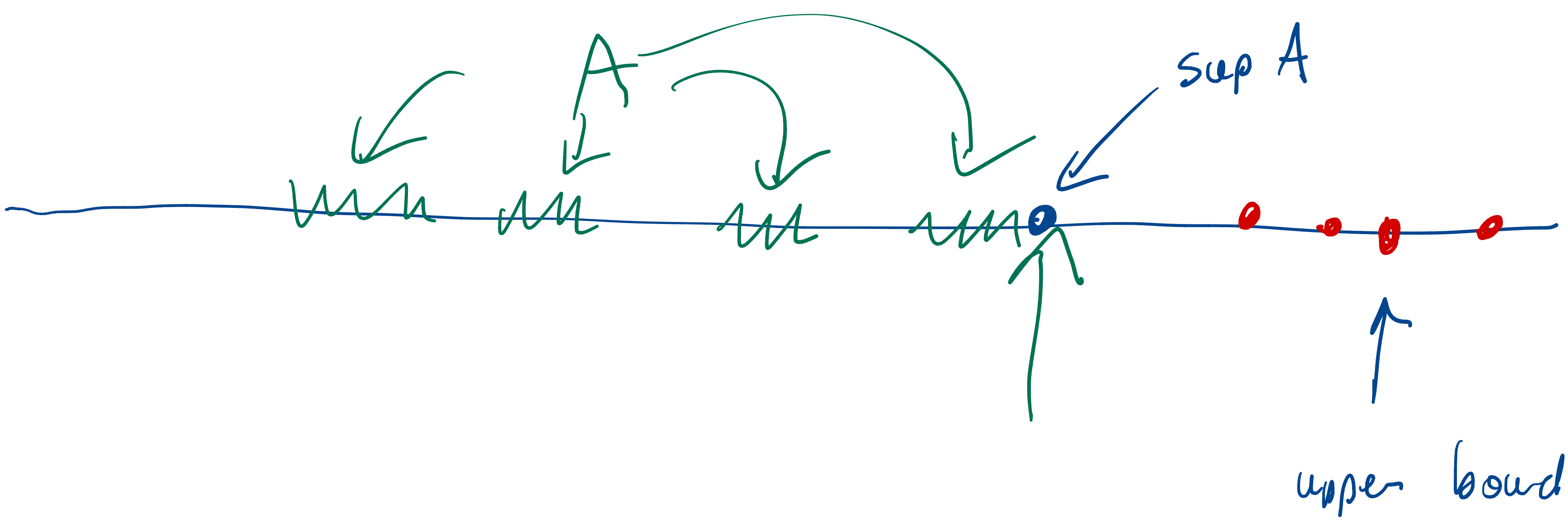
Def: Let $A \subseteq \mathbb{R}$. We say $x \in \mathbb{R}$ is a <u>supremum</u> or <u>least upper bound</u> for A if

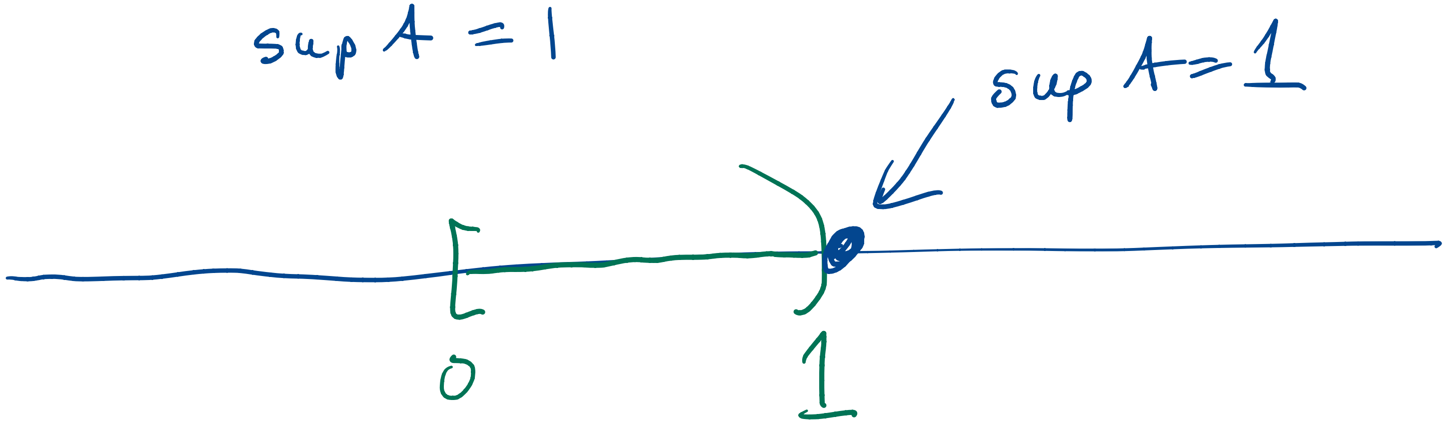
① x is an upper bound for A

② Whenever y is some upper bound for A, $x \le y$.



Eg: $A = [0, 1)$

$\sup A = 1$



To show $1 = \sup A$ we need to show two things

(1) Need to show 1 is an upper bound.

Recall $[0,1) = \{x \in \mathbb{R} : 0 \le x < 1\}$.

Hence, if $a \in A$, $a < 1$ so $a \le 1$ certainly.

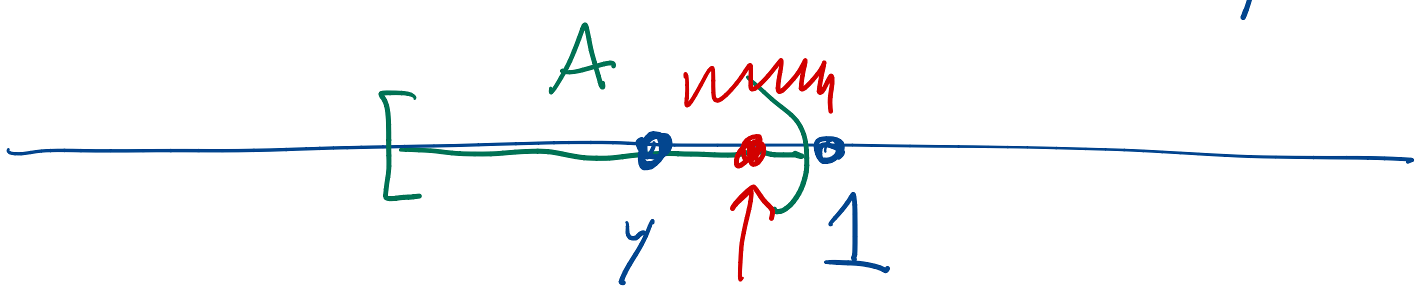
② We need to show $1 \leq$ any other upper bound.

We'll show this by the contrapositive

Direct: If y is an upper bound for A then $1 \leq y$.

If $1 > y$ then y is not an upper bound for A.

there exists $a \in A$, $a > y$.



$$\frac{1+y}{2}$$

Suppose $y < 1$. If $y < 0$ then certainly y is not an upper bound. So we can assume $0 \le y < 1$.

I claim that $a = \frac{1+y}{2} \in A$.

Indeed

$$
\begin{aligned}
0 &\le y \\
&= \frac{y+y}{2} \\
&< \underbrace{\frac{1+y}{2}}_{a} \qquad [y < 1] \\
&< \frac{1+1}{2} \\
&= 1.
\end{aligned}
$$

Hence $0 \le y < a < 1$. Thus $a \in A$ and $y < a$ so y is not an upper bound.

---

③ Axiom of Completeness

Every nonempty subset of $\mathbb{R}$ that is bounded above admits a supremum.

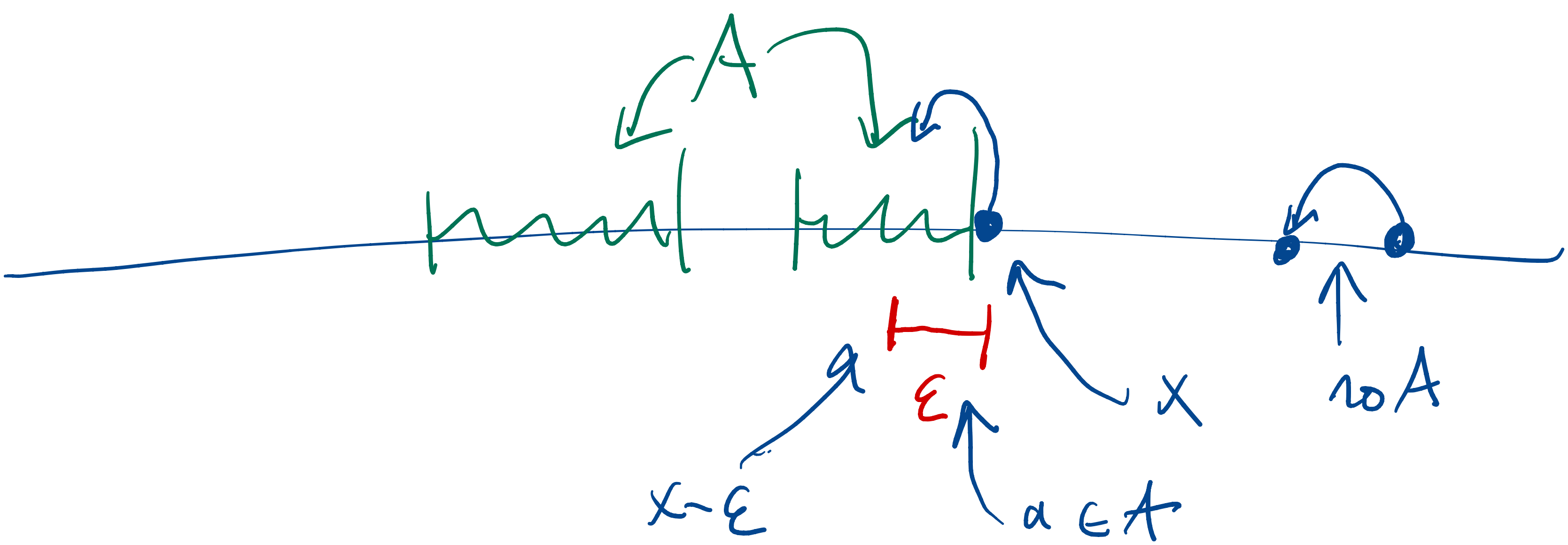
Empty set: bounded above

Technical Lemma: A number x is the supremum of a set $A \subseteq \mathbb{R}$ if and only if

① x is an upper bound for A

② For all $\varepsilon > 0$ there exists $a \in A$ such that

$$x - \varepsilon < a \leq x.$$



Revisit: $\sup [0,1) = 1$.

Pf: Evidently 1 is an upper bound for $[0,1)$.

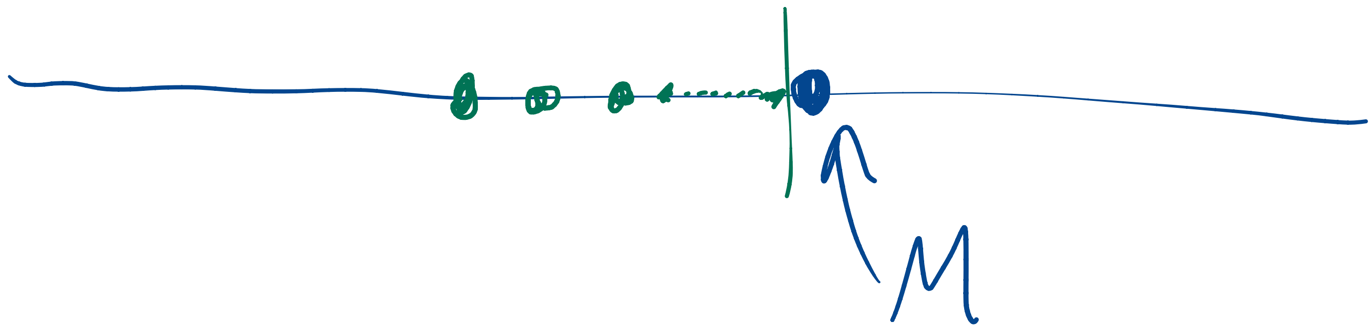
Let $\varepsilon > 0$. Pick $a = \max(0, 1 - \frac{\varepsilon}{2})$.

Then $0 \leq a < 1$ by construction, so $a \in A$.

Moreover, $1 - \varepsilon < 1 - \frac{\varepsilon}{2} = a$. $\square$

Consequences of Completeness:

Thm: $\mathbb{N}$ is not bounded above in $\mathbb{R}$.



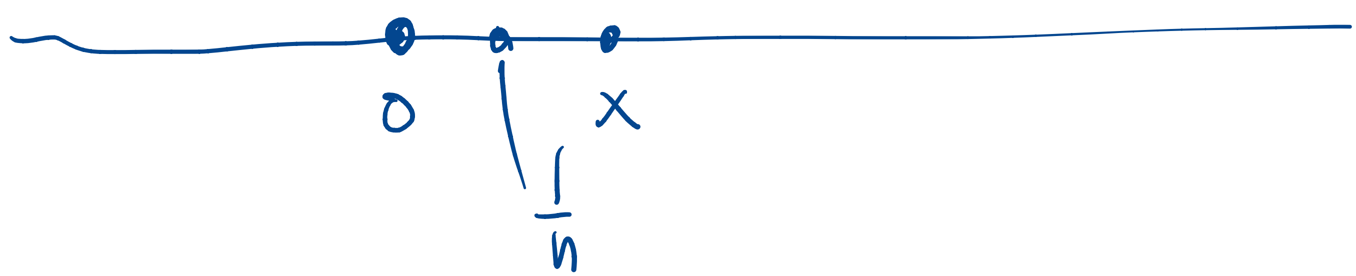
Pf: Suppose to the contrary that $\mathbb{N}$ is bounded above. Since $\mathbb{N} \neq \emptyset$ the Axiom of Completeness implies $\mathbb{N}$ admits a supremum M.

Consider $M-1$. Since M is the least upper bound for $\mathbb{N}$, $M-1$ is not an upper bound. Thus there exists $n \in \mathbb{N}$ such that $M - 1 < n$.

But then $M < n+1 \in \mathbb{N}$. But M is an upper bound for $\mathbb{N}$, which is a contradiction.

$\square$

Cor: If $x \in \mathbb{R}$ and $x > 0$ there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < x$.



Pf: Suppose $x > 0$. Then so is $1/x$.

Pick $n \in \mathbb{N}$ such that $\frac{1}{x} < n$.

But then $0 < \frac{1}{x} < n$ and by elementary arithmetic $\frac{1}{n} < x$ (using $n > 0$, $x > 0$).

"Recall" Well Ordering Property of $\mathbb{N}$

Every nonempty subset of $\mathbb{N}$ has a least element.

HW due?