Riemannian Implicit Optimism with Applications to Min-Max Problems

Christophe Roux

21.5.2025

Collaborators & References

Christophe Roux*, David Martínez-Rubio* , Sebastian Pokutta (2024). *Implicit Riemannian Optimism with Applications to Min-Max Problems*. arXiv:2403.10429

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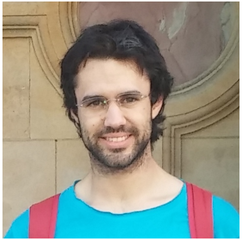

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Online Optimization

Online Optimization

Online Setup:

For rounds $t = 1, 2, \ldots, T$:

- Learner selects $x_t \in \mathcal{X}$ $\triangleright$*before observing* ℓ_t
- Environment reveals ℓ_t $\triangleright$*possibly adversarial*
- Learner pays $\ell_t(x_t)$

Goal: Minimize cumulative loss $\sum_{t=1}^{T} \ell_t(x_t)$

Regret: For a *fixed* comparator $u \in \mathcal{X}$,

$$R_T(u) = \sum_{t=1}^{T} \ell_t(x_t) - \sum_{t=1}^{T} \ell_t(u)$$

Online Gradient Descent:

$$\begin{aligned} x_{t+1} &= \arg\min_{z \in \mathcal{X}} \{\langle \nabla \ell_t(x_t), z \rangle + \frac{1}{2\eta} \|z - x_t\|^2\} \\ &= P_X(x_t - \eta \nabla \ell_t(x_t)) \end{aligned}$$

Standard Results:

- convex, Lipschitz losses ℓ_t
- compact, convex $\mathcal{X}$

$\rightarrow R_T(u) = \Theta(\sqrt{T})$

Book recommendation: *A Modern Introduction to Online Learning*, Francesco Orabona

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Online Optimization: Optimistic Methods

Can we do better?
Predictable environment, i.e., not *fully* adversarial:
$\rightarrow$ Use a hint $\tilde{\ell}_t \approx \ell_t$ to improve regret

Optimistic Online Setup:
For rounds $t = 1, 2, \ldots, T$:

- Learner chooses hint $\tilde{\ell}_t$
- Learner selects $\tilde{x}_t \in \mathcal{X}$ $\quad\triangleright$ *using the hint*
- Environment reveals ℓ_t
- Learner pays $\ell_t(\tilde{x}_t)$

Optimistic Online Gradient Descent:

$$x_{t+1} = \arg\min_{z \in \mathcal{X}} \langle \nabla \ell_t(x_t), z \rangle + \frac{1}{2\eta} \|z - x_t\|^2$$

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Results:
$R_T(u) = \mathcal{O}(D^2/\eta + \eta V_T)$

- $D \stackrel{\text{def}}{=} \text{diam}(\mathcal{X})$
- η: step size
- $V_T \stackrel{\text{def}}{=} \sum_{t=1}^T \|\nabla \ell_t(x_t) - \nabla \tilde{\ell}_t(x_t)\|_*^2$
- Good hints: $V_T = o(T)$
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Riemannian optimization

Riemannian optimization

Problem:

Given a function $f : \mathcal{M} \to \mathbb{R}$, solve

$$\min_{x \in \mathcal{M}} f(x),$$

where $\mathcal{M}$ is a Riemannian manifold.

Assumptions for this talk:

- Hadamard manifolds $\mathcal{H}$
  - Sectional curvature in $[\kappa_{\min}, 0]$
  - Uniquely geodesic (one shortest path)
- First-order methods: Oracles $\{f, \nabla f\}$
- Access to $\{\mathrm{Exp}, \mathrm{Exp}^{-1}\}$

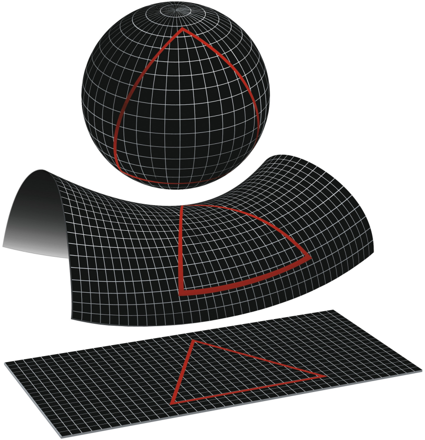


Image credit: NASA/WMAP Science Team

Why Riemannian Optimization?

Use geometric properties (Euclidean constrained $\Rightarrow$ Riemannian unconstrained)

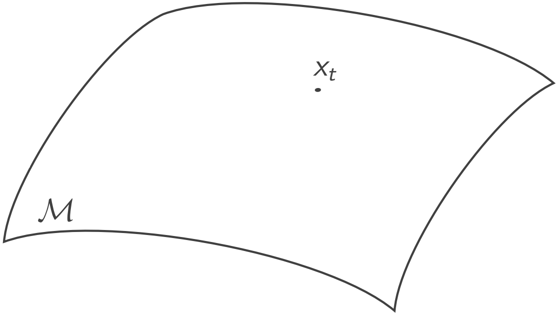
- Fitting Gaussian mixture models: SPD matrices
- DNNs with orthogonality constraints: Stiefel manifold
- Hyperbolic embeddings: Hyperbolic space
- Low-rank matrix factorization: Fixed-rank matrices

Improves problem structure:

- Non-convex *Euclidean* problems can become *geodesically* convex (g-convex) on a manifold
- Example: Operator scaling [Allen-Zhu et al., 2018]

Riemannian Gradient Descent - Key Concepts

- Point x_t on manifold $\mathcal{M}$
  point on curved surface
- Tangent Space: $\mathrm{T}_{x_t}\mathcal{M}$
  local linear approximation
- Riemannian Gradient: $\nabla f(x_t) \in \mathrm{T}_{x_t}\mathcal{M}$
  Gradient projected onto tangent space
- Exponential Map: $x_{t+1} = \mathrm{Exp}_{x_t}(-\eta \nabla f(x_t))$
  move along geodesic
- Log. Map: $\mathrm{Exp}_{x_t}^{-1}(x_{t+1}) = -\eta \nabla f(x_t)$
  vector from x_t to x_{t+1}



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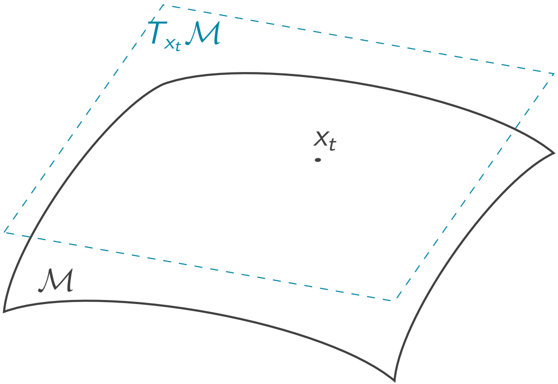

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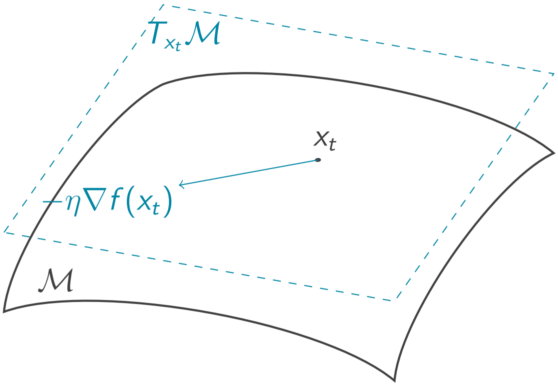

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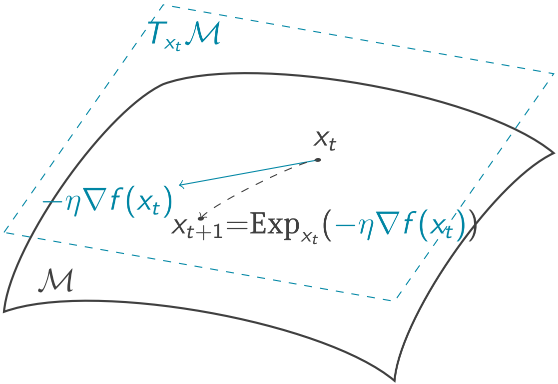

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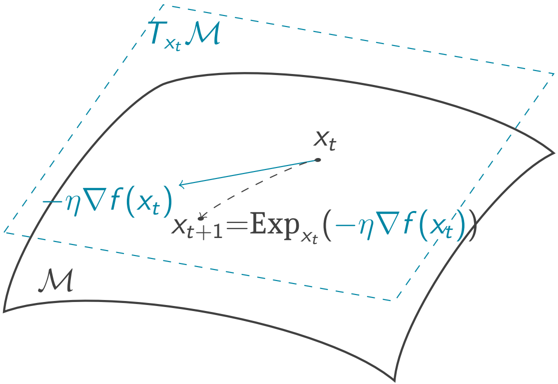


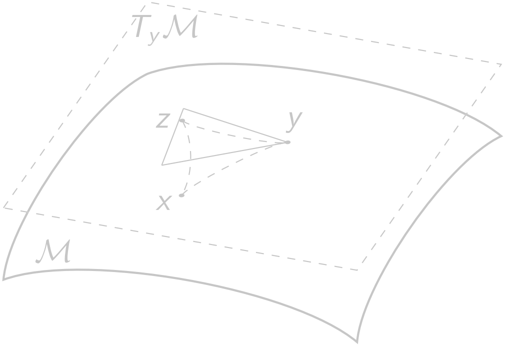
Riemannian Cosine Inequality

Euclidean cosine equality:

$$2\langle x - y, z - y\rangle = \|x - y\|^2 + \|z - y\|^2 - \|x - z\|^2$$

Riemannian Cosine Inequality: $D = \mathrm{Diam}(\triangle xyz)$, $\zeta_D = \Theta(1 + D\sqrt{|\kappa_{\min}|})$

$$2\langle \mathrm{Exp}_y^{-1}(x), \mathrm{Exp}_y^{-1}(z)\rangle_y \leq \zeta_D d(y,x)^2 + d(y,z)^2 - d(x,z)^2$$



Interpretation:

- (Informal) ζ_D measures the deformation caused by the non-linearity of the manifold
- For Hadamard manifolds: $d(x,z) \leq d_y(x,z)$
- The squared Riemannian distance function $\frac{1}{2}d(\cdot, y)^2$ is ζ_D-smooth

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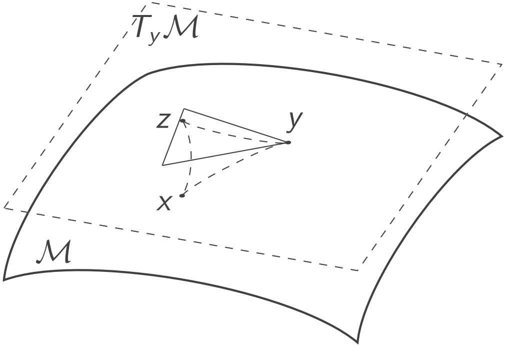

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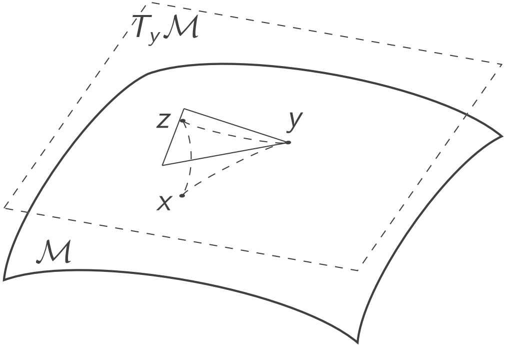

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Riemannian Optimistic Online Optimization

Prior Work

Euclidean Regret Bound:

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[Wang et al., 2023]:

- General manifolds $\mathcal{M}$, no constraints
- $R_T(u) = \mathcal{O}(\frac{D^2}{\eta} + \eta \zeta_D^2 (V_T + G^2))$
- Recurrent relationship between η and D without assuming boundedness.
- *Assume* the iterates to lie in bounded domain with diameter D

[Hu et al., 2023]:

- Hadamard manifolds $\mathcal{H}$ + constraints $\mathcal{X}$
- Two sequences, only *one* is projected to $\mathcal{X}$
- $\widetilde{\mathcal{X}} \stackrel{\text{def}}{=} \{x \in \mathcal{M} \mid d(x, \mathcal{X}) \leq \|\eta \nabla \tilde{\ell}_t(x_t)\|\}$
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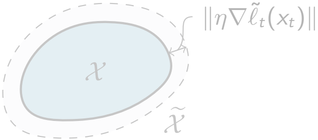

Original set $\mathcal{X}$ vs enlarged set $\widetilde{\mathcal{X}}$

- *Improper* regret: Action in $\widetilde{\mathcal{X}}$, but $u \in \mathcal{X}$
  $\widetilde{R}_T(u) = \mathcal{O}(\frac{\widetilde{D}^2}{\eta} + \eta \zeta_{\widetilde{D}} V_T)$
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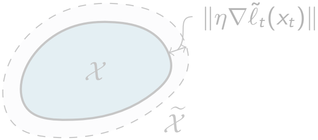

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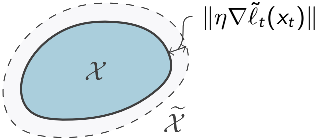

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What makes this problem hard?

Problem 1: Where to linearize?

Naive idea for optimistic update:

$$x_{t+1} = \arg\min_{z \in \mathcal{X}} \langle \nabla \ell_t(x_t), \mathrm{Exp}_{x_t}^{-1}(z) \rangle_{x_t} + \frac{1}{2\eta} d(z, x_t)^2$$

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- Linearization breaks g-convexity!
- The first update is *star g-convex* in x_t.
- But the second is *not* g-convex.
- Prior works are based on smart use of parallel transport.
- These parallel transports don't play nicely with projections.

Problem 2: Metric projections are hard

- Metric projections $P_{\mathcal{X}}(x) = \arg\min_{z \in \mathcal{X}} d(x, z)$ are hard in Riemannian geometry!
- **Positive curvature:** In general, *not* non-expansive [Wang et al., 2023a].
- **Hadamard manifolds:** First general proof of linear convergence of Projected RGD is very recent [Martínez-Rubio et al., 2023].

Riemannian Implicit Optimistic Online Gradient Descent (RIOD)

Approach: Minimize regularized *full* loss/hint function, not linearization.

Riemannian **I**mplicit **O**ptimistic Online Gradient **D**escent (**RIOD**)

$$x_0 = \tilde{x}_0 \in \mathcal{X}, \quad \forall t \in \mathbb{N}, \quad \begin{cases} \textit{Choose} & \tilde{\ell}_t \\ \textbf{Play} & \tilde{x}_t \leftarrow \operatorname{arg\,min}_{x \in \mathcal{X}} \left\{ \tilde{\ell}_t(x) + \frac{1}{2\eta} d(x, x_t)^2 \right\} \\ \textit{Observe} & \ell_t \\ \textit{Update} & x_{t+1} \leftarrow \operatorname{arg\,min}_{x \in \mathcal{X}} \left\{ \ell_t(x) + \frac{1}{2\eta} d(x, x_t)^2 \right\} \end{cases}$$

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RIOD - Main Result

Theorem:

- $\mathcal{M}$: Hadamard manifold (curvature $\kappa_{\text{min}} \leq 0$).
- $\mathcal{X}$: Compact, g-convex subset $\mathcal{X}$, $\text{diam}(\mathcal{X}) = D$.
- $\ell_t, \tilde{\ell}_t$: g-convex, differentiable in $\mathcal{X}$.

Then for any comparator $u \in \mathcal{X}$,

$$R_T(u) = \sum_{t=1}^{T} \ell_t(x_t) - \ell_t(u) \leq \frac{3D^2}{2\eta} + \eta V_T.$$

- Works for **constrained** setting $\mathcal{X}$.
- Matches **Euclidean** regret ($\mathcal{O}(D^2/\eta + \eta V_T)$) $\rightarrow$ No dependence on geometric constant ζ.
- For $\ell_t, \tilde{\ell}_t$ L-smooth in $\mathcal{X}$: Can handle **inexact** updates.
- Regret is not (oracle) complexity!

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RIOD Implementation - Smooth losses

Subproblem: $$\underbrace{\ell_t(x)}_{\substack{L\text{-smooth}\\ \text{g-convex}}} + \underbrace{\frac{1}{2\eta}d(x,x_t)^2}_{\substack{(\zeta/\eta)\text{-smooth}\\ (1/\eta)\text{-strongly g-convex}}}$$

Projected RGD:
[Martínez-Rubio et al., 2023]

$$x_{t+1} \leftarrow P_{\mathcal{X}}(\mathrm{Exp}_{x_t}(-\frac{1}{L}\nabla f(x_t)))$$

- RIOD step: $\tilde{\mathcal{O}}(\zeta(L\eta+\zeta))$
- With $\eta = \frac{1}{L}$, we get $\tilde{\mathcal{O}}(\zeta^2)$.

Composite RGD: (functions $f+g$)
[Martínez-Rubio et al., 2024]

$$x_{t+1} \leftarrow \arg\min_{y\in\mathcal{X}} \langle \nabla f(x_t), \mathrm{Exp}_{x_t}^{-1}(y)\rangle + g(y) + \frac{1}{2\gamma}d(y,x_t)^2$$

- RIOD step: $\tilde{\mathcal{O}}(L\eta)$, with $\eta = \frac{1}{L}$ we get $\tilde{\mathcal{O}}(1)$.
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Riemannian Min-Max Optimization

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$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y)$$

- $\mathcal{X} \subseteq \mathcal{M}$, $\mathcal{Y} \subseteq \mathcal{N}$ g-convex (compact) subsets.
- f is g-convex in x, g-concave in y.
- f is L-smooth in x and y.

Applications:

- Distributionally robust linear quadratic control [Taskesen et al., 2023].
- Robust version of finite-sum, g-convex optimization problem [Zhang et al., 2023, Jordan et al., 2022].

RIODA: Main Results

Theorem: RIODA converges in $\tilde{\mathcal{O}}(LR^2/\epsilon)$ iterations in the L-smooth, g-convex/g-concave case.

Total Gradient Complexity: Iterations $\times$ Subproblem cost

RIODA-PRGD: *(Constrained)*

- Improvement: $\tilde{\mathcal{O}}(\frac{LR^2}{\epsilon}\zeta^{5.5})$ to $\tilde{\mathcal{O}}(\frac{LR^2}{\epsilon}\zeta^{2})$.
- No knowledge of initial distance R, Lipschitz constant G required.
- Simpler algorithm (two loops vs five loops).

RIODA-CRGD:

- Full gradient complexity: $\tilde{\mathcal{O}}(\frac{LR^2}{\epsilon})$.
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RIOD: An *inexact, implicit, optimistic* algorithm for online constrained Riemannian optimization (Hadamard manifolds).

- Addresses key limitations of prior Riemannian online optimistic methods (in-manifold constraints, improper regret).
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Thank you! Questions?

arXiv:2403.10429

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Connection to Riemannian Lower Bounds

Known lower bound: L-smooth g-convex *minimization* $\Omega(\zeta)$ [Criscitiello and Boumal, 2023].

Our rate: $T = \tilde{\mathcal{O}}(LR^2/\epsilon)$.

Does this contradict lower bounds?

- Lower bound is in Hyperbolic space where: $f(\bar{x}) - f(x^*) \lesssim Ld(\bar{x}, x^*)^2/\zeta$.
- In other words: $\epsilon = \mathcal{O}(\frac{LR^2}{\zeta})$.
- Our rate $T = \tilde{\mathcal{O}}(LR^2/\epsilon)$ is larger than $\Omega(\zeta)$.

Takeaway Messages:

- Implicit hardness from geometry.
- Upper and lower bounds match up to log factors for Riemannian min-max problems.
- Not the case for g-convex minimization: $\tilde{\mathcal{O}}(\zeta + \sqrt{\frac{\zeta LR^2}{\epsilon}})$ vs $\Omega(\zeta)$.