Thm (Egoroff) If D is measurable and $m(D) < \infty$ → necessary

and if $\{f_k\}$ is a sequence of measurable real valued functions on D

converges pointwise a.e. to f then for every $\varepsilon > 0$

there exists a measurable set $E \subseteq D$ such that

$m(D \setminus E) < \varepsilon$ and $f_k \to f$ uniformly on E.

Littlewood's 3 Principles

1) A measurable set is almost an open set
closed
G_δ, F_σ

2) Pointwise a.e. convergence is almost uniform convergence.

3) Measurable functions are almost continuous functions.

WLOG $f_n \to f$ pointwise everywhere.

Pf: For each $n, k \in \mathbb{N}$ let

$$F_{n,k} = \{ x \in D : |f_m(x) - f(x)| \geq \frac{1}{k} \text{ for some } m \geq n \}.$$

Each $F_{n,k}$ is measurable.

For fixed k I claim $\bigcap_n F_{n,k} = \phi$.

Let $x \in D$. Since $f_\ell(x) \to f(x)$ there exists n

so that if $m \geq n$, $|f_m(x) - f(x)| < \frac{1}{k}$. Hence $x \notin F_{n,k}$.

Moreover, each $F_{n+1,k} \subseteq F_{n,k}$.

Since $\boxed{m(D) < \infty}$, continuity from above implies

$$m(F_{n,k}) \xrightarrow[n]{} m(\phi) = 0.$$

Let $\varepsilon > 0$.

Pick n_1 so that $m(F_{n_1,1}) < \varepsilon/2$.

Pick $n_2 > n_1$ so that $m(F_{n_2,2}) < \varepsilon/2^2$

Repeating this we can find increasing indices n_k

such that $m(F_{n_k,k}) < \varepsilon/2^k$.

Let $F = \bigcup_k F_{n_k, k}$. By countable subadditivity,

$m(F) < \varepsilon$. Let $E = D \setminus F$.

I claim that $f_n \to f$ uniformly on E.

Indeed, let $\eta > 0$. Pick k so that $\frac{1}{k} < \eta$.

If $x \in E$ then $x \notin F_{n_k, k}$.

Hence if $n \geq n_k$ $|f_n(x) - f(x)| < \frac{1}{k} < \eta$.

$\square$

Simple functions are measurable functions that take on finitely many values.

If the values are $a_0, a_1, \ldots, a_n$

we can write the function as

$$\sum_{j=0}^{n} a_j \chi_{E_j}$$ where E_j is the set where the function equals a_j.

Each E_j is measurable.

"standard representation"

A sum or product of simple functions is simple.

We're going to need to approximate measurable functions with simple functions.

Goal "Basic Construction"

Given a measurable function f find a sequence $\{\phi_n\}$ of simple functions with $|\phi_1| \leq |\phi_2| \leq \cdots$ and where $|\phi_n| \leq f$

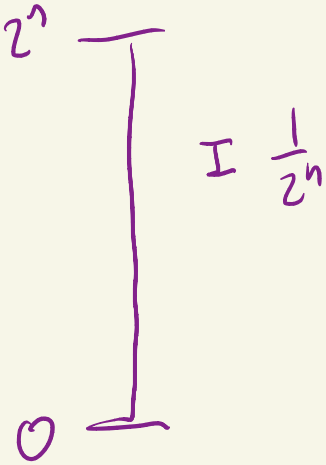
$\phi_n \to f$ pointwise

and uniformly on any set where f is bounded.

Easy case: $f \geq 0$.

$0 \leq \phi_1 \leq \phi_2 \leq \cdots$ with $\phi_n \to f$ pointwise

(and uniformly on sets where f is bounded)



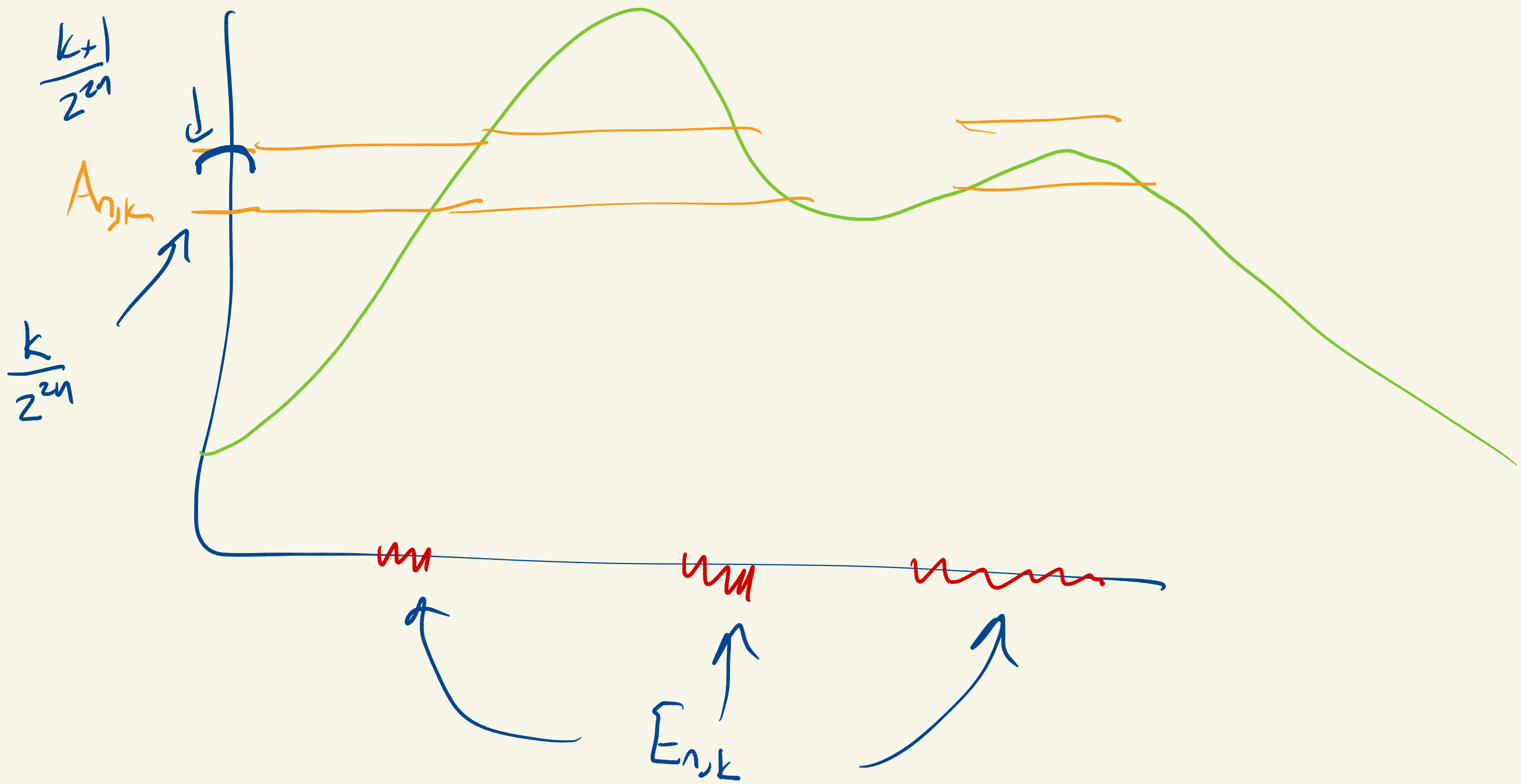
2^{2n} pieces of size 2^{-n}

$$A_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \quad 0 \le k \le 2^{2n} - 1$$

$$B_n = [2^n, \infty)$$

$$E_{n,k} = f^{-1}(A_{n,k})$$

$$F_n = f^{-1}(B_n) \qquad \text{"}f \text{ is big"}$$



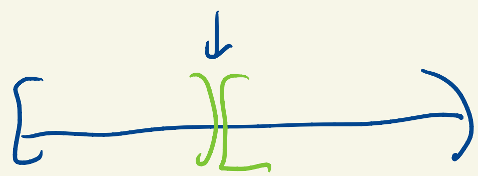
$$\varphi_n = \sum_{k=0}^{2^{2n}-1} \frac{k}{2^n} \chi_{E_{n,k}} + 2^n \chi_{F_n}$$

Claim $\varphi_{n+1} \geq \varphi_n$.

On $E_{n,k}$ $\quad \varphi_n \le f \le \varphi_n + \frac{1}{2^n}$

$A_{n,k} \quad \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$

$$\frac{k}{2^n} = \frac{2k}{2^{n+1}}$$



$\frac{2k+1}{2^{n+1}}$

$$\frac{k}{2^n} + \frac{1}{2^{n+1}} = \frac{2k}{2^{n+1}} + \frac{1}{2^{n+1}}$$

$$= \frac{2k+1}{2^{n+1}}$$

$$A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$$

On $E_{n,k}$, $\varphi_n = \frac{k}{2^n}$

and φ_{n+1} is either $\frac{2k}{2^{n+1}}$ or $\frac{2k+1}{2^{n+1}} \geq \frac{k}{2^n}$

On $E_{n,k}$, $\varphi_{n+1} \geq \varphi_n$.

On F_n, $\varphi_n = 2^n$ and $\varphi_{n+1} \geq 2^n$.

Claim: $\varphi_n \to f$ uniformly on any set where f is bounded. Suppose f is bounded on some set F.

Pick N so that $2^N \geq f$ on F.

Then if $n \geq N$ and $x \in F$ then $x \in E_{n,k}$ for some k and $\varphi_n(x) \leq f(x) \leq \varphi_n(x) + \frac{1}{2^n}$, so

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n}.$$

In particular, $\varphi_n \to f$ at any point where $f(x) < \infty$.

On the other hand, if $f(x) = \infty$ then $\varphi_n(x) = 2^n$ for all n. Since $2^n \to \infty$ the proof is complete.

General case: f

$$f_+ = \max(f, 0) \qquad f_- = \max(-f, 0) \qquad f_+ \geq 0$$

$$f = f_+ - f_- \qquad f_- \geq 0$$

$$\varphi_n^+ \nearrow f_+ \qquad \underline{\varphi_n^-} \nearrow f_-$$

$$\varphi_n = \varphi_n^+ - \varphi_n^-$$

$\varphi_n \to f$ pointwise

$\left.\begin{array}{l}\varphi_n^+ \to f_+ \\ \varphi_n^- \to f_-\end{array}\right]$ p.w.

$$|\varphi_n| = \varphi_n^+ + \varphi_n^- \leq f_+ + f_- = |f|$$

$$= \varphi_n^+ + \varphi_n^- \leq \varphi_{n+1}^+ + \varphi_{n+1}^- = |\varphi_{n+1}|$$

(and uniformly on any set where f is bounded)

Measurable functions are nearly continuous functions

$[a,b]$

Lemma: Given a measurable set E and $\varepsilon > 0$

$E \subseteq [a,b]$

there is a step function φ s.t.

$$m(\{\varphi \neq \chi_E\}) < \varepsilon$$