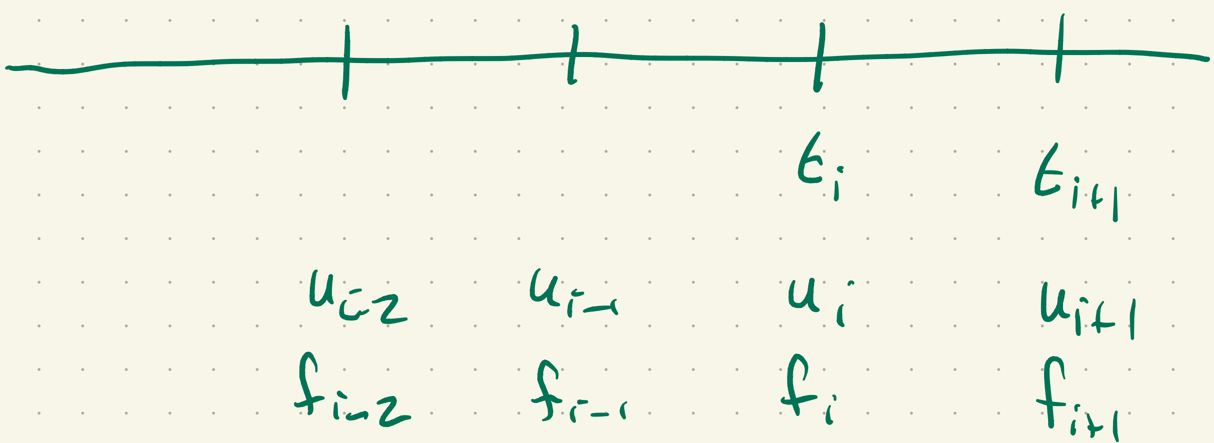
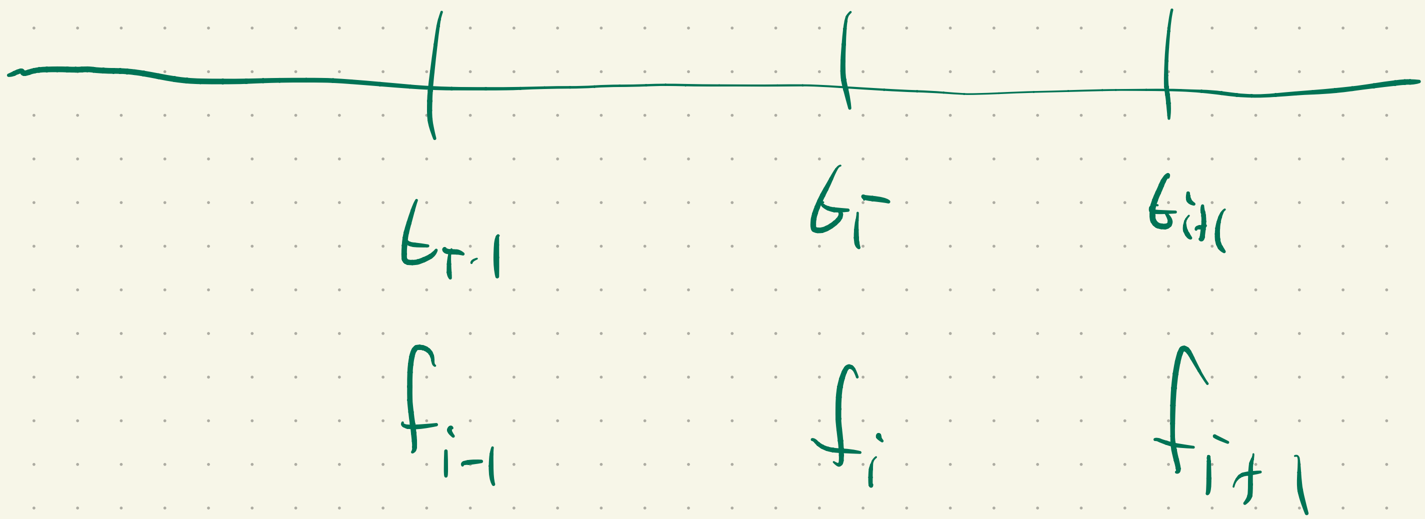
$$u_{i+1} \equiv u_i + \text{approx of } \int_{t_i}^{t_{i+1}} f(s,u)\,ds$$

Adams - Bashforth

Adams - Moulton



$$f_i = f(t_i, u_i)$$



All methods so far:

$$\alpha_k u_{i+k} + \alpha_{k-1} u_{i+k-1} + \cdots + \alpha_1 u_{i+1} + \alpha_0 u_i$$

$$= h \left[\beta_k f_{i+k} + \cdots + \beta_0 f_{i+0} \right]$$

$$u_{i+1} - u_i = h f_i \longrightarrow EM$$

$$u_{i+1} - u_i = h f_{i+1}$$

These are called Linear Multistep Methods. (LMM)

Single step: $k = 1$

Explicit: $\beta_k = 0$

Notions of stability for LMM.

a) zero stability

b) asymptotic stability (cousin A stability text focuses on this)

Def: A k step LMM is zero stable if

for a RHS $f = 0$ there exists a

constant K such that for any initial

indedendat of u / h

data $u_0, u_1, \dots, u_{k-1}$,

$$|u_i| \le K \max(|u_0|, \dots, |u_{k-1}|)$$

$$u' = \underbrace{f(t,u)}_{0}$$

$u' = 0$

$\hookrightarrow$ constant solutions

they should not grow in time.

Is Euler's method 0-stable?

$$u_{i+1} - u_i = \underbrace{h f_i}_{\hookrightarrow 0}$$

$$u_{i+1} - u_i = 0$$

↑ linear recurrence relation.

Guess a solution of $C\rho^i = u_i$

$$C\rho^{i+1} - C\rho^i = 0$$

$$\underbrace{C\rho^i}_{\text{irrelevant}}(\rho - 1) = 0$$

$\rho = 1$ $\quad u_i = C\,1^i = C$

$= u_0$

$|u_i| = |u_0|$ $\quad \forall i$, ind. of h/M

$$|u_i| \leq \frac{1}{5} |u_0|$$

$$|u_i| \leq 1 \cdot |u_0|$$

$$|u_i| \leq 5 \cdot |u_0|$$

Leapfrog:

$$u_{k+2} - u_k = \underbrace{2h\, f_{k+1}}_{=0}$$

$$u_{k+2} - u_k = 0$$

$$u_k = C\rho^k \qquad \left(C e^{\lambda t} \right)$$

$$C\rho^{k+2} - C\rho^k = 0$$

$$C\rho^k \underbrace{\left[\rho^2 - 1 \right]}_{} = 0$$

$$\rho = \pm 1$$

$$C \cdot 1^k, \qquad C(-1)^k$$

General solution: $A(1)^k + B(-1)^k$

$$u_k = A(1)^k + B(-1)^k$$

$$\left.\begin{aligned} u_0 &= A + B \\ u_1 &= A - B \end{aligned}\right] \longrightarrow \quad A = \frac{u_0 + u_1}{2} \qquad B = \frac{u_0 - u_1}{2}$$

$$|A| \le \max(|u_0|, |u_1|)$$

$$|B| \le \max(|u_0|, |u_1|)$$

$$|u_k| \le \max(|u_0|, |u_1|) + \max(|u_0|, |u_1|)$$

we can take $K = 2$ and the method is 0-stable

Def: The characteristic polynomial of a LMM is

$$\alpha_k \rho^k + \alpha_{k-1} \rho^{k-1} + \cdots + \alpha_1 \rho + \alpha_0$$

Thm A LMM is zero stable if and only if

1) All roots of the char. polynomial satisfy $|\rho| \leq 1$

2) All roots with $|\rho| = 1$ are

simple (i.e. not repeated) roots.

$\rho^2(\rho-1)$

$\hookrightarrow$ 0 is a repeated root.

If some ρ with $|\rho|>1$ is a root of the char. poly then

$$u_k = C\rho^k \text{ is a solution}$$

of recurrence relation and $|u_k| = C|\rho|^k \to \infty$.

$$u'' - 2u' + u = 0$$

$$(\lambda - 1)^2 = 0$$

e^t is a solution

te^t

$$u_{k+2} - 2u_{k+1} + u_k = 0$$

$(\rho - 1)^2 \rightarrow$ char. poly.

$\rho = 1$ only root

$C \cdot 1^k$ solutions

$C k \cdot 1^k$ another solution

$$C(k+2)(1)^{k+2} - 2C(k+1)(1)^{k+1} + Ck \cdot 1^k$$

$$= C\left[k+2-2k-2+k\right] = 0 \checkmark$$

If ρ is a repeated root

$k \rho^k$ another solution.

$$|k\rho^k| = k|\rho|^k$$

If $|\zeta| < 1$ $\quad k|\zeta|^k \to 0$

If $|\zeta| = 1$ $\quad \underline{k|\zeta|^k} \to \underline{\underline{\infty}}$

Thm: (Dahlquist)

A consistent k-step LMM is convergent if and only if it is zero stable. Moreover, if the LTE is $O(h^p)$ and if initial data are chosen so $|u_i - u(t_0 + ih)|$ are $O(h^p)$

then the solution error is $O(h^p)$.