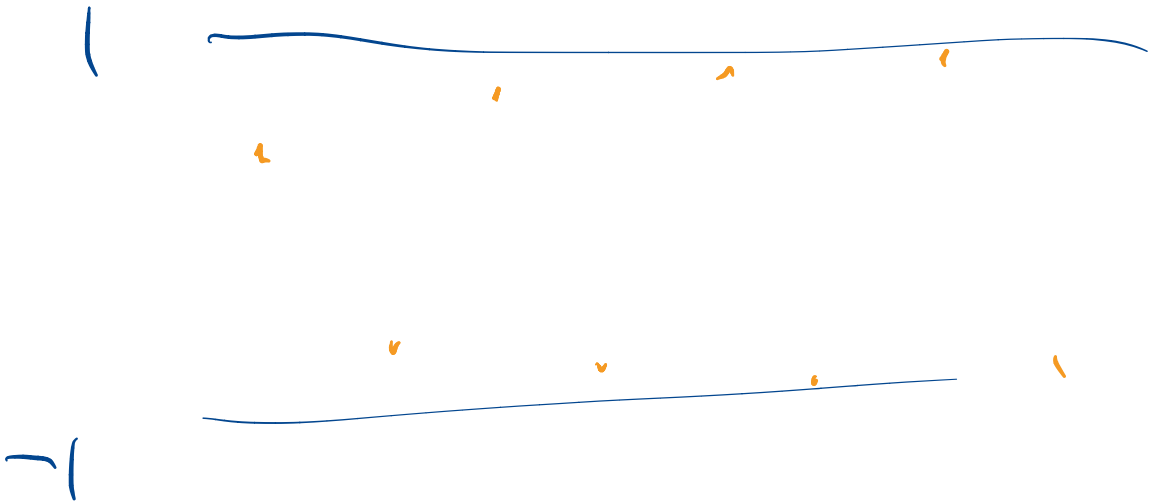


Claim: $\limsup_{n \to \infty} \frac{1}{n} = 0$

If $M \le 0$ then M is not an e.u.b.

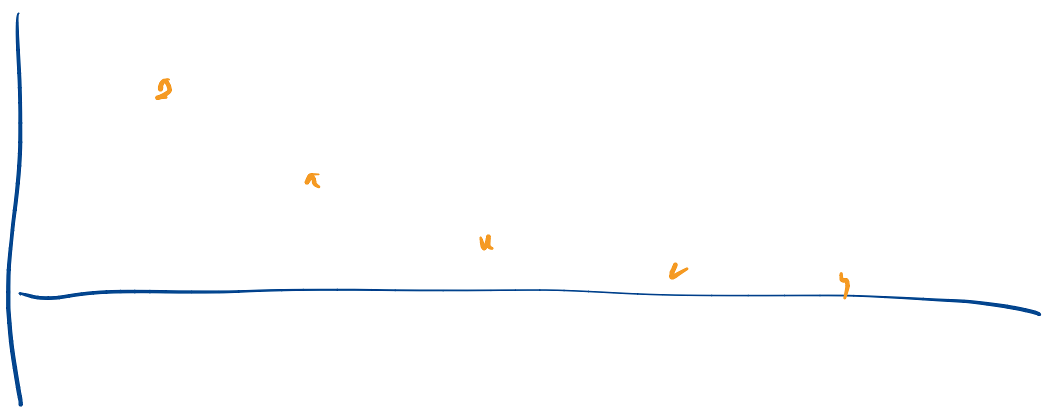
If $M > 0$ I claim M is an e.u.b.

If true the set of e.u.b is $(0, \infty]$

in which case

Suppose $M > 0$. Then there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < M$. But then if $n \geq N$ then $x_n = \frac{1}{n} \leq \frac{1}{N} < M$.

Hence M is an e.u.b.



Alt formulation:

Given (x_n) and $N \in \mathbb{N}$ we define

$$T_N = \sup \{ x_N, x_{N+1}, \dots \}$$

$T_N \geq T_{N+1}$

The sequence (T_N)

is monotone decreasing.

Def: $$\overline{\lim_{n \to \infty}} x_n = \inf_{N \geq 1} T_N$$

$$\left[= \lim_{N \to \infty} T_N \text{ with care about } \pm\infty \right]$$

I.e. $$\overline{\lim_{n\to\infty}}\, x_n = \inf_{n\geq 1} \sup_{m\geq n} x_m$$

$\limsup x_n$ via e.u.b.

Claim: These are the same.

First: each T_N is an eventual upper bound.

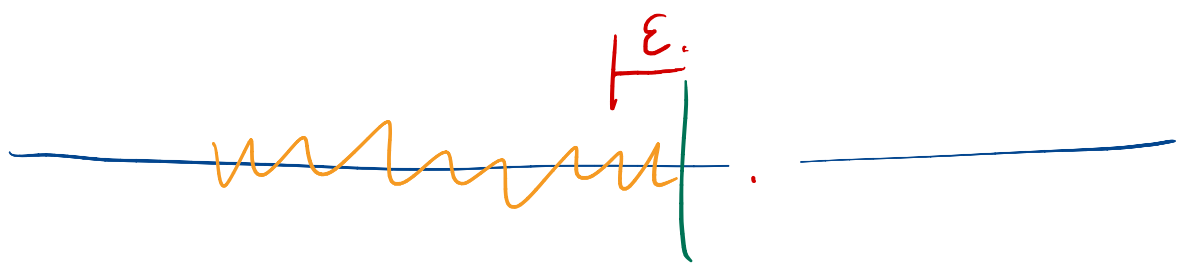
Hence $$\limsup_{n\to\infty} x_n \leq T_N.$$

Hence $$\limsup_{n\to\infty} x_n \leq \overline{\lim_{n\to\infty}}\, x_n.$$

Conversely suppose for the moment that

$$\limsup_{n\to\infty} x_n = L \in \mathbb{R}.$$

Let $\varepsilon > 0$. Then there exists an e.u.b. M with $M < L + \varepsilon$.



Then there exists N with $x_n \leq M$ if $n \geq N$.

Hence $T_N \leq M < L + \varepsilon$.

That is $\quad T_N < L + \varepsilon$, and hence

$$\inf T_N < L + \varepsilon. \quad \text{That is}$$

$$\overline{\lim_{n\to\infty}}\, x_n < \limsup_{n\to\infty} x_n + \varepsilon$$

This is true for all $\varepsilon > 0$ and hence

$$\overline{\lim_{n\to\infty}}\, x_n \leq \limsup_{n\to\infty} x_n.$$

The case $L = +\infty$ is trivial and the case $L = -\infty$ is on HW.

Analogous quantities:

m is an eventual lower bound for (x_n) if there exists $N \in \mathbb{N}$ so if $n \geq N$, then

$$m \leq x_n.$$

$$\liminf_{n\to\infty} x_n = \begin{cases} \sup \{ m : m \text{ is an e.l.b.} \} \\ \sup_{n\geq 1} \inf_{m\geq n} x_m \end{cases}$$

Exercise: If (x_n) is a sequence in $\mathbb{R}$

$\liminf_{n\to\infty} x_n = -\infty$ iff the sequence is not bounded below

$= +\infty$ iff $\lim_{n\to\infty} x_n = \infty$

(look this def up).

Lemma: $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

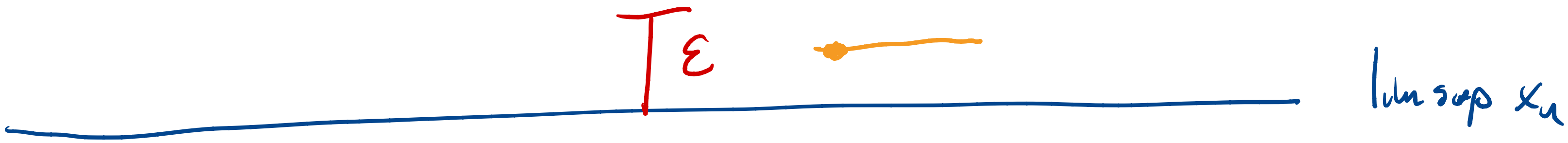
Pf: Let m, M be an e.l.b and e.u.b respectively

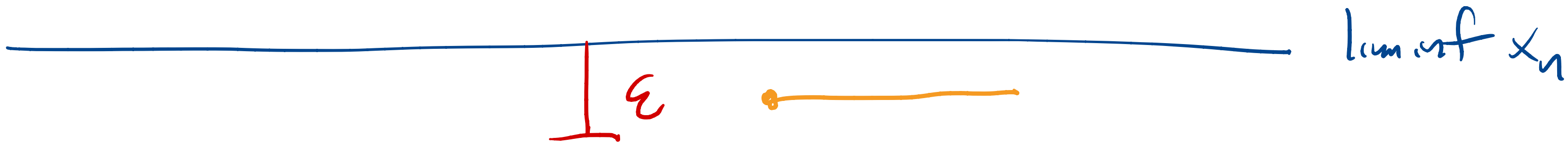
Then there exists some N with

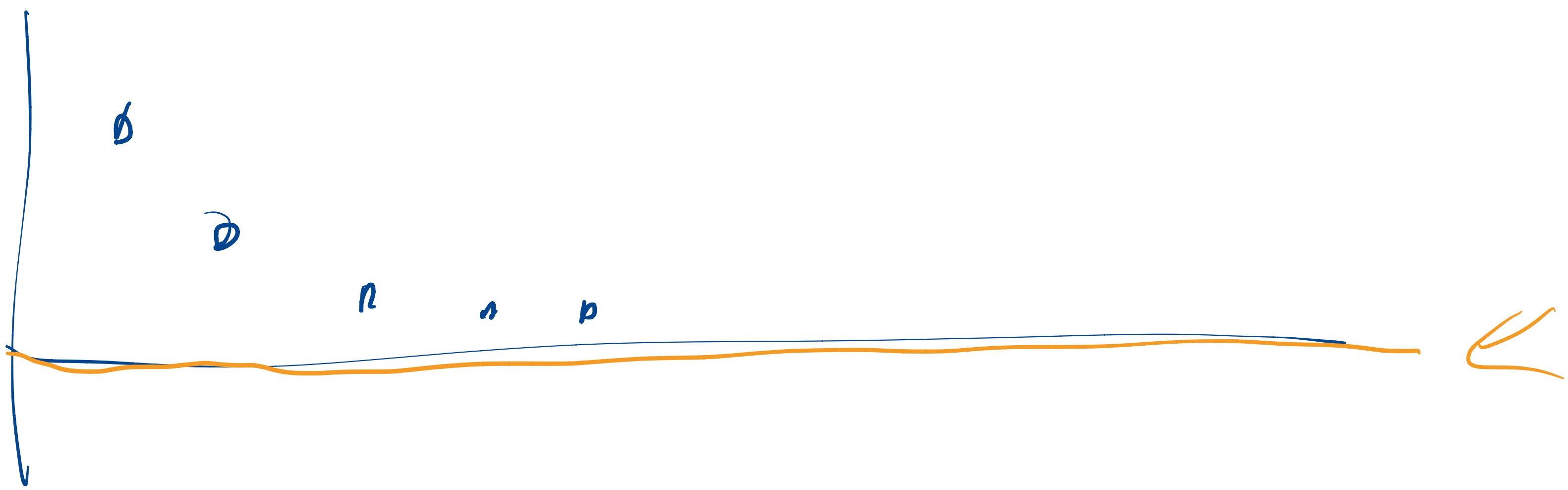
$$m \leq x_N \leq M.$$

Consequently $m \leq M$. Hence $\liminf x_n \leq M$, and

$$\liminf x_n \leq \limsup x_n.$$


ε
lim sup x_n


lim inf x_n
ε



Exercise: Let $L \in \mathbb{R}$. Then

$$\lim_{n\to\infty} x_n = L \iff \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = L.$$

Exercise: Same thing for $L = \pm\infty$.

Base p (p-adic) expansions

Let $p \in \mathbb{N}$, $p \geq 2$.

$$\mathcal{D}_p = \{0, 1, \ldots, p-1\} \qquad (\mathcal{D} \to \text{digits})$$

Given (a_k) with $a_k \in \mathcal{D}_p$

$0.a_1 a_2 a_3 \cdots$ (base p) means

$$\sum_{k=1}^{\infty} \frac{a_k}{p^k} \qquad \frac{a_1}{p} + \frac{a_2}{p^2} + \frac{a_3}{p^3} + \cdots$$

Is this well defined? Does the series converge?

Lemma: The series $\sum_{k=1}^{\infty} \frac{a_k}{p^k}$ converges to some number in $[0,1]$.

Pf: Since the series has non-negative terms we can use the $\underbrace{\text{comparison test.}}$

look me up!

Observe each $\frac{a_k}{p^k} \leq \frac{p-1}{p^k}$.

Moreover $$\sum_{k=1}^{\infty} \frac{p-1}{p^k} = (p-1) \sum_{k=1}^{\infty} \left(\frac{1}{p}\right)^k$$

$$= (p-1) \frac{1/p}{1 - 1/p}$$ (Geometric series)

$$= (p-1)\frac{1}{p-1}$$

$$= 1.$$

Thus $0 \leq \sum_{k=1}^{\infty} \frac{a_k}{p^k} \leq 1.$ $\square$