$$\|e\|_\infty = \max_{0 \le i \le M} |e_i| \qquad \ell^\infty$$

If $\|e\|_\infty = C h^p$

$$\log(\|e\|_\infty) = \log C + p \log(h)$$

For Euler's Method it looks like $p \approx 1$.

One likes higher powers of h.

h	h^2
0.1	0.01
0.01	0.0001
0.001	0.000001

Def: A finite diff method is <u>convergent</u> (in ∞ norm) if $\|e\|_\infty \to 0$ as $h \to 0$.

It is a pth order method if

$$\|e\|_\infty = O(h^p)$$

$\le Ch^p$ for some C

It looks like E.M. is first order.

In general, for a f.d. method one wants to know

a) is it convergent

b) At what rate does error vanish

Nearby question: how well does the true solution of the ODE satisfy the discrete approximation?

$$u'(t) = f(t, u)$$

$$\frac{u_{i+1} - u_i}{h} = f(t_i, u_i)$$

$$\frac{u_{i+1} - u_i}{h} - f(t_i, u_i) = 0$$

$$\downarrow$$

$$e_i = u_i - u(t_i)$$

solution error

$$\frac{u(t_{i+1}) - u(t_i)}{h} - f(t_i, u(t_i))$$

$$= \frac{u(t_{i+1}) - u(t_i)}{h} - u'(t_i)$$

$$= -\tau_i$$ local truncation error

$$\tau_i = -\frac{u''(x_i)h}{2}$$

$$= \frac{u''(x_i)}{2} h$$

Assuming u'' is bounded for true solution,

$$\frac{u''(x_i)h}{2} \to 0 \text{ as } h \to 0.$$

This is called a <u>consistent</u> finite difference method

Plug true solution into discrete approx of ODE written in units of u' and verify result → 0 as h → 0.

[consistency, convergence, stability]

Let's investigate convergence of E. M. for

$$\left.\begin{aligned} u' &= \lambda u \\ u(0) &= u_0 \end{aligned}\right] \longrightarrow u(t) = u_0 e^{\lambda t}$$

$$u_{i+1} = u_i + h\lambda u_i$$

$$= (1 + \lambda h) u_i$$

$$u(t_{i+1}) = u(t_i) + u'(t_i) \cdot h + \frac{u''(\eta_i)}{2} h^2$$

$$= u(t_i) + \lambda h u(t_i) + h \tau_i$$

$$= (1 + \lambda h) u(t_i)$$

$$e_i = u_i - u(t_i)$$

$$e_{i+1} = (1 + \lambda h) e_i - h \tau_i$$

error at t_{i+1} has two sources

a) propagation of previous error

b) generation of new error from LTE

Rule of thumb: suppose propagation of previous error is controlled

then we generate a final error of size

about $\quad M \, h \, \max \tau_i$

\# of time steps $\qquad Mh = T$

$$\|e\|_\infty \sim T \cdot \underbrace{\max \tau_i}_{\to \ \frac{u''(x_i)}{2} \cdot h}$$

$$\|e\|_\infty \sim C h$$

we observed this earlier

$$e_{i+1} = (1+\lambda h) e_i + \tau_i h$$

Initial error : e_0

$$e_1 = (1+\lambda h) e_0 + \tau_0 h$$

$$e_2 = (1+\lambda h) e_1 + \tau_1 h$$

$$= (1+\lambda h)^2 e_0 + (1+\lambda h)\tau_0 h + \tau_1 h$$

$$e_3 = (1+\lambda h)^3 e_0 + (1+\lambda h)^2 \tau_0 h + (1+\lambda h)\tau_1 h + \tau_2 h$$

$$e_k = (1+\lambda h)^k e_0 + \sum_{j=1}^{k} \tau_{k-1} h (1+\lambda h)^{k-j-1}$$

Suppose we know $\left|(1+\lambda h)^k\right| \le K$

for $0 \le k \le M$

independent of h

then

$$|e_k| \le K|e_0| + \sum_{j=1}^{k} K \cdot h |\tau_{k-1}|$$

$$\leq K|e_0| + \underbrace{M \cdot h}_{} \cdot K \max |\tau_i|$$

$$= K\left(|e_0| + T \max |\tau_i|\right)$$

Is it true $(1+\lambda h)^k \leq K$ for some K

$\lambda > 0$

$$(1+\lambda h)^k \leq (1+\lambda h)^M$$

$$= (1+\lambda h)^{T/h}$$

$$= (1+\lambda h)^{\frac{\lambda T}{\lambda h}}$$

$$= \left[(1+\lambda h)^{1/\lambda h}\right]^{\lambda T}$$

recall $\lim_{s \to 0} (1+s)^{1/s} = e$

$1+s \le e^{s}$ for all s

exercise

$$(1+s)^{1/s} \leq e$$

$$(1+\lambda h)^{k} \leq \underbrace{e^{\lambda T}}_{K}$$