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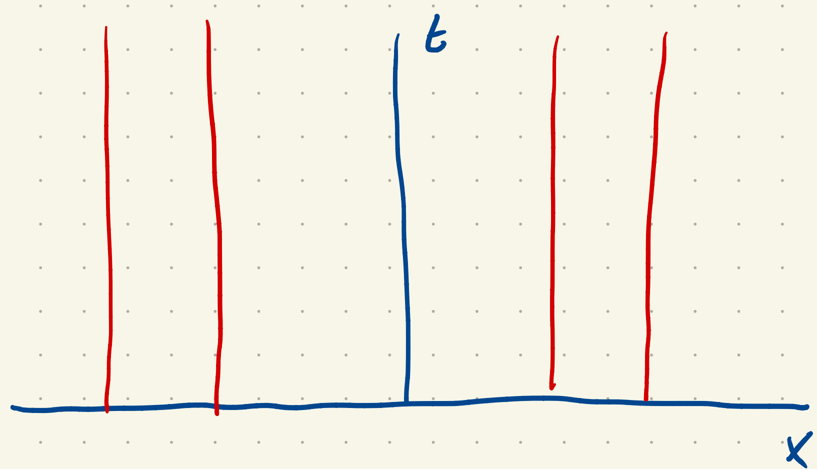

Next easiest: $v = a$, constant

$$u_t + a\,u_x = 0$$

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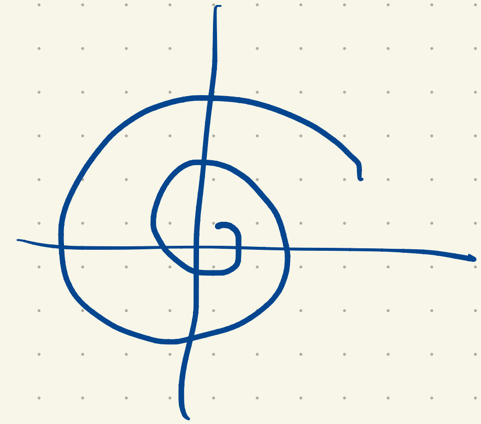

$$u_t + a\, u_x = 0$$

Method of characteristics:

find curves $\gamma(s) = (x(s), t(s))$

with u constant along γ

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Method of characteristics:

find curves $\gamma(s) = (x(s), t(s))$

with u constant along γ

$$\frac{d}{ds} u(x(s), t(s)) = \underbrace{u_t \dot{t} + u_x \dot{x}} \longrightarrow u_t + a\,u_x = 0$$

e.g.

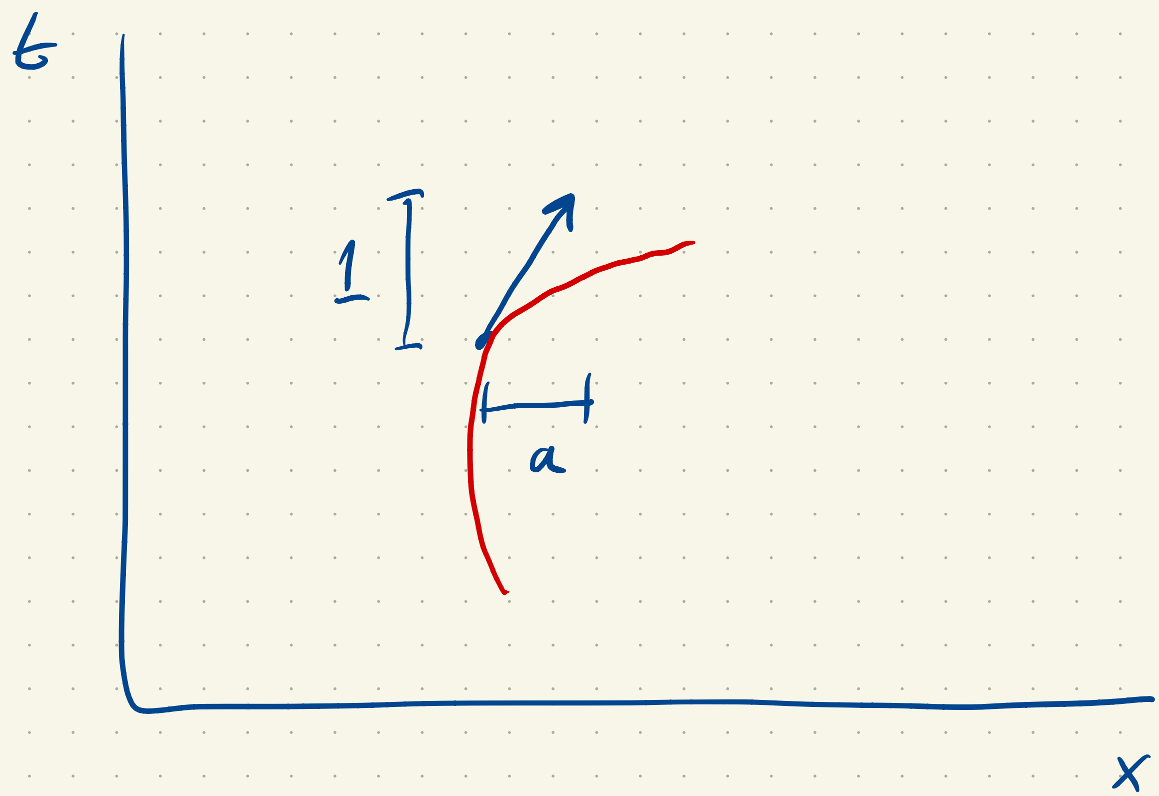
$$\dot{t} = 1$$

$$\dot{x} = a$$

Characteristic curves

$$\dot{t} = 1 \qquad s$$

$$\dot{x} = a$$



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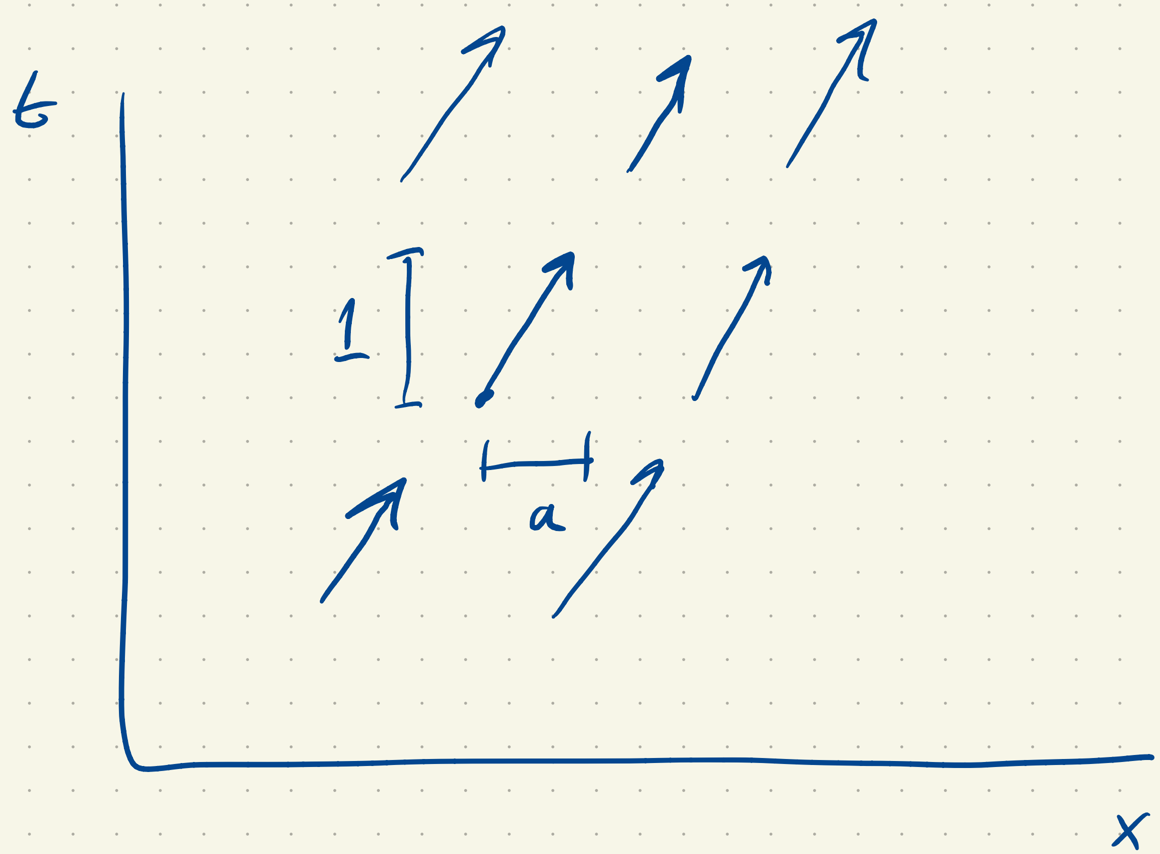

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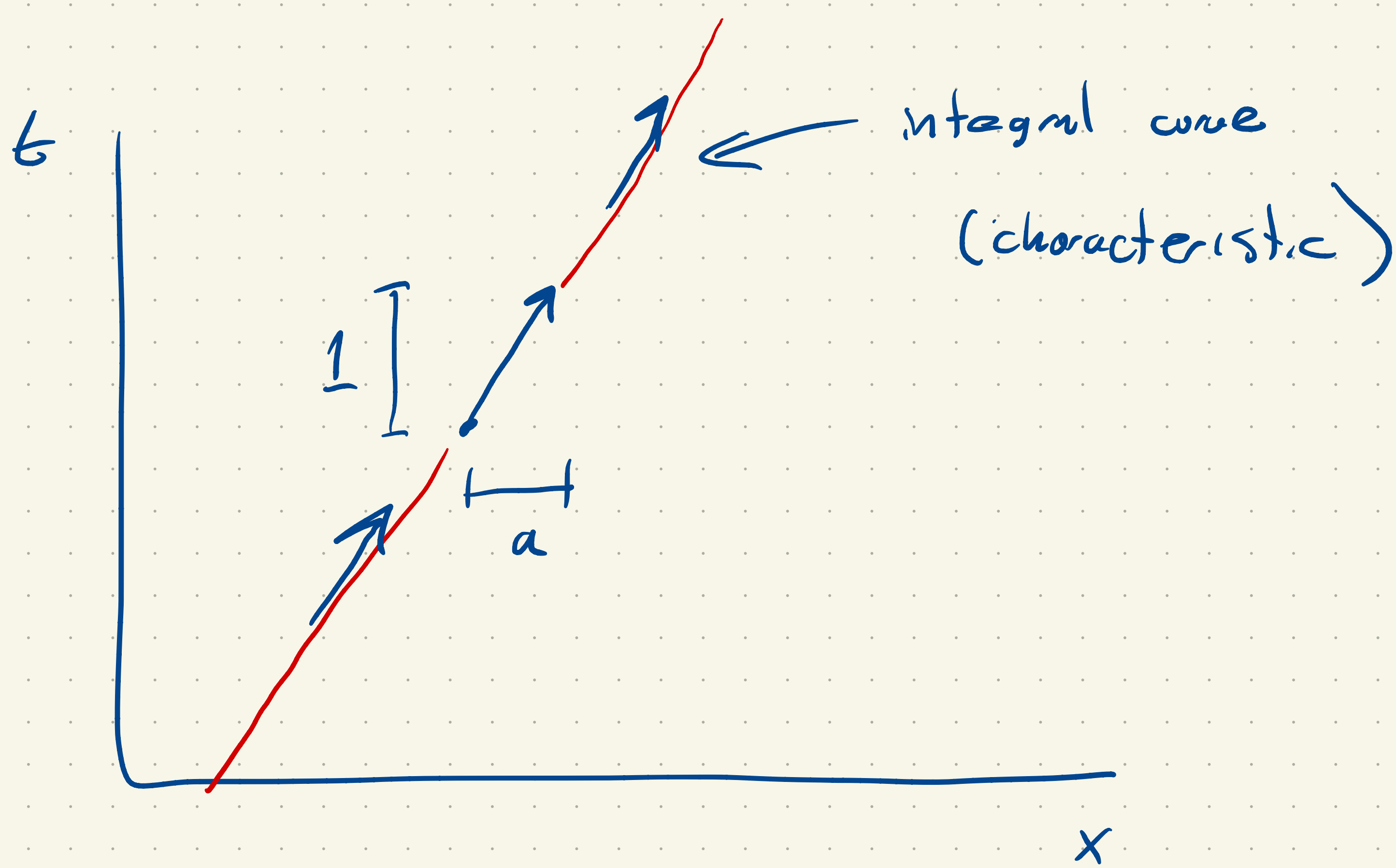

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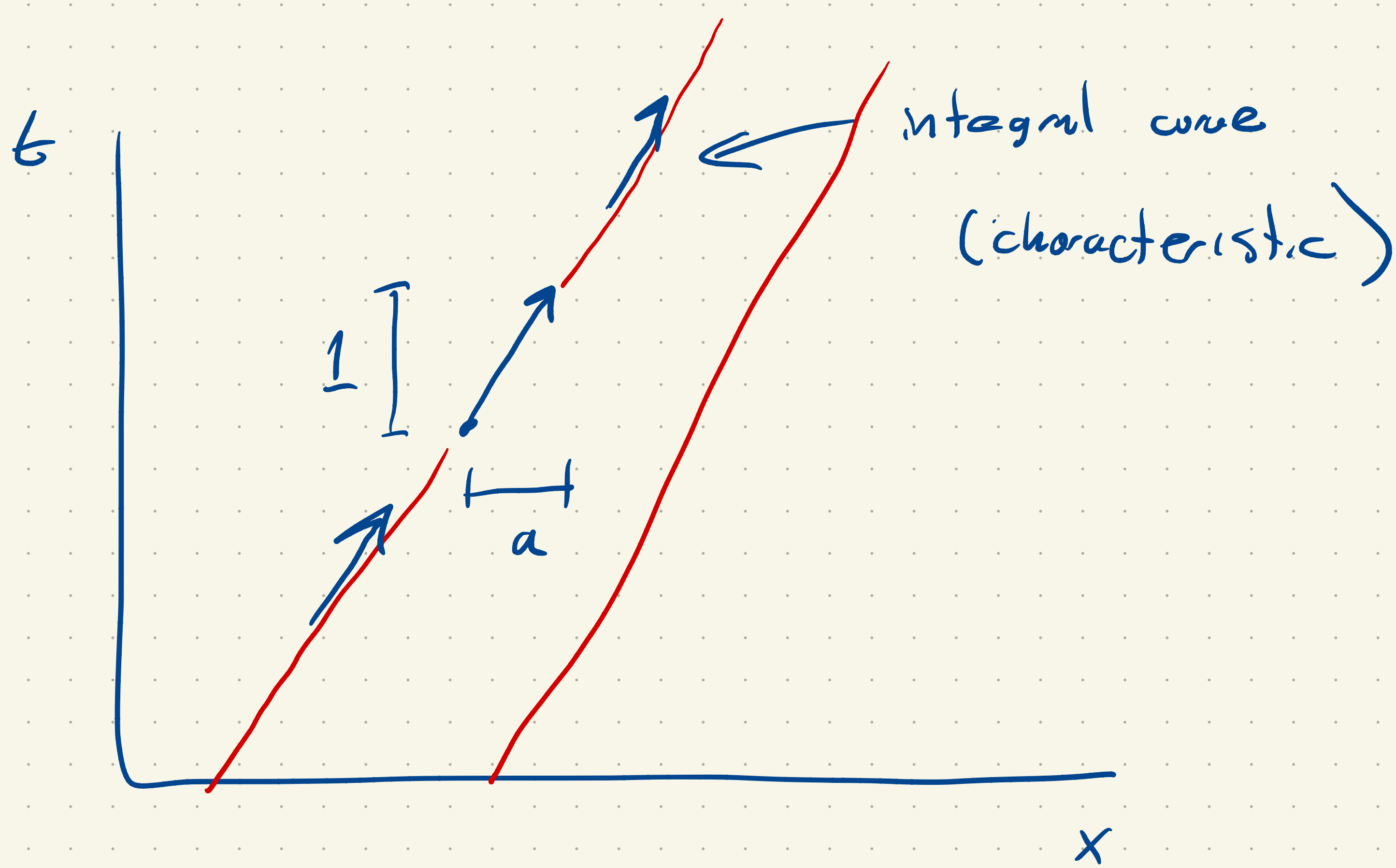


Characteristic curves

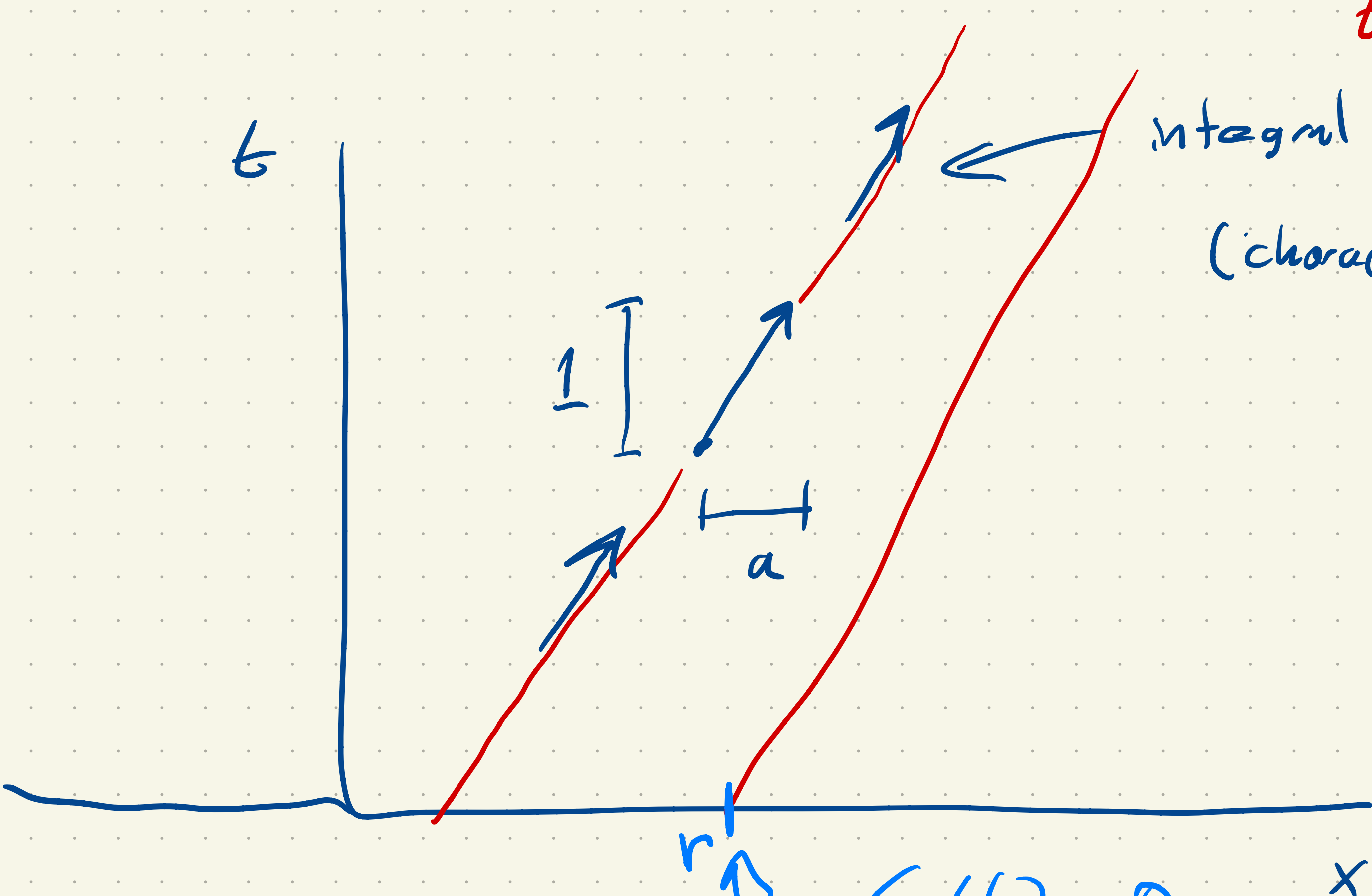
$$\dot{t} = 1$$

$$\dot{x} = a$$

$$t(s) = s$$

$$\frac{dt}{ds} = 1$$

$t = 0$ when $s = 0$



integral curve

(characteristic)

$$x(s) = as + r$$

$$\begin{cases} t(0) = 0 \\ x(0) = r \end{cases}$$

Characteristic curves

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Characteristic curves

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$$t(s) = s$$

$$x(r,s) = as + r$$

Characteristic curves

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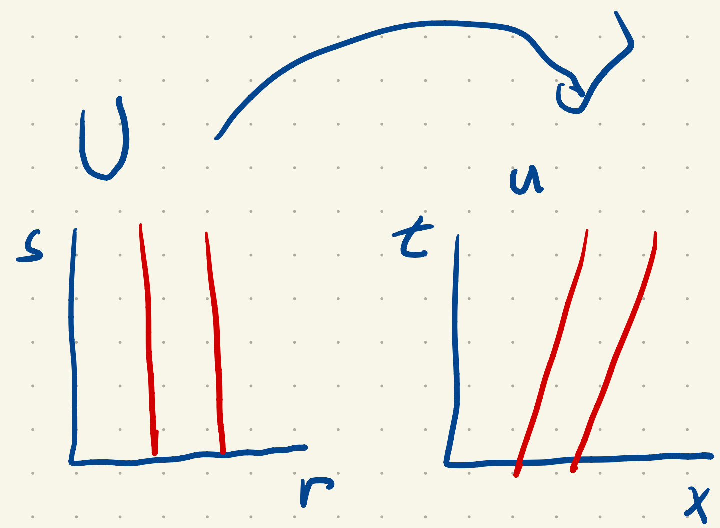
$$x(r,s) = as + r$$

$$U(r,s) = u(x(r,s), s)$$

$$\partial_s U = 0$$



Characteristic curves

$\dot{t} = 1$ $\quad t(0) = 0$

$\dot{x} = a$ $\quad x(0) = r$

$$U(r,s)$$

$$u(x,t)$$

$$t(s) = s$$

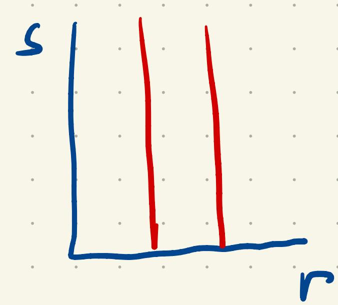
$$x(r,s) = as + r$$

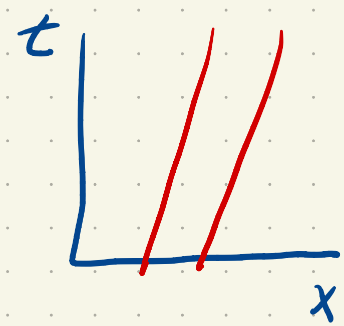
$$U\big(r(x,t), s(x,t)\big) = u(x,t)$$

$$U(r,s) = u\big(x(r,s), s\big)$$

$$\partial_s U = 0$$

$$U(r,s) = u_0(r)$$





$$U(r,s) = u(x(r,s), s)$$

$$\partial_s U = 0$$

what is $u(x,t)$?

$$U(r,s) = u_0(r)$$

$$U(r,s) = u(x(r,s),s)$$

$$\partial_s U = 0$$

$$u(x,t) = U(r(x,t),\ s(x,t))$$

$$U(r,s) = u_0(r)$$

$$= u_0(r(x,t))$$

$$t = s \qquad\qquad s = t$$

$$\Rightarrow$$

$$x = as + r \qquad\qquad r = x - at$$

$$U(r,s) = u(x(r,s), s)$$

$$\partial_s U = 0$$

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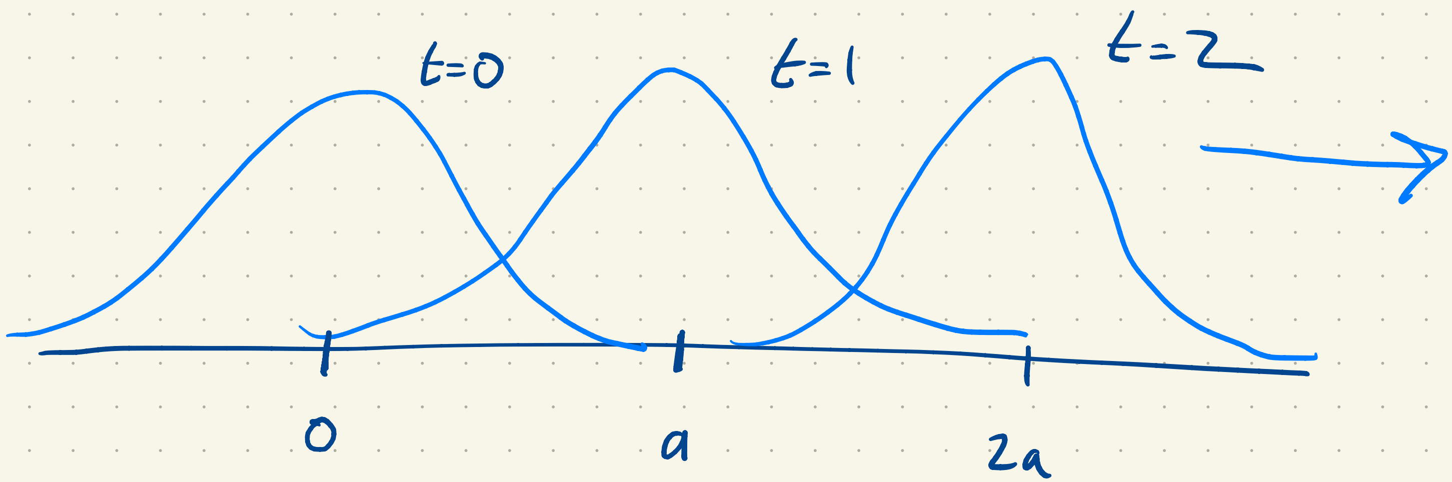
$$u(x,t) = U(r(x,t), t) = u_0(r(x,t))$$

$$= u_0(x - at)$$

$$u(x,0) = u_0(x)$$

Solution is a wave

$$u(x,t) = u_0(x-at)$$



Speed: a

Generalizations

a) $u_t + a\,u_x = f(x,t)$

e.g. $f(x,t) = x$

Generalizations

$$t = s$$
$$x = as + r$$

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$$\underbrace{\partial_s U = f(x(r,s), s) = x(r,s) = as + r}_{\text{For fixed } r \text{ an ODE for } U,\ U(r,0) = U_0(r)}$$

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e.g. $f(x,t) = x$

$$\partial_s U = f(x(r,s),s) = x(r,s) = as + r$$

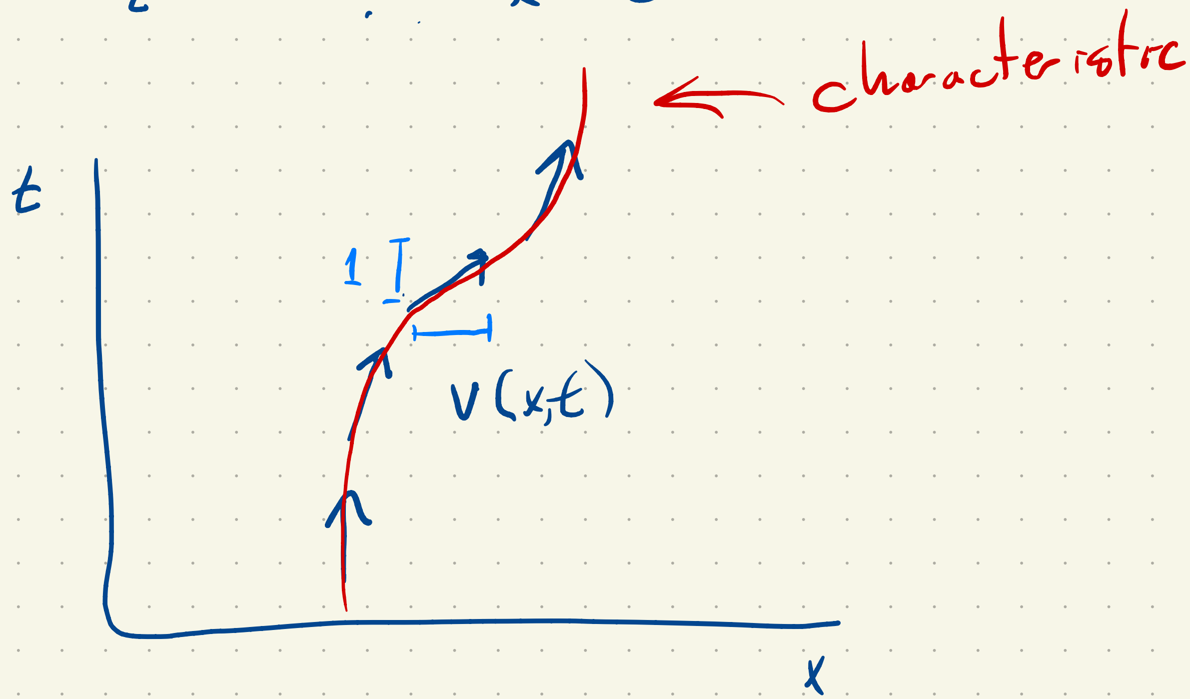
For fixed r an ODE for U, $U(r,0) = u_0(r)$

$$U(r,s) = u_0(r) + \frac{as^2}{2} + rs$$

$$u(x,t) = u_0(x-at) + a\frac{t^2}{2} + (x-at)t$$

Generalizations

b) $u_t + v(x,t)\, u_x = 0$



$\dot{t}(s) = 1 \qquad t(0) = 0$

$\dot{x}(r,s) = v(x(r,s), t(r,s)) \qquad x(r,0) = r$

$$\boxed{\frac{dx}{ds} = v(x, s) \qquad x(r,0) = r}$$

Generalizations

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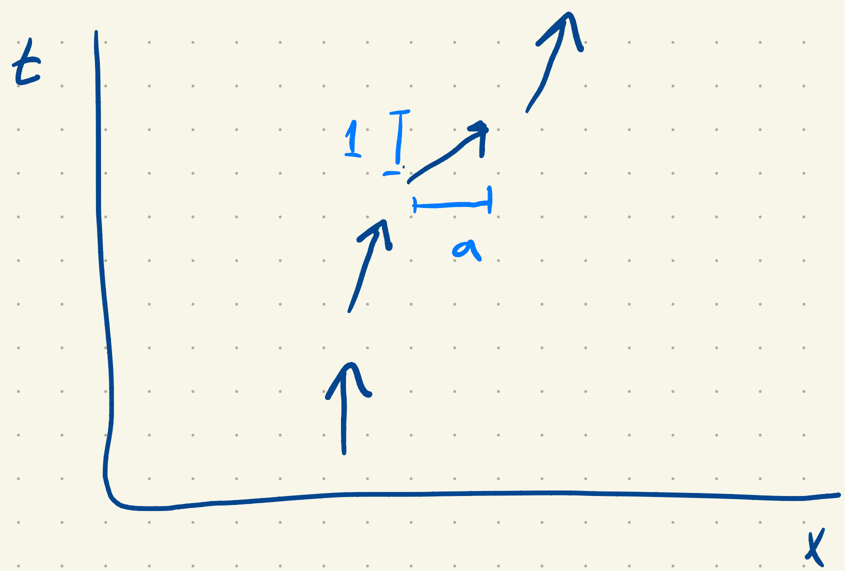

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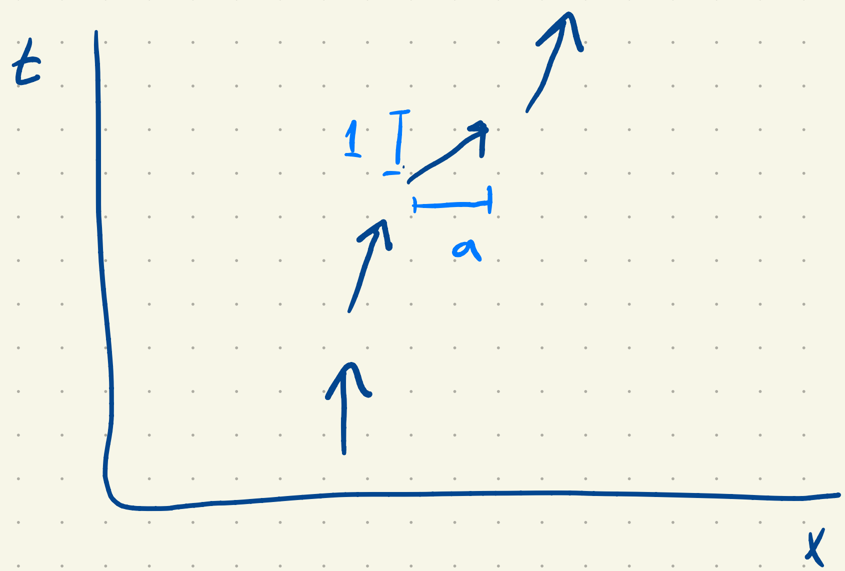

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* solve ODE for $x(r,s)$ $t(s)$

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$\frac{dx}{ds} = v(x, s)$ $\qquad x(r,0) = r$

* solve ODE for $x(r,s)$ $t(s)$

$U(r,s) = u(x(r,s), s)$

$U_s = 0$ $\qquad U(r,s) = u_0(r)$

$$u_x \cdot \frac{dx}{ds} + u_t \cdot \frac{ds}{ds} = v\,u_x + u_t = 0$$

Generalizations

b) $u_t + v(x,t)\, u_x = 0$

$\dot{t}(s) = 1 \qquad t(0) = 0$

$\dot{x}(r,s) = v(x(r,s), t(r,s)) \qquad x(r,0) = r$

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* solve ODE for $x(r,s)$ $t(s)$

$U(r,s) = u(x(r,s), s)$

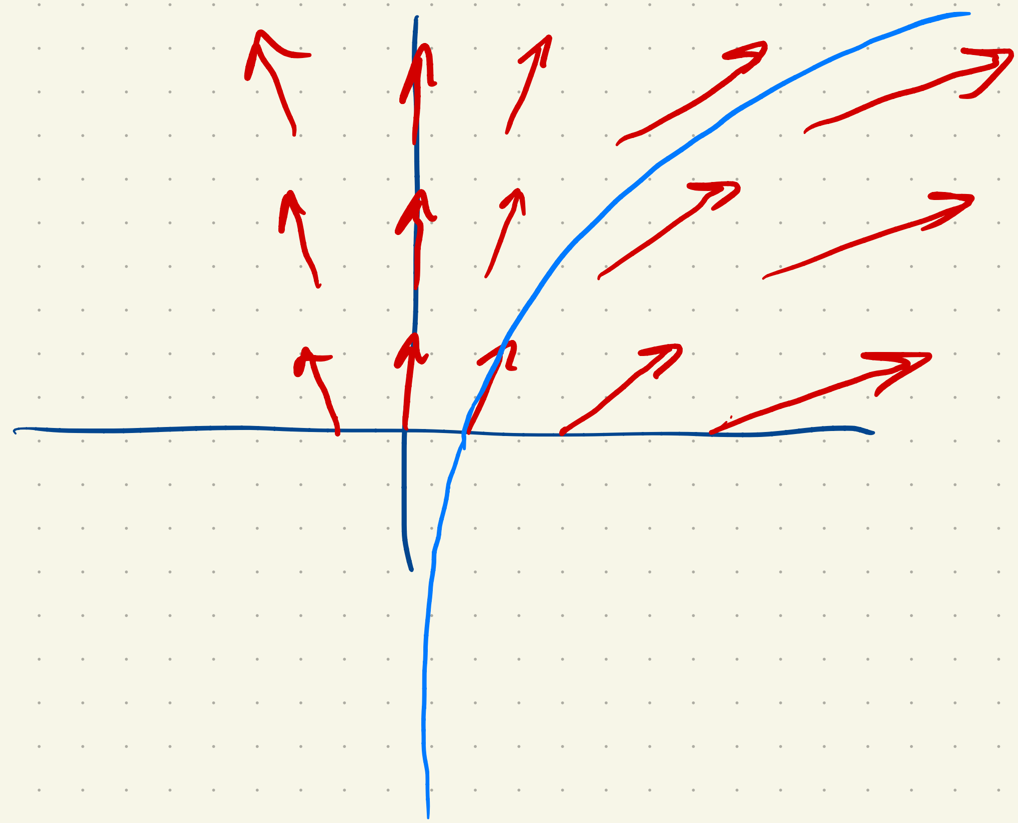
$U_s = 0 \qquad U(r,s) = u_0(r)$

* solve for r as a function of x, t instead!

$u(x,t) = u_0(r(x,t))$

Example

$$u_t + x u_x = 0$$



Example

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$t = s$

$x = re^{s}$

Example

$$u_t + x u_x = 0$$

$\dot{t} = 1 \qquad t(0) = 0$

$\dot{x} = x \qquad x(0) = r$

$t = s \qquad\longrightarrow\qquad s = t$

$x = re^{s} \qquad\qquad r = xe^{-t}$

$x = re^{t}$

$$u(x,t) = u_0(xe^{-t})$$

$$u(x,0) = u_0(xe^{-0}) = u_0(x)$$

Example

$$u_t + x u_x = 0$$

$\dot{t} = 1 \qquad t(0) = 0$

$\dot{x} = x \qquad x(0) = r$

$$\begin{aligned} t &= s \\ x &= re^{s} \end{aligned} \longrightarrow \begin{aligned} s &= t \\ r &= xe^{-t} \end{aligned}$$

$$u(x,t) = u_0(xe^{-t})$$

$u_t = -u_0' \, xe^{-t}$

$u_x = u_0' \, e^{-t}$

$u_t + xu_x = u_0'(-xe^{-t} + xe^{-t})$

$= 0$ ✓

Generalizations

c) $$u_t + \partial_x (v u) = 0$$

$$u_t + v u_x = - v_x u$$

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$x(0) = r$

$U(r,s) = u(x(r,s), s)$

Generalizations

c) $u_t + \partial_x (v u) = 0$

$$u_t + v u_x = -v_x u$$

$$t = s$$

$x(r,s)$: $\dot{x} = v(x,s)$

$x(0) = r$

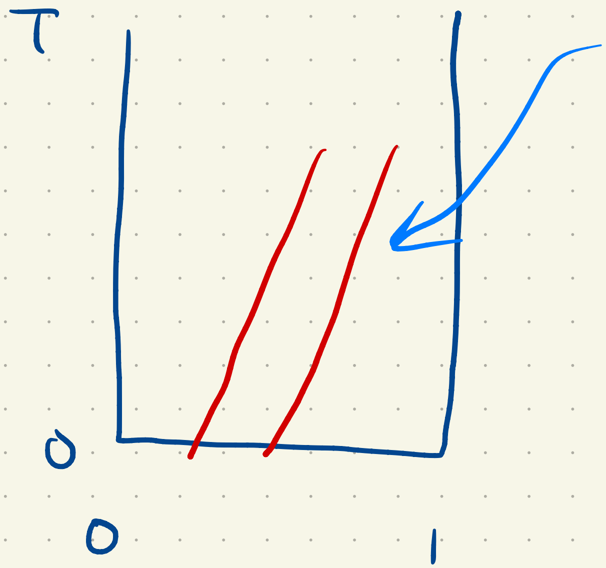
$$U(r,s) = u(x(r,s), s)$$

$$\partial_s U = f(s) U$$

$$\partial_s U = -v_x(x(r,s), s)\, U$$

ODE in s, U for each r.

Model Numerical Problem

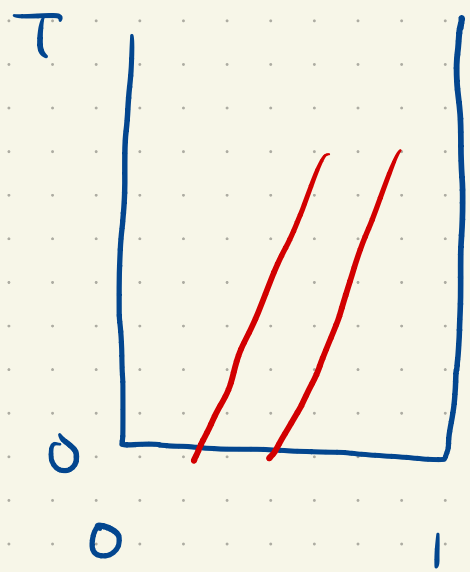


slope $1/a$

$$u_t + a\, u_x = 0$$

$$a > 0$$

Model Numerical Problem



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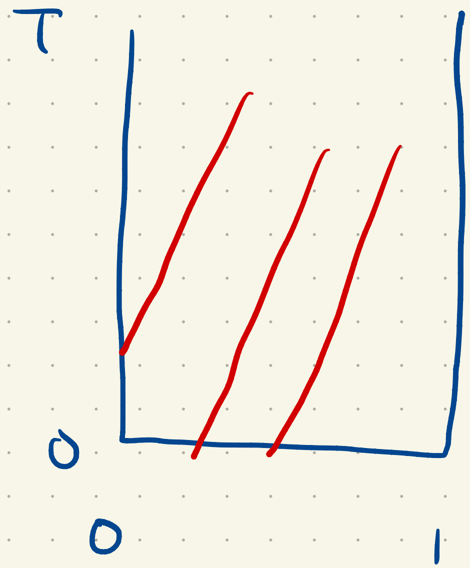

$$u_t + a\, u_x = 0$$

$$a > 0$$

$$u(x,0) = u_0(x)$$

$$u(0,t) = 0$$

Method of Lines

a) Discretize in space first