Numerical Dispersion Relations

Aim: explain qualitative observations in flaws in numerical solution of wave equation in terms of dispersion relations.

$$e^{I(kx-\omega t)}$$

Observation:

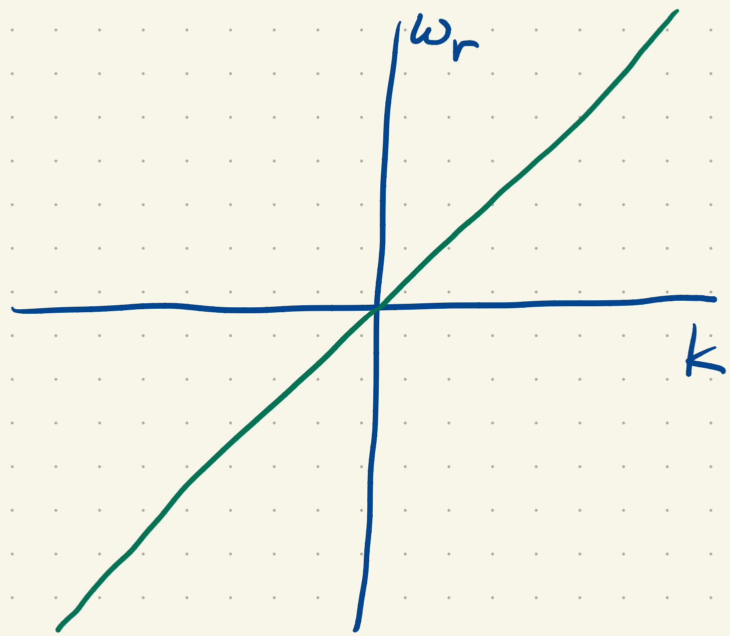
Non-dispersive equations are rare

$$(\partial_t + a\partial_x)u = -du$$

$$I(-\omega + ak) = -d$$

$$\omega - ak = -Id$$

$$v_p = \frac{\omega_r}{k} = a$$



Observation:

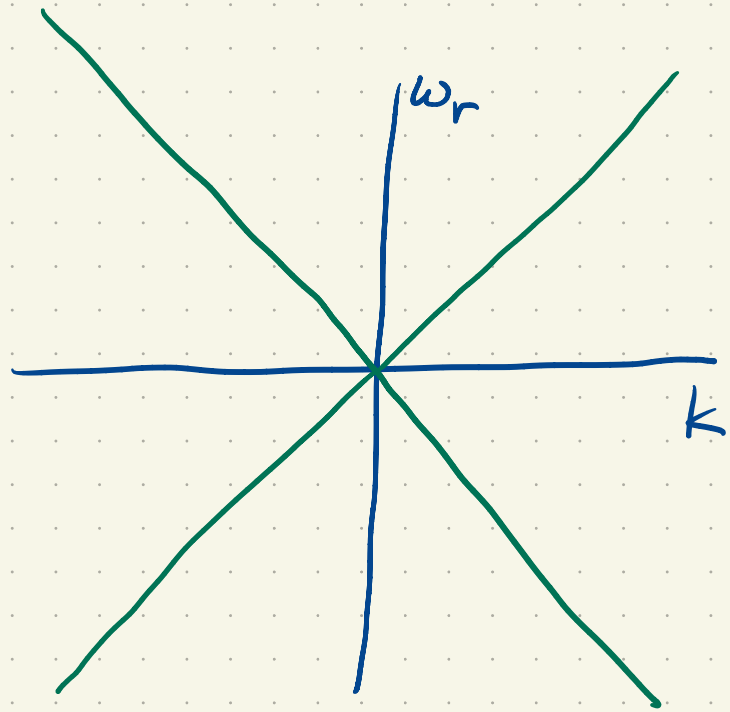
Non-dispersive equations are rare

$$(\partial_t^2 - c^2 \partial_x^2) u = 0$$

$$-\omega^2 + c^2 k^2 = 0$$

$$\omega = \pm c k$$

$$v_p = \frac{\omega_r}{k} = \pm c$$



$$u_{i,j+1} = 2u_{i,j} - \lambda^2 \left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right] - u_{i,j-1}$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \lambda^2 \left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right]$$

$$u_{i,j+1} - 2u_{ij} + u_{i,j-1} = \lambda^2 \left[u_{i+1,j} - 2u_{ij} + u_{i-1,j} \right]$$

Try $u_{i,j} = e^{I(\bar{k}x_i - \bar{\omega} t_j)}$ ($\bar{k}$ since k is taken!)

$$\left(K^j e^{Irx_i},\ r = \bar{k},\ K = e^{-I\bar{\omega}k} \right) \quad t_j = kj$$

$$u_{i,j+1} - 2u_{ij} + u_{i,j-1} = \lambda^2 \left[u_{i+1,j} - 2u_{ij} + u_{i-1,j} \right]$$

$$k j \qquad k(j+1) \qquad k(j-1)$$

Try $u_{i,j} = e^{I(\bar{k} x_i - \bar{\omega} t_j)}$ ($\bar{k}$ since k is taken!)

$$\left(K^j e^{I r x_i},\ r = \bar{k},\ K = e^{-I\bar{\omega}k} \right)$$

$$e^{-I\bar{\omega}k} - 2 + e^{+I\bar{\omega}k} = \lambda^2 \left[e^{I\bar{k}h} - 2 + e^{-I\bar{k}h} \right]$$

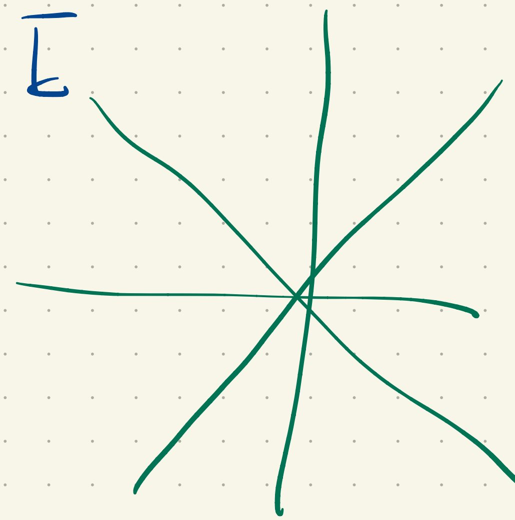
$$2\left[\cos(\bar{\omega}k) - 1\right] = \lambda^2 2 \left[\cos(\bar{k}h) - 1\right]$$

$$\sin^2\left(\frac{\bar{\omega} k}{2}\right) = \lambda^2 \sin^2\left(\frac{\bar{k} h}{2}\right) \qquad \lambda = \frac{ck}{h}$$

$$\frac{1}{k}\sin\left(\bar{\omega}\, k/2\right) = \pm c \frac{1}{h}\sin\left(\frac{\bar{k} h}{2}\right) \quad : \text{dispersive!}$$

k small relative to $\bar{\omega}$, h small relative to $\bar{k}$

$$\frac{\bar{\omega}}{2} = \pm c \frac{\bar{k}}{2} \longrightarrow \frac{\bar{\omega}}{\bar{k}} = \pm c.$$



(in the limit)

$\arcsin(\sin(\theta)) = \theta$

$$\frac{1}{k}\sin(\bar{\omega}\, k/2) = \pm c\,\frac{1}{h}\sin\left(\frac{\bar{k}h}{2}\right)$$

Assuming $-\frac{\pi}{2} \le \frac{\bar{\omega} k}{2} \le \frac{\pi}{2}$, $-\frac{\pi}{2} \le \frac{\bar{k}h}{2} \le \frac{\pi}{2}$

$$\bar{\omega} = \pm\frac{2}{k}\arcsin\left[\lambda \sin\left(\frac{\bar{k}h}{2}\right)\right]$$

$\bar{\omega} = \pm c\bar{k}$

$|\lambda| \le 1$

① $\lambda = 1$ $\quad \bar{\omega} = \pm c\bar{k}$ ✓

2. ω is real, always (non-dissapative)

3.

$$V_{np} = \frac{\bar{\omega}}{\bar{k}} = \frac{\pm 2}{k\bar{k}} \frac{h}{h} \frac{c}{c} \arcsin\left(\lambda \sin\left(\frac{\bar{k}h}{2}\right)\right)$$

$$= \frac{\pm 2c}{\lambda \bar{k} h} \arcsin\left(\lambda \sin\left(\frac{\bar{k}h}{2}\right)\right)$$

$$Z = \frac{\bar{k}h}{2} \qquad -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$V_{np} = \frac{\pm c}{\lambda z} \arcsin(\lambda \sin(z))$$

$$|\lambda| \leq 1$$

$$\arcsin(x) = x + \frac{x^3}{6} + O(x^5)$$

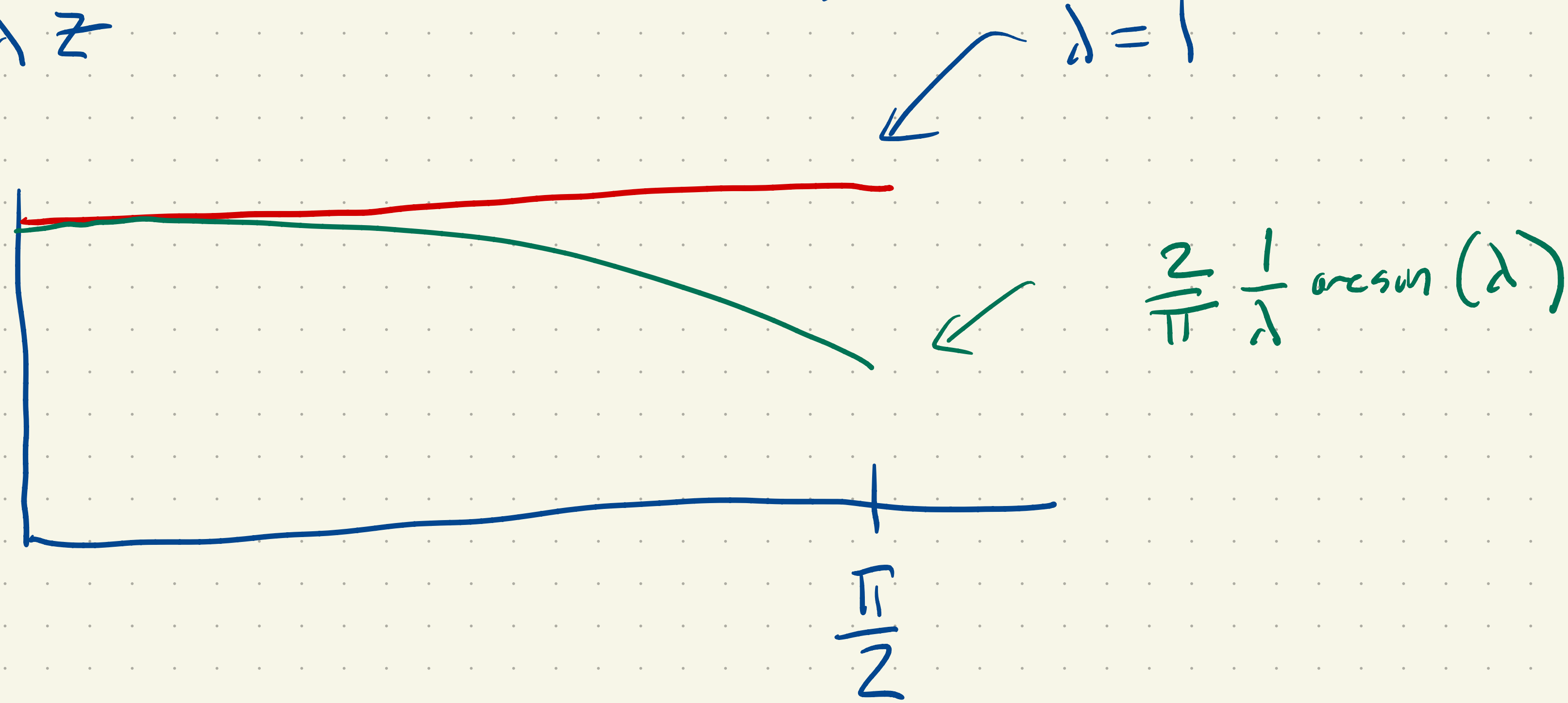
$$\sin(z) = z - \frac{z^3}{6} + O(z^5)$$

$$V_{np} = \pm c \left[\frac{1}{\lambda z} \left[\lambda z - \frac{\lambda z^3}{6} + \frac{\lambda^3 z^3}{6} + O(z^5) \right] \right]$$

$$= \pm c \left[1 - \frac{1}{6}(1-\lambda^2) z^2 + O(z^4) \right]$$

$$v_{np} = \pm c \left[1 - \frac{1}{24}(1-\lambda^2)(\bar{k}h)^2 + O(h^4) \right]$$

$$\frac{1}{\lambda z} \arcsin(\lambda \sin(z))$$



$$\bar{\omega} = \pm \frac{2}{k} \arcsin\left[\lambda \sin\left(\frac{\bar{k} h}{2} \right) \right]$$

$$v_{ng} = \frac{\partial \bar{\omega}}{\partial \bar{k}} = \frac{2}{k} \frac{1}{\sqrt{1 - \lambda^2 \sin^2\left(\frac{\bar{k}h}{2}\right)}} \cdot \lambda \cos\left(\frac{\bar{k}h}{2}\right) \frac{h}{2}$$

$$= c \frac{\cos(\bar{k}h/2)}{\sqrt{1 - \lambda^2 \sin^2(\bar{k}h/2)}} \qquad |\lambda|^2 \leq 1$$

$$= c \frac{\cos(\bar{k}h/2)}{\sqrt{(1-\lambda^2) + \lambda^2 \cos^2(\bar{k}h/2)}}$$

$$v_{ng} = c \frac{\cos(z)}{\sqrt{(1-\lambda^2)+\lambda^2\cos^2(z)}}$$

$$> \cos^2(z) \quad \text{if} \quad 0 \leq \lambda < 1$$

Group velocity is also $< c$.

Upwind:

$$\tau_{i,j} = \underbrace{-\frac{1}{2} a h (1-\lambda)}_{-\varepsilon} u_{xx}$$

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Suppose $u_t + au_x = \varepsilon u_{xx}$.

Substitute into upwind approx for its "LTE"

Upwind:

$$\tau_{i,j} = \underbrace{-\frac{1}{2}ah(1-\lambda)}_{-\varepsilon} u_{xx}$$

$$u_t + a u_x = 0$$

Suppose $u_t + a u_x = \varepsilon u_{xx}$.

Substitute into upwind approx for its "LTE"

$$u(x, t+k) = u + u_t k \quad u_{tt}\frac{k^2}{2} + u_{ttt}\frac{k^3}{6} + O(k^4)$$

$$u_t = -a u_x + \varepsilon u_{xx}$$

$$u_{tt} = -a \partial_x (u_t) + \varepsilon u_{xx}$$

$$= -a \partial_x \left(-a u_x + \varepsilon u_{xx} \right) + \varepsilon u_{xx}$$

$$= a^2 u_{xx} - a \varepsilon u_{xxx} + \varepsilon u_{xx}$$

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$$O(hk)$$

$$\frac{u(x,t+k) - u(x,t)}{k} = u_t + \frac{a^2 k}{2} u_{xx} + O(\varepsilon k) + O(k^2)$$

$$a \, \frac{u(x,t) - u(x-h,t)}{h} = a \left[u_x - \frac{u_{xx}}{2} h + O(h^2) \right]$$

LTE:

$$u_t + a u_x + \underbrace{\left[\frac{a^2 k}{2} - \frac{ah}{2}\right] u_{xx}}_{\frac{ah}{2}\left[\frac{ak}{h} - 1\right] = -\varepsilon} + O(h^2) + O(k^2) + O(hk)$$

$$\varepsilon = \frac{ah}{2}[1 - \lambda]$$

$$\frac{ah}{2}\left[1 - \frac{ak}{h}\right]$$

$$\underbrace{u_t + a u_x - \varepsilon u_{xx}}_{= 0} + O(h^2) + O(k^2) + O(hk)$$

Upshot:

The solution of $u_t + a u_x = \varepsilon u_{xx}$

better satisfies the numerical method.

The εu_{xx} term makes the PDE act like

heat equation, gives diffusion.

Same game for LW:

$$\tau_{i,j} = \frac{1}{6}\left[ah^2 - a^2k^2\right] u_{xxx} + \cdots \qquad \frac{ak}{h}$$

$$u_t + au_x = \underbrace{-\frac{1}{6}\left[ah^2 - a^3k^2\right] u_{xxx}}$$ solves discrete better

effect is dispersion.

$$-I\bar{\omega} + aI\bar{k} = \frac{ah^2}{6}\left[1 - \lambda^2\right] I^3 \bar{k}^3$$

$$\bar{\omega} = a\bar{k} - \frac{ah^2}{6}(1-\lambda^2)\bar{k}^3$$

$$= a\bar{k}\left[1 - \frac{(h\bar{k})^2}{6}(1-\lambda^2)\right]$$

dispersive, speeds $< a$.