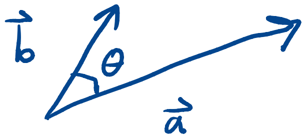
Supplement: Dot Product and Angles:

Goal:



$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta}$$

0) Notation: $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

1) Observe $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2.$

2) Compute

$$|\vec{a} - \vec{b}| = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$

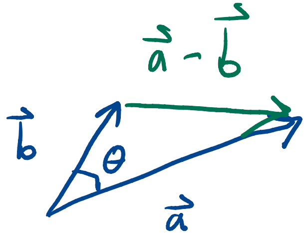
$$= \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

$$= |\vec{a}|^2 - 2\,\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$$

3) Solve for $\vec{a}\cdot\vec{b}$:

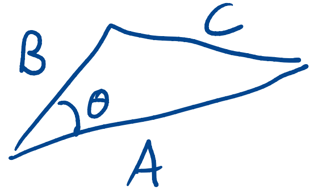
$$\vec{a}\cdot\vec{b} = \frac{1}{2}\left[\,|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}-\vec{b}|^2\,\right]$$

4) Ponder this diagram:



$\vec{b} + (\vec{a}-\vec{b}) = \vec{a}$ ✓

5) Law of Cosines



$$2AB\cos\theta = A^2 + B^2 - C^2$$

6) Combine steps 3), 4), 5) with $A = |\vec{a}|$, $B = |\vec{b}|$, $C = |\vec{a} - \vec{b}|$:

$$\vec{a}\cdot\vec{b} = \frac{1}{2}\left[\,|\vec{a}|^2 + |\vec{b}|^2 - |\vec{a}-\vec{b}|^2\,\right]$$

$$= \frac{1}{2}\left[\,A^2 + B^2 - C^2\,\right]$$

$$= \frac{1}{2}\left[\,2AB\cos\theta\,\right] \qquad \text{(by step 5)}$$

$$= |\vec{a}|\,|\vec{b}|\cos\theta.$$

That is,

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$ 🎉 .