1. Carothers 2.21 (Solution by David Maxwell)

Consider a number $a$with base 3 expansion $0.a_1a_2\dots a_n11$. By problem 1.21, a has exactly one other base 3 expansion, namely $0.a_1a_2\dots a_n1022\cdots$. Both of these expansions contain a 1. On the other hand, Theorem 2.25 implies every element of the Cantor set has a base 3 expansion where all the digits are 0 or 2. Hence $a \notin \Delta$.

2. Carothers 2.22 (Solution by former student Mason Brewer)

Show that Δ contains no (nonempty) open intervals. In particular, show that if $x, y \in \Delta$ with $x < y$, then there is some $z \in [0, 1] \setminus \Delta$ with $x < z < y$. (It follows from this that Δ is *nowhere dense*, which is another way of saying that Δ is "small.")

Solution:
Let $x, y \in \Delta$ with $x < y$. Then let $x = 0.a_1a_2a_3...$ and $y = 0.b_1b_2b_3...$ be the base-3 expansions of x and y without any values of 1. Since they are not equal, there exists some i that is the first decimal place where x and y disagree. In other words, $a_n = b_n$ for $n < i$, and $a_i < b_i$, which must be the case because $x < y$. Thus it must be the case that $a_i = 0$ and $b_i = 2$ in order to satisfy $a_i < b_i$. Now define $z = 0.c_1c_2c_3...$ such that $c_n = a_n = b_n$ for $n < i$, and $c_n = 1$ for $n \geq i$. Note that since z has more than a single decimal equal to 1, it must be that $z \notin \Delta$, which implies that $z \neq x, y$. Since $a_i < c_i < b_i$, we know that $x < z < y$.

3. Carothers 2.25 (Solution by David Maxwell)

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1$ if $x \in \Delta$, and $g(x) = 0$ otherwise. At which points of $\mathbb{R}$ is g continuous?

Solution:
Let $A = \mathbb{R} \setminus \Delta$. We claim that g is continuous exactly on A.

First, suppose $x \in A$, and let $\epsilon > 0$. Notice that Δ is a closed set, being an intersection of closed sets, and hence A, its complement, is open. Thus there is a $\delta > 0$ such that if $|x - y| < \delta$ then $y \in A$. But then $g(x) = g(y) = 0$ and $|g(x) - g(y)| = 0 < \epsilon$. Hence g is continuous at x

On the other hand, suppose $x \notin A$, so $x \in \Delta$. By the previous problem, for each $n \in \mathbb{N}$, we can find $x_n \in (x - 1/n, x + 1/n)$ such that $x_n \notin \Delta$. Now $|x - x_n| < 1/n$, so $x_n \to x$. And $g(x_n) = 0$ for all n whereas $g(x) = 1$. Since $g(x_n) \not\to g(x)$, we conclude that g is not continuous at x.

4. Carothers 2.16 (Solution by former student Lander Ver Hoef)

The *algebraic numbers* are those real or complex numbers that are the roots of polynomials having *integer* coefficients. Prove that the set of algebraic numbers is countable. [Hint: First show that the set of polynomials having integer coefficients is countable.]

Solution:
First, observe that for a given n, there is a natural surjective mapping from the ordered

n-tuple of integers with a non-zero final element (that is, an element of $\mathbb{Z}\times\mathbb{Z}\times\cdots\times\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})$) to the set of polynomials of degree $n-1$ with integer coefficients, given by

$$f(a_1, a_2, \ldots, a_n) = a_1 + a_2x + \cdots + a_nx^{n-1},$$

where $a_n \neq 0$. Thus, the set P_n of all polynomials of degree n with integer coefficients is countable.

The set P of all polynomials with integer coefficients is the union of the P_n across n, and the countable union of countable sets is, itself, countable, so P is countable.

Each polynomial in P has countably many roots, so there is a surjective mapping from $P \times \mathbb{N}$ to the set of algebraic numbers, defined by (p, n) being mapped to the nth root of the polynomial p. Thus, there are only countably many algebraic numbers.

5. Carothers 3.7 (Solution by former student Max Heldman)

Let $f : [0, \infty) \to [0, \infty)$ be increasing and satisfy $f(0) = 0$, and $f(x) > 0$ for all $x > 0$. If $f(x)$ also satisfies $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$, then $f \circ d$ is a metric whenever d is a metric. Each of the following conditions is sufficient to ensure that $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$:

a) f has a second derivative satisfying $f'' \leq 0$;

b) f has a decreasing first derivative;

c) $f(x)/x$ is decreasing for $x > 0$;

Solution:
We first show that (a) $\implies$ (b) $\implies$ (c), and then prove that (c) is sufficient. To show (a) $\implies$ (b), we prove the contrapositive. Suppose $f'(x + h) > f'(x)$ for some $h > 0$. Then by the Mean Value Theorem there exists $c \in (x, x + h)$ such that $f''(c) = \frac{f'(x+h)-f'(x)}{h} > 0$.

For (b) $\implies$ (c), it is sufficient to show that $\frac{d}{dx}\left(\frac{f(x)}{x}\right) = \frac{f'(x)x-f(x)}{x^2} \leq 0$ for $x > 0$. Observe that by Taylor's Theorem we have $f(x) = f(0) + f'(tx)x$, where $t \in [0, 1]$. Since f' is decreasing and $f(0) = 0$,

$$f(x) = f(0) + f'(tx)x = f'(tx)x \geq f'(x)x.$$

To complete the proof, suppose $f(x)/x$ is decreasing for $x > 0$. Let $x \geq y > 0$. Then $\frac{f(x+y)}{x+y} \leq \frac{f(x)}{x}$, and $\frac{f(x)}{x} \leq \frac{f(y)}{y}$, that is, $f(x)y \leq f(y)x$. Thus

$$f(x + y) \leq \frac{(x + y)f(x)}{x} = \frac{f(x)x + f(x)y}{x} \leq \frac{f(x)x + f(y)x}{x} = f(x) + f(y).$$

6. Carothers 3.15 (Solution by David Maxwell)

Show that a set A is bounded if and only if the diameter of the set is finite.

Solution:
Suppose A is bounded. So we can pick $x \in A$ and $R > 0$ such that $A \subseteq B_R(x)$. But then if $a, b \in A$, $d(a,b) \le d(a,x) + d(x,b) < 2R$. Hence

$$\operatorname{diam}(A) = \sup\{d(a,b) : a, b \in A\} \le 2R.$$

So A has finite diameter.

Conversely, suppose A has finite diameter D. By hypothesis, A is not empty; let $x \in A$. For any $y \in A$, $d(y,x) \le \operatorname{diam}(A) = D < 2D$. Thus $A \subseteq B_{2D}(x)$ and A is bounded.