Last Class:

Θ Methods:

$$\vec{u}_{j+1} = \vec{u}_j + \theta\lambda D\vec{u}_j + (1-\theta)\lambda D\vec{u}_{j+1} + \vec{f}_j$$

$\theta = 0$ B.E

$\theta = 1$ Explicit

Applied VN analysis which suggested stability if

$$(2\theta - 1)\lambda \leq \frac{1}{2} \qquad \text{B}$$

Discrete Max principle $\qquad \lambda\theta \leq \frac{1}{2} \qquad \text{A}$

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Applied VN analysis which suggested stability if

$$(2\theta - 1)\lambda \leq \frac{1}{2}$$

Discrete Max principle $\qquad \lambda\theta \leq \frac{1}{2}$

$\lambda \leq \frac{1}{2}$

$k \leq \frac{1}{2} h^2$

Last Class:

Θ Methods:

$O(k^2) + O(h^2)$

$$\vec{u}_{j+1} = \vec{u}_j + \theta \lambda D \vec{u}_j + (1-\theta)\lambda D \vec{u}_{j+1} + \vec{f}_j$$

$\theta = 0$ B.E

$\theta = 1$ Explicit

always satisfied if $0 \leq \theta \leq \frac{1}{2}$

$\downarrow$

Applied VN analysis which suggested stability if

$(2\theta - 1)\lambda \leq \frac{1}{2}$ (B)

Discrete Max principle $\lambda\theta \leq \frac{1}{2}$ (A)

In fact: A $\Rightarrow$ B

$\lambda\theta \leq \frac{1}{2}$ A

$(2\theta - 1)\lambda =$

In fact: A $\Rightarrow$ B

$$(2\theta - 1)\lambda = \theta\lambda + \overbrace{(\theta - 1)\lambda}^{\leq 0} \leq \theta\lambda \leq \frac{1}{2}$$

$0 \leq \theta \leq 1$

$-1 \leq \theta - 1 \leq 0$

Convergence under strong condition A $\theta\lambda \le \frac{1}{2}$

$U_{i,j}$

$$U_{i,j+1}\left(1+2(1-\theta)\lambda\right) = (1-\theta)\lambda\left(U_{i-1,j+1} + U_{i+1,j+1}\right)$$

$$+\lambda\theta\left(U_{i-1,j} + U_{i+1,j}\right)$$

$$+(1-2\theta\lambda)U_{i,j} + kf_{i,j}$$

Convergence under strong condition A

$U_{i,j}$

$$U_{i,j+1}\left(1+2(1-\theta)\lambda\right) = (1-\theta)\lambda\left(U_{i-1,j+1} + U_{i+1,j+1}\right)$$
$$+ \lambda\theta\left(U_{i-1,j} + U_{i+1,j}\right)$$
$$+ (1-2\theta\lambda)U_{i,j} + kf_{i,j}$$

$$u_{i,j+1}\left(1+2(1-\theta)\lambda\right) = (1-\theta)\lambda\left(u_{i-1,j+1} + u_{i+1,j+1}\right)$$
$$+ \lambda\theta\left(u_{i-1,j} + u_{i+1,j}\right)$$
$$+ (1-2\theta\lambda)u_{i,j} + kf_{i,j} + k\tau_{i,j}$$

$$\theta\lambda \le \frac{1}{2} \Rightarrow 1-2\theta\lambda \ge 0$$

$$E_{i,j+1} \cdot \left(1+2(1-\theta)\lambda\right) = (1-\theta)\lambda\left(E_{i+1,j+1} + E_{i-1,j+1}\right) + \theta\lambda\left(E_{i+1,j} + E_{i-1,j}\right) + (1-2\theta\lambda)E_{i,j} - k\tau_{i,j}$$

$$\ge 0$$

$$E_{i,j+1} \cdot \left(1+2(1-\theta)\lambda\right) = (1-\theta)\lambda\left(E_{i+1,j+1} + E_{i-1,j+1}\right) + \theta\lambda\left(E_{i+1,j} + E_{i-1,j}\right) + (1-2\theta\lambda) E_{i,j} - k\,\tau_{i,j}$$

$$E_j = \max_i |E_{i,j}| \quad \left(\|\vec{E}_j\|_\infty \right)$$

$$E_{j+1}\left(1+2(1-\theta)\lambda\right) \le 2(1-\theta)\lambda\, E_{j+1} + 2\theta\lambda\, E_j + \underbrace{(1-2\theta\lambda)}_{\text{used } \theta\lambda \le 1/2} E_j + k\,\tau_j$$

$$\tau_j = \max_i |\tau_{i,j}|$$

$$E_0 \le \max_j E_j$$

$$E_{j+1} \le E_j + k\tau_j$$

Exercise: $\max_j E_j \le E_0 + Mk \max_j \tau_j$

$$= E_0 + T\tau$$

$$\boxed{E_0 \le E_0 + T\tau} \quad \tau = \max_{i,j} |\tau_{i,j}|$$

$$\max_j \max_i |E_{i,j}| = \max_{i,j} |U_{i,j} - u_{i,j}| \qquad E_0 = 0$$

$$\max_{i,j} |U_{i,j} - u_{i,j}| \le T \max_{i,j} |\tau_{i,j}|$$

$$E_{j+1} \leq E_j + k\tau_j$$

Exercise: $\max\limits_j E_j \leq E_0 + Mk \max\limits_j \tau_j$

$$= E_0 + T\tau$$

$$\tau = \max_{i,j} |\tau_{i,j}|$$

I.e. $\max\limits_{i,j} \left| U_{i,j} - u_{i,j} \right| \leq T \max\limits_{i,j} \left| \tau_{i,j} \right|$

$$\theta\lambda \le \frac{1}{2}$$

$$E_{j+1} \le E_j + k\tau_j \qquad k \le \frac{1}{2\theta} h^2$$

Exercise: $\max_j E_j \le E_0 + Mk \max_j \tau_j$

$$= E_0 + T\tau$$

$$\tau = \max_{i,j} |\tau_{i,j}|$$

I.e. $\max_{i,j} |U_{i,j} - u_{i,j}| \le T \max_{i,j} |\tau_{i,j}|$

So $\tau_{i,j} = O(k) + O(h^2) \Rightarrow \max_{i,j} |U_{i,j} - u_{i,j}| \to 0$

Next: convergence using only $(2\theta - 1)\lambda \le \frac{1}{2}$ (B)

under a different notion of error.

$\vec{U}_j$ $\quad U_{i,j}$

$$\vec{E}_j = \vec{U}_j - \vec{u}_j \leftarrow \text{N-dimensional vector}$$

$\vec{u}_j$ $\quad u_{i,j}$

we've been working with

$$\| \vec{E}_j \|_\infty = \max_i | E_{i,j} |$$

Other vector norms:

$$1 \le p < \infty \qquad x \in \mathbb{R}^n$$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Other vector norms:

$$1 \leq p < \infty$$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p \right]^{1/p}$$

$p = 1$ $\qquad \displaystyle\sum_{i=1}^{\infty} |x_i|$

Other vector norms:

$$1 \leq p < \infty$$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

$p = 1$ $\quad \sum_{i=1}^{\infty} |x_i|$

$$\|x\|_\infty = \max |x_i|$$

$p = 2$ $\quad \left[\sum_{i=1}^{\infty} |x_i|^2\right]^{1/2}$

Other vector norms:

$1 \le p < \infty$ → $0 \le \frac{1}{p} \le 1$

$1, 2, \text{Inf}$

$$\|x\|_p = \left[\sum_{i=1}^{n} |x_i|^p\right]^{1/p}$$

norm(x, ↑)

$p=1$ $\quad \sum_{i=1}^{\infty} |x_i|$

$$\|x\|_\infty = \max |x_i|$$

$p=2$ $\quad \left[\sum_{i=1}^{\infty} |x_i|^2\right]^{1/2}$

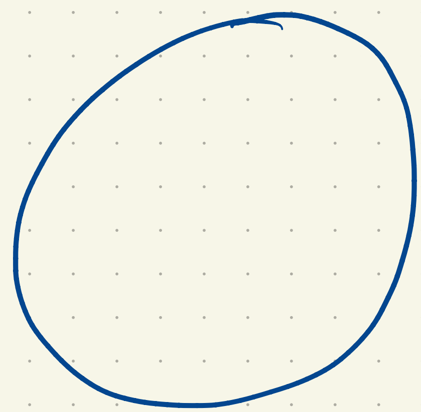
Exercise: $\|x\|_p \to \|x\|_\infty$

as $p \to \infty$

Unit Balls : $\{x : \|x\|_p \leq 1\} = B_p(r)$

$\|x\|_2 \leq 1$

Unit Balls: $\{x : \|x\|_p \leq 1\} = B_p(r)$

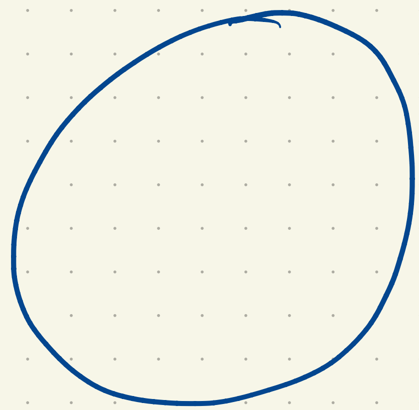


$\|x\|_2 = 1$

Unit Balls: $\{x : \|x\|_p \leq 1\} = B_p(r)$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\|x\|_\infty$

$\max(|x_1|, |x_2|)$

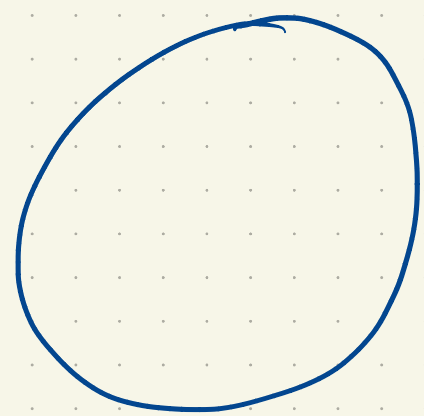


$\|x\|_2 = 1$

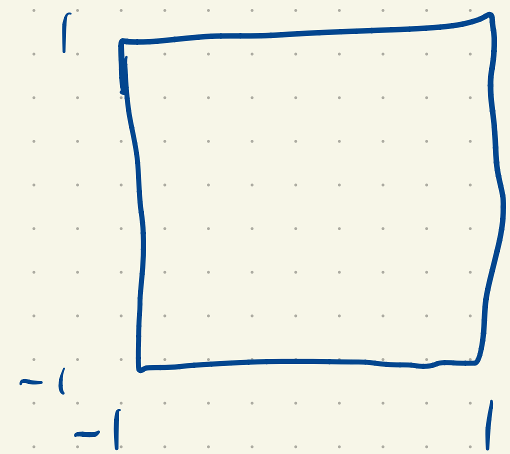
$\|x\|_\infty = 1$

$\to |x_1| = 1, |x_2| \leq 1$

Unit Balls : $\{x : \|x\|_p \leqslant 1\} = B_p(r)$



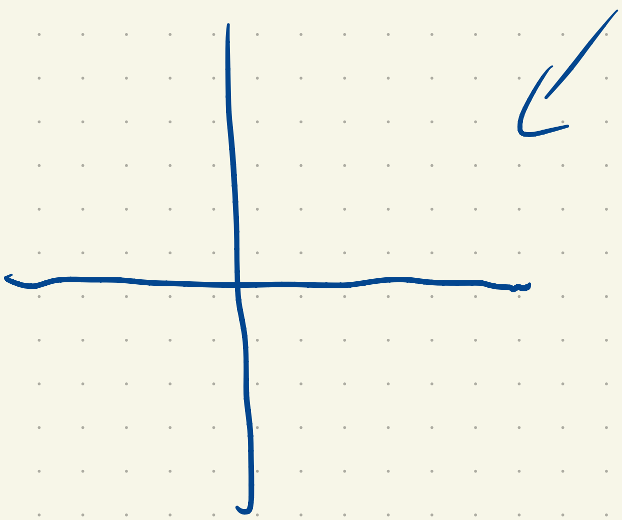
$\|x\|_2 = 1$

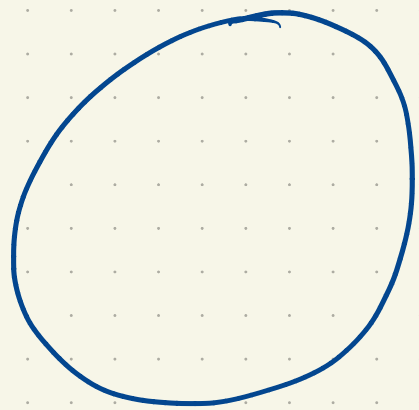


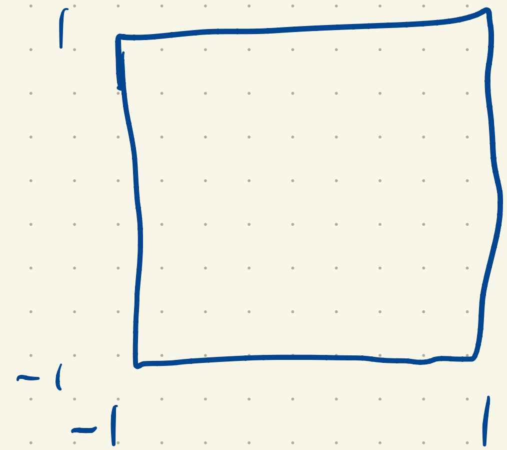
$\|x\|_\infty = 1$

Unit Balls: $\{x : \|x\|_p \leq 1\} = B_p(r)$

$|x_1| + |x_2| = 1 \longrightarrow x_1 + x_2 = 1$

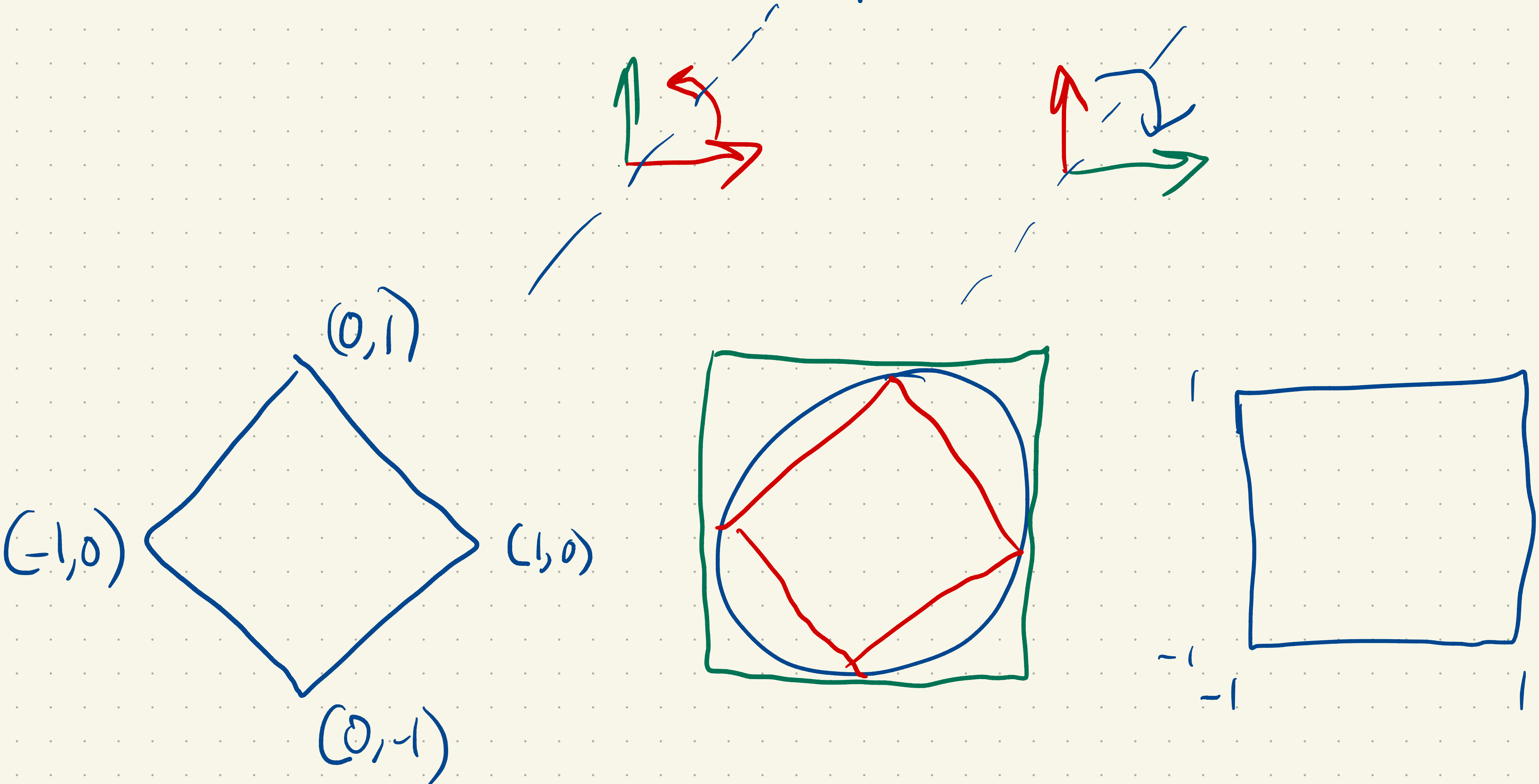






$\|x\|_1 = 1$ $\|x\|_2 = 1$ $\|x\|_\infty = 1$

Unit Balls: $\{x : \|x\|_p \leq 1\} = B_p(r)$



$\|x\|_1 = 1$ $\|x\|_2 = 1$ $\|x\|_\infty = 1$

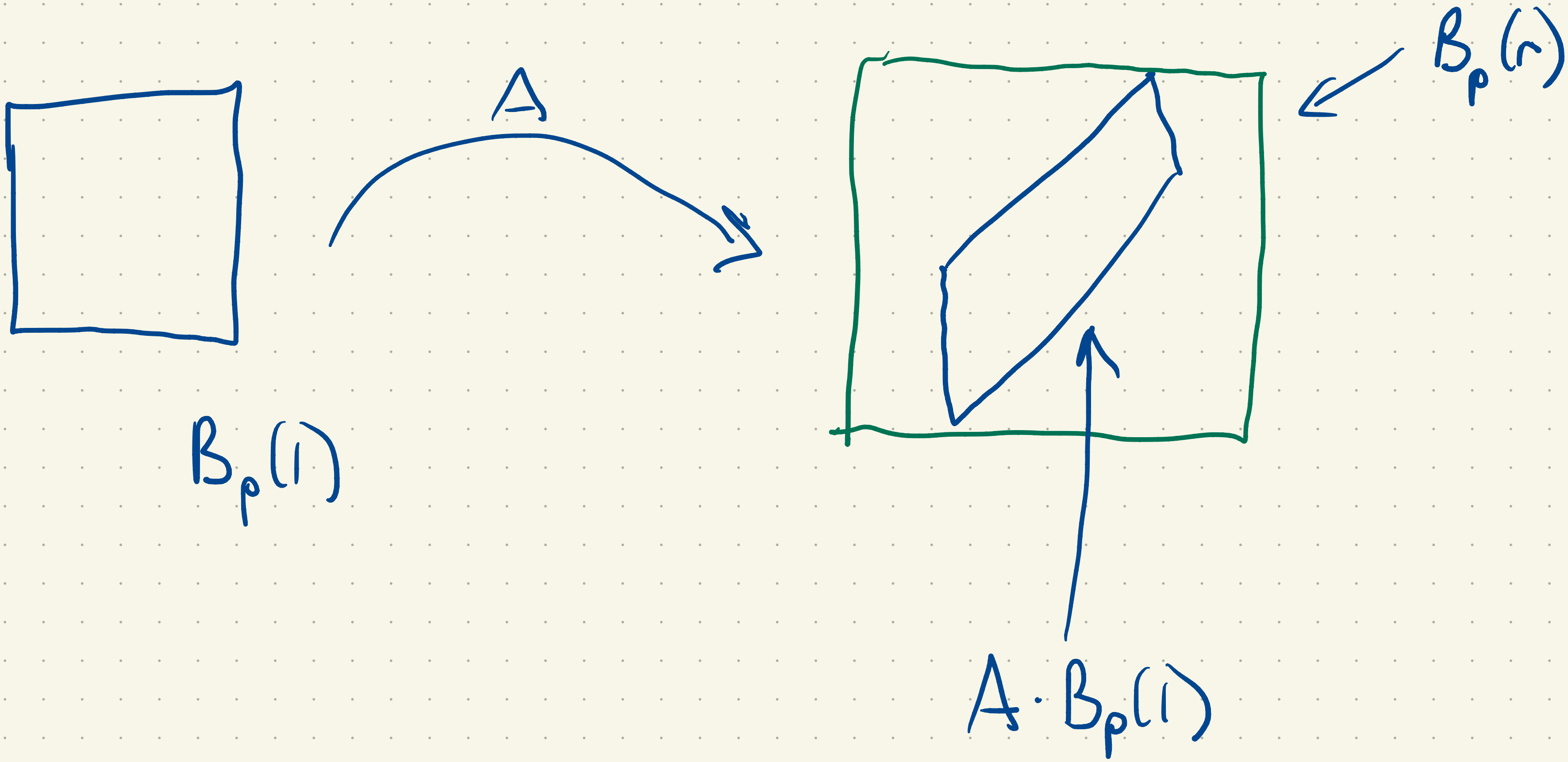
$$\|A+B\|_p \leq \|A\|_p + \|B\|_p$$

$$\det(A+B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\det: GL(\mathbb{R}^n) \to \mathbb{R}$$

Matrix Norms A: $n \times n$ matrix



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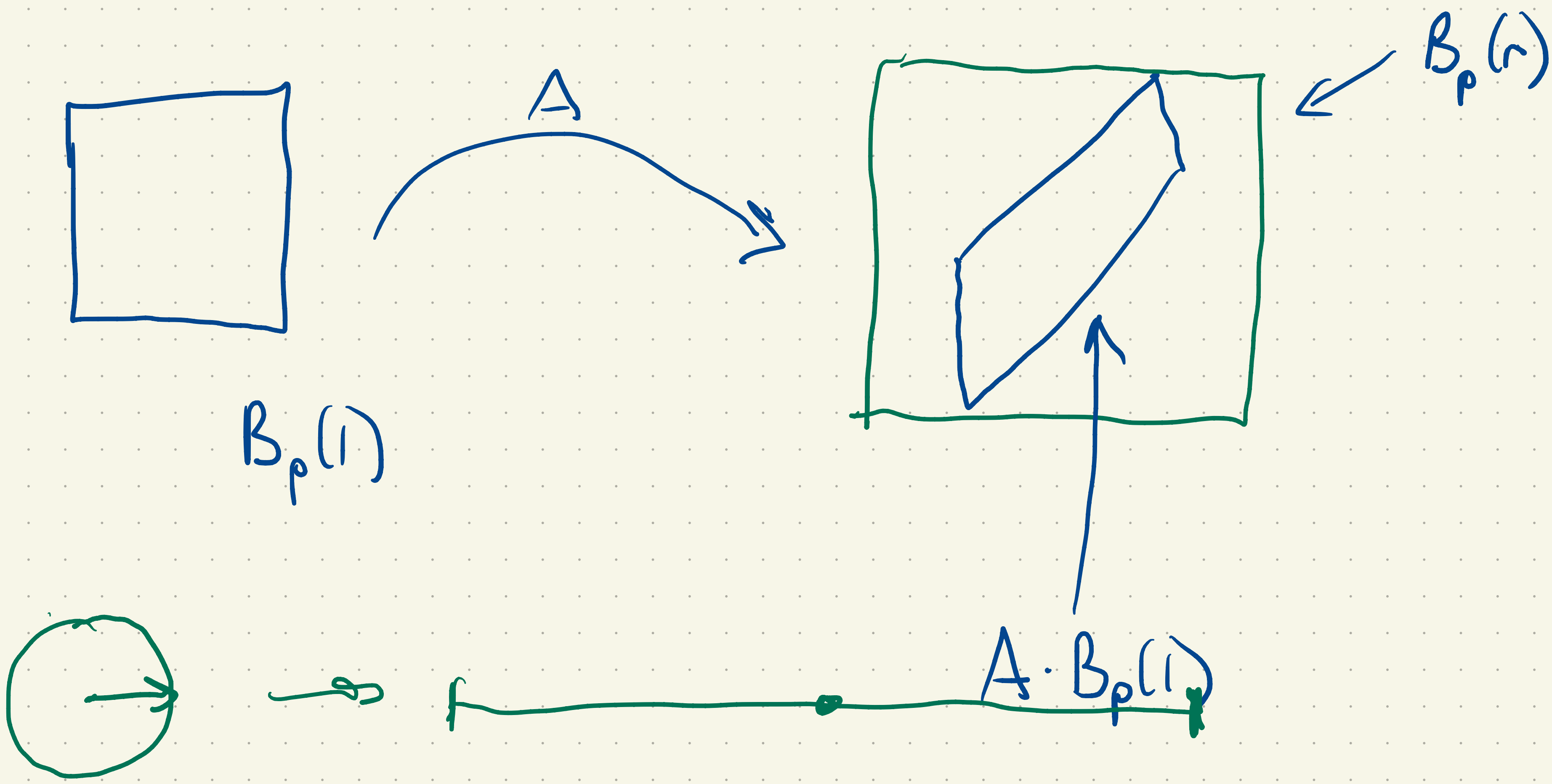

$\|A\|_p$ is smallest r so $A\,B_p(1) \subseteq B_p(r)$