Math F310 Final Exam Fall 2020

Name: ______________________

Student Id: ______________________

Rules:

You have 120 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

One half-page of notes is permitted, as is a straight edge. No other aids, including calculators or books, are permitted.

Turn off anything that might go beep during the exam.

If you need extra space for scratch work, there is a blank page at the end of the exam.

Good luck!

Problem	Possible	Score
1	15	
2	15	
3	10	
4	15	
5	10	
6	10	
7	10	
8	10	
9	5	
10	5	
11	10	
EC1	5	
EC2	2	
Total	115	

1. (15 points)

a. [10 points] Suppose we wish to perform polynomial interpolation of $f(x) = 1/x$ at the points $x = 1$, $2, 3$ and 4. Set up the linear system to compute the interpolating polynomial in Vandermonde form. Your answer should indicate the form of the interpolating polynomial, up to unknown coefficients, and should fully set up the linear system to solve for the unknown coefficients. DO NOT SOLVE THE SYSTEM! DO NOT SIMPLIFY!

Continued....

b. [5 points] Write down the Lagrange form of the interpolating polynomial. DO NOT SIMPLIFY!

c. [Extra Credit: 5 points] Repeat part a) but now set up the system for computing the polynomial in Newton form.

2. (15 points)

a. [10 points] Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Perform LU-decomposition with partial pivoting to find a permutation matrix P and lower- and upper-triangular matrices L and U such that

$$PA = LU.$$

Continued....

b. [5 points] Use the decomposition from part **a.** to solve $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b}^T = [2, 1]$.

3. (10 points)

Write a MATLAB function that performs Newton's method to approximate the solution of $f(x) = 0$. Your function should take the function `f`, its derivative `df`, and the initial approximation `x0` as inputs. It should stop sucessfully if x is found with $|f(x)| < 10^{-6}$. It should report an error if more than 500 iterations take place.

4. (15 points)

a. [4 points] Compute the 1-norm, 2-norm and ∞-norm of $\mathbf{v} = [1, 2, 3]^T$.

b. [4 points] Compute the ∞-norm of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 2 & 1 \end{pmatrix}$$

c. [5 points] Compute the condition number of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$$

with respect to the 1-norm.

Continued....

c. [2 points] Consider the 11x11 Vandermonde matrix

$$V = \begin{pmatrix} x_0^{10} & x_0^9 & \cdots & x_0 & 1 \\ x_1^{10} & x_1^9 & \cdots & x_1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{10}^{10} & x_{10}^9 & \cdots & x_{10} & 1 \end{pmatrix}$$

where $x_0 = 0, x_1 = 0.1, x_2 = 0.2, \cdots, x_{10} = 1$.

MATLAB computes that V has a condition number of about 10^8 with respect to the ∞-norm.

Suppose we solve

$$V\mathbf{c} = \mathbf{b}$$

where $b_k = \sin(k\pi/10)$ for $0 \le k \le 10$. MATLAB gives the following answer

```
octave:7> V\b
ans =

  -0.024766311705429
   0.123831558561227
  -0.043482754776624
  -0.569058332380669
  -0.014338506126532
   2.554812227827973
  -0.001004485292244
  -5.167575914115260
  -0.000010470804109
   3.141592988811666
   0.000000000000000
```

To how many digits do you trust this answer? To answer this question, draw a line through the MATLAB output above separating the digits you trust from the digits you do not.

d. Extra Credit (2 points) The computation in part **c.** is solving an polynomial interpolation problem. What is the function being interpolated, and what are the sample points?

5. (10 points)

Consider the inner product on polynomials on $[-1,1]$ given by

$$\langle p, q\rangle = \int_{-1}^{1} (1+x^3)p(x)q(x)\ dx.$$

Apply the Gram-Schmidt algorithm to polynomials $p_0(x) = 1$ and $p_1(x) = x$ to find orthonormal polynomials $q_0(x)$ and $q_1(x)$ where q_0 is a multiple of p_0 and q_1 is a linear combination of p_0 and p_1.

6. (10 points)

Find coefficients A, B and C such that the approximation

$$f'(x) \approx Af(x) + Bf(x+h) + Cf(x+2h)$$

has the highest possible order acurracy, and determine what order of accuracy this is.

7. (10 points)

Consider the function

$$f(x) = \begin{cases} 4 - 3x + x^3 & 0 \le x < 1 \\ 6 - 9x + 6x^2 - x^3 & 1 \le x \le 4. \end{cases}$$

Verify that this is a cubic spline through the points $(0,4)$, $(1,2)$, and $(4,2)$. Do not worry about determining what boundary conditions may apply.

8. (10 points)

Recall the error formula for polynomial interpolation. If $f \in C^{n+1}[a,b]$, $a \le x_0 < x_1 < \cdots < x_n \le b$, and if p is a degree n polynomial that interpolates f at $x_0, \ldots, x_n$, then

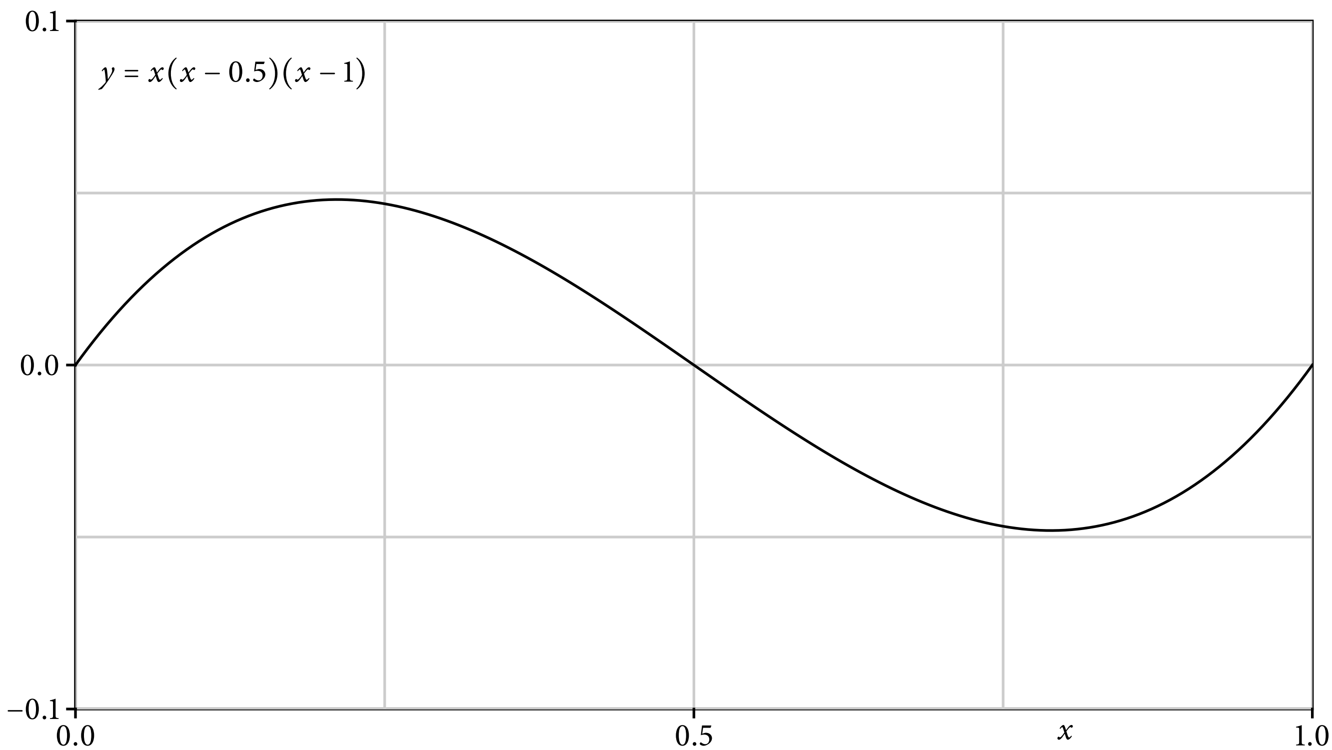
$$f(x) = p(x) + E(x)$$

for $x \in [a,b]$ where

$$E(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{k=0}^{n} (x - x_k)$$

and where ξ_x depends on x and lies in $[a,b]$.

Let $p(x)$ be the quadratic interpolant of $f(x) = e^{x/2}$ on $[0,1]$ at the sample points 0, 0.5 and 1. Use the error formula to estimate the maximum value of $|f(x) - e^x|$ for $0 \le x \le 1$. You may find the graph below of $x(x-0.5)(x-1)$ to be helpful.



9. (5 points)

In physics and engineering applications one often sees the approximation

$$\sin(x) \approx x$$

for small values of x. Approximate the size of the error in this approximation assuming $|x| \le 0.1$.

10. (5 points)

Consider the snippit of code:

```
x=zeros(0,n);
for k=1:n
  for j=1:n
    x(k) = x(k) + A(k,j)*b(j);
  end
end

d=0
for k=1:n
  d = x(k)*x(k)
end
```

How many floating point operations are performed by this code? For full credit, you should give a brief justification, in part so that if your answer is in error, partial credit can be assigned. Your answer should be a formula that depends on n.

11. (10 points)

Consider an approximate integration rule on $[-1,1]$ of the form

$$\int_{-1}^{1} f(x)\, dx \approx A_0 f(-1/2) + A_1 f(0) + A_2 f(1/2).$$

where the A_k's are weights. Suppose we would like this rule to be exact for polynomials of degree 2 or less. Set up a system to solve for the weights A_k. DO NOT SOLVE THE SYSTEM.

[BLANK PAGE FOR SCRATCH WORK]