) + A_{i+1}\mathrm{SRemU}_i(P, Q) \\
\mathrm{SRemV}_{i+1}(P, Q) &= -\mathrm{SRemV}_{i-1}(P, Q) + A_{i+1}\mathrm{SRemV}_i(P, Q)
\end{aligned}
$$

for $0 \le i \le k$ where k is the least non-negative integer such that $\mathrm{SRemS}_{k+1}(P, Q) = 0$.

The **extended signed remainder sequence** $\mathrm{Ex}(P, Q)$ of P and Q is $\mathrm{Ex}_0(P, Q), \dots, \mathrm{Ex}_k(P, Q)$ with

$$\mathrm{Ex}_i(P, Q) = (\mathrm{SRemS}_i(P, Q), \mathrm{SRemU}_i(P, Q), \mathrm{SRemV}_i(P, Q))$$

The proof uses the following lemma

Lemma For $0 \le i \le k + 1$

$$\mathrm{SRemS}_i(P, Q) = \mathrm{SRemU}_i(P, Q)P + \mathrm{SRemV}_i(P, Q)Q$$

Let $d_i = \deg(\mathrm{SRemU}_i(P, Q))$. For $1 \le i \le k$, $\deg(\mathrm{SRemV}_{i+1}(P, Q)) = q - d_i$, and $\deg(\mathrm{SRemV}_{i+1}(P, Q)) = p - d_i$.

Proof: It is easy to verify by induction on i that, for $0 \leq i \leq k+1$

$$\mathrm{SRemS}_i(P,Q) = \mathrm{SRemU}_i(P,Q)P + \mathrm{SRemV}_i(P,Q)Q$$

Note that $d_i < d_{i-1}$. The proof of the claim on the degrees proceeds by induction. Clearly since

$$\begin{aligned}
\mathrm{SRemU}_2(P,Q) &= -1 \\
\mathrm{SRemU}_3(P,Q) &= -\mathrm{Quo}(\mathrm{SRemS}_1(P,Q),\mathrm{SRemP}_2(P,Q))
\end{aligned}$$

$$\begin{aligned}
\deg(\mathrm{SRemU}_2(P,Q)) &= q - d_1 \\
\deg(\mathrm{SRemU}_3(P,Q)) &= q - d_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
\deg(\mathrm{SRemV}_2(P,Q)) &= p - d_1 \\
\deg(\mathrm{SRemV}_3(P,Q)) &= p - d_2
\end{aligned}$$

Using the definitions of $\mathrm{SRemU}_{i+1}(P,Q)$, $\mathrm{SRemV}_{i+1}(P,Q)$ and the induction hypothesis, we get

$$\begin{aligned}
\deg(\mathrm{SRemU}_{i-1}(P,Q)) &= q - d_{i-2} \\
\deg(\mathrm{SRemU}_i(P,Q)) &= q - d_{i-1} \\
\deg(A_{i+1}\mathrm{SRemU}_i(P,Q)) &= d_{i-1} - d_i + q - d_{i-1} \\
&= q - d_i > q - d_{i-2}
\end{aligned}$$

Hence, $\deg(\mathrm{SRemU}_{i+1}) = q - d_i$, Simlarly,

$$\begin{aligned}
\deg(\mathrm{SRemV}_{i-1}(P,Q)) &= p - d_{i-2} \\
\deg(\mathrm{SRemV}_i(P,Q)) &= p - d_{i-1} \\
\deg(A_{i+1}\mathrm{SRemV}_i(P,Q)) &= d_{i-1} - d_i + p - d_{i-1} \\
&= p - d_i > p - d_{i-2}
\end{aligned}$$

Hence, $\deg(\mathrm{SRemV}_{i+1}) = p - d_i$.

Proof: The claim follows from the Lemma and Proposition since $\mathrm{SRemS}_k(P,Q)$ is the gcd of P and Q, taking

$$U = \mathrm{SRemU}_k(P,Q), V = \mathrm{SRemV}_k(P,Q)$$

and noting that $p - d_{k-1} < p - g$, $q - d_{k-1} < q - g$.

The extended signed remainder sequence also provides a least common multiple of P and Q.

Proposition *The equality*

$$\mathrm{SRemU}_{k+1}(P,Q)P = -\mathrm{SRemV}_{k+1}(P,Q)Q$$

holds and $\mathrm{SRemU}_{k+1}(P,Q)P = -\mathrm{SRemV}_{k+1}(P,Q)Q$ *is a least common multiple of P and Q.*

Proof: Since $d_k = \deg(\gcd(P,Q))$, $\deg(\mathrm{SRemU}_{k+1}(P,Q)) = q - d_k$, $\deg(\mathrm{SRemV}_k(P.Q)) = p - d_k$ and

$$\mathrm{SRemU}_{k+1}(P,Q)P + \mathrm{SRemV}_{k+1}(P,Q)Q = 0$$

it follows that

$$\mathrm{SRemU}_{k+1}(P,Q)P - \mathrm{SRemV}_{k+1}(P,Q)Q$$

is a common multiple of P and Q of degree $p + q - d_k$, hence a least common multiple of P and Q.

Definition [Greatest common divisor of a family] A **greatest common divisor of a finite family of polynomials** is a divisor of all the polynomials in the family that is also a multiple of any polynomial that divides every polynomial in the family. A greatest common divisor of a family can be obtained inductively on the number of elements of the family by

$$\begin{aligned} \gcd(\emptyset) &= 0 \\ \gcd(\mathcal{P} \cup \{P\}) &= \gcd(P, \gcd(\mathcal{P})) \end{aligned}$$

Note that

- $x \in C$ is a root of every polynomial in $\mathcal{P}$ if and only if it is the root of $\gcd(\mathcal{P})$
- $x \in C$ is not a root of any polynomial in $\mathcal{Q}$ if and only if it is not a root of $\prod_{Q \in \mathcal{Q}} Q$ (with the convention that the proeuct of the empy family is 1).
- every root of P in C is a root of Q if and only if $\gcd(P, Q^{deg(\mathcal{P})})$ (with the convention that $Q^{deg(0)} = 0$).

WIth these observations the following lemma is clear:

Lemma *If $\mathcal{P}$,$\mathcal{Q}$ are two finite subsets of D[X], then there is an $x \in C$ such that*

$$\left(\bigwedge_{P \in \mathcal{P}} P(x) = 0 \right) \wedge \left(\bigwedge_{Q \in \mathcal{Q}} Q(x) \neq 0 \right)$$

if and only if

$$\deg(\gcd(\gcd(\mathcal{P}), \prod_{Q \in \mathcal{Q}} Q^d)) \neq \deg(\gcd(\mathcal{P}))$$

where d is any integer greater than $\deg(\gcd(\mathcal{P}))$.

Note that when $\mathcal{Q} = \emptyset$, since $\prod_{Q \in \emptyset} Q = 1$, the lemma says that there is an $x \in C$ such that $\bigwedge_{P \in \mathcal{P}} P(x) = 0$ if and only if $\deg(\deg(\mathcal{P})) \neq 0$. Note also that when $\mathcal{P} = \emptyset$, the lemma says that there is an $x \in C$ such that $\bigwedge_{Q \in \mathcal{Q}} Q(x) \neq 0$ if and only if $\deg(\prod_{Q \in \mathcal{Q}} Q) \geq 0$, i.e. $1 \neq \mathcal{Q}$.

B.1.4 Berger and Schwichtenberg [Berg95]

The Greatest Common Divisor: A Case Study for Program Extraction from Classical Proofs.

Theorem

$$\forall a_1, a_2 (0 < a_2 \rightarrow \exists k_1, k_2 (abs(k_1 a_1 - k_2 a_2) | a_1 \wedge abs(k_1 a_1 - k_2 a_2) | a_2 \wedge 0 < abs(k_1 a_1 - k_2 a_2)))$$

Proof *Let a_1, a_2 be given and assume $0 < a_2$. The ideal (a_1, a_2) generated from a_1, a_2 has a least positive element c, since $0 < a_2$. This element has a representation $c = abs(k_1 a_1 - k_2 a_2)$ with $k_1, k_2 \in \mathbb{N}$. It is a common divisor of a_1 and a_2 since otherwise the remainder $f(a_i, c)$ would be a smaller positive element of the ideal. The number $c \in (a_1, a_2)$ dividing a_1 and a_2 is the greatest common divisor since any common divisor of a_1 and a_2 must also be a divisor of c.*

```
(lambda (a1)
 (lambda (a2)
   ((((((nat-rec-at '(arrow nat (arrow nat (start nat nat))))
        (lambda (k1) (lambda (k2) (cons n000 n000))))
      (lambda (n)
       (lambda (w)
        (lambda (k1)
         (lambda (2)
          ((((if-at '(star nat nat))
             ((<-strict-nat 0) r2))
               ((w L21) L22))
               (((((if-at '(star nat nat))
                 ((<-strict-nat 0) r1))
                 ((w L11 L12))
                (cons k1 k2))))))))))
            ((plus-nat a2) 1))
          0)
        1)))
```

Here we have manually introduced r1,r2,L11,L12,L21,L22 for somewhat lengthy terms cor-
responding to our abbreviations r_1, $\overrightarrow{\to l_1}$. The unbound variable n000 appearing in the base
case is a dummy variable used by the system when it is asked to produce a realizing term for
the instance $\bot \to \exists k A(k)$ of ex-falso-quodlibet. In our case, when the existential quantifier
is of type nat one might as well pick the constant 0 (as we did in the text).

B.1.5   Cardelli [Card85]

Cardelli states that **coercions** are a form of ad-hoc polymorphism. Axiom allows ML-style
type inference in the interpreter. Declaring a function

```
f(n) == n + 1
```

without specifying types is possible. The function types are inferred when used and a type-
specialized version of the function is compiled. If called with different argument types it
will compile a new type-specialized version of the function as needed. Raises the distinction
between **equivalence** and **inclusion** as it occurs in subtypes (p483). The FUN language
defines **Quantified Types** as

QuantifiedType ::=

$\forall$ A, Type		Universal Quantification
$\exists$ A. Type		Existential Quantification
$\forall$ A $\subseteq$ Type. Type	$\exists$ A $\subseteq$ Type. Type	Bounded Quantification

See p516 for the Classification of Type Systems diagram.

B.1.6   Clarke [Clar91]

Clarke shows several proofs

B.1.7 Crocker [Croc14]

Primary specification constructs

pre*(expression-list)*	*Declares preconditions*
post*(expression-list)*	*Declares postconditions*
returns*(expression)*	*Declares the value returned by a function. Equivalent to* **post(result) ==** *expression except that recursion is permitted in expression*
assert*(expression-list)*	*Asserts conditions*
invariant*(expression-list)*	*Used in class declarations to declare class invariants, and in* **typedef** *declarations to declare constraints*
keep*(expression-list)*	*Declares loop invariants*
decrease*(expression-list)*	*Declares loop variant or recursion variant expressions*
writes*(lvalue-expression-list)*	*Declares what non-local variables the function modifies. If a function is declared without a writes-clause, then a default writes-clause is constructed based on the signature of the function*
assumes*(expression-list)*	*Declares predicates to be assumed without proof*
ghost*(expression-list)*	*Declares ghost variables, functions parameters, etc.*

Additional specification expressions

exists *identifier* **in** *expression :- predicate*	*Existential quantification over the elements of expression which must be an array or an abstract collection type*
exists *type identifier := predicate*	*Existential quantification over all values of type*
forall *identifier* **in** *expression :- predicate*	*Universal quantification over the elements of expression, which must be an array or an abstract collection type*
forall *type identifier :- predicate*	*Universal quantification over all values of type*
for *identifier* **in** *expression1* **yield** *expression2*	*Applies the mapping function expression2 to each element of collection expression1, yielding a new collection*
those *identifer* **in** *expression1 :- predicate*	*Selects those elements of collection expression1 for which predicate is true*
that *identifier* **in** *expression1 :- predicate*	*Selects the single element of the collection expression1 for which predicate is true*
expression1 **in** *expression2*	*Shorthand for* **exists** *id* **in** *expression2 :- id == expression1, where id is a new identifier*
expression **holds** *member*	*expression must have union type, and member must be a member of that type. Yields true if and only if the value of expression was defined by assignment or initialization through member*
disjoint*(lvalue-expression-list)*	*Yields true if and only if no two objects in the expression list have overlapping storage. Typically used in preconditions to state that parameters passed by pointer or reference refer to distinct objects*
operator **over** *expression*	*Left-fold operator over collection expression. Used to express e.g. summation of the elements of an array*
old*(expreesion)*	*When used in a postcondition, this refers to the value of expression when the function was entered. When used in a loop invariant, it refers to the value of expression just before the first iteration of the loop.*

B.1.8   Davenport [Dave02]

B.1.9   Davenport [Dave12b]

B.1.10   Davenport [Dave18]

The following definition is valid whenever we have a concept of division.

Definition *h is said to be a greatest commmon divisor, or g.c.d, of f and g if, and only if*

> 1. *h divides both f and g*
>
> 2. *if h' divides both f and g, then h' divides h*

This definition clearly extends to any number of arguments. The g.c.d is normally written gcd(f,g).

Note that we have defined a g.c.d. whereas it is more common to talke of the g.c.d. However 'a' is correct. We normally say that 2 is the g.c.d of 4 and 6, but in fact -2 is equally a g.c.d. of 4 and 6.

Definition *if $a = u * b$ where u is a unit, we say that a and b are associates.*

Proposition *if h and h' are greatest common divisors of a and b, they are associates.*

Example *(Greatest common divisors need not exist) Consider the set of all integers with $\sqrt{-5}$. 2 clearly divides both 6 and $2 + 2\sqrt{-5}$. However, so does $1 + \sqrt{-5}$ (since $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$), yet there is no multiple of both 2 and $1 + \sqrt{-5}$ which divides both.*

Definition *A ring R is said to be an integral domain if there is a neutral element 1 such that $1 * a = a$ and, whenever $a * b = 0$ at least one of a and b are zero.*

Definition *An integral docmain is which any two elements have a greatest common divisor is known as a g.c.d. domain.*

If R is a g.c.d domain, then the elements of the field of fractions can be simplified by cancelling a g.c.d. between numerator and denomiator, oftan called "reducing to lowest terms". While this simplifies fractions, it does not guarantee that they are normal or canonical. One might think that $\frac{0}{1}$ was the unique representation of zero required for normality, but what of $\frac{0}{-1}$? Equally, $\frac{-1}{2} = \frac{1}{-2}$, and in general we have to remove the ambiguity caused by units. In the case of rational numbers, we do this automatically by making the denominator positive, but the general case is more difficult.

Definition h is said to be a least common multiple, or l.c.m. of f and g if, and only if

> 1. *both f and g divide h*
>
> 2. *if both f and g divide h', the h divides h'*

This definition clearly extends to any number of arguments. The l.c.m. is normally written lcm(f,g).

Proposition *If gcd(f,g) exists, then $fg/gcd(f, g)$ is a least common multiple of f and g.*

This result is normally written as $fg = gcd(f, g)lcm(f, g)$, but this is only true up to associates. We should also note that this result does not extend to any number of arguments: in general $fgh \neq gcd(f, g, g)lcm(f, g, h)$.

B.1.11   Davis [Davi09]

Davis creates a verified proof checker by creating a simple, obviously correct "level 1" checker based on a few rules. A "level 2" checker is implemented and verified by level 1. This bootsrap process continues up to level 11 which has significant power. Every level 11 proof can be reduced to a level 1 proof, giving a solid foundation.

B.1.12   Filliatre [Fill03]

A formal method to establish software correctness can involve several steps. The first one is the specification. A second one is a method to generate some proof obligations. And a third one is a framework to establish their validity.

Type theory identifies types with propositions and terms with proofs, through the widely known Curry-Howard isomorphism. There is no real difference between the usual first-order objects of the mathematical discourse – such as naturals, sets, and so forth – and the proof objects. The natural 2 is a first-order object of type nat*, and a proof that 2 is even is a first-order object of type even(2). One can define a function f taking as arguments a natural n and a proof that n is even, and its type would be something like $\forall n : \mathtt{nat}.even(n) \to \tau$. Such a function represents a partial function on naturals, where the proof of even(n) may be seen as a precondition. Similarly, one can define a function returning a proof term. For instance, the function f could return a natural p and a proof that $n = 2 \times p$. Finally, the type of f will look like*

$$\forall n : \mathtt{nat}.even(n) \to \exists p : \mathtt{nat}.n = 2 \times p$$

where the proof of $n = 2 \times p$ may be seen as a postcondition. More generally, a type of the form

$$forall x : nat.P(x) \to \exists y : nat.Q(x,y)$$

is the type of a function with a precondition P and a postcondition Q. Building a term of this type is exactly like building a function together with a proof of its correctness, and consequently type theory appears as naturally suited for the proof of purely functional programs.

We propose an interpretation of the Hoare triple $\{P\}e\{Q\}$ as a proof of the above proposition and then define a systematic construction of this proof from a given annotated program, where the lacking proof terms are the so-called proof obligations.

The Coq **Correctness** *tactic takes an annotated program as argument and generates a set of goals, which are logical propositions to be proved by the user. Given an annotated program e, the tactic* **Correctness** *applies the following steps:*

  1. *It determines the type of computation of κ of e by the typing algorithm*

  2. *proposition $\hat{\kappa}$ is computed and declared as a goal*

  3. *The partial proof term $\hat{e}$ is computed following Definition 8 and is given to the proof engine, using the* Refine *tactic developed on purpose, and each hole in $\hat{e}$ leads to a subgoal*

  4. *Once the proofs are completed, the program is added to the environment and may be used in other programs*

Some features have been added to simplify the specification of programs – mainly, the possibility of inserting labels in the programs and referring in annotations to the value that

a reference had at the program points corresponding to those labels. In particular, a loop invariant may mention the values of the references at some point before the loop using such a label.

B.1.13   Frege [Freg1891]

Function and Concept seminal paper

Frühwirht [Frue91] details optimistic type systems for logic programs.

B.1.14   Gurevich [Gure12]

What is an Algorithm?

Turing's analysis of computation was a stroke of genius. The analysis is extremely famous and yet it is often misunderstood.

Some people think that every computable function, total or partial, can be computed by a Turing machine. This is not so, and here are some counter-examples. Consider Euclid's algorithm for computing the greatest common divisor $d = gcd(a, b)$ of two natural numbers a, b.

```
let M = max(a,b), m = min(a,b)
while M > m do
  M,m := max(M-m,m, min(M-m,m)
d := M
```

The gcd function on natural numbers is of course Turing computable, but the algorithm was also applied – in theory and in practice – to the lengths of segments of a straight line, which gives rise to a computable partial function (the algorithm does not terminate if the two given lengths are incommensurate) that is not Turing computable because you cannot place an arbitrary length on the Turing tape. More generally, the functions computed by ruler-and-conpass algorithms are not Turing computable. And let us emphasize that ruler and compass were practical tools in ancient Greece and that a number of rule and compass algorithms were practical algorithms.

Moschovakis [Mosc84] discusses Euclid's algorithm for the greatest common divisor of two natural numbers. Then he says:

Following the drive of the discussion, we might be expected at this point to simply identify the Euclidean algorithm with the funcitonal gcd. We will not go quite that far, because the time0honored intuitive concept of algorithm carries many linguistic and intensional connotations (some of them tied up with implementations) with which we have not concerned ourselves. Instead we will make the weaker (and almost trivial) claim that the functional gcd embodies all the essential mathematical properties of the Euclidean algorithm.

B.1.15   Harrison [Harr98, p13]

There are several examples of computer algebra results which may be checked relatively easily:

- *factoring polynomials (or numbers)*
- *finding GCDs of polynomials (or numbers)*

- *solving equations (algebraic, simultaneous, differential,...)*
- *finding antiderviatives*
- *finding closed forms for summations*

In most cases the certificate is simply the answer. An exception is the GCD, where a slightly more elaborate certificate is better for our purposes. If we ask to find the GCD of $x^2 - 1$ and $x^5 + 1$ using the gcd function, for example, the respons is $x + 1$. How can this result be checked? It's certainly straightforward to check that this is a common divisor. If we don't want to code polynomial division ourselves in HOL, we can call the `divide` function, and then simply verify the quotient as above. But how can we prove that $x + 1$ is the greatest common divisor[1] At first sight, there is no easy way, short of replicating something like the Euclidean algorithm inside the logic (although that isn't really a difficult prospect).

However, a variant GCD algorithm, called gcdex will, given polynomials p and q, produce not just the GCD d, but also two other polynomials r and s such that $d = pr + qs$. (Indeed, the coefficients in this sort of Bezout identity follow easily from the Euclidean GCD algorithm.) For example, applied to $x^2 - 1$ and $x^5 + 1$ we get the following equation:

$$(-x^3 - x)(x^2 - 1) + 1(x^5 + 1) = x + 1$$

This again can be checked easily, and from that, the fact that $x + 1$ is the greatest common divisor follows by an easily proved theorem, since obviously any common factor of $x^2 - 1$ and $x^5 + 1$ must, by the above equation, divide $x + 1$ too. So here, given a certificate slightly more elaborate than simply the answer, easy and efficient checking is possible.

B.1.16   Hoare [Hoar87]

Let us suppose first that they agree to confine attention to positive whole numbers (excluding zero). The required relationship between the parameters (x,y) and the result (z) may be formalized as follows:

```
D1.1  z divides x
D1.2  z divides y
D1.3  z is the greatest of the set of numbers satisfying both conditions
D1.4  "p divides q" means "there exists a positive whole number w
      such that pw=q"
D1.5  "p is the greatest member of the set S" means "p is in S, and no
      member of S is strictly greater than p"
```

We need to check that for every pair of positive numbers x and y there exists a number z with the properties specified in D1.3. A proof of this has three steps.

```
P1.1 The number one is a divisor of every number. So it is a common
     divisor of every pair of numbers. This shows that the set of
     common divisors of two numbers is non-empty.
P1.2 Each number is its own greatest divisor, so every set of
     divisors is finite. The common subset of any two finite sets is
     also finite. So the set of common divisors of two nubers is
     both finite and non-empty.
P1.3 Every finite non-empty set of integers has a greatest member.
```

---

[1] The use of "greatest" is a misnomer: in a general ring we say that a is the GCD of b and c iff it is a common divisor, and any other common divisor of b and c divides a. For example, both 2 and -2 are GCDs of 8 and 10 over $\mathbb{Z}$.

```
So the maximum used to define the greatest common divisor
always exists.
```

Here is an idealized logic program to compute the greatest common divisor of two positive integers. To help in checking its correctness, it has been designed to preserve as far as possible the structure and clarity of the original requirements. We assume that "isproduct" and "differsfrom" are available as built-in predicates on positive integers.

```
L2.1 isdivisor(x,z) if there exists a w not greater than x such that
     isproduct(z,w,x)
L2.2 iscommutative(x,y,z) if isdivisor(x,z) and isdivisor(y,z)
L2.3 isgcd(x,y,z) if iscommondiv(x,y.z) and for all w from z to x
     isnotcommondiv(x,y,z)
L2.4 isnotcommondiv(x,y,z) if isnotdiv(x,z) or isnotdiv(y,z)
L2.5 isnotdiv(x,z) if for all w from 1 to x isnotproduct(z,w,x)
L2.6 isnotproduct(z,w,x) if isproduct(z,w,y) and differsfrom(y,x)
```

This program is a great deal more complicated than the requirements specification in the preious section. The obvious reason is that the absence of negation in the programming language requires explicit programming of a search through all possibilities before a negative answer is given. In order to ensure termination a finite range for each search has to be specified, and setting this limit requires knowledge of the application domain. For example, in L2.3 we rely on the fact that the common divisor of two numbers cannot exceed either number.

When restricted from using disjunction and negation the algebraic equations have to be derived as needed by mathematical reasoning from the whole of the original specification. For gcd we see

```
L3.1 The greatest divisor of x is x. So the greatest common divisor
     of x and x is also x.
        x = gcd(x,x) for all x
L3.2 If z divides x and y, it also divides x+y. So every common
     divisor of x and y is also a common divisor of x+y and y.
     Similarly, every common divisor of x+y and y is also a common
     divisor of x and y. So the greatest of these identical sets
     of common divisors are the same.
        gcd(x,y) = gcd(x+y,y) for all x,y
L3.3 Every common divisor of x and y is also a common divisor of
     y and x.
        gcd(x,y) = gcd(y,x) for all x,y
```

But are the laws a complete specification, in the sense that there is aonly one function satisfying them? Or do we need to look for more laws? A proof of completeness has to show that for any given positive numerals p and q there is a numeral r such that the equation

$$r = gcd(p, q)$$

can be proved solely from the algebraic specification and the previously known laws of arithmetic.

This can be shown by mathematical induction: We assume the result for all p and q strictly less than N, and prove it for all p and q less than or equal to N. For such numbers, four cases can be distinguished.

1. *Both p and q are strictly less than N. In this case, what we have to prove is the same as the induction hypothesis, which may be assumed without proof.*

2. *Both p and q are equal to N. Then the result*

$$N = gcd(p, q)$$

   is proved immediately by law L3.1

3. *$p = N$ and $q < N$. It follows that $p - q$ is positive and less than N. By the induction hypothesis, there is an r such that*

$$r = gcd(p - q, q)$$

   is deducible from the algebraic laws. One application of L3.2 then gives

$$r = gcd(p - q) + q, q)$$

   which by the laws of arithmetic leads to the required conclusion

$$r = gcd(p, q)$$

4. *$p < n$ and $q = N$. Then there is an r such that*

$$r = gcd(q, p)$$

   is provable in the same way as in case (3) described above. One application of L3.3 then gives

$$r = gcd(p, q)$$

   That concludes the proof that the algebraic specification is complete.

Clearly there is no structural correspondence between the three clauses of the algebraic specification and the five clauses expressing the original requirement. As a result, some mathematical ingenuity and labour has been needed to prove that the two orthogonal specifications describe (and completely describe) the same function. This labor could be avoided by simply leaving out the original formalization of requirements in the general notations of mathematics, and by starting instead within the more restricted equational framework of algebra.

But this would be a mistake. The purpose of the specification is to tell the user of a subroutine the properties of the result it produces, and to do so in a manner conducive to the wider objectives of the program as a whole. Clearly, the user of a subroutine to compute the greatest common divisor will be very directly interested in the fact that the result of every subroutine call divides each of its two arguments exactly. But the algebraic law tells us only that the same result have been obtained if the two arguments had been permuted (L3.3) or added together (L3.2) before the call. These facts by themselves seem a lot less directly useful.

It would also be a mistake to regard the different specification, the abstract one and the algebraic one, as rivals or even as alternatives. They are both needed; they are essentially complementary, and the can be used for different purposes.

Here is a functional program to compute the greatest common divisor of positive integers.

F4.1	$gcd(x, y) = x$	if $x = y$
F4.2	$gcd(x, y) = gcd(x - y, y)$	if $x > y$
F4.3	$gcd(x, y) = gcd(y, x)$	if $x < y$

To compute the greatest common divisor of 10 and 6:

$$
\begin{array}{rll}
gcd(10,6) & = & gcd(4,6) \quad \text{by F4.2} \\
& = & gcd(6,4) \quad \text{by F4.3} \\
& = & gcd(2,4) \quad \text{by F4.2} \\
& = & gcd(4,2) \quad \text{by F4.3} \\
& = & gcd(2,2) \quad \text{by F4.2} \\
& = & 2 \qquad\quad\ \text{by F4.1}
\end{array}
$$

However, the responsibility for controlling this goal-directed behavior is placed upon the programmer, who has to prove that there is no infinite chain of substitutions. For example, as an algebraic formula

$$gcd(x,y) = gcd(y,x)$$

is quite correct, but if this is executed as part of a functional program, it leads to an infinite chain of substitutions. In the program shown above, this cycle is broken by ensuring that the dangerous substitution is made only when y is strictly greater than x.

We can optimize the search for the greatest common divisor algebraically.

- *L5.1 If z divides x, the $2z$ divides $2x$. So if z is the greatest common divisor of x and y, then $2z$ is a common divisor of $2x$ and $2y$. It is therefore not greater than their greatest common divisor*

$$2gcd(x,y) < gcd(2x,2y)$$

  Conversely, if z is the greatest common divisor of $2x$ and $2y$, then z is even and $z/2$ is a common divisor of x and y.

$$gcd(2x,y2)/2 \le gcd(x,y)$$

  From these two inequalities ti follows that

$$2gcd(x,y) = gcd(2x,2y)$$

- *L5.2 All divisors of an odd number are odd, and if an odd number divides $2x$ is also divides x. If y is odd, the greatest common divisor of $2x$ and y is odd, so it is also a common divisor of x and y*

$$gcd(2x,y) \le gcd(x,y) \quad \text{if } y \text{ is odd}$$

  Conversely, every divisor of x divides $2x$

$$gcd(x,y) \le gcd(2x,y)$$

  From these two inequalities it follows that

$$gcd(2x,y) = gcd(x,y) \quad \text{if } y \text{ is odd}$$

- *L5.3 If both x and y are odd, and x is greater than y, it follows that $x - y$ is positive and even. So under these conditions*

$$gcd(x,y) = gcd((x-y)/2,x)$$

When these equations are coded as a functional program, it becomes clear that the number of operations required when the argument of size 2^N has been reduced to about N.

A Procedural program has some assertions, generally specified as preconditions, invariants, and postconditions.

- *P6.1 $x > 0 \wedge y > 0$ as a precondition*
- *P6.2 $Z = gcd(x, y)$ as a postcondition*
- *P6.3 $s^N gcd(X, Y) = gcd(x, y)$. The task of the first part is to make P6.3 true on termination. That is easily accomplished by just one multiple assignment*

$$N, Z, Y := 0, x, y$$

which can be proven by substitution

$$2^0 gcd(x, y) = gcd(x, y)$$

- *P6.4 We can show that*
$$2^N Z = gcd(x, y)$$

This would be obviously true if N were already zero. If N is non-zero, it can be made closer to zero by subtracting one. But ath would amek P6.4 false, and therefore useless. Fortunately, the truth of P6.4 can easily be restored if every subtraction of one from N is accompanied by a doubling of Z. This can be proven by

$$n > 0 \wedge 2^N Z = gcd(x, y) \rightarrow 2^{N-1}(2Z) = gcd(x, y)$$

Since termination is obvious, we have proved the correctness of the loop

while $N > 0$ **do** $N, Z := N - 1, 2X$

On termination of this loop, the value of N is zero and P6.4 is still true. Consequently, the postcondition of the whole program has been established.

Having completed the first and last of the three tasks, the time has come to confess that the middle task is the most difficult. Its precondition is P6.3 and its postcondition is P6.4. The task can be split into four subtasks, in accordance with the following series of intermediate assertions.

P6.3 $\wedge$ (X odd $\vee$ Y odd)
P6.3 $\wedge$ (Y add)
P6.3 $\wedge$ (Y odd) $\wedge$ X=Y

B.1.17   Jenks [Jenk84b]

Overview of Scratchpad.

B.1.18   Kifer [Kife91]

Typed Predicate Calculus giving declarative meaning to logic programs with type declarations and type inference.

B.1.19   Manna and Waldinger [Mann78a]

Greatest Common Divisor of Two Numbers

In the previous two examples, we have applied the intermittent-assertion method to programs involving only one loop. The following program, which computes the greatest common divisor

(gcd) of two positive integers, is introduced to show how the intermittent-assertion method is applied to a program with more complex loop structure.

We define $gcd(y, y)$, where x and y are positive integers, as the greatest integer that divides both x and y, that is

$$gcd(x, y) = max\{u : u \mid x and u \mid y\}$$

For instance, $gcd(9, 12) = 3$ and $gcd(12, 25) = 1$. The program is

```
        input(x,y)
start:
more:   if x = y
        then finish: output(y)
        else reducex: if x > y
                      then x <- x - y
                           goto reducex
              reducey: if y > x
                      then y <- y - x
                           goto reducey
              goto more
```

This program is motivated by the following properties of the gcd:

$$gcd(x, y) = y \text{ if } x = y$$
$$gcd(x, y) = gcd(x - y, y) \text{ if } x > y$$
$$gcd(x, y) = gcd(x, y - x) \text{ if } y > x$$

We would like to use the intermittent-assertion method to prove the total correctness of this program. The total correctness can be expressed as follows:

Theorem*: If sometime $x = a$, $y = b$ and $b > 0$ at start then sometime $y = gcd(a, b)$ at finish.*

To prove this theorem, we need a lessa that describes the internal behaavior of the program.

Lemma*: If*
sometime $x = a$, $y = b$, and $a, b > 0$ at more *or*
sometime $x = a$, $y = b$, and $a, b > 0$ at reducex *or*
sometime $x = a$, $y = b$, and $a, b > 0$ at reducey,
then sometime $y = gcd(a, b)$ at finish

To show that the lemma implies the theorem, we assume that

sometime $x = a$, $y - b$, and $a, b > 0$ at start.

Then control passes to more, *so that*

sometime $x = a$, $y - b$, and $a, b > 0$ at more.

But then the lemma implies that

sometime $y = gcd(a, b)$ at finish,

which is the desired conclusion of the theorem.

It remains to prove the lemma. We suppose

sometime $x = a$, $y = b$, and $a, b > 0$ at more *or*
sometime $x = a$, $y = b$, and $a, b > 0$ at reducex *or*
sometime $x = a$, $y = b$, and $a, b > 0$ at reducey,

The proof proceeds by induction on $a + b$; we assume inductively that the lemma holds whenever $x = a'$ and $y = b'$, where $a' + b' < a + b$, and show that

sometime $y = gcd(a, b)$ at `finish`.

We must distinguish between three cases.

Case $a = b$. *Regardless of whether control is at* `more`, `reducex`, *or* `reducey`, *control passes to* `finish` *with $y = b$, so that*

sometime $y = b$ at `finish`

But in this case $b = gcd(a, b)$, by a given property of the gcd function, so we have

sometime $y = gcd(a, b)$ at `finish`.

which is the desired conclusion of the lemma.

Case $a > b$. *Regardless of whether control is at* `more`, `reducex`, *or* `reducey`, *control reaches* `reducex` *and passes around the top inner loop, resetting x to $a - b$, so that*

sometime $x = a - b$ and $y = b$ at `reducex`

For simplicity, let us denote $a - b$ and b by a' and b', respectively. Note that

$a', b' > 0$
$a + b > a' + b'$, *and*
$gcd(a', b') = gcd(a - b, b) = gcd(a, b)$

This last condition follows by a given property of the gcd.

Because $a', b' > 0$ and $a + b > a' + b'$, the induction hypothesis implies that

sometime $y = gcd(a', b')$ at `finish`.

i.e., by the third condition above,

sometime $y = gcd(a, b)$ at `finish`.

Case $b > a$. *This case is disposed of in a manner symmetric to the previous case.*

This concludes the proof of the lemma. The total correctness of the program is thus established.

It is not difficult to prove the partial correctness of the above program by using the conventional invariant-assertion method. For instance, to prove that the program is partially correct with respect to the input specification

$x_0 > 0$ and $y_0 > 0$

and output specification

$y = gcd(x_0, y_0)$

(where x_0 and y_0 are the initial values of x and y), we can use the same invariant assertion

$x, y > 0$ and $gcd(x, y) = gcd(x_0, y_0)$

at each of the labels `more`, `reducex` *and* `reducey`.

In contrast, the termination of this program is awkward to prove by the conventional well-founded ordering method, because it is possible to pass from `more` *to* `reducex`, `reducex` *to* `reducey`, *or from* `reducey` *to* `more` *without changing any of the program variables. One of the simplest proofs of the termination of the gcd program by this method involves taking the well-founded set to be the pairs of nonnegative integers ordered by the regular lexicographic ordering. When the expresion corresponding to the loop labels are taken to be*

$(x + y, 2)$ at `more`
if $x \neq y$ then $(x + y, 1)$ else $(x + y, 4)$ at `reducex`, and
if $x < y$ then $(x + y, 0)$ else $(x + y, 3)$ at `reducey`

it can be shown that their successive values decrease as control passes from one loop label to the next. Although this method is effective, it is not the most natural in establishing the termination of the gcd program.

B.1.20 Meshveliani [Mesh16a]

Prejudice 1*: "Proof by contradiction is not possible in constructive mathematics"*

In fact: it is possible – when the relation has a **decision algorithm***.*

Example*: In most domains in computer algebra the equality relation has a decision algorithm* `_=?_`*. Respecitvely, a program of the kind.*

$$\text{case x } =? \text{ y of } \backslash \{(yes x \approx y) \to \dots ; (\text{ no } x \neq y) \to \dots \}$$

actually applies the excluded third law to this relation.

Prejudice 2*: "Programs in the verified programming tools (like Coq, Agda) do not provide a proof itself, instead they provide an algorithm to build a proof witness for each concrete data".*

I claim: they also provide a proof in its ordinary meaning (this is so in Agda, and I expect, the same is with Coq).

B.1.21 Myreen [Myre10]

GCD is used as a benchmark example of a verified JIT Compiler on p 111.

B.1.22 Neuper [Neup13]

The case study was motivated by a master's thesis at RISC Linz, which implemented a CA algorithm for the greatest common divisor of multivariate polynomials in SML [Mein13].

B.1.23 Nordström, Petersson, and Smith [Nord90]

In recent years several formalisms for program construction have been introduced. One such formalism is the type theory developed by Per Martin-L¨of. It is well suited as a theory for program construction since it is possible to express both specifications and programs within the same formalism. Furthermore, the proof rules can be used to derive a correct program from a specification as well as to verify that a given program has a certain property. This book contains an introduction to type theory as a theory for program construction.

As a programming language, type theory is similar to typed functional languages such as Hope [18] and ML [44], but a major difference is that **the evaluation of a well-typed program always terminates***. In type theory it is also possible to write specifications of programming tasks as well as to develop provably correct programs. Type theory is therefore more than a programming language and it should not be compared with programming languages, but with formalized programming logics such as LCF and PL/CV.*

A proposition is interpreted as a set whose elements represent the proofs of the proposition. Hence, a false proposition is interpreted as the empty set and a true proposition as a non-empty set. Chapter 2 contains a detailed explanation of how the logical constants correspond to sets, thus explaining how a proposition could be interpreted as a set. A set cannot only be viewed as a proposition; it is also possible to see a set as a problem description. This possibility is important for programming, because if a set can be seen as a description of a problem, it can, in particular, be used as a specification of a programming problem. When a set is seen as a problem, the elements of the set are the possible solutions to the problem; or similarly if we see the set as a specification, the elements are the programs that satisfy the specification. Hence, set membership and program correctness are the same problem in type theory, and because all programs terminate, correctness means total correctness

B.1.24   O'Donnell [Odon81]

Several different definitions of the correctness of a system of reasoning, which are equivalent for the Predicate Calculus, are not equivalent for the Hoare language.

Consistency refers to the inability of a system to derive an explicit contradiction, while the stronger notion of soundness says that everything derived in a system is in some sense true.

Consistency is defined as:

- *Assume that a relation **contradictory**(ϕ) has been defined on finite sets ϕ of formulae in a language so that **contradictory**(ϕ) captures the intuitive notion that the formulae in ϕ are explicitly contradictory.*

- *A logical system of reasoning is **strongly consistent** if it is not possible to prove all of the formulae in a set ϕ such that **contradictory**(ϕ).*

- *A logical system of reasoning is **weakly consistent** if it is not possible to prove a single formula F such that **contradictory**(ϕ).*

Strong consistency certainly implies weak consistency.

In Hoare logics, two formulae $A\{P\}B$ and $C\{Q\}D$ cannot be combined with a symbol like &. So weak and strong consistency might not be equivalent for systems of reasoning in Hoare languages.

Strong consistency, for some reasonable definition of contradictory, is intuitively a necessary condition for the correctness of a logical system, but it is not in general a sufficient condition, since a system might prove a formula which is false but does not contradict any other provable formula.

- *A set of formulae ϕ **implies** a formula F if F is true in every world in which all the formulae in ϕ are true.*

- *A logical system is **theorem sound** if every provable formula is true*

- *A logical system is **inferentially sound** if, for every set of formulae ϕ and every formulae F, if F can be proved from assumptions in ϕ, the ϕ implies F.*

In any system where contradictory formulae cannot all be true, theorem soundness implies strong consistency. By letting ϕ be the empty set, we see that inferential soundness implies theorem soundness.

ALthough theorem soundness seems at first glance to be enough for an intuitive claim of correctness, this weaker form of soundness only justifies the theorems of a system, not the

methods of reasoning.

*Defintion: A Hoare triple $A\{P\}B$ is a **true partial correctness formula** if, whenever the program segment P begins execution with its first command, in a state for which A is true, and P terminates normally by executing its last command, then B is true of the resulting final state.*

The soundness of rules for function definition is a slippery issue when function bodies fail, since the normal interpretation of the Predicate Calculus does not allow for partial functions.

Partial correctness reasoning in the Hoare language is very natural for programs with only conditionals and loops for control structures, but not for programs with defined functions and/or GoTos. Defined functions tangle partial correctness and termination together to such an extent that it is no longer convenient to separate them. Since it is essential to prove termination anyway, we should use total correctness logics for reasoning about function definitions.

B.1.25 Scott and Strachey [Scot71]

Monotonic functions on complete lattices always have fixed points. They even have least fixed points. This makes their use most convenient for our purposes. Actually the functions we use – and which are appropriate to computational theory – hae an even stronger property: they are continuous. We shall assume this stronger property but shall not go into the technical details in this paper. The reader should only be assured that normal functions are automatically continuous.

What does all this theory have to do with the subject of semantics? Step by step the relevance is this: Commands (programs) are naturally thought of as defining state transformations. Recursive commands require partial functions. Solving for these partial functions is just finding (minimal) fixed points in certain functional equations. In general the existence of fixed points is justified by a lattice-theoretic argument. Therefore, if we can see the connection between lattices and partial functions, the relevance of the theory will be established.

B.1.26 Smolka [Smol89a]

details the foundations for relational logic programming with polymorphically order-sorted data types.

B.1.27 Strub, Pierre Yves

```
Formal Proofs and Decision Procedures  (Strub, Pierre Yves)
https://www.youtube.com/watch?v=YgOoDNIT8A8
year = "2016"


Prop is a type containing other types
The inhabitants of Prop are proof types or propositions
If A:Prop and B:Prop are proof types then A->B is a proof type (implication)
A proof type is valid if there exists a program of that type
(fun (x:A) => x) is a proof of (A -> A)
```

```
Elimination of the \forall connector
   \forall x,P
   --------
   P{x \mapsto t}
 We want to express a proof of (\forall x,P) by a function taking a t
 and returning a proof of P(t)
 The result type depends on the input
    Dependent typ: type that depends on a value


Arithmetic (P. Cregut)
Real Arithmetic (L Pottier)
First Order Logic (P Corbineau)
Polynomial Systems (B Barras, B Gregoire, A Mahboubi)
External Oracles (N Ayache JC Filliatre)


two terms are computationally equal are considered identical

Fact fact1 A B C : (A -> B -> C) -> (A -> B) -> A -> C
fact1 = fun (A B C : Type) (HABC : A -> B -> C) (HAB : A -> B) (HA : A) =>
HABC HA (HAB HA) : forall A B C : Type, (A -> B -> C) -> (A -> B) -> A -> C

Fact fact2 A B C : (A -> B -> C) -> (A -> B) -> A -> C
fact2 = fun (A B C : Type) (X : A -> B -> C) (X0 : A -> B) (X1 : A) =>
let X2 := X X1 in let X3 := X0 X1 in unkeyed (X2 X3)
  : forall A B C : (A -> B -> C) -> (A -> B) -> A -> C

Fact fact3 (m n:Z) : (1 + 2 * m <> 2 * n)%Z
fact3 - fun (m n : Z) (H : (1 + 2 * M)%Z = (2 * n)%Z) =>
let H0 :=
  fact_Zmult comm 2 m (fun x : Z => (1 + x + - (2 * n))%Z = 0%Z -> False)
    (fast_Zplus_comm 1 (m * 2)
      (fun x : Z => (x + - (2 * n))%Z = 0%Z -> False)
      (fast _Zmult_comm 2 n (fun x :Z => (M * 2 + 1 - x)%Z = 0%Z -> False)
        (fast_Zopp_mult disr r n 2
          (fun x : Z => (m * 2 + 1 + x)%Z = 0%Z -> False)
  (fast_Zplus_comm (m * 2 + 1) (n * -2)
    (fun x : Z => x = 0%Z -> False)
    (fun Omega0 : (n * -2 + (m * 2 + 1))%Z = 0%Z =>
      let H0 := erefl Gt in
      (let H1 := erefl Gt in
       fun auxiliary_1 : (1 > 0)%Z =>
       OMEGA4 1 2 (n * -1 + (m * 1 + 0)) auxiliary_1 H1
         (fast_OMEGA11 n (-1) (m * 1 + 0) 1 2
  (eq^- 0%Z)
    (fast_OMEGA11 m 1 0 1 2
      (fun x : Z => (n * (-1 * 2) + x)%Z = 0%Z)

Fact fact4: 2 + 2 = 4
fact4 =
 eq_ind_r (eq^ - 4)
  (eq_ind_r (fun _pattern_value : nat => _pattern_value_.+1 = 4)
      (eq_ind_r (fun _pattern_value : nat => _pattern_value_.+2 = 4) (erefl 4) (addn0 2))
      (addnS 2 0))
  (addnS 2 1) : 2 + 2 = 4
```

```
 \forall x,x+0 = x    \forall x y, x+S(y)=S(x+y)


2+2 = 4 is proved by

 |- \forall x,x+S(y)=S(x+y)          |- \forall x,x+S(y)=S(x+y)
 --------------------            --------------------
   |- 2+2 = S(2+1)                      |- S(2+1) = S(S(2+0))           |- \forall x,x+0 = x
   ----------------------------------------       ----------------
                              |- 2+2 = S(2+0)                             |- S(2+0) = 4
      ----------------------------------------
            |- 2+2 = 4
```

```
Calculus of Construction (Coquand, Huet, 1985)
  \beta convertibility
Calculus of Inductive Construction (Coquand, Paulin, 1990)
  \beta convertibility + recursors for inductive types
Extensional Calculus of Construction (Oury 05)
  \beta + convertible
  "any terms which can be proven equal are convertible"
   (BUT proof checking isn't decidable because convertibility is not decidable)
Coq Modulo Theory (TCS'08, CSL'10, LICS'11) (Strub, Pierre Yves)
  add decision procedure to conversion rule
Calculus of Presburger Constructions
  \beta conversions (function evaluation)
  recursor for natural numbers (def. by induction of functions and types)
  a decision procedure for Presburger arithmetic in its conversion
  an extraction from the proof environment of arithmtic equations
  whose proof checking is decidable


    \Gamma |- t:T   \Gamma |- T':s'   T ~r T'
    -------------------------------
          \Gamma |- t:T'


  where ~r will include
    \beta-convertibility (function evaluation)
    recursor for natural numbers (fixpoint+match evaluation)
    validity entailment of the Presburger arithmetic
       At least, the conversion must consider equal
 any pair of Presburger-equal algebraic terms
  Algebraic terms are all terms built from 0, S,+ and variables with the right arity
  This solves our motivation examples
      list n + m ~ list m + n
  because n + m and m + n are algebraic and P-equal
  BUT this does not define a sound logic
```

B.1.28 Sutor [Suto87]

Type inference and coercion in Scratchpad II.

B.1.29   Wijngaarden [Wijn68, Section 6, p95]

ALGOL 68 has carefully defined rules for coercion, using dereferencing, deproceduring, widening, rowing, uniting, and voiding to transform values to the type required for further computation.

B.1.30   McAllester, D. and Arkondas, K., [Mcal96]

"Primitive recursion is a well known syntactic restriction on recursion definitions which guarantees termination. Unfortunately many natural definitions, such as the most common definition of Euclid's GCD algorithm, are not primitive recursive. Walther has recently given a proof system for verifying termination of a broader class of definitions. Although Walther's system is highly automatible, the class of acceptable definitions remains only semi-decidable. Here we simplify Walther's calculus and give a syntactic criteria generalizes primitive recursion and handles most of the examples given by Walthar. We call the corresponding class of acceptable defintions "Walther recursive".",

Appendix A

Untyped Lambda in Common Lisp

See Garret [Garr14]

This is a compiler for the untyped lambda calculus.

— **lambda** —

```
(defmacro ulambda (args body)
  (if (and args (atom args)) (setf args (list args)))
  (if (and (consp body) (consp (car body))) (push 'funcall body))
  (if (null args)
    body
    '(lambda (&rest args1)
  (let ((,(first args) (first args1)))
    (declare (ignorable ,(first args)))
    (flet ((,(first args) (&rest args2) (apply ,(first args) args2)))
      (if (rest args1)
        (apply (ulambda ,(rest args) ,body) (rest args1))
        (ulambda ,(rest args) ,body)))))))
```

———-

The factorial function.

— **factorial** —

```
(ulambda (n f)
  (n (ulambda (c i)
       (i (c (ulambda (f x)
               (i f (f x))))))
     (ulambda x f)
     (ulambda x x)))
```

———-

The factorial of 10. factorial of 3, add 4, compute factorial of that.

— (—

```
factorial10}
(ulambda ()
  ((ulambda (f s) (f (s (s (s (s (f (s (s (s (ulambda (f x) x))))))))))))
   (ulambda (n f) (n (ulambda (c i) (i (c (ulambda (f x) (i f (f x))))))
                   (ulambda x f) (ulambda x x)))
   (ulambda (n f x) ((n f) (f x)))
   '1+ 0))
```

———-

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> **Link:** *https://hott.github.io/book/nightly/ hott-letter-1075-g3c53219.pdf*

[Adam01]   *Andrew A. Adams, Martin Dunstan, Hanne Gottlieben, Tom Kelsey, Ursula Martin, and Sam Owre. Computer Algebra meets Automated Theorem Proving: Integrating Maple and PVS. In Theorem proving in higher order logics, TPHOLs 2001, pages 27–42, 2001.*

> **Abstract:** *We describe an interface between version 6 of the Maple computer algebra system with the PVS automated theorem prover. The interface is designed to allow Maple users access to the robust and checkable proof environment of PVS. We also extend this environment by the provision of a library of proof strategies for use in real analysis. We demonstrate examples using the interface and the real analysis library. These examples provide proofs which are both illustrative and applicable to genuine symbolic computation problems.*

[Avig14]   *Jeremy Avigad. LEAN proof of GCD, 2014.*

> **Link:** *http://github.com/leanprover/lean2/blob/ master/library/data/nat/gcd.lean*

[Avig16]   *Jeremy Avigad. LEAN github repository, 2016.*

> **Link:** *http://github.com/leanprover*

[Back81]   *R.J.R Back. On Correct Refinement of Programs. J. Computer and System Sciences, 23(1):49–68, 1981.*

> **Abstract:** *The stepwise refinement technique is studied from a mathematical point of view. A relation of correct refinement between programs is defined, based on the principle that refinement steps should be correctness preserving. Refinement between programs will therefore depend on the criterion of program correctness used. The application of the refinement relation in showing the soundness of different techniques for refining programs is discussed. Special attention is given to the use of abstraction in program construction. Refinement with respect to partial and total correctness will be studied in more detail, both for deterministic*

and nondeterministic programs. The relationship between these refinement relations and the approximation relation of fixpoint semantics will be studied, as well as the connection with the predicate transformers used in program verification.

[Ball95]   Clemens Ballarin, Karsten Homann, and Jacques Calmet. *Theorems and Algorithms: An Interface between Isabelle and Maple.* In ISSAC 95, pages 150–157. ACM, 1995.

> **Abstract:** *Solving sophisticated mathematical problems often requires algebraic algorithms and theorems. However, there are no environments integrating theorem provers and computer algebra systems which consistently provide the inference capabilities of the first and the powerful arithmetic of the latter systems. As an example for such a mechanized mathematics environment we describe a prototype implementation of an interface between Isabelle and Maple. It is achieved by extending the simplifier of Isabelle through the introduction of a new class of simplification rules called evaluation rules in order to make selected operations of Maple available, and without any modification to the computer algebra system. Additionally, we specify syntax translations for the concrete syntax of Maple which enables the communication between both systems illustrated by some examples that can be solved by theorems and algorithms*
>
> **Link:** *https://pdfs.semanticscholar.org/077e/606f92b4095637e624a9efc942c5c63c4bc2.pdf*

[Barr18]   Bruno Barras and Benjamin Werner. *Coq in Coq.*

> **Abstract:** *We formalize the definition and the metatheory of the Calculus of Constructions (CC) using the proof assistant Coq. In particular, we prove strong normalization and decidability of type inference. From the latter proof, we extract a certified Objective Caml program which performs type inference in CC and use this code to build a small-scale certified proof-checker.*
>
> **Comment:** *https://github.com/coq-contribs/coq-in-coq*
>
> **Link:** *http://www.lix.polytechnique.fr/Labo/Bruno.Barras/publi/coqincoq.pdf*

[Basu10]   Saugata Basu, Richard Pollack, and Marie-Francoise Roy. *Algorithms in Real Algebraic Geometry.* Springer-Verlag, 2010, 978-3-642-06964-2.

[Bate85]   Joseph L. Bates and Robert L. Constable. *Proofs as Programs.* ACM TOPLAS, 7(1), 1985.

> **Abstract:** *The significant intellectual cost of programming is for problem solving and explaining, not for coding. Yet programming systems offer mechanical assistance for the coding process exclusively. We illustrate the use of an implemented program development system, called PRL ('pearl'), that provides automated assistance with the difficult part. The problem and its explained solution are seen as formal objects in a constructive logic of the data domains. These formal explanations can be*

> executed at various stages of completion. The most incomplete explanations resemble applicative programs, the most complete are formal proofs.

[Berg95]   U. Berger and H. Schwichtenberg. *The Greatest Common Divisor: A Case Study for Program Extraction from Classical Proofs.* LNCS, *1158:36–46, 1995.*

[Blak96]   Bob Blakley. *The Emperor's Old Armor.* In Proc. 1996 New Security Paradigms Workshop. *ACM, 1996.*

[Bold07]   Sylvie Boldo and Jean-Christophe Filliatre. *Formal Verification of Floating-Point programs.*

> **Link:** http://www-lipn.univ-paris13.fr/CerPAN/files/ARITH.pdf

[Bold07a]   Sylvie Boldo and Jean-Christophe Filliatre. *Formal Verification of Floating-Point programs.*

> **Abstract:** *This paper introduces a methodology to perform formal verification of floating-point C programs. It extends an existing tool for verification of C programs, Caduceus, with new annotations for specific floating-point arithmetic. The Caduceus first-order logic model for C programs is extended accordingly. Then verification conditions are obtained in the usual way and can be discharged interactively with the Coqa proof assistant, using an existing Coq formalization of floating-point arithmetic. This methodology is already implemented and has been successfully applied to several short floating-point programs, which are presented in this paper.*

> **Link:** http://www.lri.fr/~filliatr/ftp/publis/caduceus-floats.pdf

[Bold11]   Sylvie Boldo and Claude Marche. *Formal verification of numerical programs: from C annotated programs to mechanical proofs.* Mathematics in Computer Science, *5:377–393, 2011.*

> **Abstract:** *Numerical programs may require a high level of guarantee. This can be achieved by applying formal methods, such as machine-checked proofs. But these tools handle mathematical theorems while we are interested in C code, in which numerical computations are performed using floating-point arithmetic, whereas proof tools typically handle exact real arithmetic. To achieve this high level of confidence on C programs, we use a chain of tools: Frama-C, its Jessie plugin, Why and provers among Coq, Gappa, Alt-Ergo, CVC3 and Z3. This approach requires the C program to be annotated; each function must be precisely specified, and we prove the correctness of the program by proving both that it meets its specifications and that no runtime error may occur. The purpose of this paper is to illustrate, on various examples, the features of this approach.*

> **Link:** https://hal.archives-ouvertes.fr/hal-00777605/document

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              Link: http://frenchy64.github.io/2018/04/07/
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              Abstract: *New myths about formal methods are gaining tacit*
              acceptance both outside and inside the system-development com-
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              their observations of industrial projects. The myths include: for-
              mal methods delay the development process; they lack tools;
              they replace traditional engineering design methods; they only
              apply to software; are unnecessary; not supported; and formal
              methods people always use formal methods.

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              Abstract: *Fashioning complex conceptual constructs is the*
              essence; accidental tasks arise in representing the constructs in
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              Abstract: *In this paper, we present the formal verification of*
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              computing Groebner bases of polynomial ideals. This work is car-
              ried out in the ACL2 system and shows how verified Computer
              Algebra can be achieved in an executable logic.

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              Abstract: *Peano natural numbers, defintions of operations*

Link: https://coq.inria.fr/library/Coq.Init.Nat.
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> **Abstract:** *Proofs in the Nuprl system, an implementation of
> a constructive type theory, yield correct-by-construction pro-
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> tracting efficient and readable programs from inductive proofs.
> The resulting extracted programs are in a form suitable for use
> in hierarchical verifications in that they are amenable to clean
> partial evaluation via extensions to the Nuprl rewrite system.
> The method is based on two elements: specifications written
> with careful use of the Nuprl set-type to restrict the extracts
> to strictly computational content; and on proofs that use in-
> duction tactics that generate extracts using familiar fixed-point
> combinators of the untyped lambda calculus. In this paper the
> methodology is described and its application is illustrated by
> example.*

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> **Abstract:** *There is a strong analogy between proving theorems
> in mathematics and writing programs in computer science. This
> paper is devoted to an analysis, from the perspective of this anal-
> ogy, of proof in mathematics. We will argue that while the Hilber-
> tian notion of proof has few chances to change, future proofs will
> be of various types, will play different roles, and their truth will
> be checked differently. Programming gives mathematics a new
> form of understanding. The computer is the driving force be-
> hind these changes.*

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> in programming languages, present a model of typed, polymor-
> phic programming languages that reflects recent research in type
> theory, and examine the relevance of recent research to the de-
> sign of practical programming languages. Object-oriented lan-
> guages provide both a framework and a motivation for exploring
> the interaction among the concepts of type, data abstraction,
> and polymorphism, since they extend the notion of type to data
> abstraction and since type inheritance is an important form of
> polymorphism. We develop a λ-calculus-based model for type
> systems that allows us to explore these interactions in a sim-
> ple setting, unencumbered by complexities of production pro-
> gramming languages. The evolution of languages from untyped
> universes to monomorphic and then polymorphic type systems
> is reviewed. Mechanisms for polymorphism such as overloading,*

coercion, subtyping, and parameterization are examined. A unifying framework for polymorphic type systems is developed in terms of the typed λ-calculus augmented to include binding of types by quantification as well as binding of values by abstraction. The typed λ-calculus is augmented by universal quantification to model generic functions with type parameters, existential quantification and packaging (information hiding) to model abstract data types, and bounded quantification to model subtypes and type inheritance. In this way we obtain a simple and precise characterization of a powerful type system that includes abstract data types, parametric polymorphism, and multiple inheritance in a single consistent framework. The mechanisms for type checking for the augmented λ-calculus are discussed. The augmented typed λ-calculus is used as a programming language for a variety of illustrative examples. We christen this language Fun because fun instead of λ is the functional abstraction keyword and because it is pleasant to deal with. Fun is mathematically simple and can serve as a basis for the design and implementation of real programming languages with type facilities that are more powerful and expressive than those of existing programming languages. In particular, it provides a basis for the design of strongly typed object-oriented languages

[Cast16]   Pierre Casteran and Mattieu Sozeau. *A Gentle Introduction to Type Classses and Relations in Coq*, 2016.

Link: http://www.labri.fr/perso/casteran/CoqArt/TypeClassesTut/typeclassestut.pdf

[Chli15]   Adam Chlipala. *Certified Programming with Dependent Types*. MIT Press, 2015, 9780262026659.

Link: http://adam.chlipala.net/cpdt/cpdt.pdf

[Chli17]   Adam Chlipala. *Formal Reasoning About Programs*. MIT, 2017.

Abstract: *Briefly, this book is about an approach to bringing software engineering up to speed with more traditional engineering disciplines, providing a mathematical foundation for rigorous analysis of realistic computer systems. As civil engineers apply their mathematical canon to reach high certainty that bridges will not fall down, the software engineer should apply a different canon to argue that programs behave properly. As other engineering disciplines have their computer-aided design tools, computer science has proof assistants, IDEs for logical arguments. We will learn how to apply these tools to certify that programs behave as expected. More specifically: Introductions to two intertangled subjects: the Coq proof assistant, a tool for machine-checked mathematical theorem proving; and formal logical reasoning about the correctness of programs.*

Link: http://adam.chlipala.net/frap/frap_book.pdf

[Chli17a]   Adam Chlipala. *Coming Soon: Machine-Checked Mathematical Proofs in Everyday Software and Hardware Development*, 2017.

Abstract: *Most working engineers view machine-checked mathematical proofs as an academic curiosity, if they have ever heard of the concept at all. In contrast, activities like testing, debugging, and code review are accepted as essential. They are woven into the lives of nearly all developers. In this talk, I will explain how I see machine-checked proofs enabling new everyday activities for developers of computer software and hardware. These activities have the potential to lower development effort dramatically, at the same time as they increase our assurance that systems behave correctly and securely. I will give a cosmological overview of this field, answering the FAQs that seem to stand in the way of practicality; and I will illustrate the principles with examples from projects that you can clone from GitHub today, covering the computing stack from digital hardware design to cryptographic software and applications. Today's developers of computer software and hardware are tremendously effective, compared to their predecessors. We have found very effective ways of modularizing and validating our work. The talk is about ammunition for these activities from a perhaps-unexpected source. Modularity involves breaking a complex system into a hierarchy of simpler pieces, which may be written and understood separately. Structured programming (e.g., using loops and conditionals instead of gotos) helps us read and understand parts of a single function in isolation, and data abstraction lets us encapsulate important functionality in objects, with guarantees that other code can only access the private data by calling public methods. That way, we can convince ourselves that the encapsulated code upholds certain essential properties, regardless of which other code it is linked with. Systematic unit testing also helps enforce contracts for units of modularity. Each of these techniques can be rerun automatically, to catch regressions in evolving systems, and catch those regressions in a way that accurately points the finger of responsibility to particular modules. Validation is an important part of development that encompasses testing, debugging, code review, and anything else that we do to raise our confidence that the system behaves as intended. Experienced engineers know that validation tends to take up the majority of engineering effort. Often that effort involves mentally taxing activities that would not otherwise come up in coding. One example is thinking about test-case coverage, and another is including instrumentation that produces traces to consult during debugging. It is not hard for working developers to imagine great productivity gains from better ways to break systems into pieces or raise our confidence in those pieces. The claim I will make in this talk is that a key source of such insights has been neglected: machine-checked mathematical proofs. Here the basic functionality is an ASCII language for defining mathematical objects, stating theorems about them, and giving proofs of theorems. Crucially, an algorithm checks that purported proofs re-*

ally do establish the theorems. By going about these activities in the style of programming, we inherit usual supporting tools like IDEs, version control, continuous integration, and automated build processes. But how could so esoteric a task as math proofs call for that kind of tooling, and what does it have to do with building real computer systems? I will explain a shared vision to that end, developed along with many other members of my research community. Let me try to convince you that all of the following goals are attainable in the next 10 years.

- *We will have complete computer systems implementing moderately complex network servers for popular protocols, proved to implement those protocols correctly, from the level of digital circuits on up. We will remove all deployed code (hardware or software) from the trusted computing base, shifting our trust to much smaller specifications and proof checkers.*

- *Hobbyists will be able to design new embedded computing platforms by mixing and matching open-source hardware and software components, also mixing and matching the proofs of these components, guaranteeing no bugs at the digital-abstraction level or higher, with no need for debugging.*

- *New styles of library design will be enabled by the chance to attach a formal behavioral specification to each library. For instance, rank-and-file programmers will able to assemble their own code for cryptographic protocols, with code that looks like reference implementations in Python, but getting performance comparable to what experts handcraft in assembly today. Yet that benefit would come with no need to trust that library authors have avoided bugs or intentional backdoors, perhaps even including automatic proofs of cryptographic security properties.*

Main technical topics to cover to explain my optimism:

- *The basic functionality of proof assistants and why we should trust their conclusions*

- *How to think about system decomposition with specifications and proofs, including why, for most components, we do not need to worry about specification mistakes*

- *The different modes of applying proof technology to check or generate components*

- *The engineering techniques behind cost-effective proof authoring for realistic systems*

- *A hardware case study: Kami, supporting component-based digital hardware authoring with proofs*

- *A software case study: Fiat Cryptography, supporting correct-by-construction auto-generation of fast code for*

elliptic-curve cryptography

- *Pointers to where to look next, if you would like to learn more about this technology*

Link: *https://media.ccc.de/v/34c3-9105-coming_soon_machine-checked_mathematical_proofs_in_everyday_software_and_hardware_development*

[Clar91]   Edmund Clarke and Xudong Zhao. *Analytica – A Theorem Prover in Mathematica, 1991.*

Link: *http://www.cs.cmu.edu/~emc/papers/Conference%20Papers/Analytica%20A%20Theorem%20Prover%20in%20Mathematica.pdf*

[Cons98]   Robert L. Constable and Paul B. Jackson. *Towards Integrated Systems for Symbolic Algebra and Formal Constructive Mathematics, 1998.*

Abstract: *The purpose of this paper is to report on our efforts to give a formal account of some of the algebra used in Computer Algebra Systems (CAS). In particular, we look at the concepts used in the so called 3rd generation algebra systems, such as Axiom[4] and Weyl[9]. It is our claim that the Nuprl proof development system is especially well suited to support this kind of mathematics.*

Link: *http://www.nuprl.org/documents/Constable/towardsintegrated.pdf*

[Coqu16]   Thierry Coquand, Gérard Huet, and Christine Paulin. *The COQ Proof Assistant, 2016.*

Link: *https://coq.inria.fr*

[Coqu16a]   Thierry Coquand, Gérard Huet, and Christine Paulin. *COQ Proof Assistant Library Coq.ZArith.Znumtheory, 2016.*

Link: *https://coq.inria.fr/library/Coq.ZArith.Znumtheory.html*

[Coqu86]   Thierry Coquand and Gérard Huet. *The Calculus of Constructions. Technical Report 530, INRIA Centre de Rocquencourt, 1986.*

Abstract: *The Calculus of Constructions is a higher-order formalism for constructive proofs in natural deduction style. Every proof is a λ-expression, typed with propositions of the underlying logic. By removing types we get a pure λ-expression, expressing its associated algorithm. Computing this λ-expression corresponds roughly to cut-elimination. It is our thesis that (as already advocated by Martin-Lof) the Curry-Howard correspondence between propositions and types is a powerful paradigm for Computer Science. In the case of Constructions, we obtain the notion of a very high-level functional programming language, with complex polymorphism well-suited for modules specification. The notion of type encompasses the usual notioin of data type, but allows as well arbitrarily complex algorithmic specifications. We develop the basic theory of a Calculus of Construc-*

tions, and prove a strong normalization theorem showing that all computations terminate. Finally, we suggest various extensions to stronger calculi.

Link: *https://hal.inria.fr/inria-00076024/document*

[Croc14]   David Crocker. *Can C++ Be Made as Safe as SPARK?* In Proc 2014 HILT, *2014, 978-1-4503-3217-0.*

Abstract: *SPARK offers a way to develop formally-verified software in a language (Ada) that is designed with safety in mind and is further restricted by the SPARK language subset. However, much critical embedded software is developed in C or C++ We look at whether and how benefits similar to those offered by the SPARK language subset and associated tools can be brought to a C++ development environment.*

[Cyph17]   *Cypherpunks. Chapter 4: Verification Techniques, 2017.*

Abstract: *Wherein existing methods for building secure systems are examined and found wanting*

Link: *http://www.cypherpunks.to/~peter/04_verif_techniques.pdf*

[Dave02]   *James H. Davenport. Equality in computer algebra and beyond.* J. Symbolic Computing, *34(4):259–270, 2002.*

Abstract: *Equality is such a fundamental concept in mathematics that, in fact, we seldom explore it in detail, and tend to regard it as trivial. When it is shown to be non-trivial, we are often surprised. As is often the case, the computerization of mathematical computation in computer algebra systems on the one hand, and mathematical reasoning in theorem provers on the other hand, forces us to explore the issue of equality in greater detail.In practice, there are also several ambiguities in the definition of equality. For example, we refer to $\mathbb{Q}(x)$ as "rational functions", even though $\frac{x^2-1}{x-1}$ and $x+1$ are not equal as functions from $\mathbb{R}$ to $\mathbb{R}$, since the former is not defined at $x=1$, even though they are equal as elements of $\mathbb{Q}(x)$. The aim of this paper is to point out some of the problems, both with mathematical equality and with data structure equality, and to explain how necessary it is to keep a clear distintion between the two.*

Link: *http://www.calculemus.net/meetings/siena01/Papers/Davenport.pdf*

[Dave12b]   *James H. Davenport. Small Algorithms for Small Systems.* ACM Comm. in Computer Algebra, *46(1), 2012.*

[Dave18]   *James H Davenport.* Computer Algebra: Systems and Algorithms for Algebraic Computation (updated). *Davenport, 2018.*

Link: *http://staff.bath.ac.uk/masjhd/JHD-CA.pdf*

[Davi09]   *Jared Curran Davis.* A Self-Verifying Theorem Prover. *PhD thesis, University of Texas at Austin, 2009.*

Abstract: *Programs have precise semantics, so we can use mathematical proof to establish their properties. These proofs are often too large to validate with the usual social process of mathematics, so instead we create and check them with theorem-proving software. This software must be advanced enough to make the proof process tractable, but this very sophistication casts doubt upon the whole enterprise: who verifies the verifier? We begin with a simple proof checker, Level 1, that only accepts proofs composed of the most primitive steps, like Instantiation and Cut. This program is so straightforward the ordinary, social process can establish its soundness and the consistency of the logical theory it implements (so we know theorems are always true). Next, we develop a series of increasingly capable proof checkers, Level 2, Level 3, etc. Each new proof checker accepts new kinds of proof steps which were not accepted in the previous levels. By taking advantage of these new proof steps, higher-level proofs can be written more concisely than lower-level proofs, and can take less time to construct and check. Our highest-level proof checker, Level 11, can be thought of as a simplified version of the ACL2 or NQTHM theorem provers. One contribution of this work is to show how such systems can be verified. To establish that the Level 11 proof checker can be trusted, we first use it, without trusting it, to prove the fidelity of every Level n to Level 1: whenever Level n accepts a proof of some ϕ, there exists a Level 1 proof of ϕ. We then mechanically translate the Level 11 proof for each Level n into a Level n 1 proofthat is, we create a Level 1 proof of Level 2s fidelity, a Level 2 proof of Level 3s fidelity, and so on. This layering shows that each level can be trusted, and allows us to manage the sizes of these proofs. In this way, our system proves its own fidelity, and trusting Level 11 only requires us to trust Level 1.*

[Davi15]   Jared Davis and Magnus O. Myreen. *The Reflective Milawa Theorem Prover is Sound.* J. Automated Reasoning, 55(2):117–183, 2015.

Abstract: *This paper presents, we believe, the most comprehensive evidence of a theorem provers soundness to date. Our subject is the Milawa theorem prover. We present evidence of its soundness down to the machine code. Milawa is a theorem prover styled after NQTHM and ACL2. It is based on an idealised version of ACL2s computational logic and provides the user with high-level tactics similar to ACL2s. In contrast to NQTHM and ACL2, Milawa has a small kernel that is somewhat like an LCF-style system. We explain how the Milawa theorem prover is constructed as a sequence of reflective extensions from its kernel. The kernel establishes the soundness of these extensions during Milawas boot-strapping process. Going deeper, we explain how we have shown that the Milawa kernel is sound using the HOL4 theorem prover. In HOL4, we have formalized its logic, proved the logic sound, and proved that the source code for the Milawa kernel (1,700 lines of Lisp) faithfully implements this logic. Go-*

*ing even further, we have combined these results with the x86
machine-code level verification of the Lisp runtime Jitawa. Our
top-level theorem states that Milawa can never claim to prove
anything that is false when it is run on this Lisp runtime.*

Link: https://www.cl.cam.ac.uk/~mom22/jitawa/

[Demi79]   *Richard A. DeMilo, Richard J. Lipton, and Alan J. Perlis. Social Processes
and Proofs of Theorems and Programs. Communications of the ACM,
22(5):271–280, 1979.*

> **Abstract:** *It is argued that formal verifications of programs, no
> matter how obtained, will not play the same key role in the devel-
> opment of computer science and software engineering as proofs
> do in mathematics. Furthermore the absence of continuity, the
> inevitability of change, and the complexity of specification of
> significantly many real programs make the formal verification
> process difficult to justify and manage. It is felt that ease of for-
> mal verification should not dominate program language design.*

[Dijk70]   *Edsger W. Dijkstra. Concern for Correctness as a Guiding Principle for
Program Composition, 1970.*

> **Comment:** *EWD288*

> **Link:**   [https://www.cs.utexas.edu/users/EWD/](https://www.cs.utexas.edu/users/EWD/transcriptions/EWD02xx/EWD288.html)
> [transcriptions/EWD02xx/EWD288.html](https://www.cs.utexas.edu/users/EWD/transcriptions/EWD02xx/EWD288.html)

[Dijk71]   *E.W. Dijkstra. A Short Introduction to the Art of Programming, 1971.*

> **Comment:** *EWD316*

[Dijk72a]   *E.W. Dijkstra. Notes on Structured Programming. In Structured Pro-
gramming, pages 1–82. Academic Press, 1972.*

[Dijk83]   *Edsger Dijkstra. Fruits of Misunderstanding, 1983.*

> **Link:**   [https://www.cs.utexas.edu/users/EWD/](https://www.cs.utexas.edu/users/EWD/transcriptions/EWD08xx/EWD854.html)
> [transcriptions/EWD08xx/EWD854.html](https://www.cs.utexas.edu/users/EWD/transcriptions/EWD08xx/EWD854.html)

[Dola16]   *Stephen Dolan. Algebraic Subtyping. PhD thesis, University of Cambridge,
2016.*

> **Abstract:**   *Type inference gives programmers the benefit of
> static, compile-time type checking without the cost of manually
> specifying types, and has long been a standard feature of func-
> tional programming languages. However, it has proven difficult
> to integrate type inference with subtyping, since the unification
> engine at the core of classical type inference accepts only equa-
> tions, not subtyping constraints. This thesis presents a type sys-
> tem combining ML-style parametric polymorphism and subtyp-
> ing, with type inference, principal types, and decidable type sub-
> sumption. Type inference is based on biunification , an analogue
> of unification that works with subtyping constraints. Making this
> possible are several contributions, beginning with the notion of
> an 'extensible' type system, in which an open world of types is
> assumed, so that no typeable program becomes untypeable by
> the addition of new types to the language. While previous for-*

mulations of subtyping fail to be extensible, this thesis shows that adopting a more algebraic approach can remedy this. Using such an approach, this thesis develops the theory of biunification, shows how it is used to infer types, and shows how it can be efficiently implemented, exploiting deep connections between the algebra of regular languages and polymorphic subtyping.

Link: *https://www.cl.cam.ac.uk/~sd601/thesis.pdf*

[Dola17] Stephen Dolan and Alan Mycroft. *Polymorphism, Subtyping, and Type Inference in MLsub.* ACM SIGPLAN Notices, 52(1):60–72, 2017.

Abstract: *We present a type system combining subtyping and ML-style parametric polymorphism. Unlike previous work, our system supports type inference and has compact principal types. We demonstrate this system in the minimal language MLsub, which types a strict superset of core ML programs. This is made possible by keeping a strict separation between the types used to describe inputs and those used to describe outputs, and extending the classical unification algorithm to handle subtyping constraints between these input and output types. Principal types are kept compact by type simplification, which exploits deep connections between subtyping and the algebra of regular languages. An implementation is available online.*

Link: *https://www.cl.cam.ac.uk/~sd601/papers/mlsub-preprint.pdf*

[Domi18] Domipheus. *Designing a CPU in VHDL,* 2018.

Abstract: *A VHDL CPU*

Link: *http://labs.domipheus.com/blog/tpu-series-quick-links*

[Dura14] Antonio J. Duran, Mario Perez, and Juan L. Varona. *The Misfortunes of a Trio of Mathematicians Using Computer Algebra Systems. Can We Trust in Them?* Notices of the AMS, 61(10):1249–1252, 2014.

Link: *www.ams.org/notices/201410/rnoti-p1249.pdf*

[Fefe95] Solomon Feferman. *Definedness,* 1995.

Abstract: *Questions of definedness are ubiquitous in mathematics. Informally, these involve reasoning about expressions which may or may not have a value. This paper surveys work on logics in which such reasoning can be carried out directly, especially in computational contexts. It begins with a general logic of 'partial terms', continues with partial combinatory and lambda calculi, and concludes with an expressively rich theory of partial functions and polymorphic types, where termination of functional programs can be established in a natural way.*

Link: *https://math.stanford.edu/~feferman/papers/definedness.pdf*

[Fent93] Norman Fenton. *How Effective Are Software Engineering Methods.* J. Systems Software, 22:141–148, 1993.

Abstract: *For 25 years, software engineers have sought methods which they hope can provide a technological fix for the software crisis. Proponents of specific methods claim that their use leads to significantly improved quality and productivity. Such claims are rarely, if ever, backed up by hard evidence. We show that often, where real empirical evidence does exist, the results are counter to the views of the so-called experts. We examine the impact on the software industry of continuing to place our trust in unproven, and often revolutionary, methods. The very poor state of the art of empirical assessment in software engineering can be explained in part by inappropriate or inadequate use of measurement. Numerous empirical studies are flawed because of their poor experimental design and lack of adherence to proper measurement principles.*

[Fetz88]   James H. Fetzer. *Program Verification: The Very Idea.* Communications of the ACM, *31(9):1048–1063, 1988.*

Abstract: *The notion of program verification appears to trade upon an equvocation. Algorithms, as logical structures, are appropriate subjects for deductive verification. Programs, as causal models of those structures, are not. The success of program verification as a generally applicable and completely reliable method for guaranteeing program performance is not even a theoretical possibility.*

[Fill03]   Jean-Christophe Filliatre. *Verification of Non-Functional Programs using Interpretations in Type Theory.* J. Functional Programming, *13(4):709–745, 2003.*

Abstract: *We study the problem of certifying programs combining imperative and functional features within the general framework of type theory. Type theory is a powerful specification language, which is naturally suited for the proof of purely functional programs. To deal with imperative programs, we propose a logical interpretation of an annotated program as a partial proof of its specification. The construction of the corresponding partial proof term is based on a static analysis of the effects of the program which excludes aliases. The missing subterms in the partial proof term are seen as proof obligations, whose actual proofs are left to the user. We show that the validity of those proof obligations implies the total correctness of the program. This work has been implemented in the Coq proof assistant. It appears as a tactic taking an annotated program as argument and generating a set of proof obligations. Several nontrivial algorithms have been certified using this tactic.*

[Fill13]   Jean-Christophe Filliatre and Andrei Paskevich. *Why3 – Where Programs Meet Provers.* LNCS, *7792, 2013.*

Abstract: *We present Why3, a tool for deductive program verification, and WhyML, its programming and specification language. WhyML is a first-order language with polymorphic types,*

pattern matching, and inductive predicates. Programs can make use of record types with mutable fields, type invariants, and ghost code. Verification conditions are discharged by Why3 with the help of various existing automated and interactive theorem provers. To keep verification conditions tractable and comprehensible, WhyML imposes a static control of aliases that obviates the use of a memory model. A user can write WhyML programs directly and get correct-by-construction OCaml programs via an automated extraction mechanism. WhyML is also used as an intermediate language for the verification of C, Java, or Ada programs. We demonstrate the benefits of Why3 and WhyML on non-trivial examples of program verification.

Link: *https://hal.inria.fr/hal-00789533/document*

[Floy67]   Robert W. Floyd. *Assigning Meanings to Programs.* In Proc. Symp. in Applied Mathematics, *pages 19–32. American Mathematical Society, 1967.*

[Frad08]   Maria Joao Frade. *Calculus of Inductive Construction. Software Formal Verification.* MFES, *2008.*

Link: *http://www4.di.uminho.pt/~jno/mfes/0809/ SFV-CIC.pdf*

[Freg1891]   Gottlob Frege. *Function and Concept, 1891.*

Link: *http://fitelson.org/proseminar/frege_fac.pdf*

[Frue91]   Thom Fruehwirth, Ehud Shapiro, Moshe Y. Vardi, and Eyal Yardeni. *Logic programs as types for logic programs.* In Proc. Sixth Annual IEEE Symp. on Logic in Comp. Sci., *pages 300–309. IEEE, 1991.*

Abstract: *Type checking can be extremely useful to the program development process. Of particular interest are descriptive type systems, which let the programmer write programs without having to define or mention types. We consider here optimistic type systems for logic programs. In such systems types are conservative approximations to the success set of the program predicates. We propose the use of logic programs to describe types. We argue that this approach unifies the denotational and operational approaches to descriptive type systems and is simpler and more natural than previous approaches. We focus on the use of unary-predicate programs to describe types. We identify a proper class of unary-predicate programs and show that it is expressive enough to express several notions of types. We use an analogy with 2-way automata and a correspondence with alternating algorithms to obtain a complexity characterization of type inference and type checking. This characterization was facilitated by the use of logic programs to represent types.*

[Garr14]   Ron Garret. *The Awesome Power of Theory, 2014.*

Link: *http://www.flownet.com/ron/lambda-calculus. html*

[Glas02]   Robert L. Glass. *The Proof of Correctness Wars.* Communications of the ACM, *45(8):19–21, 2002.*

[Gord06]   Mike Gordon, Juliano Iyoda, Scott Owens, and Konrad Slind. *Automatic Formal Synthesis of Hardware from Higher Order Logic.* Electronic Notes in Theoretical Computer Science, *145:27–43, 2006.*

> **Abstract:** *A compiler that automatically translates recursive function definitions in higher order logic to clocked synchronous hardware is described. Compilation is by mechanised proof in the HOL4 system, and generates a correctness theorem for each function that is compiled. Logic formulas representing circuits are synthesised in a form suitable for direct translation to Verilog HDL for simulation and input to standard design automation tools. The compilation scripts are open and can be safely modified: synthesised circuits are correct-by-construction. The synthesisable subset of higher order logic can be extended using additional proof-based tools that transform definitions into the subset.*

[Gord89]   Michael J.C. Gordon. *Mechanizing Programming Logics in Higher Order Logic.* In Current Trends in Hardware Verification and Automated Theorem Proving. *Springer, 1989.*

> **Abstract:** *Formal reasoning about computer programs can be based directly on the seman tics of the programming language, or done in a special purpose logic like Hoare logic. The advantage of the first approach is that it guarantees that the formal reasoning applies to the language being used (it is well known, for example, that Hoare's assignment axiom fails to hold for most programming languages). The advantage of the second approach is that the proofs can be more direct and natural. In this paper, an attempt to get the advantages of both approaches is described. The rules of Hoare logic are mechanically derived from the semantics of a simple imperative programming language (using the HOL system). These rules form the basis for a simple program verifier in which verification conditions are generated by LCF -style tactics whose validations use the derived Hoare rules. Because Hoare logic is derived, rather than postulated, it is straight tforw ard to mix seman tic and axiomatic rea- soning. It is also forward to combine the constructs of Hoare logic with other application-specific notations. This is briefly illustrated for various logical constructs, including termination statements, VDM-style 'relational' correctness specifications, weakest precondition statements and dynamic logic formulae. The theory underlying the work presented here is well known. Our contribution is to propose a way of mechanizing this theory in a way that makes certain practical details work out smoothly .*

[Gure12]   Yuri Gurevich. *What Is An Algorithm?* LNCS, *7147:31–42, 2012.*

> **Abstract:** *We attempt to put the title problem and the Church-Turing thesis into a proper perspective and to clarify some common misconcep- tions related to Turings analysis of computation. We examine two approaches to the title problem, one well-known among philosophers and another among logicians.*

[Gutt95]   Joshua D. Guttman, John D. Ramsdell, and Mitchell Wand. VLISP: A
             Verified Implementation of Scheme. Lisp and Symbolic Computation, 8:5–
             32, 1995.

> **Abstract:** *The VL!SP project showed how to produce a compre-*
> *hensively verified implementation for a programming language,*
> *namely Scheme. This paper introduces two more detailed studies*
> *on VLISP [13, 21). It summarizes the basic techniques that were*
> *used repeatedly throughout the effort. It presents scientific con-*
> *clusions about the applicability of the these techniques as well*
> *as engineering conclusions about the crucial choices that allowed*
> *the verification to succeed.*

[Hall90]   Anthony Hall. 7 Myths of Formal Methods. IEEE Software, 7(5):11–19,
             1990.

> **Abstract:** *Formal methods are difficult, expensive, and not*
> *widely useful, detractors say. Using a case study and other real-*
> *world examples, this article challenges such common myths.*

[Hamm95]   Richard Hamming. Hamming, 'You and Your Research', 1995.

> **Link:** <https://www.youtube.com/watch?v=a1zDuOPkMSw>

[Hard13]   David S. Hardin, Jedidiah R. McClurg, and Jennifer A. Davis. Creating
             Formally Verified Components for Layered Assurance with an LLVM to
             ACL2 Translator.

> **Abstract:** *This paper describes an effort to create a library of*
> *formally verified software component models from code that have*
> *been compiled using the Low-Level Virtual Machine (LLVM) in-*
> *termediate form. The idea is to build a translator from LLVM to*
> *the applicative subset of Common Lisp accepted by the ACL2*
> *theorem prover. They perform verification of the component*
> *model using ACL2's automated reasoning capabilities.*

> **Link:** <http://www.jrmcclurg.com/papers/law_2013_paper.pdf>

[Hard14]   David S. Hardin, Jennifer A. Davis, David A. Greve, and Jedidiah R.
             McClurg. Development of a Translator from LLVM to ACL2.

> **Abstract:** *In our current work a library of formally verified*
> *software components is to be created, and assembled, using the*
> *Low-Level Virtual Machine (LLVM) intermediate form, into sub-*
> *systems whose top-level assurance relies on the assurance of the*
> *individual components. We have thus undertaken a project to*
> *build a translator from LLVM to the applicative subset of Com-*
> *mon Lisp accepted by the ACL2 theorem prover. Our translator*
> *produces executable ACL2 formal models, allowing us to both*
> *prove theorems about the translated models as well as validate*
> *those models by testing. The resulting models can be translated*
> *and certified without user intervention, even for code with loops,*
> *thanks to the use of the def::ung macro which allows us to de-*
> *fer the question of termination. Initial measurements of concrete*
> *execution for translated LLVM functions indicate that perfor-*

mance is nearly 2.4 million LLVM instructions per second on a typical laptop computer. In this paper we overview the translation process and illustrate the translator's capabilities by way of a concrete example, including both a functional correctness theorem as well as a validation test for that example.

Link: <http://arxiv.org/pdf/1406.1566>

[Harp13]   Robert Harper. *15.819 Homotopy Type Theory Course, 2013.*

Link: <http://www.cs.cmu.edu/~rwh/courses/hott>

[Harr98]   J. Harrison and L. Thery. *A Skeptic's approach to combining HOL and Maple.* J. Autom. Reasoning, *21(3):279–294, 1998.*

Abstract: *We contrast theorem provers and computer algebra systems, pointing out the advantages and disadvantages of each, and suggest a simple way to achieve a synthesis of some of the best features of both. Our method is based on the systematic separation of search for a solution and checking the solution, using a physical connection between systems. We describe the separation of proof search and checking in some detail, relating it to proof planning and to the complexity class NP, and discuss different ways of exploiting a physical link between systems. Finally, the method is illustrated by some concrete examples of computer algebra results proved formally in the HOL theorem prover with the aid of Maple.*

Link: <http://www.cl.cam.ac.uk/~jrh13/papers/cas.ps.gz>

[Hoar69]   C. A. R. Hoare. *An Axiomatic Basis for Computer Programming.* CACM, *12(10):576–580, 1969.*

Abstract: *In this paper an attempt is made to explore the logical foundations of computer programming by use of techniques which were first applied in the study of geometry and have later been extended to other branches of mathematics. This involves the elucidation of sets of axioms and rules of inference which can be used in proofs of the properties of computer programs. Examples are given of such axioms and rules, and a formal proof of a simple theorem is displayed. Finally, it is argued that important advantages, both theoretical and practical, may follow from a pursuance of these topics*

Link: <https://www.cs.cmu.edu/~crary/819-f09/Hoare69.pdf>

[Hoar87]   Charles Antony Richard Hoare. *An Overview of Some Formal Methods for Program Design.* Computer, *20(9), 1987.*

[Hoar96]   C.A.R Hoare. *How did software get so reliable without proof?* LNCS, *1051, 1996.*

Abstract: *By surveying current software engineering practice, this paper reveals that the techniques employed to achieve reliability are little different from those which have proved effective in*

all other branches of modern engineering: rigorous management of procedures for design inspection and review; quality assurance based on a wide range of targeted tests; continuous evolution by removal of errors from products already in widespread use; and defensive programming, among other forms of deliberate over-engineering. Formal methods and proof play a small direct role in large scale programming; but they do provide a conceptual framework and basic understanding to promote the best of current practice, and point directions for future improvement.

[Homa94]   Karsten Homann and Jacques Calmet. *Combining Theorem Proving and Symbolic Mathematical Computing.* LNCS, *958:18–29, 1994.*

Abstract: *An intelligent mathematical environment must enable symbolic mathematical computation and sophisticated reasoning techniques on the underlying mathematical laws. This paper disscusses different possible levels of interaction between a symbolic calculator based on algebraic algorithms and a theorem prover. A high level of interaction requires a common knowledge representation of the mathematical knowledge of the two systems. We describe a model for such a knowledge base mainly consisting of type and algorithm schemata, algebraic algorithms and theorems.*

[Hugh90]   John Hughes. *Why Functional Programming Matters.* In Research Topics in Functional Programming, *pages 17–42. Addison-Wesley, 1990.*

Abstract: *As software becomes more and more complex, it is more and more important to structure it well. Well-structured software is easy to write and to debug, and provides a collection of modules that can be reused to reduce future programming costs. In this paper we show that two fea- tures of functional languages in particular, higher-order functions and lazy evaluation, can contribute significantly to modularity. As examples, we manipulate lists and trees, program several numerical algorithms, and implement the alpha-beta heuristic (an algorithm from Artificial Intelligence used in game-playing programs). We conclude that since modularity is the key to successful programming, functional programming offers important advantages for software development.*

[Igar75]   Shigeru Igarashi, Ralph L. London, and David C. Luckham. *Automatic Program Verification I: A Logical Basis and Its Implementation.* Acta Informatica, *4(2):145–182, 1975.*

Abstract: *Defining the semantics of programming languages by axioms and rules of inference yields a deduction system within which proofs may be given that programs satisfy specifications. The deduction system herein is shown to be consistent and also deduction complete with respect to Hoare's system. A subgoaler for the deduction system is described whose input is a significant subset of Pascal programs plus inductive assertions. The output is a set of verification conditions or lemmas to be proved. Several*

*non-trivial arithmetic and sorting programs have been shown
to satisfy specifications by using an interactive theorem prover
to automatically generate proofs of the verification conditions.
Additional components for a more powerful verification system
are under construction.*

[Jack95]   Paul Bernard Jackson. Enhancing the NUPRL Proof Development System
and Applying it to Computational Abstract Algebra. *PhD thesis, Cornell
University, 1 1995.*

Abstract: *This thesis describes substantial enhancements that
were made to the software tools in the Nuprl system that are
used to interactively guide the production of formal proofs. Over
20,000 lines of code were written for these tools. Also, a cor-
pus of formal mathematics was created that consists of roughly
500 definitions and 1300 theorems. Much of this material is of a
foundational nature and supports all current work in Nuprl. This
thesis concentrates on describing the half of this corpus that is
concerned with abstract algebra and that covers topics central to
the mathematics of the computations carried out by computer
algebra systems. The new proof tools include those that solve lin-
ear arithmetic problems, those that apply the properties of order
relations, those that carry out inductive proof to support recur-
sive definitions, and those that do sophisticated rewriting. The
rewrite tools allow rewriting with relations of differing strengths
and take care of selecting and applying appropriate congruence
lemmas automatically. The rewrite relations can be order rela-
tions as well as equivalence relations. If they are order relations,
appropriate monotonicity lemmas are selected. These proof tools
were heavily used throughout the work on computational alge-
bra. Many examples are given that illustrate their operation and
demonstrate their effectiveness. The foundation for algebra in-
troduced classes of monoids, groups, ring and modules, and in-
cluded theories of order relations and permutations. Work on
finite sets and multisets illustrates how a quotienting operation
hides details of datatypes when reasoning about functional pro-
grams. Theories of summation operators were developed that
drew indices from integer ranges, lists and multisets, and that
summed over all the classes mentioned above. Elementary fac-
torization theory was developed that characterized when cancel-
lation monoids are factorial. An abstract data type for the op-
erations of multivariate polynomial arithmetic was defined and
the correctness of an implementation of these operations was
verified. The implementation is similar to those found in current
computer algebra systems. This work was all done in Nuprl's
constructive type theory. The thesis discusses the appropriate-
ness of this foundation, and the extent to which the work relied
on it.*

Keyword: *axiomref, CAS-Proof, printed*

[Jenk84b]   *Richard D. Jenks. A primer: 11 keys to New Scratchpad. In Proc. EU-*

ROSAM ISSAC 1984, *pages 123–147. Springer-Verlag, 1984, 0-387-13350-X.*

Abstract: *This paper is an abbreviated primer for the language of new SCRATCHPAD, a new implementation of SCRATCH-PAD which has been under design and development by the Computer Algebra Group at the IBM Research Center during the past 6 years. The basic design goals of the new SCRATCHPAD language and interface to the user are to provde:*

- *a "typeless" interactive language suitable for on-line solution of mathematical problems by novice users with little or no programming required, and*

- *a programming language suitable for the formal description of algorithms and algebraic structures which can be compiled into run-time efficient object code.*

The new SCRATCHPAD language is introduced by 11 keys with each successive key introducing an additional capability of the language. The language is thus described as a "concentric" language with each of the 11 levels corresponding to a language subset. These levels are more than just a pedagogic device, since they correspond to levels at which the system can be effectively used. Level 1 is sufficient for naive interactive use; levels 2-8 progressively introduce interactive users to capabilities of the language; levels 9-11 are for system programmers and advanced users. Levesl 2, 4, 6, and 7 give users the full power of LISP with a high-level language; level 8 introduces "type declarations;" level 9 allows polymorphic functions to be defined and compiled; levels 10-11 give users an Ada-like facility for defining types and packages (those of new SCRATCHPAD are dynamically constructable, however). One language is used for both interactive and system programming language use, although several freedomes such as abbreviation and optional type-declarations allowed at top-level are not permitted in system code. The interactive language (levels 1-8) is a blend of original SCRATCHPAD [GRJY75], some proposed extensions [JENK74], work by Loos [LOOS74], SETL [DEWA79], SMP [COWO81], and new ideas; the system programming language (levels 1-11) superficially resembles Ada but is more similar to CLU [LISK74] in its semantic design. This presentation of the language in this paper omits many details to be covered in the SCRATCHPAD System Programming Manual [SCRA84] and an expanded version of this paper will serve as a primer for SCRATCHPAD users [JESU84].

[Juds15]   *Thomas W. Judson. Abstract Algebra: Theory and Applications. Website, 2015.*

Link: <http://abstract.ups.edu/aata/colophon-1.html>

[Kaes88]   *Stefan Kaes. Parametric Overloading in Polymorphic Programming Languages. LNCS, 300:131–144, 1988.*

Abstract: *The introduction of unrestricted overloading in lan-*

*guagues with type systems based on implicit parametric poly-
morphism generally destroys the principal type property: namely
that the type of every expression can uniformly be represented
by a single type expression over some set of type variables. As a
consequence, type inference in the presence of unrestricted over-
loading can become a NP-complete problem. In this paper we
define the concept of parametric overloading as a restricted form
of overloading which is easily combined with parametric poly-
morphism. Parametric overloading preserves the principal type
property, thereby allowing the design of efficient type inference
algorithms. We present sound type deduction systems, both for
predefined and programmer defined overloading. Finally we state
that parametric overloading can be resolved either statically, at
compile time, or dynamically, during program execution.*

[Kama15]   Fairouz Kamareddine, Joe Wells, Christoph Zengler, and Henk Barendregt.
              Computerising Mathematical Text, 2015.

Abstract: *Mathematical texts can be computerised in many
ways that capture differing amounts of the mathematical mean-
ing. At one end, there is document imaging, which captures
the arrangement of black marks on paper, while at the other
end there are proof assistants (e.g. Mizar, Isabelle, Coq, etc.),
which capture the full mathematical meaning and have proofs
expressed in a formal foundation of mathematics. In between,
there are computer typesetting systems (e.g. Latex and Presen-
tation MathML) and semantically oriented systems (e.g. Content
MathML, OpenMath, OMDoc, etc.). In this paper we advocate
a style of computerisation of mathematical texts which is flex-
ible enough to connect the diferent approaches to computerisa-
tion, which allows various degrees of formalsation, and which
is compatible with different logical frameworks (e.g. set theory,
category theory, type theory, etc.) and proof systems. The basic
idea is to allow a man-machine collaboration which weaves hu-
man input with machine computation at every step in the way.
We propose that the huge step from informal mathematics to
fully formalised mathematics be divided into smaller steps, each
of which is a fully developed method in which human input is
minimal.*

[Katz75]   Shmuel Katz and Zohar Manna. *A Closer Look at Termination*. Acta
              Informatica, 5:333–352, 1975.

Abstract: *Several methods for proving that computer programs
terminate are presented and illustrated. The methods considered
involve (a) using the 'no-infinitely-descending-chain' property of
well-founded sets (Floyd's approach), (b) bounding a counter as-
sociated with each loop (loop approach), (c) showing that some
exit of each loop must be taken (exit approach), or (d) induct-
ing on the structure of the data domain (Burstall's approach).
We indicate the relative merit of each method for proving ter-
mination or non-termination as an integral part of an automatic*

verification system.

[Kife91]   *Michael Kifer and James Wu. A First-order Theory of Types and Polymorphism in Logic Programming. In Proc Sixth Annual IEEE Symp. on Logic in Comp. Sci., pages 310–321, 1991.*

> **Abstract:** *A logic called typed predicate calculus (TPC) that gives declarative meaning to logic programs with type declarations and type inference is introduced. The proper interaction between parametric and inclusion varieties of polymorphism is achieved through a construct called type dependency, which is analogous to implication types but yields more natural and succinct specifications. Unlike other proposals where typing has extra-logical status, in TPC the notion of type-correctness has precise model-theoretic meaning that is independent of any specific type-checking or type-inference procedure. Moreover, many different approaches to typing that were proposed in the past can be studied and compared within the framework of TPC. Another novel feature of TPC is its reflexivity with respect to type declarations; in TPC, these declarations can be queried the same way as any other data. Type reflexivity is useful for browsing knowledge bases and, potentially, for debugging logic programs.*

[Kreb17]   *Robbert Jan Krebbers. The CH_2O formalization of ISO C11, 2017.*

> **Link:** <http://robbertkrebbers.nl/research/ch2o/>

[Lamp02]   *Leslie Lamport. Specifying Systems. Addison-Wesley, 2002, 0-321-14306-X.*

> **Link:** <http://research.microsoft.com/en-us/um/people/lamport/tla/book-02-08-08.pdf>

[Lamp14]   *Leslie Lamport. How to Write a 21st Century Proof, 2014.*

> **Abstract:** *A method of writing proofs is described that makes it harder to prove things that are not true. The method, based on hierarchical structuring, is simple and practical. The author's twenty years of experience writing such proofs is discussed.*
>
> **Link:** <http://lamport.azurewebsites.net/pubs/proof.pdf>

[Lamp14a]   *Leslie Lamport. Talk: How to Write a 21st Century Proof, 2014.*

> **Comment:** *2nd Heidelberg Laureate Forum Lecture Tuesday Sep 23, 2014*
>
> **Link:** <http://hits.mediasite.com/mediasite/Play/29d825439b3c49f088d35555426fbdf81d>

[Lamp16]   *Leslie Lamport. TLA+ Proof System, 2016.*

> **Abstract:** *Demonstration of Euclid Algorithm Proof in TLA+*
>
> **Link:** <https://tla.msr-inria.inria.fr/tlaps/content/Documentation/Tutorial/The_example.html>

[Lamp81]   *Leslie Lamport and Susan Owicki. Program Logics and Program Verification. LNCS, 131:197–199, 1981.*

[Lero09]   *Xavier Leroy. A Formally Verified Compiler Back-end.* Logic in Computer Science, *43(4):363–446, 2009.*

> **Abstract:** *This article describes the development and formal verification (proof of semantic preservation) of a compiler back-end from Cminor (a simple imperative intermediate language) to PowerPC assembly code, using the Coq proof assistant both for programming the compiler and for proving its soundness. Such a verified compiler is useful in the context of formal methods applied to the certification of critical software: the verification of the compiler guarantees that the safety properties proved on the source code hold for the executable compiled code as well.*

[Mann78]   *Zohar Manna and Richard Waldinger. The Logic of Computer Programming.* IEEE Trans. on Software Engineering, *4(3), 1978.*

> **Abstract:** *Techniques derived from mathematical logic promise to provide an alternative to the conventional methodology for constructing, debugging, and optimizing computer programs. Ultimately, these techniques are intended to lead to the automation of many of the facets of the programming process. This paper provides a unified tutorial exposition of the logical techniques, illustrating each with examples. The strengths and limitations of each technique as a practial programming aid are assessed and attempts to implement these methods in experimental systems are discussed.*

[Mann78a]   *Zohar Manna and Richard Waldinger. Is 'sometime' sometimes better than 'always'?* CACM, *21(2), 1978.*

> **Abstract:** *This paper explores a technique for proving the correctness and termination of programs simultaneously. This approach, the intermittent-assertion method, involves documenting the program with assertions that must be true at some time when control passes through the corresponding point, but that need not be true every time. The method, introduced by Burstall, promises to provide a valuable complement to the more conventional methods. The intermittent-assertion method is presented with a number of examples of correctness and termination proofs. Some of these proofs are markedly simpler than their conventional counterparts. On the other hand, it is shown that a proof of correctness or termination by any of the conventional techniques can be rephrased directly as a proof using intermittent assertions. Finally it is shown how the intermittent assertion method can be applied to prove the validity of program transformations and the correctness of continuously operating systems.*

[Mano03]   *Panagiotis Manolios and J Strother Moore. Partial Functions in ACL2.* J. of Automated Reasoning, *31:107–127, 2003.*

> **Abstract:** *We describe a method for introducing 'partial functions' into ACL2, that is, functions not defined everywhere. The function 'definitions' are actually admitted via the encapsulation principle: the new function symbol is constrained to satisfy*

the appropriate equation. This is permitted only when a witness function can be exhibited, establishing that the constraint is satisfiable. Of particular interest is the observation that every tail recursive definition can be witnessed in ACL2. We describe a macro that allows the convenient introduction of arbitrary tail recursive functions, and we discuss how such functions can be used to prove theorems about state machine models without reasoning about clocks or counting the number of steps until termination. Our macro for introducing partial functions also permits a variety of other recursive schemes, and we briefly illustrate some of them.

[Mart79]   P. Martin-Lof. *Constructive Mathematics and Computer Programming.* In Proc. 6th Int. Congress for Logic, Methodology and Philosophy of Science, pages 153–179. North-Holland, 1979.

Abstract: *If programming is understood not as the writing of instructions for this or that computing machine but as the design of methods of computation that it is the computer's duty to execute (a difference that Dijkstra has referred to as the difference between computer science and computing science), then it no longer seems possible to distinguish the discipline of programming from constructive mathematics. This explains why the intuitionistic theory of types, which was originally developed as a symbolism for the precise codification of constructive mathematics, may equally well be viewed as a programming language. As such it provides a precise notation not only, like other programming languages, for the programs themselves but also for the tasks that the programs are supposed to perform. Moreover, the inference rules of the theory of types, which are again completely formal, appear as rules of correct program synthesis. Thus the correctness of a program written in the theory of types is proved formally at the same time as it is being synthesized.*

[Maso86]   Ian A. Mason. *The Semantics of Destructive Lisp. Center for the Study of Language and Information,* 1986, 0-937073-06-7.

Abstract: *Our basic premise is that the ability to construct and modify programs will not improve without a new and comprehensive look at the entire programming process. Past theoretical research, say, in the logic of programs, has tended to focus on methods for reasoning about individual programs; little has been done, it seems to us, to develop a sound understanding of the process of programming – the process by which programs evolve in concept and in practice. At present, we lack the means to describe the techniques of program construction and improvement in ways that properly link verification, documentation and adaptability.*

[Mcal96]   D. McAllester and K. Arkondas. *Walther Recursion.* In CADE 13. Springer-Verlag, 1996.

Abstract: *Primitive recursion is a well known syntactic restric-*

tion on recursion definitions which guarantees termination. Unfortunately many natural definitions, such as the most common definition of Euclid's GCD algorithm, are not primitive recursive. Walther has recently given a proof system for verifying termination of a broader class of definitions. Although Walther's system is highly automatible, the class of acceptable definitions remains only semi-decidable. Here we simplify Walther's calculus and give a syntactic criteria generalizes primitive recursion and handles most of the examples given by Walthar. We call the corresponding class of acceptable defintions "Walther recursive".

[Mein13] Diana Meindl. *Implementation of an Algorithm Computing the Greatest Common Divisor for Multivariate Polynomials*. Master's thesis, RISC Linz, 2013.

[Mesh16a] Sergei D. Meshveliani. *Provable programming of algebra: particular points, examples*, 2016.

> **Abstract:** *It is discussed an experiance in provable programming of a computer algebra library with using a purely functional language with dependent tyhpes (`Agda`). There are given several examples illustrating particular points of implementing the approach of constructive mathematics.*
>
> **Link:** http://www.botik.ru/pub/local/Mechveliani/provProgExam.zip

[Mosc84] Y.N. Moschovakis. *Abstract Recursion as a Foundation of the Theory of Algorithms*. LNCS, 1104:289–364, 1984.

> **Abstract:** *The main object of this paper is to describe an abstract, (axiomatic) theory of recursion and its connection with some of the basic, foundational questions of computer science.*

[Moyx10] Yannick Moy and Angela Wallenburg. *Tokeneer: Beyond Formal Program Verification*, 2010.

> **Abstract:** *Tokeneer is a small-sized (10 kloc) security system which was formally developed and verified by Praxis at the request of NSA, using SPARK technology. Since its open-source release in 2008, only two problems were found, one by static analysis, one by code review. In this paper, we report on experiments where we systematically applied various static analysis tools (compiler, bug-finder, proof tools) and focused code reviews to all of the SPARK code (formally verified) and supporting Ada code (not formally verified) of the Tokeneer Project. We found 20 new problems overall, half of which are defects that could lead to a system failure should the system be used in its current state. Only two defects were found in SPARK code, which confirms the benefits of applying formal verification to reach higher levels of assurance. In order to leverage these benefits to code that is was not formally verified from the start, we propose to associate static analyses and dynamic analyses around a common expression of properties and constraints. This is the goal of starting project Hi-Lite, which involves AdaCore and Altran*

Praxis together with several industrial users and research labs.

Link: *http://www.cse.chalmers.se/~angelaw/papers/ERTS2010.pdf*

[Murr18] *Toby Murray and P.C. van Oorcchot. BP: Formal Proofs, the Fine Print and Side Effects.*

> **Abstract:** *Given recent high-profile successes in formal verification of security-related properties (e.g. for seL4), and the rising popularity of applying formal methods to cryptographic libraries and security protocols like TLS, we revisit the meaning of security-related proofs about software. We re-examine old issues, and identify new questions that have escaped scrutiny in the formal methods literature. We consider what value proofs about software systems deliver to end-users (e.g. in terms of net assurance benefits), and at what cost in terms of side effects (such as changes made to software to facilitate the proofs, and assumption-related deployment restrictions imposed on software if these proofs are to remain valid in operation). We consider in detail, for the first time to our knowledge, possible relationships between proofs and side effects. To make our discussion concrete, we draw on tangible examples, experience, and the literature.*

> **Comment:** *Version: 26 June 2018 to appear in IEEE SecDev 2018*

> **Link:** *https://people.eng.unimelb.edu.au/tobym/papers/secdev2018.pdf*

[Myre09] *Magnus O. Myreen and Michael J.C. Gordon. Verified LISP Implementations on ARM, x86 and PowerPC. LNCS, 5674:359–374, 2009.*

> **Abstract:** *This paper reports on a case study, which we believe is the first to produce a formally verified end-to-end implementation of a functional programming language running on commercial processors. Interpreters for the core of McCarthys LISP 1.5 were implemented in ARM, x86 and PowerPC machine code, and proved to correctly parse, evaluate and print LISP s-expressions. The proof of evaluation required working on top of verified implementations of memory allocation and garbage collection. All proofs are mechanised in the HOL4 theorem prover.*

[Myre09a] *Magnus O. Myreen, Konrad Slind, and Michael J.C. Gordon. Extensible Proof-Producing Compilation. LNCS, 5501:2–16, 2009.*

> **Abstract:** *This paper presents a compiler which produces machine code from functions defined in the logic of a theorem prover, and at the same time proves that the generated code executes the source functions. Unlike previously published work on proof-producing compilation from a theorem prover, our compiler provides broad support for user-defined extensions, targets multiple carefully modelled commercial machine languages, and does not require termination proofs for input functions. As a case study, the compiler is used to construct verified interpreters for*

a small LISP-like language. The compiler has been implemented in the HOL4 theorem prover.

[Myre10]   Magnus O. Myreen. *Verified Just-In-Time Compiler on x86.* ACM SIGLAN Notices - POPL'10, *45(1):107–118, 2010.*

> **Abstract:** *This paper presents a method for creating formally correct just-in- time (JIT) compilers. The tractability of our approach is demonstrated through, what we believe is the first, verification of a JIT compiler with respect to a realistic semantics of self-modifying x86 machine code. Our semantics includes a model of the instruction cache. Two versions of the verified JIT compiler are presented: one generates all of the machine code at once, the other one is incremental i.e. produces code on-demand. All proofs have been performed inside the HOL4 theorem prover.*

[Myre11]   Magnus O. Myreen and Jared Davis. *A Verified Runtime for a Verified Theorem Prover.* NCS, *6898:265–280, 2011.*

> **Abstract:** *Theorem provers, such as ACL2, HOL, Isabelle and Coq, rely on the correctness of runtime systems for programming languages like ML, OCaml or Common Lisp. These runtime systems are complex and critical to the integrity of the theorem provers. In this paper, we present a new Lisp runtime which has been formally verified and can run the Milawa theorem prover. Our runtime consists of 7,500 lines of machine code and is able to complete a 4 gigabyte Milawa proof effort. When our runtime is used to carry out Milawa proofs, less unverified code must be trusted than with any other theorem prover. Our runtime includes a just-in-time compiler, a copying garbage collector, a parser and a printer, all of which are HOL4-verified down to the concrete x86 code. We make heavy use of our previously developed tools for machine-code verification. This work demonstrates that our approach to machine-code verification scales to non-trivial applications.*

> **Link:** *https://www.cl.cam.ac.uk/~mom22/jitawa/*

[Myre12]   Magnus O. Myreen. *Functional Programs: Conversions between Deep and Shallow Embeddings.* LNCS, *7406:412–417, 2012.*

> **Abstract:** *This paper presents a method which simplifies verification of deeply embedded functional programs. We present a technique by which proof-certified equations describing the effect of functional programs (shallow embeddings) can be automatically extracted from their operational semantics. Our method can be used in reverse, i.e. from shallow to deep embeddings, and thus for implementing certifying code synthesis: we have implemented a tool which maps HOL functions to equivalent Lisp functions, for which we have a verified Lisp runtime. A key benefit, in both directions, is that the verifier does not need to understand the operational semantics that gives meanings to the deep embeddings.*

> **Link:** *https://www.cl.cam.ac.uk/~mom22/jitawa/*

[Myre14]   Magnus O. Myreen and Jared Davis. *The Reflective Milawa Theorem Prover is Sound.* LNAI, pages 421–436, 2014.

> **Abstract:** *Milawa is a theorem prover styled after ACL2 but with a small kernel and a powerful reflection mechanism. We have used the HOL4 theorem prover to formalize the logic of Milawa, prove the logic sound, and prove that the source code for the Milawa kernel (2,000 lines of Lisp) is faithful to the logic. Going further, we have combined these results with our previous verification of an x86 machine-code implementation of a Lisp runtime. Our top-level HOL4 theorem states that when Milawa is run on top of our verified Lisp, it will only print theorem statements that are semantically true. We believe that this top-level theorem is the most comprehensive formal evidence of a theorem provers soundness to date.*

[Neup13]   Walther Neuper. *Computer Algebra implemented in Isabelle's Function Package under Lucas-Interpretation – a Case Study,* 2013.

> **Link:** *http://ceur-ws.org/Vol-1010/paper-09.pdf*

[Nord90]   Bengt Nordström, Kent Petersson, and Jan M. Smith. *Programming in Martin-Löf's Type Theory.* Oxford University Press, 1990.

[OCAM14]   unknown. *The OCAML website.*

> **Link:** *http://ocaml.org*

[Odon81]   Michael J. O'Donnell. *A Critique of the Foundations of Hoare-style Programming Logics.* LNCS, 131:349–374, 1981.

> **Abstract:** *Much recent discussion in computing journals has been devoted to arguments about the feasibility and usefulness of formal verification methods for increasing confidence in computer programs. Too little attention has been given to precise criticism of specific proposed systems for reasoning about programs. Whether such systems are to be used for formal verification, by hand or automatically, or as a rigorous foundation for imformal reasoning, it is essential that they be logically sound. Several popular rules in the Hoare language are in fact not sound. These rules have been accepted because they have not been subjected to sufficiently strong standards of correctness. This paper attempts to clarify the different technical definitions of correctness of a logic, to show that only the strongest of these definitions is acceptable for Hoare logic, and to correct some of the unsound rules which have appeared in the literature. The corrected rules are given merely to show that it is possible to do so. Convenient and elegant rules for reasoning about certain programming constructs will probably require a more flexible notation than Hoare's.*

[Peyt17]   Simon Peyton-Jones. *Escape from the ivory tower: the Haskell journey,* 2017.

> **Link:** *https://www.youtube.com/watch?v=re96UgMk6GQ*

[Pfei12]   Greg Pfeil. *Common Lisp Type Hierarchy,* 2012.

Link: http://sellout.github.io/2012/03/03/
common-lisp-type-hierarchy

[Pier15]   Benjamin C. Pierce, Chris Casinghino, Marco Gaboardi, Michael Green-
berg, Catalin Hritcu, Vilhelm Sjöberg, and Brent Yorgey. Software Foun-
dations, 2015.

> **Abstract:** *This electronic book is a course on Software Founda-*
> *tions, the mathematical underpinnings of reliable software. Top-*
> *ics include basic concepts of logic, computer-assisted theorem*
> *proving, the Coq proof assistant, functional programming, op-*
> *erational semantics, Hoare logic, and static type systems. The*
> *exposition is intended for a broad range of readers, from ad-*
> *vanced undergraduates to PhD students and researchers. No spe-*
> *cific background in logic or programming languages is assumed,*
> *though a degree of mathematical maturity will be helpful. The*
> *principal novelty of the course is that it is one hundred per cent*
> *formalized and machine-checked: the entire text is literally a*
> *script for Coq. It is intended to be read alongside an interactive*
> *session with Coq. All the details in the text are fully formalized*
> *in Coq, and the exercises are designed to be worked using Coq.*
> *The files are organized into a sequence of core chapters, cover-*
> *ing about one semester's worth of material and organized into*
> *a coherent linear narrative, plus a number of appendices cover-*
> *ing additional topics. All the core chapters are suitable for both*
> *upper-level undergraduate and graduate students.*

[Pier17]   Benjamin Pierce. DeepSpec Summer School, Coq Intensive, Part 1 (July
13,2017), 2017.

Link: https://www.youtube.com/watch?v=jG61w5pOc2A

[Reid17]   Alastair Reid. How can you trust formally varified software, 2017.

> **Abstract:** *Formal verification of software has finally started to*
> *become viable: we have examples of formally verified microker-*
> *nels, realistic compilers, hypervisors, etc. These are huge achieve-*
> *ments and we can expect to see even more impressive results in*
> *the future but the correctness proofs depend on a number of as-*
> *sumptions about the Trusted Computing Base that the software*
> *depends on. Two key questions to ask are: Are the specifications*
> *of the Trusted Computing Base correct? And do the implemen-*
> *tations match the specifications? I will explore the philosophical*
> *challenges and practical steps you can taek in answering that*
> *question for one of the major dependencies: the hardware your*
> *software runs on. I will describe the combination of formal veri-*
> *fication and testing that ARM uses to verify the processor spec-*
> *ification and I will talk about our current challenge: getting the*
> *specification down to zero bugs while the architecture continues*
> *to evolve.*

Link: https://media.ccc.de/v/34c3-8915-how_can_you_
trust_formally_verified_software#t=12

[Sant95]   Philip S. Santas. A Type System for Computer Algebra. J. Symbolic

Computation, *19(1-3):79–109, 1995.*

> **Abstract:** *This paper presents a type system for support of subtypes, parameterized types with sharing and categories in a computer algebra environment. By modeling representation of instances in terms of existential types, we obtain a simplified model, and build a basis for defining subtyping among algebraic domains. The inheritance at category level has been formalized; this allows the automatic inference of type classes. By means of type classes and existential types we construct subtype relations without involving coercions. A type sharing mechanism works in parallel and allows the consistent extension and combination of domains. The expressiveness of the system is further increased by viewing domain types as special case of package types, forming weak and strong sums respectively. The introduced system, although awkward at first sight, is simpler than other proposed systems for computer algebra without including some of their problems. The system can be further extended in other to support more constructs and increase its flexibility.*

[Scot71]   Dana S. Scott and C. Strachey. *Towards a Mathematical Semantics for Computer Languages.* Proc. Symp. on Computers and Automata, *21, 1971.*

> **Abstract:** *Compilers for high-level languages are generally constructed to give a complete translation of the programs into machine lanugage. As machines merely juggle bit patterns, the concepts of the original language may be lost or at least obscured during this passage. The purpose of a mathematical semantics is to give a correct and meaningful correspondence between programs and mathematical entities in a way that is entirely independent of an implementation. This plan is illustrated in a very elementary method with the usual idea of state transformations. The next section shows why the mathematics of functions has to be modified to accommodate recursive commands. Section 3 explains the modification. Section 4 introduces the environments for handling variables and identifiers and shows how the semantical equations define equivalence of programs. Section 5 gives an exposition of the new type of mathematical function spaces that are required for the semantics of procedures when these are allowed in assignment statements. The conclusion traces some of the background of the project and points the way to future work.*

[Sho08]   Victor Shoup. *A Computational Introduction to Number Theory.*

> **Link:** <http://shoup.net/ntb/ntb-v2.pdf>

[Site74]   Richard L. Sites. *Some Thoughts on Proving Clean Termination of Programs.* technical report STAN-CS-74-417, Stanford University, 1974.

> **Abstract:** *Proof of clean termination is a useful sub-goal in the process of proving that a program is totally correct. Clean termination means that the program terminates (no infinite loops) and that it does so normally, without any execution-time seman-*

tic errors (integer overflow, use of undefined variables, subscript out of range, etc.). In contrast to proofs of correctness, proof of clean termination requires no extensive annotation of a program by a human user, but the proof says nothing about the results calculated by the program, just that whatever it does, it terminates cleanly. Two example proofs are given, of previously published programs: TREESORT3 by Robert Floyd, and SELECT by Ronald L. Revest and Robert Floyd.

[Smol89a]   G. Smolka. Logic Programming over Polymorphically Order-Sorted Types. *PhD thesis, Fachbereich Informatik, Universitat Kaiserslautern, 1989.*

[Soze08]   *Mattieu Sozeau and Nicolas Oury. First-Class Type Classes.* Lecture Notes in Computer Science, *5170:278–293, 2008.*

Abstract: *Type Classes have met a large success in Haskell and Isabelle, as a solution for sharing notations by overloading and for specifying with abstract structures by quantificaiton on contexts. However, both systems are limited by second-class implementations of these constructs, and these limitations are only overcome by ad-hoc extensions to the respective systems. We propose an embedding of type classes into a dependent type theory that is first-class and supports some of the most popular extensions right away. The implementation is correspondingly cheap, general, and integrates well inside the system, as we have experimented in Coq. We show how it can be used to help structured programming and proving by way of examples.*

Link: [https://www.irif.fr/~sozeau/research/](https://www.irif.fr/~sozeau/research/publications/First-Class_Type_Classes.pdf)
[publications/First-Class_Type_Classes.pdf](https://www.irif.fr/~sozeau/research/publications/First-Class_Type_Classes.pdf)

[Spit11]   *Bas Spitters and Eelis van der Weegen. Type Classes for Mathematics in Type Theory.* Math. Struct. Comput. Sci., *21(4):795–825, 2011.*

Abstract: *The introduction of first-class type classes in the Coq system calls for a re-examination of the basic interfaces used for mathematical formalisation in type theory. We present a new set of type classes for mathematics and take full advantage of their unique features to make practical a particularly flexible approach that was formerly thought to be infeasible. Thus, we address traditional proof engineering challenges as well as new ones resulting from our ambition to build upon this development a library of constructive analysis in which any abstraction penalties inhibiting efficient computation are reduced to a minimum. The basis of our development consists of type classes representing a standard algebraic hierarchy, as well as portions of category theory and universal algebra. On this foundation, we build a set of mathematically sound abstract interfaces for different kinds of numbers, succinctly expressed using categorical language and universal algebra constructions. Strategic use of type classes lets us support these high-level theory-friendly definitions, while still enabling efficient implementations unhindered by gratuitous indirection, conversion or projection. Algebra thrives on the inter-*

play between syntax and semantics. The Prolog-like abilities of type class instance resolution allow us to conveniently define a quote function, thus facilitating the use of reflective techniques.

Link: *https://arxiv.org/pdf/1102.1323.pdf*

[Stac17]   *StackExchange. How do Gap generate the elements in permutation groups, 2017.*

Link: *http://math.stackexchange.com/questions/ 1705277/how-do-gap-generate-the-elements-in-permutation-groups*

[Suto87]   *Robert S. Sutor and Richard D. Jenks. The type inference and coercion facilities in the Scratchpad II interpreter.* SIGPLAN Notices, *22(7):56–63, 1987, 0-89791-235-7.*

Abstract: *The Scratchpad II system is an abstract datatype programming language, a compiler for the language, a library of packages of polymorphic functions and parametrized abstract datatypes, and an interpreter that provides sophisticated type inference and coercion facilities. Although originally designed for the implementation of symbolic mathematical algorithms, Scratchpad II is a general purpose programming language . This paper discusses aspects of the implementation of the interpreter and how it attempts to provide a user friendly and relatively weakly typed front end for the strongly typed programming language.*

Comment: *IBM Research Report RC 12595 (#56575)*

[Talp92]   *Jean-Pierre Talpin and Pierre Jouvelot. The Type and Effect Discipline. In* Conf. on Logic in Computer Science. *Computer Science Press, 1992.*

Abstract: *The type and effect discipline is a new framework for reconstructing the principal type and the minimal effect of expressions in implicitly typed polymorphic functional languages that support imperative constructs. The type and effect discipline outperforms other polymorphic type systems. Just as types abstract collections of concrete values, effects denote imperative operations on regions. Regions abstract sets of possibly aliased memory locations. Effects are used to control type generalization in the presence of imperative constructs while regions delimit observable side-effects. The observable effects of an expression range over the regions that are free in its type environment and its type; effects related to local data structures can be discarded during type reconstruction. The type of an expression can be generalized with respect to the variables that are not free in the type environment or in the observable effect.*

[Ther01]   *Laurent Théry. A Machine-Checked Implementation of Buchberger's Algorithm.* Journal of Automated Reasoning, *26:107–137, 2001.*

Abstract: *We present an implementation of Buchberger's algorithm that has been proved correct within the proof assistant Coq. The implementation contains the basic algorithm plus two standard optimizations.*

[Thur94]   William P. Thurston. *On Proof and Progress in Mathematics.* Bulletin AMS, *30(2), April 1994.*

>   **Link:** *http://www.ams.org/journals/bull/1994-30-02/ S0273-0979-1994-00502-6/S0273-0979-1994-00502-6.pdf*

[Tros13]   *Anne Trostle. An Algorithm for the Greatest Common Divisor, 2013.*

>   **Link:** *http://www.nuprl.org/MathLibrary/gcd/*

[Tuho18]   *Henri Tuhola. An another MLsub study, 2018.*

>   **Comment:** *Subtyping does not solve the problem of coercion*

>   **Link:** *http://boxbase.org/entries/2018/mar/12/ mlsub-study/*

[Wadl88]   *Philip Wadler and Stephen Blott. How to Make Ad-hoc Polymorphism Less Ad hoc. In* Proc 16th ACM SIGPLAN-SIGACT Symp. on Princ. of Prog. Lang, *pages 60–76, 1988, 0-89791-294-2.*

>   **Abstract:** *This paper presents type classes, a new approach to ad-hoc polymorphism. Type classes permit overloading of arithmetic operators such as multiplication, and generalise the "eqtype variables" of Standard ML Type classes extend the Hindley/Milner polymorphic type system, and provide a new approach to issues that arise in object-oriented programming, bounded type quantification, and abstract data types. This paper provides an informal introduction to type classes, and defines them formally by means of type inference rules*

>   **Link:** *http://202.3.77.10/users/karkare/courses/ 2010/cs653/Papers/ad-hoc-polymorphism.pdf*

[Wijn68]   *A. van Wijngarrden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.T. Meertens, and R.G. Fisker. Revised Report on the Algorithmic Language ALGOL 68, 1968.*

>   **Link:** *http://www.eah-jena.de/~kleine/history/ languages/algol68-revisedreport.pdf*

[Wiki14a]   *ProofWiki. Euclidean Algorithm.*

>   **Link:** *http://proofwiki.org/wiki/Euclidean_Algorithm*

[Wiki14b]   *ProofWiki. Division Theorem.*

>   **Link:** *http://proofwiki.org/wiki/Division_Theorem*

[Wiki17]   *Wikipedia. Calculus of constructions, 2017.*

>   **Link:** *https://en.wikipedia.org/wiki/Calculus_of_ constructions*

[WikiED]   *Wikipedia. Euclidean Domain, 2017.*

>   **Link:** *https://en.wikipedia.org/wiki/Euclidean_ domain*

[Wilk85a]   *Maurice Wilkes.* Memoirs of a Computer Pioneer. *MIT Press, 1985.*

[Zdan14]   *Steve Zdancewic and Milo M.K. Martin. Vellvm: Verifying the LLVM.*

>   **Link:** *http://www.cis.upenn.edu/~stevez/vellvm*

[Zeil10]   Doron Zeilberger. *Against Rigor*. LNCS, *6167:262–262, 2010.*

> **Abstract:** *The ancient Greeks gave (western) civilization quite a few gifts, but we should beware of Greeks bearing gifts. The gifts of theatre and democracy were definitely good ones, and perhaps even the gift of philosophy, but the 'gift' of the so-called 'axiomatic method' and the notion of 'rigorous' proof did much more harm than good. If we want to maximize Mathematical Knowledge, and its Management, we have to return to Euclid this dubious gift, and give-up our fanatical insistence on perfect rigor. Of course, we should not go to the other extreme, of demanding that everything should be non-rigorous. We should encourage diversity of proof-styles and rigor levels, and remember that nothing is absolutely sure in this world, and there does not exist an absolutely rigorous proof, nor absolute certainty, and 'truth' has many shades and levels.*

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