The goal of this worksheet is for you to be able to write a Matlab code that performs 4^{th} order numerical integration using polynomial interpolation.

1. By hand compute $\int_a^b x^n \, dx$ for each $n \in \mathbb{N}$.

2. Given five sample points $x_1, \ldots, x_5 \in [a, b]$ we will approximate integration on $[a, b]$ by a rule

$$\int_a^b f(x) \, dx = Q(f) = \sum_{j=1}^{5} A_j f(x_j). \tag{1}$$

We demand that

$$Q(p) = \int_a^b p(x) \, dx \tag{2}$$

whenever p is a polynomial of degree at most 4. By hand, write down what this condition implies in the case $p(x) = 1$. That is, substitutue $f(x) = 1$ in in equation (2) and obtain an equation relating the A_j's.

3. By substituting $p(x) = x^k$ for $1 \le k \le 4$, we obtain 4 more equations relating the A_j's. Write down these equations, along with your equation from problem 2, in matrix form. That is, you should have an equation of the form

$$B \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{pmatrix} = \mathbf{b},$$

and your job is to determine the 5×5 matrix B and the right-hand side vector $\mathbf{b}$.

4. Write a Matlab function `quad4th` that takes as input

- A function f.
- The interval end points a and b.
- An array of five points x, the five sample points

and returns $Q(f)$. So the function should have the form

```
function Q = quad4th(f,a,b,x)
    ...
end
```

Your function should set up the matrix B and the right-hand side $\mathbf{b}$ from problem 3 and then use MATLAB's backslash operator to compute the weights A_j. It should then evaluate Q using equation (1).

Pro tip: You may find the function `arrayfun` to be helpful. Given a function f and an array x,

```
arrayfun(f,x)
```

returns an array that is the same size as `x` an contains f evaluated at each entry of `x`.

5. Test that your function seems to work by applying it to approximate $\int_0^\pi \sin(x)\ dx$ with 5 equally spaced sample points that include the interval endpoints. The true answer is 2, of course. Your answer should be correct to about 10^{-3}.

6. The matlab command `rand(1,5)` will generate 5 random numbers between 0 and 1. Use your function `quad4th` with 5 randomly selected sample points in $[0,1]$ to approximate $\int_0^1 x^5\ dx$. How big is the error? Try this a number of times with randomly selected sample points and oberve the smallest error you see.

7. Use your function `quad4th` with 5 equally-space sample points including 0 and 1 to approximate $\int_0^1 x^5\ dx$. How big is the error?

8. What is the biggest integer n such that $\int_0^1 x^n\ dx = Q(x^n)$ when using equally-spaced sample points?

9. For an interval $[-1,1]$ there are special sample points, Gauss-Lobotto points given by

$$\mathbf{x} = \left[-1, -\sqrt{3/7}, 0, \sqrt{3/7}, 1\right].$$

For any other interval $[a,b]$ the Gauss-Lobotto points are obtained by translation and scaling, i.e.

$$a + (\mathbf{x}+1)(b-a)/2$$

Using the Gauss-Lobotto sample points on the interval $[0,1]$ determine the largest value of n such that $\int_0^1 x^n\,dx = Q(x^n)$.

10. Use the Gauss-Lobotto sample points on $[0,\pi]$ to approximate $\int_0^\pi \sin(x)\,dx$. Compare the error with the error using equally-spaced sample points.

11. When using equally spaced sample points,

$$\int_a^b f(x)\,dx = Q(f) + Kf^{(6)}(\eta)$$

for some $\eta \in [a,b]$. Determine K for the interval $[0,1]$ by subsituting $f(x) = x^6$ in the above formula. Then use this value to estimate the error in approximating $\int_0^1 \sin(\pi x)\,dx$ using equally spaced sample points. Is true error less than the estimate you just computed?

12. Use a technique similar to the previous problem to estimate the error in approximating $\int_0^1 \sin(\pi x)\,dx$ using Gauss-Lobatto sample points.