Alternative Perspective on CFL

(Absolute Stability!)

$0 \leq \lambda \leq 1$ $\quad \frac{ak}{h}$

$-1 \leq \lambda \leq 1$

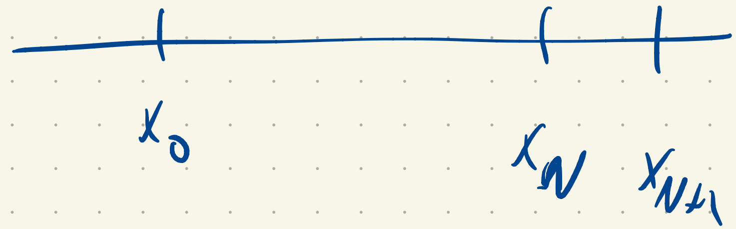
We applied Forward Euler to

$$\vec{u}\,' = -\frac{a}{h} D \vec{u} \qquad D = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \end{bmatrix}$$

So what are the eigenvalues of $-\frac{a}{h} D$?

Do they lie in region of abs stab?

Periodic:



$$D_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

Try eigenvector of form $v_j = e^{I r x_j}$

Periodic:

$$D_p \vec{v} = \lambda \vec{v}$$

$$D_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

Try eigenvector of form $v_j = e^{Irx_j}$

$$v_j - v_{j-1} = e^{Irx_j} - \underbrace{e^{Irx_{j-1}}}_{x_j - h} = e^{Irx_j}\left[1 - e^{-Irh}\right]$$

Periodic:

$$D_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

Try eigenvector of form $v_j = e^{Irx_j}$

$$v_j - v_{j-1} = e^{Irx_j} - e^{Irx_{j-1}} = e^{Irx_j}\left[1 - e^{-Irh}\right]$$

$$v_j - v_{j-1} = K v_j \Rightarrow 1 - e^{-Irh} = K$$

$$v_0 - v_N = K v_0 \Rightarrow 1 - e^{+Irl} = K = 1 - e^{-Irh}$$

$$l = x_N - x_0 = x_N$$

$$v_i = e^{Irx_i}$$

$$v_0 - v_N = K v_0 \Rightarrow \underline{\underline{1 - e^{+Irl}}} = K = \underline{\underline{1 - e^{-Irh}}}$$

$$l = x_N - x_0 = x_N$$

So $$e^{Ir(l+h)} = 1$$

$$r = \frac{2\pi}{\underbrace{l+h}_{L}} n \qquad 0 \leq n < N$$

$$v_0 - v_N = k v_0 \Rightarrow 1 - e^{+Irl} = K = 1 - e^{-Irh}$$

So $e^{I r(l+h)} = 1$

$$r = \frac{2\pi}{\underbrace{l+h}_{L}} n \qquad 0 \leq n < N$$

eigenvalues: $1 - e^{-Irh}$ $\qquad r = \frac{2\pi}{L} n$

$$1 - e^{-I\theta_n} \qquad \theta = rh = 2\pi \frac{h}{L} n$$

$$-\frac{a}{h} D \rightarrow \text{eigenvalues}$$

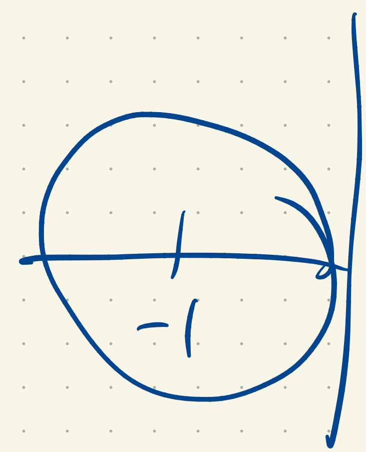
$$\underbrace{-\frac{a}{h}\left[1-e^{-Irh}\right]}_{\eta_n} \qquad r = \frac{2\pi}{L} n$$

$-\frac{a}{h} D \rightarrow$ eigenvalues

$$\underbrace{-\frac{a}{h}\left[1 - e^{-Irh}\right]}_{\eta_n} \qquad r = \frac{2\pi}{L} n$$

Need $z = k\eta_n$ to lie in region of absolute stability for all n

$$|-1 + z| \leq 1$$



$$|-1+z|^2 \leq 1^2$$

$$(z-1)(\bar{z}-1) \leq 1$$

$$|z|^2 - 2\mathrm{Re}\, z \leq 0$$

$$z = -\frac{ak}{h}\left(1 - e^{-I\theta_n}\right) = -\lambda\left(1 - e^{-I\theta_n}\right)$$

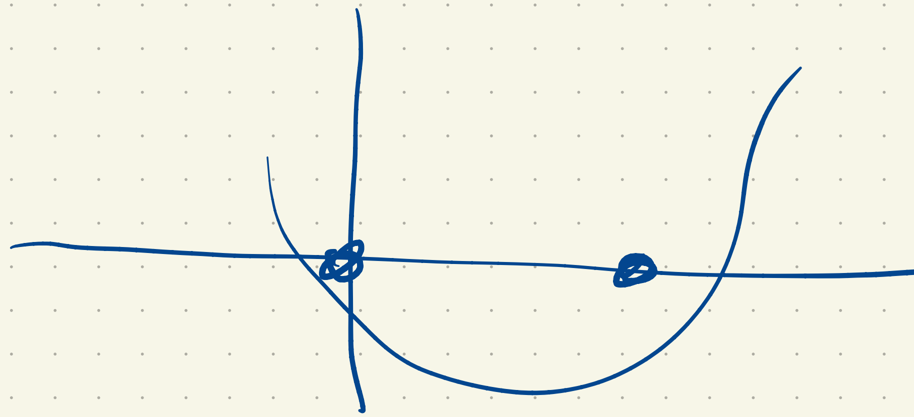
$$|z|^2 = \lambda^2\, 2\left(1 - \cos\theta_n\right)$$

$$\mathrm{Re}\, z = -\lambda\left(1 - \cos\theta_n\right)$$

$$\left[2\lambda^2 - 2\lambda\right](1-\cos\theta_n) \le 0$$

??

$$\lambda^2 - \lambda \le 0$$



$$0 \le \lambda \le 1$$

[Exactly the CFL condition!]

Alt:

$$u' = -\frac{a}{h} D$$

$$\vec{u}_{j+1} = B\vec{u}_j \qquad B = I - \lambda D$$

eigenvalues: $1 - \lambda\left(1 - e^{I\theta_n}\right)$ for some θ_n

$$1 + z$$

same analysis: $\left|1+z\right| \le 1$ if $0 \le \lambda \le 1$

Evolution of Eigenvectors

$$K = 1 - e^{-I\theta_n} = R e^{I\phi}$$

$$K e^{Irx} = R e^{I(rx + \phi)}$$

$$= R e^{Ir(x + \phi/r)}$$

Evolution of Eigenvectors

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a) scale by R

b) translate by $-\phi/r$

Evolution of Eigenvectors

$$B = I - \lambda D$$

$$u_{j+1} \rightarrow (1 - \lambda K)\, u_j \qquad e^{-Irx_i} = v_i$$

$$1 - \lambda K = R e^{I\phi}$$

$$R e^{I\phi} e^{-Irx_i} = R e^{-Irx_i + I\phi} = R e^{-Ir\left(x_i - \frac{\phi}{r}\right)}$$

a) scale by R — should be 1

b) translate by ϕ/r — should be ak

$$1 - \lambda k = 1 - \lambda\left(1 - e^{-I\theta}\right)$$

$$= (1-\lambda) + \lambda e^{-I\theta}$$

$$|1 - \lambda k|^2 = (1-\lambda)^2 + \lambda(1-\lambda)\left(e^{-I\theta} + e^{I\theta}\right) + \lambda^2$$

$$= (1-\lambda)^2 + 2\lambda(1-\lambda)\cos\theta + \lambda^2$$

$$= 1 - 2\lambda + 2\lambda^2 + 2\lambda(1-\lambda)\cos\theta$$

$$= 1 - 2\lambda(1-\lambda)\left[1 - \cos\theta\right]$$

$$= 1 - 4\lambda(1-\lambda)\sin^2(\theta/2)$$

$$R = \sqrt{1 - 4\lambda(1-\lambda)\sin^2(\theta/2)}$$

$0 \le \lambda \le 1$

$$\simeq 1 - 2\lambda(1-\lambda)\sin^2\left(\frac{\theta}{2}\right)$$

θ small

$\theta = -2\pi \frac{h}{L} n$

Damping by an $O(h^2)$ term.

$a = 1$

$N = M$

$N = 40 \quad M = 200$

LTE

Lax Wendroff

$O(h) + O(k)$

$O(h^2) + O(k^2)$

$u_t + a u = 0$