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$$\omega_r = ak$$

$$\omega_i = -b$$

Phase velocity: a

$$e^{\omega_i t}\, e^{Ik(\longrightarrow)}$$

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$\omega_r = ak$ $\qquad$ Phase velocity: a

$\omega_i = -b$

$\omega_i = -b \qquad e^{\omega_i t} = e^{-bt}$

decay.

dissapation

Vocab:

ω, k

$\omega_k = ck$

Phase velocity: $\frac{\omega_r}{k}$

waves spread out

Dispersive: phase velocity depends on k

ω_i: > 0 unstable

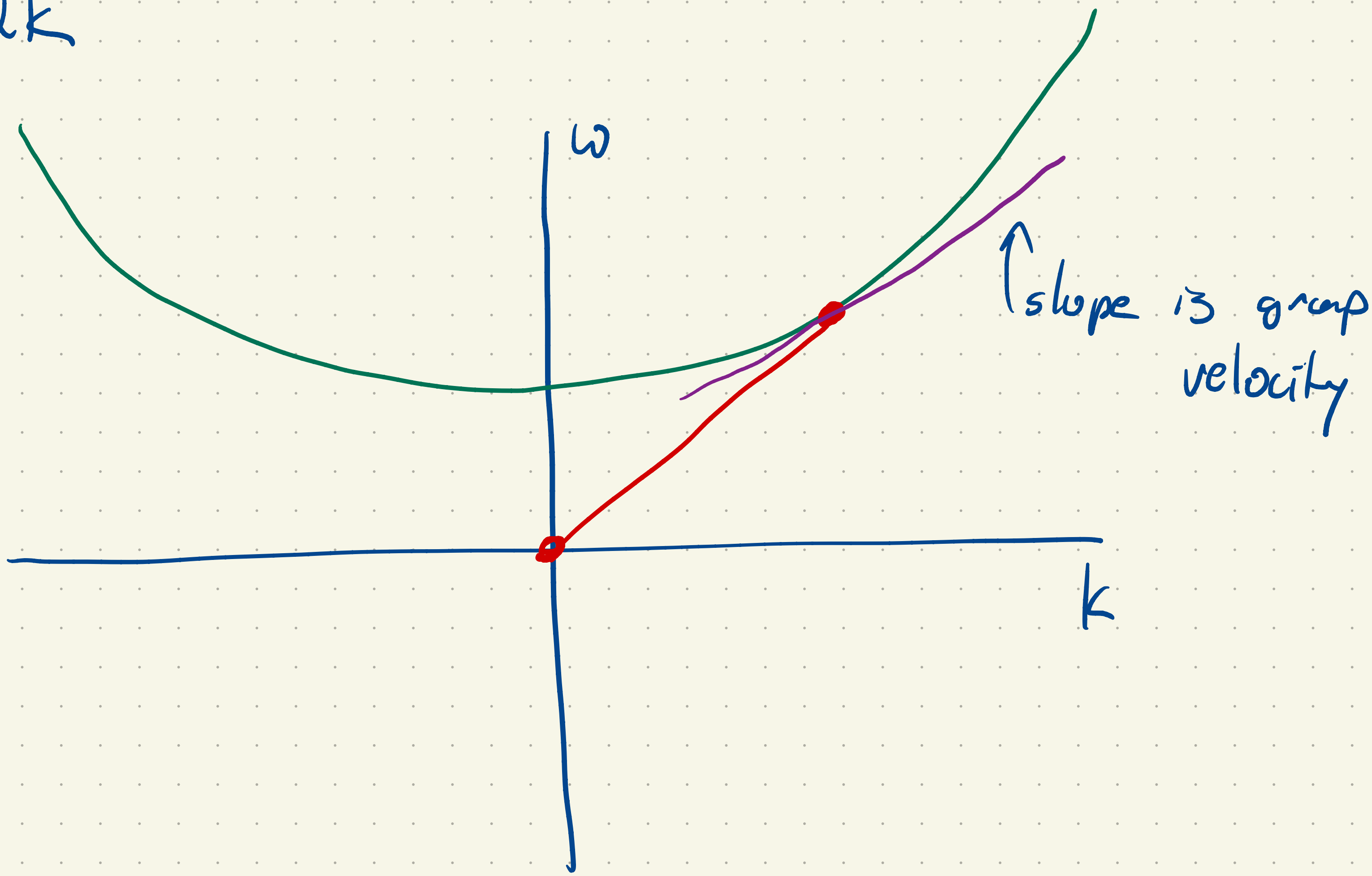
$= 0$ stable (non-dissapative)

< 0 dissapative

waves decay

Related idea: Group Velocity

$$v_g(k) = \frac{d\omega}{dk} \qquad \left(\text{rather than } v_p = \frac{\omega}{k}\right)$$



Where's the group?

Two superimposed waves k_0, $\overbrace{k_0 + \Delta k}^{k_1}$ Δk small

$$\omega_0 = \omega(k_0)$$

$$\omega_1 = \omega(k_1) \approx \omega(k_0) + v_g \Delta k$$

$$e^{I(k_0 x - \omega_0 t)} + e^{I(k_1 x - \omega_1 t)}$$

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Δk small

$$e^{I(k_0 x - \omega_0 t)} + e^{I(k_1 x - \omega_1 t)}$$

$$\underline{\underline{e^{IA} + e^{IB}}} = e^{I\left(\frac{A+B}{2}\right)}\left[e^{I\left(\frac{A-B}{2}\right)} + e^{-I\left(\frac{A-B}{2}\right)}\right]$$

$$= e^{I(A+B)/2}\, 2\cos\left(\frac{A-B}{2}\right)$$

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$$A = k_0 x - \omega_0 t$$

$$B = k_1 x - \omega_1 t$$

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$$A = k_0 x - \omega_0 t$$

$$B = k_1 x - \omega_1 t$$

$$\frac{A+B}{2} = \left(\frac{k_0 + k_1}{2}\right)x - \frac{\omega_0 + \omega_1}{2} t$$

$$= \left(k_0 + \frac{\Delta k}{2}\right)x - \left(\omega_0 + \frac{\Delta\omega}{2}\right)t$$

$$k_1 = k_0 + \Delta k$$

$$\omega_1 \approx \omega_0 + \underbrace{v_g \Delta k}_{\Delta\omega}$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

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$$k_1 = k_0 + \Delta k$$

$$\omega_1 \approx \omega_0 + \underbrace{v_g \Delta k}_{\Delta\omega}$$

$$\frac{A-B}{2} = +\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t$$

$$\approx \frac{\Delta k}{2}x - \frac{v_g \Delta k}{2}t$$

$$\omega_1 \approx \omega_0 + v_g \Delta k$$

$$\cos(A) + \cos(B)$$

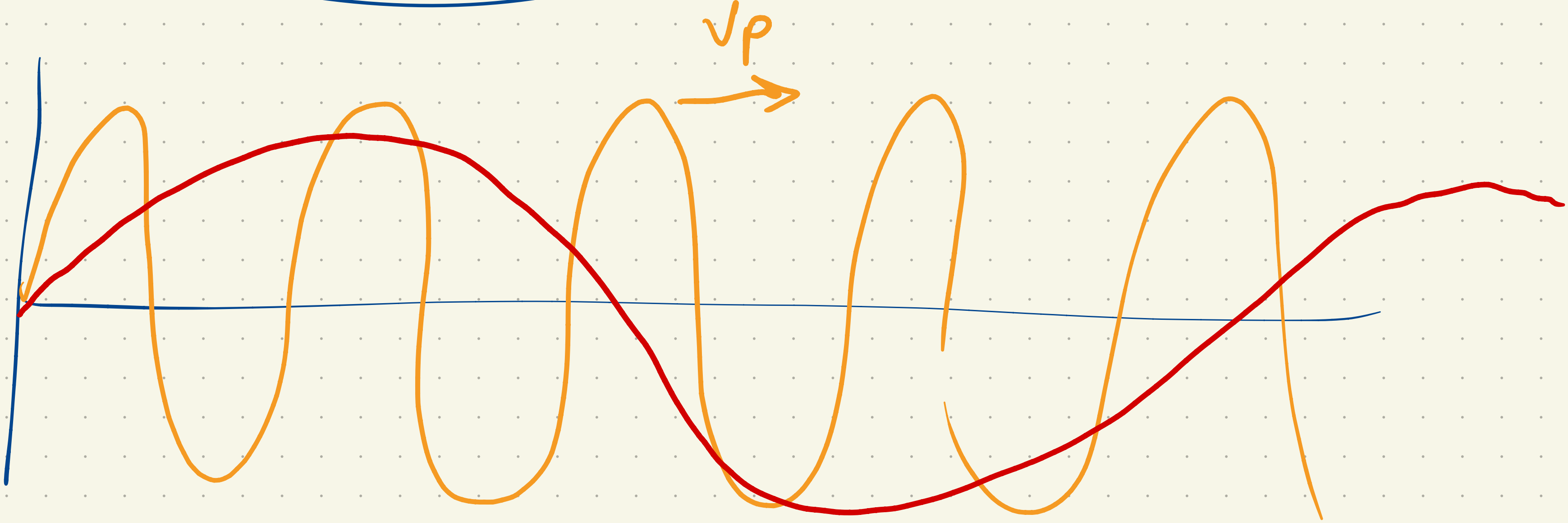
$$\approx 2\cos\left(\left(\frac{k_0+k_1}{2}\right)x - \left(\frac{\omega_0+\omega_1}{2}\right)t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

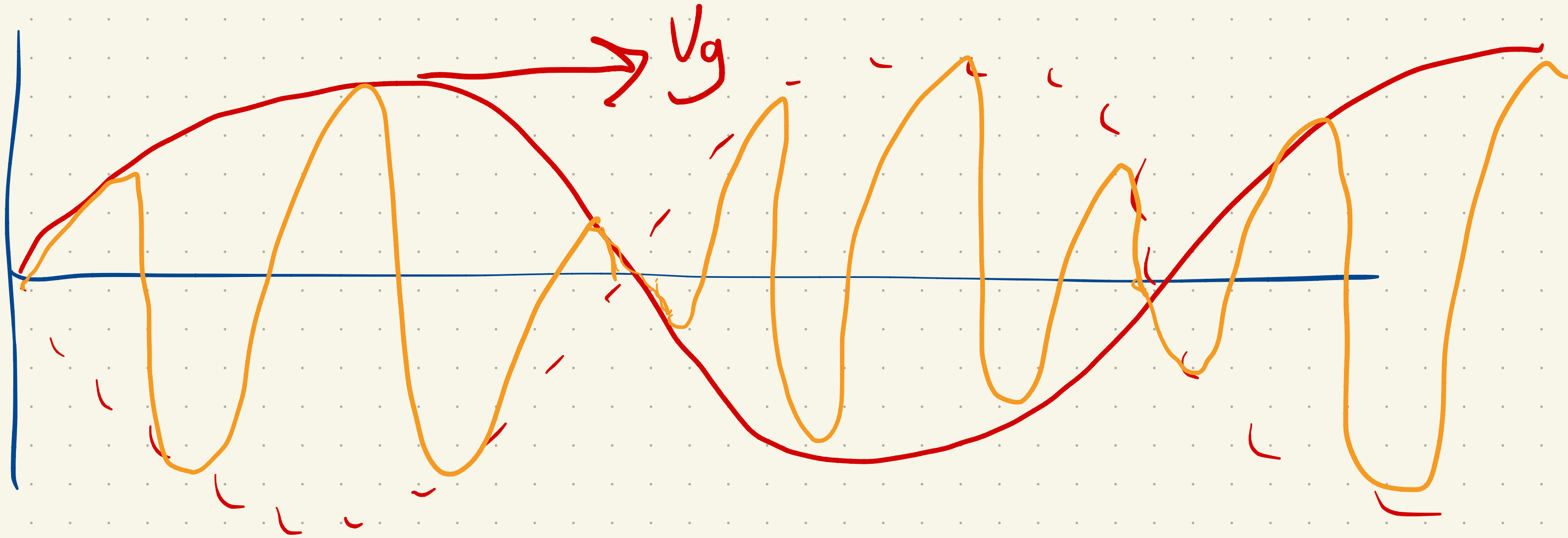
$$\omega_1 \approx \omega_0 + v_g \Delta k$$

$$\cos(A) + \cos(B)$$

$$= 2\cos\left(\left(\frac{k_0+k_1}{2}\right)x - \left(\frac{\omega_0+\omega_1}{2}\right)t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

$$= \underbrace{2\cos\left(\left(\frac{k_0+k_1}{2}\right)x - \left(\frac{\omega_0+\omega_1}{2}\right)t\right)}\cos\left(\frac{\Delta k}{2}\left(x - v_g t\right)\right)$$

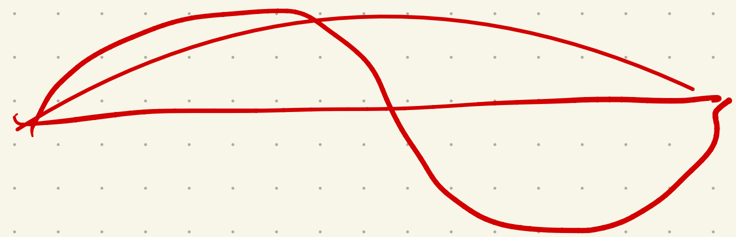




v_g

$\omega_1 \approx \omega_0 + v_g \Delta k$

$\cos(A) + \cos(B)$



$$= 2\cos\left(\left(\frac{k_0+k_1}{2}\right)x - \left(\frac{\omega_0+\omega_1}{2}\right)t\right)\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$$

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Product of two waves

1) with wave number $\frac{k_0+k_1}{2}$, ~~speed~~ freq $\frac{\omega_0+\omega_1}{k_0+k_1}$

2) with wave number $\frac{\Delta k}{2}$, speed v_g

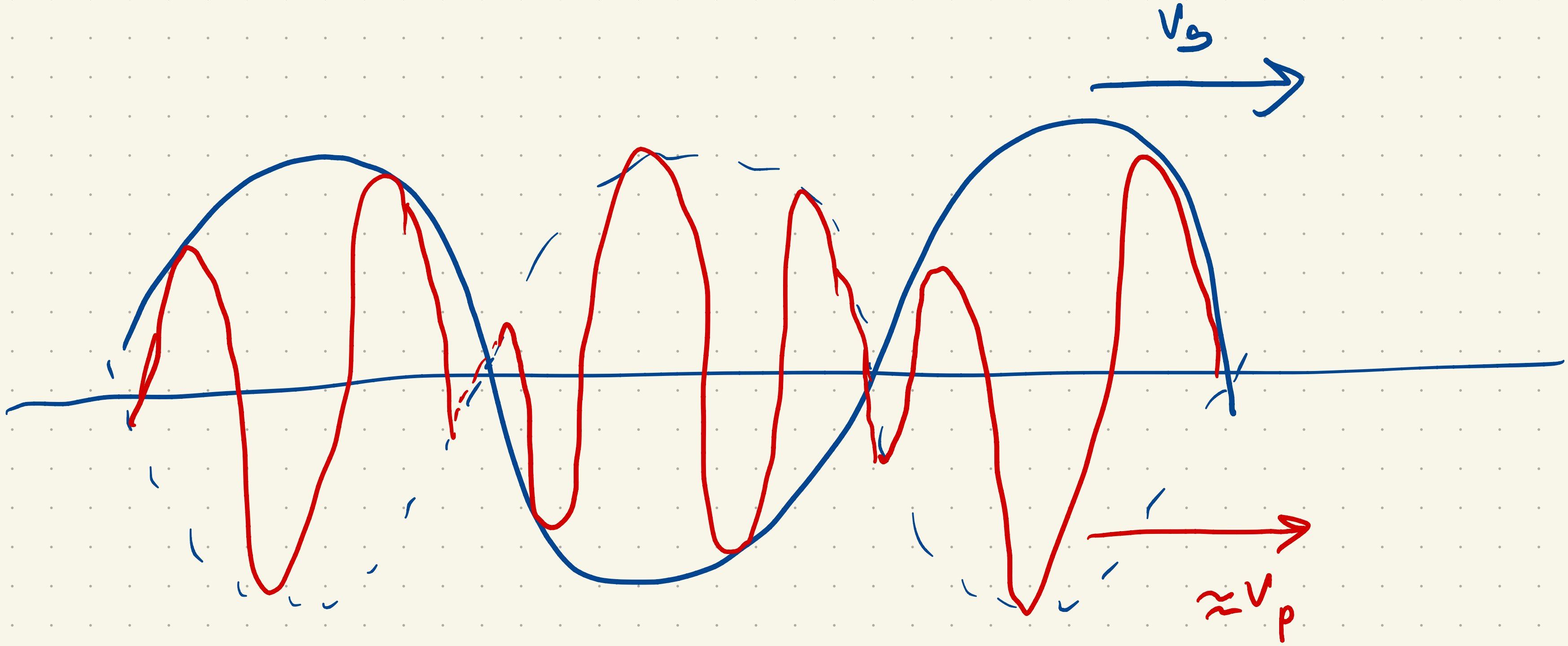
↑ long wave length

↳ group vel!

$$\frac{k_0+k_1}{2} = k_0 + \frac{\Delta k}{2}$$

$$\frac{\omega_0+\omega_1}{k_0+k_1} = \frac{\omega_0 + \Delta\omega/2}{k_0 + \Delta k/2} = \frac{\omega_0}{k_0}\left[1 + \frac{\Delta\omega}{2\omega_0} - \frac{\Delta k}{2k_0} + \cdots\right]$$

$$= \frac{\omega_0}{k_0}\left[1 + \frac{\Delta k}{2\omega_0}\left[v_g - v_p\right] + \cdots\right]$$



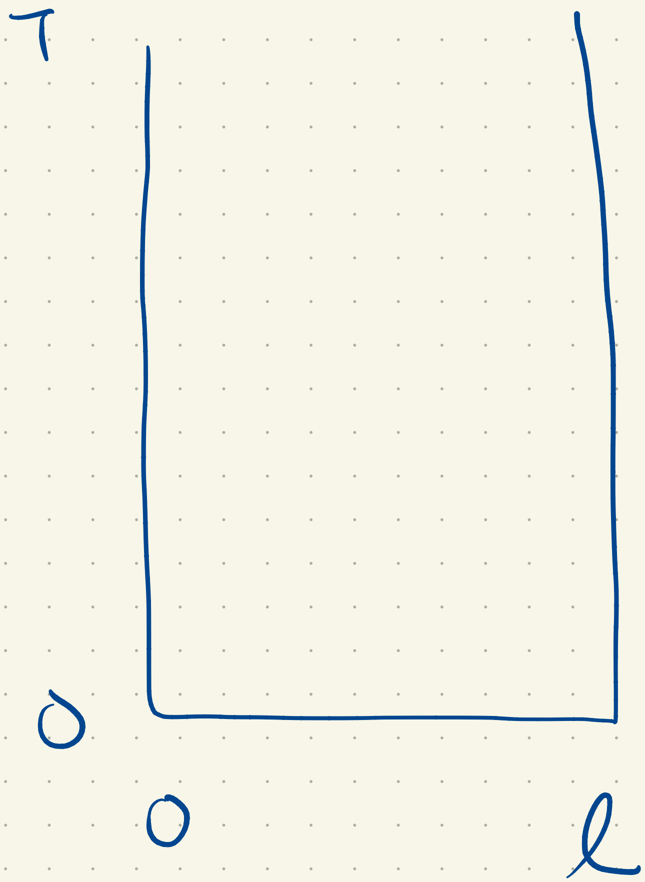
v_g
$\approx v_p$

Wave Equation Numerics

$$u_{tt} = c^2 u_{xx}$$

$$u\big|_{x=0} = 0 \;, \quad u\big|_{x=\ell} = 0$$

$$u(x,0) = u_0 \;, \quad u_t(x,0) = v$$



M+1 nodes

$k = T/M$

N+2 nodes

N unknowns

$h = \ell/N+1$

Centered Differences

$$u_{i,j} \approx u(x_i, t_j)$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = \frac{c^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right]$$

LTE: $u_{tttt}(x_i, \eta_j)\,k^2 - c^2 u_{xxxx}(\xi_i, t_j)\,h^2$

Order of accuracy: $O(k^2) + O(h^2)$

$$u_{i,j+1} = 2u_{i,j} + \frac{k^2c^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] - u_{i,j-1}$$

$$\left[u_{0,j} = 0,\ u_{N+1,j} = 0\right]$$

$$\vec{u}_{j+1} = A\vec{u}_j - \vec{u}_{j-1}$$

$$A = 2I + \lambda^2 D$$

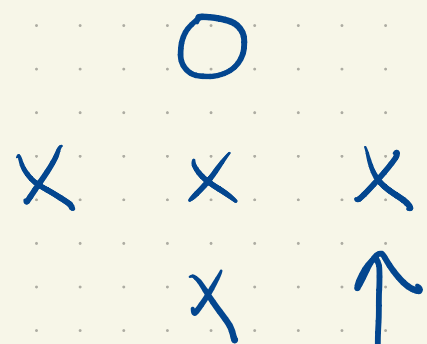
$$\begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & & \ddots & \\ & & 1 & -2 \end{bmatrix}$$

$$\lambda = \frac{kc}{h}$$

$$\vec{u}_2 = A\vec{u}_1 - \vec{u}_0$$

$$u_{i,j+1} = 2u_{i,j} + \frac{k^2c^2}{h^2}\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] - u_{i,j-1}$$

Stencil



Initial condition ?

$$\vec{u}_1 = ?$$

$\vec{U}_1$

One approach:

u_0, v given

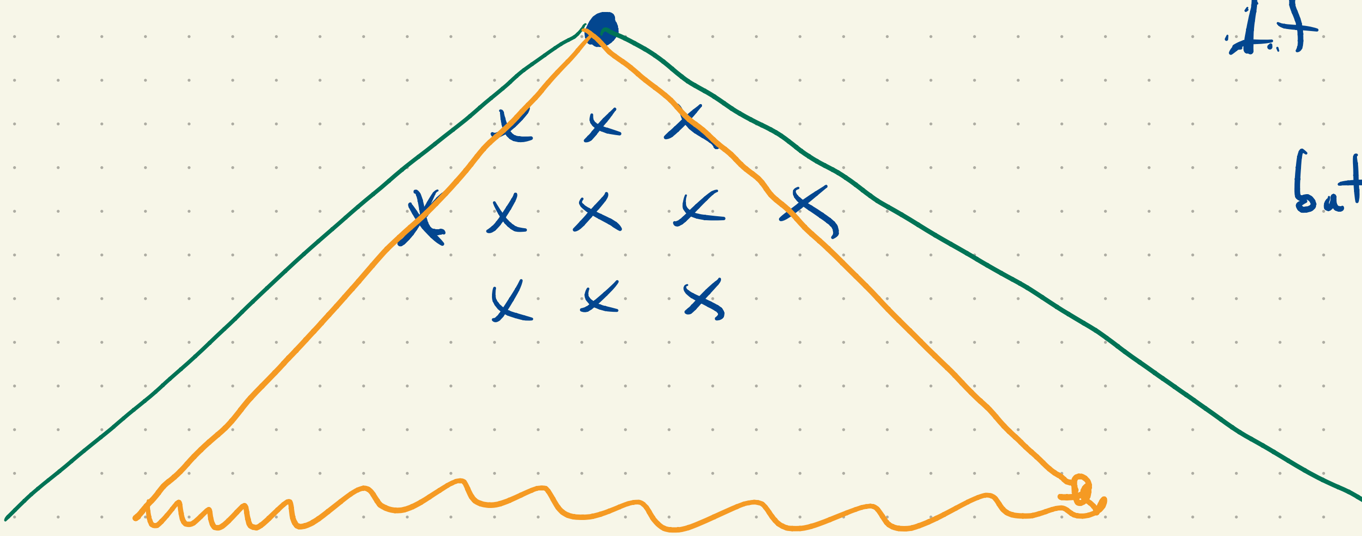
$$u(x_i, t_1) = u(x_i, k) = u(x_i, 0) + k\, u_t(x_i, 0) + O(k^2)$$

$$u_{i,1} = u_0(x_i) + v(x_i)\, k \quad ?$$

[We'll revisit]

Domain of Dependence

$u_{tt} = c^2 u_{xx}$



If $h, k \to 0$

but $\frac{h}{k} \leq \eta c$ $\quad \eta < 1$

no hope of convergence

CFL: $\frac{h}{k} \geq c$ $\qquad \lambda = \frac{ck}{h}$ so $\frac{ck}{h} \leq 1$

$\lambda \leq 1$