Last class $\|\cdot\|_{2,\infty}$ convergence assumption

$$\|B^{-1}\|_2 \leq 1$$

$$\|B^{-1}A\|_2 \leq 1$$

For a symmetric matrix

$$\|A\|_2 = \sigma(A) = \max(|\lambda_1|, \ldots, |\lambda_n|)$$

spectral radius

$$B\vec{u}_{j+1} = A\vec{u}_j + \vec{f}_j$$

Last class $\|\cdot\|_{2,\infty}$ convergence assuming

$$\|B^{-1}\|_2 \le 1 \quad \longleftarrow \quad \text{always}$$

$$\|B^{-1}A\|_2 \le 1 \quad \longleftarrow \quad (2\theta-1)\lambda \le \frac{1}{2}$$

For a symmetric matrix

$$\|A\|_2 = \sigma(A) = \max(|\lambda_1|, \dots, |\lambda_n|)$$

spectral radius

Back to our claims:

$$\|B^{-1}\|_2 \leq 1$$

$$\|B^{-1}A\|_2 \leq 1$$

a) Compute eigenvectors/values; show $|\lambda_i| \leq 1$ if $\lambda(2\theta-1) \leq 1/2$

b) show are symmetric

eigenvectors $v_1, \ldots, v_n$ of D

$$Dv_j = \lambda_j v_j$$

$$\lambda_j = -4 \sin^2\left(\frac{rh}{2}\right) \qquad r = j\pi$$

A: $I + \theta\lambda D$

$$Dv = Kv$$

$$(I + D)\vec{v} = I\vec{v} + D\vec{v}$$
$$= \vec{v} + K\vec{v}$$
$$= (1 + K)\vec{v}$$

eigenvectors $v_1, \dots, v_n$ of D

$$D v_j = \lambda_j v_j$$

$$\lambda_j = -4 \sin^2\left(\frac{rh}{2}\right) \qquad r = j\pi$$

A: $\quad I + \theta\lambda D \qquad\qquad 1 - 4\theta\lambda \sin^2(rh/2)$

eigenvectors $v_1, \dots, v_n$ of D

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A: $\quad I + \theta\lambda D \qquad\qquad 1 - 4\theta\lambda \sin^2(rh/2)$

B: $\quad I - (1-\theta)\lambda D$

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B: $\quad I - (1-\theta)\lambda D \qquad 1 + 4(1-\theta)\lambda \sin^2(rh/2)$

B^{-1}: $\quad \left(I - (1-\theta)\lambda D\right)^{-1} \qquad B^{-1} B v = v$

$$B^{-1} \alpha v = v \qquad B^{-1} v = \frac{1}{\alpha} v$$

$B^{-1}A$

eigenvectors $v_1, \dots, v_n$ of D

$$Dv_j = \lambda_j v_j$$

$$\lambda_j = -4\sin^2\left(\frac{rh}{2}\right) \qquad r = j\pi$$

A: $\quad I + \theta\lambda D \qquad 1 - 4\theta\lambda\sin^2(rh/2)$

B: $\quad I - (1-\theta)\lambda D \qquad 1 + 4(1-\theta)\lambda\sin^2(rh/2)$

B^{-1}: $\quad (I - (1-\theta)\lambda D)^{-1} \qquad (1 + 4(1-\theta)\lambda\sin^2(rh/2))^{-1}$

B^{-1} : eigenvalues: $\dfrac{1}{1+4(1-\theta)\lambda\sin^2(rh/2)}$

$$\left|\frac{1}{1+4(1-\theta)\lambda\sin^2(rh/2)}\right| \leq 1 \quad \checkmark$$

$$\sigma(B^{-1}) \leq 1$$

If B^{-1} is symmetric, $\|B^{-1}\|_2 \leq 1$.

$B^{-1}A$: eigenvalues $\dfrac{1 - 4\theta\lambda \sin^2(\nu h/2)}{1 + 4(1-\theta)\lambda \sin^2(\nu h/2)}$

$\|B^{-1}A\|_2 = \sigma(B^{-1}A)$

≤ 1

Want $-1 \leq \uparrow \leq 1$

$$-1 - 4(1-\theta)\lambda \overbrace{\sin^2\left(\frac{\nu h}{2}\right)}^{S} \leq 1 - 4\theta\lambda \overbrace{\sin^2\left(\frac{\nu h}{2}\right)}^{S}$$

$$-\quad (1-\theta)\lambda S + \theta\lambda S \leq \frac{1}{2}$$

$$(2\theta - 1)\lambda S \leq 1/2 \longrightarrow \boxed{(2\theta-1)\lambda \leq \frac{1}{2}}$$

$B^{-1}A$: eigenvalues $$\frac{1 - 4\theta\lambda \sin^2(rh/2)}{1 + 4(1-\theta)\lambda \sin^2(rh/2)}$$

Want $-1 \le \uparrow \le 1$

$$-\left(1 + 4(1-\theta)\lambda \sin^2\left(\frac{rh}{2}\right)\right) \le 1 - 4\theta\lambda \sin^2(rh/2)$$

$$(2\theta - 1)\lambda \le \frac{1}{2}$$

Last: Are they symmetric?

$A^T = A$

$I^T = I$

$I + \lambda \theta D$

Obviously A, B are symmetric

Claim: B^{-1} is symmetric, as is $B^{-1}A$

$$(B^{-1})^T = (B^T)^{-1} = B^{-1} \checkmark$$

$$(B^{-1})^T = (B^T)^{-1} = B^{-1} \checkmark$$

$$(B^{-1}A)^T$$

$$= A^T (B^{-1})^T$$

$$= AB^{-1}$$

$$(B^{-1}A)^T = (AB^{-1})^T = (B^{-1})^T A^T$$

$$= B^{-1}A$$

B^{-1}, A have a common basis of eigenvectors!

Crank-Nicholson

$$\theta = \frac{1}{2}$$

$$\tau_{i,j} = O(h^2) + O(k^2)$$

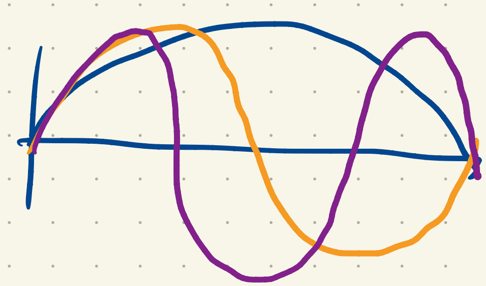
Convergence in $\|\cdot\|_{2,\infty}$ $\forall$ choices of h, k

What's not to like?

DEMO

To understand this, need to understand Fourier series better.

$$\int_0^1 \sin(k\pi x)\sin(j\pi x)\,dx = \begin{cases} 1/2 & k = j \\ 0 & k \neq j \end{cases} \qquad k, j \in \mathbb{N}$$



$$e_k(x) = \sqrt{2}\sin(k\pi x)$$

$$\int_0^1 e_k(x)\,e_j(x)\,dx = \delta_{jk}$$

$\sqrt{2}\sin(k\pi x)$

Suppose $u_0 = \sum_{k=1}^{n} b_k e_k(x)$

$$u(t,x) = \sum_{k=1}^{n} b_k e^{-k^2\pi^2 t} e_k(x)$$

$u_t = u_{xx}$

$u|_{x=0} = u|_{x=1} = 0$

$$u|_{t=0} = \sum_{k=1}^{n} b_k e_k(x).$$

Suppose $u_0 = \sum_{k=1}^{n} b_k e_k(x)$.

If I don't tell you b_k, can you compute it anyway?

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If I don't tell you b_k, can you compute it anyway?

$$\int_0^1 u_0(x) e_j(x)\,dx = \sum_{k=1}^{n} \int_0^1 b_k u_k(x)\, e_j(x)\,dx$$

$$= \sum_{k=1}^{n} b_k \delta_{kj} = b_j \qquad (!)$$

What if $u_0(x)$ is merely Riemann integrable?

We can still define

$$b_k = \int_0^1 u_0(x)\, e_k(x)\, dx$$ $\to$ Fourier sine coeffs

$\sqrt{2}\, b_k = c_k$

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$$

What if $u_0(x)$ is merely Riemann integrable?

We can still define

$$b_k = \int_0^1 u_0(x)\, e_k(x)\, dx$$ → Fourier sine coeffs.

$\sqrt{2}\, b_k = c_k$

To what extent is

$$u_0(x) = \sum_{k=1}^{\infty} b_k e_k(x)$$

what does this mean?