If $|\zeta| < 1$ $\quad k|\zeta|^k \to 0$

If $|\zeta| = 1$ $\quad \underline{k|\zeta|^k} \to \underline{\underline{\infty}}$ $\quad \infty$

Thm: (Dahlquist)

A consistent k-step LMM is convergent if and only if it is zero stable. Moreover, if the LTE is $O(h^p)$ and if initial data are chosen so $|u_i - u(t_0 + ih)|$ are $O(h^p)$

then the solution error is $O(h^p)$.

$$u' = 0$$

$\uparrow$

Exercise: A LMM is consistent iff

$$\alpha_k + \alpha_{k-1} + \cdots + \alpha_0 = 0$$

$$k\alpha_k + (k-1)\alpha_{k-1} + \cdots + \alpha_1 + 0 \cdot \alpha_0 = \beta_k + \cdots + \beta_0$$

$$\alpha_k u_{i+k} + \cdots + \alpha_0 u_{i+0} = h\left(\beta_k f_{k+i} + \cdots + \beta_0 f_{k+0}\right)$$

and as a consequence every consistent 1-step LMM is zero stable.

So: consistent + 1 step $\Rightarrow$ convergent

2^{nd} notion of stability asymtotic stability.

$u' = \lambda u \qquad (\lambda < 0)$

$\longrightarrow \quad u' = -7u$

$\longrightarrow$ exp. <u>decay</u>

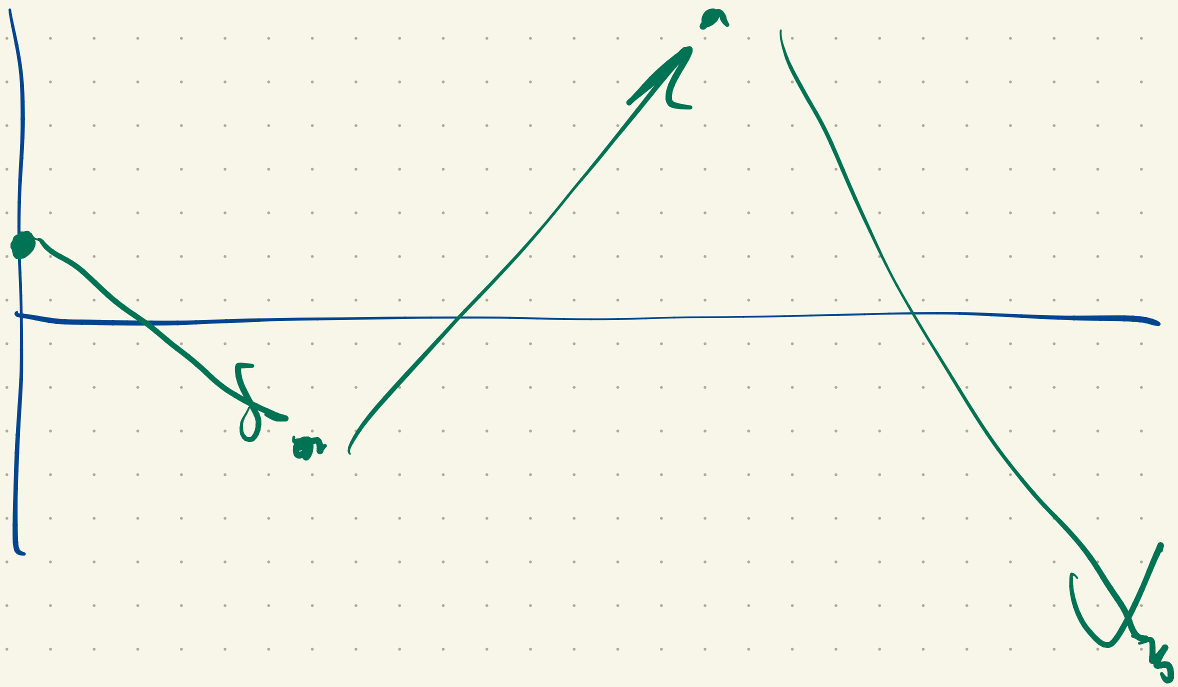
$$u_{k+1} = (1+\lambda h)\, u_k$$

$$u_k = \underbrace{(1+\lambda h)^k}\, u_0$$

$$\left|(1+\lambda h)^k\right| = \left|1+\lambda h\right|^k$$

$$\leq (1+|\lambda| h)^k$$

$$\leq e^{|\lambda| T}$$



Consider $u' = \cos(t)$

$u(0) = 1.2$

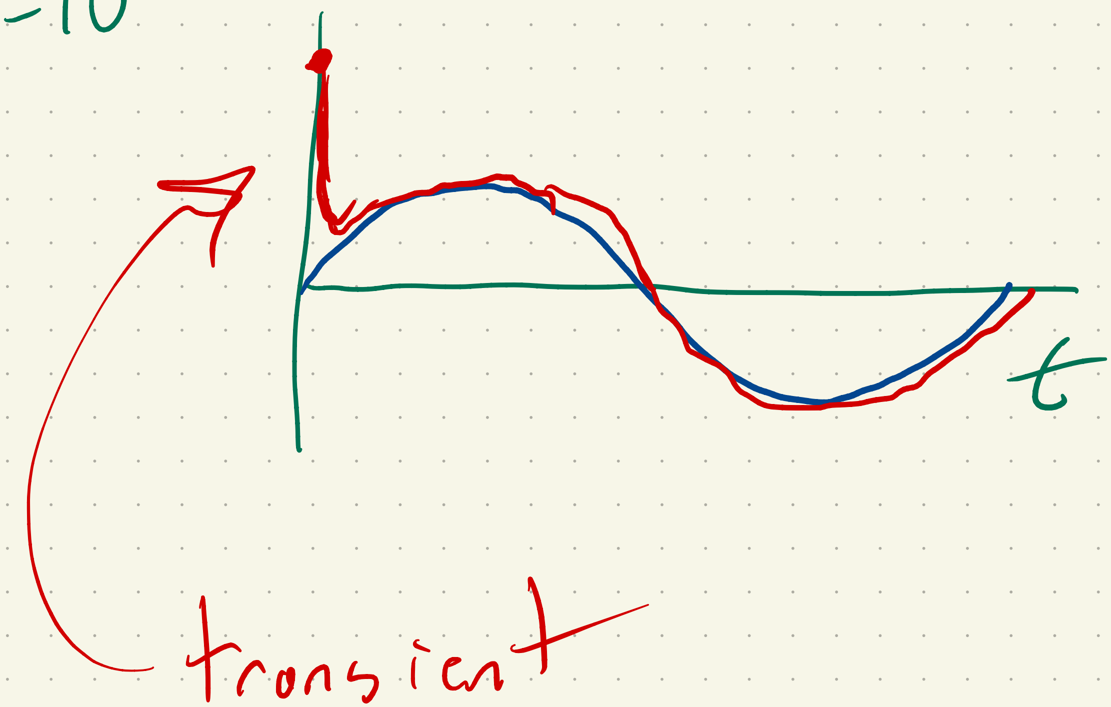
$u(t) = 1.2 + \sin(t)$

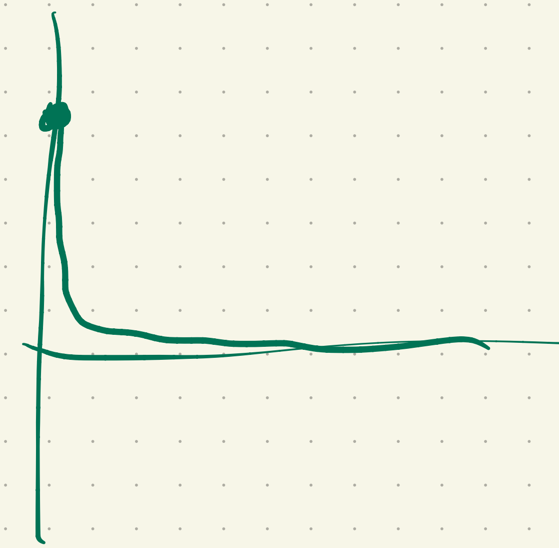
$$u' = -\lambda\left(u - \sin(t)\right) + \cos(t)$$

$$u(0) = 1.2$$

$$u(t) = 1.2\, e^{-\lambda t} + \sin(t)$$

$\lambda = 10$

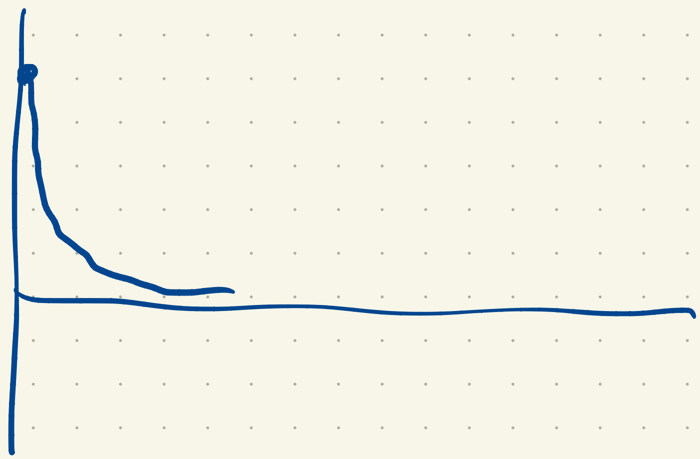




What's the deal? Two time scales

$\sin(t)$ is changing on a scale $\Delta t = O(1)$

$e^{-\lambda t}$ $\Delta t \approx 1/|\lambda|$



$$\vec{u}' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -40 \end{bmatrix} \vec{u}$$

$u' = -\lambda u$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} -u_1 \\ -20u_2 \\ -40u_3 \end{bmatrix}$$

$$\vec{u}(t) = \begin{bmatrix} A e^{-t} \\ B e^{-20t} \\ C e^{-40t} \end{bmatrix}$$

$$\vec{u}' = \underbrace{A}_{\text{matrix}} \vec{u}$$

Suppose $\vec{x}$ is an eigenvector of A with eigenvalue α.

$$A\vec{x} = \alpha\vec{x}$$

A solution of $\vec{u}' = A\vec{u}$ is

$$\vec{u} = c\vec{x}e^{\alpha t}$$

$$\vec{u}' = \alpha c\vec{x}e^{\alpha t} = \alpha\vec{u}$$

$$A\vec{u} = A(c\vec{x}e^{\alpha t}) = ce^{\alpha t}A\vec{x}$$

$$= ce^{\alpha t}\alpha\vec{x}$$

$$= \alpha(ce^{\alpha t}\vec{x})$$

$= \alpha \vec{u}$

If α_1 and α_2 are two diffent eigenvalues of A with eigenvectors $\vec{x}_1$, $\vec{x}_2$ then

$\vec{x}_1 e^{\alpha_1 t}$ $\vec{x}_2 e^{\alpha_2 t}$

solutions

$c_1 e^{\alpha_1 t} \vec{x}_1 + c_2 e^{\alpha_2 t} \vec{x}_2$ solution $\forall$ c_1, c_2

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -40 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -40 \end{bmatrix}$$

Euler's Method:

$\lambda \in \mathbb{R}$

$$u' = \lambda u$$

$$u_{k+1} = (1 + \lambda h) u_k$$

$$u_k = (1+\lambda h)^k u_0$$

When do we see a decaying solution?

$$|1+\lambda h| < 1$$

$$-1 < 1+\lambda h < 1$$

$$-2 < \lambda h < 0$$

If $\lambda = -20$ we need

$$-2 < -20h < 0$$ to get decay.

$$\frac{1}{10} > h$$

If $\lambda = -200$ we need

$$\frac{1}{100} > h$$ to get decay.

For reasons related to systems (which can have complex eigenvalues) we will consider the case $\lambda \in \mathbb{C}$.

$$u' = \lambda u \qquad \lambda \in \mathbb{C}$$

$$u = A e^{\lambda t}$$ still solves

$$\lambda = a + bi \qquad a, b \in \mathbb{R}$$

$$e^{\lambda t} = e^{(a+bi)t} = e^{at} e^{bit}$$

$$= e^{at}\underbrace{\left(\cos(bt) + i\sin(bt)\right)}$$

If $a < 0$ the solution decays

$\mathrm{Re}(\lambda)$

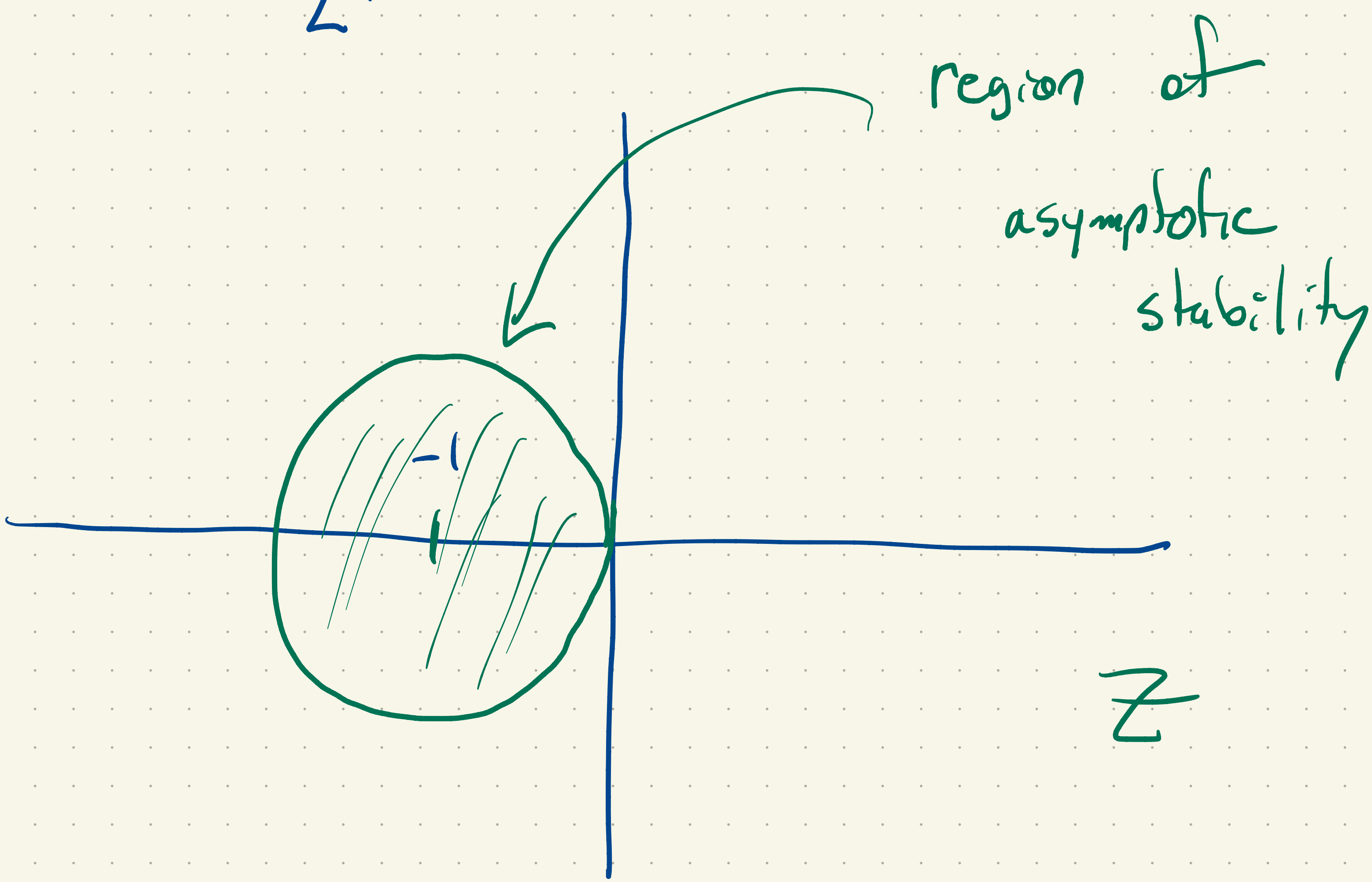
$\in \mathbb{C}$

$u' = \lambda u$

$$u_{k+1} = (1 + \lambda h)u_k$$

when do I see decay?

$$\left| 1 + \underbrace{\lambda h}_{z} \right| < 1$$



A~~bsolute~~ stability:
symptotic

Apply the LMM to $u' = \lambda u$

The method is absolutely stable for a step size h if for any initial data the solution $u_k \longrightarrow 0$.

We say z is in the region of

absolute sability if $z = \lambda h$

and the method is a.s. for stepsize h.