LTE

$u_t + a u_x$

$$u_t + a u_x = f$$

$$\frac{u_{i,j+1} - u_{i,j}}{k} + a\left[\frac{u_{i,j} - u_{i-1,j}}{h}\right] - f_{i,j} = 0$$

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$$u(x_i, t_{j+1}) = u(x_i, t_j) + u_t(x_i, t_j)k + u_{tt}(x_i, s_{j+1})\frac{k^2}{2}$$

$$u(x_{i-1}, t_j) = u(x_i, t_j) - u_x(x_i, t_j)h + u_{xx}(\xi_i, t_j)h^2/2$$

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$$u(x_{i-1}, t_j) = u(x_i, t_j) - u_x(x_i, t_j)h + u_{xx}(\xi_i, t_j)h^2/2$$

$$\to u_t(x_i, t_j) + a\, u_x(x_i, t_j) + u_{tt}(-)\frac{k}{2} - a\, u_{xx}(-)\frac{h}{2} - f(x_i, t_j)$$

$$\tau = O(k) + O(h) \quad [\text{if } u_{tt}, u_{xx} \text{ exist!}]$$

Proof of convergence:

$$U_{i,j+1} = (1-\lambda) U_{i,j} + \lambda U_{i-1,j}$$

$$u_{i,j+1} = (1-\lambda) u_{i,j} + \lambda u_{i-1,j} + k\tau_{i,j}$$

Proof of convergence:

$$1-\lambda+\lambda=1$$

$$|1-\lambda|+|\lambda|\neq 1$$

$$U_{i,j+1}=(1-\lambda)U_{i,j}+\lambda U_{i-1,j}$$

$$|E_{i,j}|\leq S$$

$$u_{i,j+1}=(1-\lambda)u_{i,j}+\lambda u_{i-1,j}+k\tau_{i,j}$$

$$|E_{i-1,j}|\leq S$$

$$E_{i,j+1}=(1-\lambda)E_{i,j}+\lambda E_{i-1,j}+k\tau_{i,j}$$

$$|E_{i,j+1}|\leq S$$

$$|E_{i,j+1}|=|\underline{(1-\lambda)}E_{i,j}|+|\underline{\lambda}E_{i-1,j}|+k|\tau_{i,j}|$$

Proof of convergence:

$$U_{i,j+1} = (1-\lambda)U_{i,j} + \lambda U_{i-1,j} \qquad -1 \le \lambda \le 1$$

$$u_{i,j+1} = (1-\lambda)u_{i,j} + \lambda u_{i-1,j} + k\tau_{i,j}$$

$0 \le \lambda \le 1$!!

$$E_{i,j+1} = (1-\lambda)E_{i,j} + \lambda E_{i-1,j} - k\tau_{i,j}$$

$$|E_{i,j+1}| = |(1-\lambda)E_{i,j}| + |\lambda E_{i-1,j}| + k|\tau_{i,j}|$$

$$\le \underbrace{|(1-\lambda)|}_{\le 2}\,|E_{i,j}| + \underbrace{|\lambda|}_{\le 1}\,|E_{i-1,j}| + k|\tau_{i,j}|$$

$$\max_i |E_{i,j+1}| \le \max_i |E_{i,j}| + k\tau \longrightarrow \max |\tau_{i,j}|$$

$$\|E_{j+1}\|_\infty \le \|E_j\|_\infty + k\tau$$

1 here is critical!

$\geq$

$a \geq 0$ but not const.

$0 \le \lambda \le 1$

needed for stability

$$\max_i |E_{i,j+1}| \leq \max_i |E_{i,j}| + k\tau \quad \max|\tau_{i,j}|$$

$$\|E_{j+1}\|_\infty \leq \|E_j\|_\infty + k\tau \qquad h, k \to 0$$

1 here is critical!

What if 1.1?

$$\max_i |E_{i,j+1}| \le \max_i |E_{i,j}| + k\tau \to \max|\tau_{i,j}|$$

$$\|E_{j+1}\|_\infty \le \|E_j\|_\infty + k\tau$$

$\uparrow$

$\underline{1}$ here is critical!

$h, k \to 0$

What if 1.1?

$$\|E_1\| \le (1.1)\|E_0\| + k\tau$$

M

$$\|E_M\| \le (1.1)^M \underbrace{\|E_0\|}_{7}$$

$$\|E_2\| \le (1.1)\|E_1\| + k\tau$$

$$\le (1.1)^2\|E_0\| + (1.1)k\tau + k\tau$$

$$\|E_j\| \le (1.1)^j\|E_0\| + \left[\sum_{s=}^{j-1} (1.1)^s\right] k\tau$$

$$\|E_{j+1}\| \le \|E_j\| + k\tau$$

$$\|E_1\| \le \|E_0\| + k\tau$$

$$\max_j \|E_j\| \le \|E_0\| + T\tau$$

$$\|E_2\| \le \|E_1\| + k\tau$$

$$\le \|E_0\| + 2k\tau$$

$$\to 0$$

$$O(h) + O(k)$$

$$\|E_m\| \le \|E_0\| + \underbrace{Mk}_{T}\tau$$

$$\left(\sum_{s=0}^{j-1} r^s\right)(1-r) = 1 - r^j$$

$$\frac{1-(1.1)^j}{1-(1.1)}k = \frac{(1.1)^j - 1}{1.1-1} \cdot \frac{T}{M}$$

$0 \cdot \infty$ $$\frac{(1.1)^M}{M} \longrightarrow \infty$$

Instead: $\|E_j\|_\infty \leq \|E_0\|_\infty + Mk\tau = \|E_0\|_\infty + T\tau$

Alternative Perspective on CFL

$0 \le \lambda \le 1$ μ

(Absolute Stability!)

$-1 \le \lambda \le 1$

We applied Forward Euler to

$$\vec{u}' = -\frac{a}{h} D u \qquad D = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \end{bmatrix}$$

So what are the eigenvalues of $-\frac{a}{h} D$?

Do they lie in region of abs stab?