Preamble: There is a total of **65** points on this exam; not every problem is equally weighted. Please write out your proofs and your statements of theorems and definitions as clearly as you can. In grading, however, some consideration will be made for the exam's time constraints. Good luck!

1. [14 points]

Define the following.

a) $\limsup_{n\to\infty} x_n$

b) The linear map $T : V \to W$ is bounded.

c) The set A is dense in the metric space X.

d) A (Lebesgue) measurable function.

e) A compact set.

f) A totally bounded subset of a metric space.

g) A complete metric space.

2. [16 points]

State the following theorems.

a) The Dominated Convergence Theorem.

b) The Monotone Convergence Theorem.

c) The Arzelá Ascoli Theorem.

d) The Weierstrass Approximation Theorem.

e) The Weierstrass M-test.

f) The long-winded theorem connecting derivatives and uniform convergence.

g) Fatou's Lemma

h) Continuity from Above for Lebesgue Measure.

3. [8 points]

a) In class we gave seven desired properties of a length function. Name them.

b) Define (Lebesgue) outer measure.

c) Which of the seven properties of part **(a)** are satisfied by outer measure?

d) State the definition a measurable subset of $\mathbb{R}$. You must give the definition from class, not Carother's definition.

4. [4 points]

a) Show, carefully, that $\int_1^\infty 1/s^2 \, ds = 1$.

b) Show, carefully, that $\int_1^\infty \frac{\sin(1/s)}{s^2} \, ds = 1 - \cos(1)$.

5. [6 points]

Let K be a compact subset of the metric space X. Suppose $f : X \to Y$ is continuous. Show that $f(K)$ is compact.

6. [4 points]

Precisely state any three results that encapsulate the idea that a measurable set is "nice". At least one of these conditions must refer to a finite collection of intervals.

7. [5 points]

Suppose (f_n) is a sequence of functions in $B(X)$ converging to f and that each f_n is continuous at $x \in X$. Show that f is continuous at x.

8. [4 points]

Suppose f is Lipschitz continuous with constant K. Show that for any set A, $m^*(f(A)) \leq Km^*(A)$. If you need it, you may use the fact that if $B \subseteq \mathbb{R}$ and $\operatorname{diam}(B) \leq r$, then there is an interval of length r that contains B.

9. [4 points]

Suppose f and g are continuous function from X to Y, that A is dense in X, and that $f = g$ on A. Show that $f = g$.

10. [Extra Credit: 3 points]

On the take-home final, you proved that every Riemann integrable function is continuous almost everywhere. In fact the converse is true, though we did not have time to prove this. With this in mind, show that if F is a closed set and $m(F) = 0$, then χ_F is Riemann integrable.