Exercise 1.4.7: Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ contradicts the assumption that $\alpha = \sup T$.

Proof. We first observe that if $x > 0$ and $x^2 > 2$ then x is an upper bound for T. Indeed, if x is not an upper bound for T, there is an $a \in T$ with $x < a$. But then, since $x > 0$ and $a > 0$, $x^2 < a^2 \le 2$.

Suppose $\alpha = \sup T$ and suppose to produce a contradiction that $\alpha^2 > 2$. Our strategy is to show that we can find $n \in \mathbb{N}$ such that $\alpha - 1/n > 0$ and $(\alpha - 1/n)^2 > 2$. Assuming we have done this, the proof is completed, for then $(\alpha - 1/n)$ is an upper bound less than $\alpha = \sup T$, a contradiction.

First, observe that $\alpha > 0$ as $\sup T \ge 1 > 0$. Now, let $\epsilon = \alpha^2 - 2$ and pick $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \min\left(\alpha, \frac{\epsilon}{2\alpha}\right); \tag{1}$$

this is possible since $\epsilon > 0$ and since $\alpha > 0$. Since $1/n < \alpha$, we know $\alpha - 1/n > 0$. Since $\alpha > 0$ and since $1/n < \epsilon/(2\alpha)$ it follows that

$$-\frac{2\alpha}{n} > -\epsilon$$

and hence

$$\begin{aligned}\left(\alpha - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}\\ &\ge \alpha^2 - \frac{2\alpha}{n}\\ &> \alpha^2 - \epsilon\\ &= 2.\end{aligned}$$

That is, $(\alpha - 1/n)^2 > 2$. Since we have already noted $\alpha - 1/n > 0$, we are done. □

Exercise Supplemental 1: Give a from-scratch proof of the following facts:

(a) If $f : A \to B$ has an inverse function g, then f is injective.

(b) If $f : A \to B$ has an inverse function g, then f is surjective.

Proof (a). Suppose $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. But $g(f(a)) = a$ for all $a \in A$. Hence $a_1 = a_2$. □

Proof (b). Let $b \in B$ and let $a = g(b)$. Then, since g is the inverse of f, $f(g(b)) = b$. Thus f is surjective. □

Exercise Supplemental 2: Show that the sets $[0, 1)$ and $(0, 1)$ have the same cardinality.

Let $x_n = 1/(n + 1)$.

Define $f : [0, 1) \to (0, 1)$ by

$$f(x) = \begin{cases} x_{n+1} & x = x_n \\ x_1 & x = 0 \\ x & \text{otherwise.} \end{cases}$$

To show that f is surjective consider $x \in (0, 1)$. Then either $x = x_n$ for some n or not. If $x \neq x_n$ for any n, then (noting that $x \in (0, 1)$ and is in the domain of f) we have $f(x) = x$. Otherwise either $x = x_n$ for some $n > 1$, or $x = x_1$. If $x = x_1$, then $f(0) = x$. Otherwise, $f(x_{n-1}) = x_n = x$.

We now show that f is injective. Let $A = \{x_n : n \in \mathbb{N}\}$, $B = \{0\}$ and $C = (0, 1) \setminus A$.

Suppose $a, b \in A$ and $f(a) = f(b)$. Then $a = x_n$ and $b = x_m$ for some n, m and $f(a) = x_{n+1} = 1/(n + 2)$ and $f(b) = x_{m+1} = 1/(m + 2)$. So $n + 2 = m + 2$ and $n = m$. That is, $a = b$.

Suppose $a, b \in B$ and $f(a) = f(b)$. Then $a, b = 0$. So $a = b$.

Suppose $a, b \in C$ and $f(a) = f(b)$. Then $f(a) = a$ and $f(b) = b$. So $a = f(a) = f(b) = b$.

We have now shown that if $f(a) = f(b)$, and if a and b are both in A or B or C, then $a = b$. On the other hand, if $a \in A$ and $b \in B$, then $f(a) = x_n$ for some $n > 1$ while $f(b) = x_1$. So $a \neq b$. If $a \in A$ and $b \in C$ then $f(a) = x_n$ for some $n > 1$ and $f(b) = b$. Since $b \in C$, $b \neq x_n$ for any n, so $a \neq b$. Finally, if $a \in B$ and $b \in C$ then $f(a) = x_1$ and $f(b) = b \neq x_1$.

Exercise 1.5.10 (a) (c): (Wait until after Wednesday to start this one)

(a) Let $C \subseteq [0, 1]$ be uncountable. Show that there exists $a \in (0, 1)$ such that $C \cap [a, 1]$ is uncountable.

(c) Determine, with proof, if the same statement remains true replacing uncountable with infinite.

Solution:

(a) Suppose each $C \cap [a, 1]$ is at most countable. Define $A_0 = \{0\}$ and for $n \in \mathbb{N}$ define $A_n = C \cap [1/n, 1]$. We claim that $C \subseteq \cup_{n=0}^{\infty} A_n$. Assuming this is true, it follows that C is at most countable since each set in the countably infinite union is at-most countable.

Suppose $x \in C$ and $x \neq 0$. Pick $N \in \mathbb{N}$ such that $1/N < x$; this is possible since $x > 0$. Then $x \in C$ and $1/N < x$ so $x \in C_{1/N} = A_N$. Hence $x \in \cup_{n=0}^{\infty} A_n$.

(b) The claim is false. Consider $C = \{1/n : n \in \mathbb{N}\}$. For each $a \in (0, 1)$, $C \cap [a, 1] = \{1/n : n \leq 1/a$, which is a finite set. But C is evidently infinite.

Exercise Supplemental 3: (Wait until after Wednesday to start this one) Suppose for each $k \in \mathbb{N}$ that A_k is at most countable. Use the fact that $\mathbb{N} \times \mathbb{N}$ is countably infinite to show that $\cup_{k=1}^{\infty} A_k$ is at most countable. Hint: take advantage of surjections.

Proof. The result is trivial if each A_k is empty, so we may assume there is some $a_* \in \cup_k A_k$. For each $k \in \mathbb{N}$ such that A_k is nonemtpy, let $f_k : \mathbb{N} \to A_k$ be a surjection. Now define $f : \mathbb{N} \times \mathbb{N} \to \cup_k A_k$ by

$$f(k,n) = \begin{cases} f_k(n) & A_k \neq \emptyset \\ a_* & A_k = \emptyset. \end{cases}$$

We claim that f is a surjection. Indeed, suppose $a \in \cup_k A_k$. Then $a \in A_n$ for some n. Since f_n is a surjection, there exists $j \in \mathbb{N}$ such that $f_n(j) = a$. But then $f(n, j) = f_n(j) = a$.

Since $\mathbb{N} \times \mathbb{N}$ is a countably infinite set, and since $f : \mathbb{N} \times \mathbb{N} \to \cup_k A_k$ is a surjection, $\cup_k A_k$ is at most countable. □