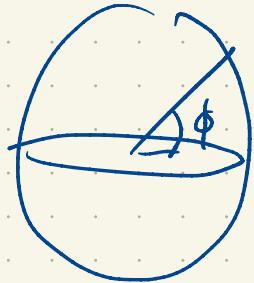


Alt:



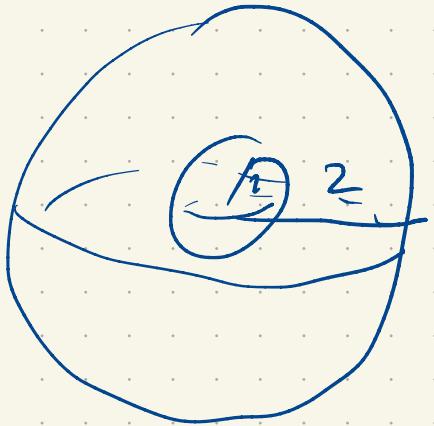
$$z = \rho \sin \phi$$

$$x = \rho \cos \phi \cos \theta$$

$$y = \rho \cos \phi \sin \theta$$

$$dV = \rho^2 \cos \phi \, d\rho \, d\theta \, d\phi$$

e.g.



ϵ

$$\iiint_{\epsilon} z^2 dV$$

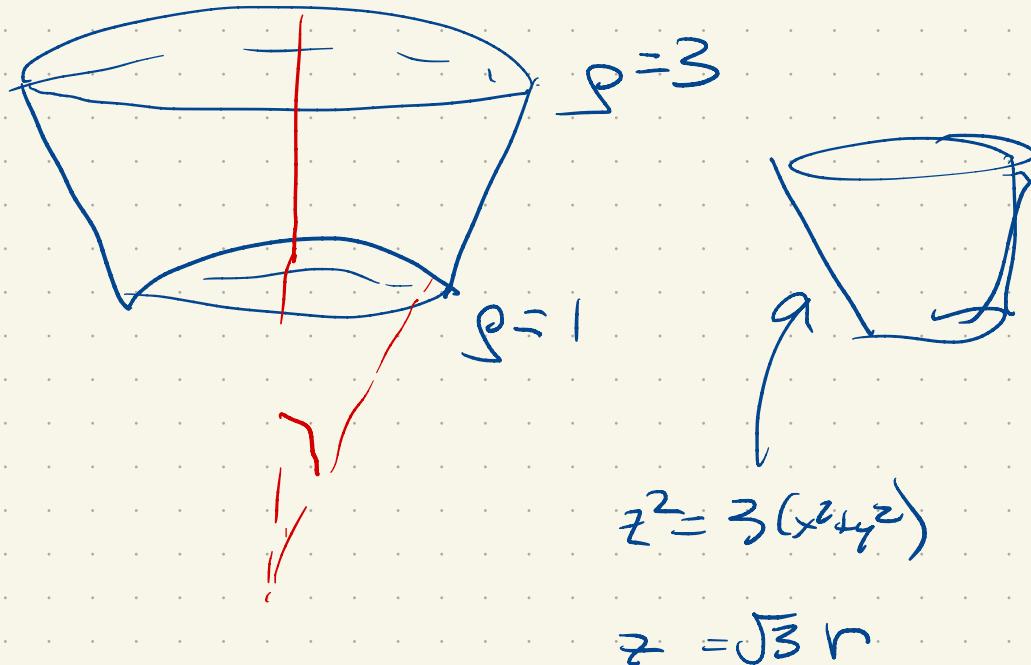
$$\int_0^{2\pi} \int_0^{\pi} \int_1^2 z^2 \cos^2 \phi \, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{2\pi} \int_0^{\pi} \frac{\rho^5}{5} \Big|_1^2 \cos^2 \phi \sin \phi \, d\phi \, d\theta$$

$$2\pi \left(\frac{2^5 - 1}{5} \right) \int_0^{\pi} \cos^2 \phi \sin \phi \, d\phi \quad u = \cos \phi$$

$$\boxed{\frac{124}{15}\pi}$$

e.g



$$z^2 = 3(x^2 + y^2)$$

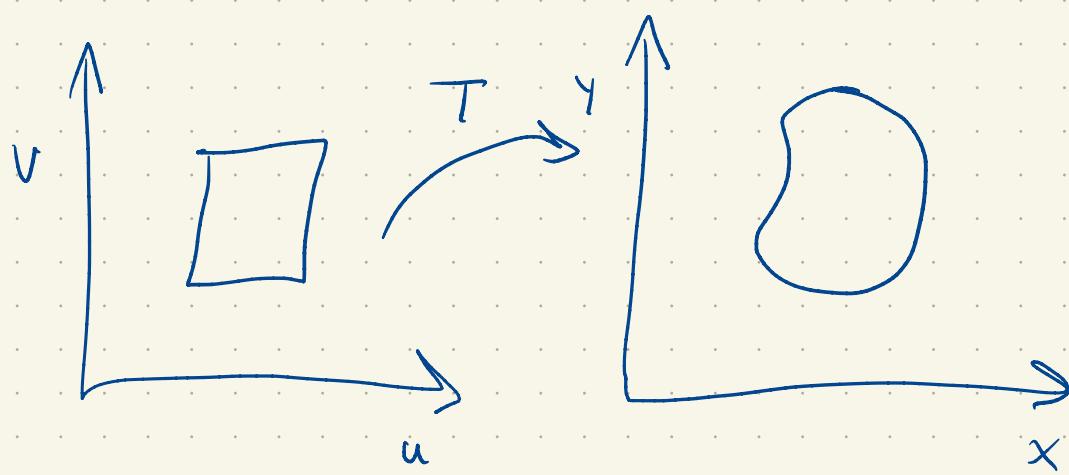
$$z = \sqrt{3} r$$

Compute the volume.

$$\iiint_E 1 dV = \int_0^{2\pi} \int_1^3 \int_0^? 1 \sin\phi \, d\phi \, dr \, d\theta$$

$$\sqrt{3} \left[\frac{\pi}{6} \right]_{\rho=2}^{z=\sqrt{3}} = \frac{2\pi}{3} (2 - \sqrt{3})$$

Change of variables



$$T(r, \theta) = (r \cos \theta, r \sin \theta)$$

We have an interval in $x-y$ coords we want to express
in $u-v$ coords

A diagram illustrating a small rectangular element in the $u-v$ plane, labeled R_{uv} , with dimensions Δu and Δv . A red arrow indicates its orientation. This element is mapped by a transformation $T(u, v)$ to a curved element in the $x-y$ plane, labeled R_{xy} , with area $\Delta x \Delta y$. A red arrow indicates the orientation of this curved element.

$$\text{given: } \left\langle \frac{\partial x}{\partial u} \Delta u, \frac{\partial y}{\partial u} \Delta u \right\rangle \Delta u$$

$$= \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle \Delta u$$

$$\text{given: } \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \right\rangle \Delta v$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} \Delta u & \frac{\partial y}{\partial u} \Delta u \\ \frac{\partial x}{\partial v} \Delta v & \frac{\partial y}{\partial v} \Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \Delta u & \Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$$

$$\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \rightarrow \text{Jacobian matrix}$$

This is the real analog of the derivative.

Area of R_{xy} is related to the determinant of the matrix above, up to sign.

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| \quad \text{Jacobian determinant}$$

Moral: Area of $R_{uv} \approx \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u \Delta v$

$$dx dy = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

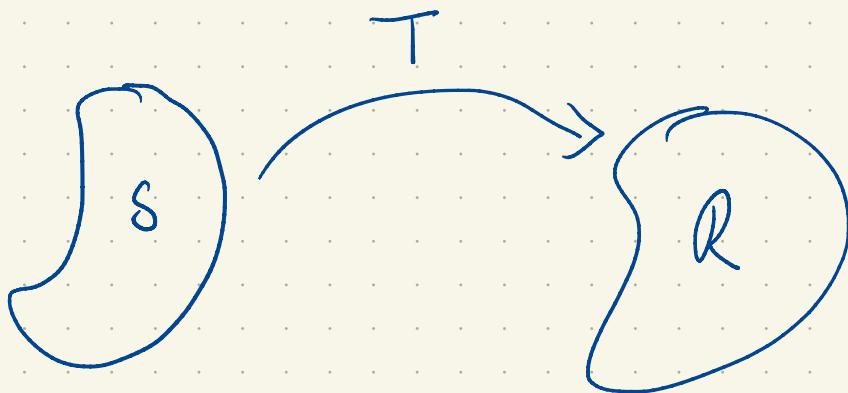
e.g. $x = r \cos \theta$

$$y = r \sin \theta$$

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

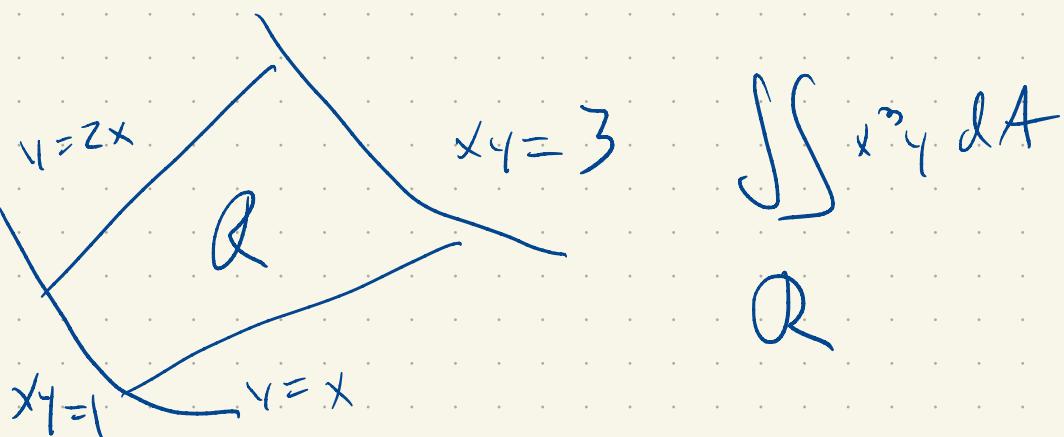
$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{bmatrix} \cos^2 \theta + r \sin^2 \theta \end{bmatrix} \right| = r \quad \checkmark$$

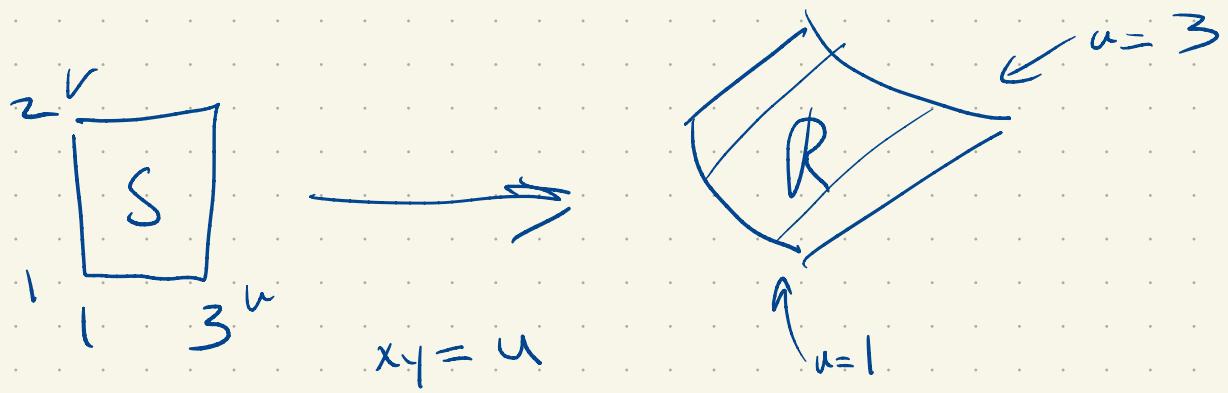
$$dxdy = r dr d\theta$$



$$\iint_R f(x,y) dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v)$$

$$\int_{u(a)}^{u(b)} f(u) du = \int_a^b f(u(w)) \underbrace{u'(w)}_{\frac{du}{dw}} dw$$





$$x = u$$

$$y = v$$

$$x = \sqrt{uv}$$

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{uv}}, \quad \frac{\partial x}{\partial v} = -\frac{1}{2} \sqrt{u} v^{-\frac{1}{2}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2} \sqrt{\frac{v}{u}}, \quad \frac{\partial y}{\partial v} = \frac{1}{2} \sqrt{\frac{u}{v}}$$

$$J = \frac{1}{4} \left[\frac{1}{r} + \frac{1}{r} \right] = \frac{1}{2} \sqrt{J}$$

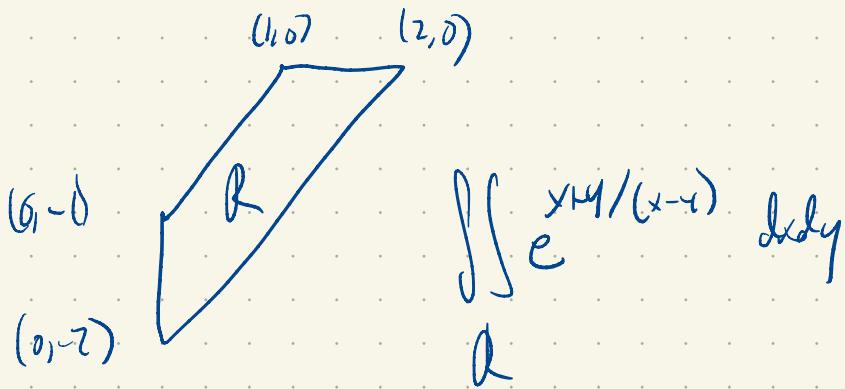
$$x^3 y = \frac{u^{3/2}}{v^{3/2}} \sqrt{u} \sqrt{v} = \frac{u^2}{v}$$

$$\int_1^2 \int_1^3 \frac{u^2}{v} \frac{1}{2} \frac{1}{r} du dv = \int_1^2 \left[\frac{u^3}{6v^2} \right]_1^3 dr$$

$$= \frac{1}{6} [26] \int_1^2 \frac{1}{v^2} dv = \frac{26}{6} \left(\frac{1}{v} \right)_1^2$$

$$= \frac{26}{6} \left[\frac{1}{2} + 1 \right]$$

$$= \frac{13}{6}$$

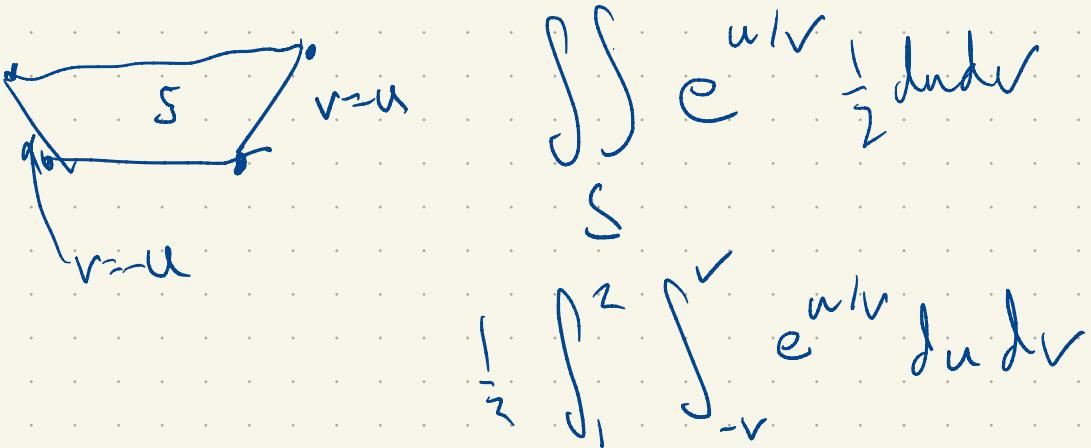


$$\begin{aligned} u &= x+y \\ v &= x-y \end{aligned} \quad \rightarrow \quad \begin{aligned} \frac{u+v}{2} &= x \\ \frac{u-v}{2} &= y \end{aligned}$$

$$\begin{aligned} \frac{\partial x}{\partial u} &= \frac{1}{2} & \frac{\partial x}{\partial v} &= \frac{1}{2} \\ \frac{\partial y}{\partial u} &= \frac{1}{2} & \frac{\partial y}{\partial v} &= -\frac{1}{2} \end{aligned}$$

$$J = \begin{vmatrix} -\frac{1}{4} & -\frac{1}{4} \end{vmatrix} = \frac{1}{2}$$

$$\begin{aligned} (1,0) &\rightarrow (1,1) & (6,-1) &\rightarrow (-1,1) \\ (2,0) &\rightarrow (3,2) \end{aligned}$$



$$\frac{1}{2} \int_1^2 v e^{uv} \left. \begin{array}{l} u \\ u = v \end{array} \right| v \, dv$$

$$\frac{1}{2} \int_1^2 v [e^{-v} - e^{-1}] \, dv$$

$$\frac{1}{2} (e^{-1} - e) \left. \frac{v^2}{2} \right|_1^2$$

$$\sinh(1) \left[2 - \frac{1}{2} \right] = \frac{3}{4} \sinh(1)$$