Implicit Method (Backwards Euler)

Replace $u_t(x_i, t_j) \approx \dfrac{u(x_i, t_j + k) - u(x_i, t_j)}{k}$

with $\dfrac{u(x_i, t_j) - u(x_i, t_j - k)}{k}$

Implicit Method (Backwards Euler)

Replace $u_t(x_i,t_j) \approx \underline{u(x_i,t_j+k)-u(x_i,t_j)}$

with $$\frac{u(x_i,t_j)-u(x_i,t_j-k)}{k}$$

$\hat{\vec{u}}_{j+1} \Leftarrow_{\text{Euler}}$

$$\vec{u}_{j+1} = \vec{u}_j + \frac{k}{h^2} D\vec{u}_{j+1} + k\vec{f}_j$$

tridiagonal!

$$\overbrace{(I-\lambda D)}\vec{u}_{j+1} = \vec{u}_j + k\vec{f}_j$$

$O(N)$

$$u_{j+1} = \underbrace{(I - \lambda D)^{-1}}_{\text{Don't form this! Just solve: } O(N).} \left[\vec{u}_j + k \vec{f}_j \right]$$

Don't form this! Just solve:

$O(N)$.

$A \backslash b$

$\downarrow$

$L\, U$ $\quad O(N^2)$

$O(N^3)$

Fourier Analysis

$$u_{i,j} = \quad q^j V_i \qquad\qquad V_i = e^{\sqrt{r}\,x_i}$$

Fourier Analysis

$$(I - \lambda D)\vec{u}_{j+1} = \vec{u}_j + k\vec{f}_j$$

$$u_{i,j} = q^j v_i \qquad v_i = e^{Jrx_i}$$

$$q^{j+1}\left[-\lambda e^{-Jhr} + (1+2\lambda) - \lambda e^{Jhr}\right] e^{Jrx_i} = e^{Jrx_i} q^j$$

Fourier Analysis

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$$q\left[1 + \lambda\left[2 - 2\cos(hr)\right]\right] = 1$$

Fourier Analysis

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$$q\left[1 + \lambda\left[2 - 2\cos(hr)\right]\right] = 1$$

$$q\left[1 + 2\lambda \underbrace{(1-\cos(hr))}_{2\sin^2\left(\frac{hr}{2}\right)}\right] = 1$$

Amplification factor:

$$g = \frac{1}{1 + 4\lambda \sin^2\left(\frac{h\nu}{2}\right)} \leq 1$$

(regardless of λ)

$$\lambda = \frac{k}{h^2}$$

$$\frac{k}{h^2} \leq \frac{1}{2}$$

θ - methods

$0 \leq \theta \leq 1$

$$\vec{u}_{j+1} = \vec{u}_j + \theta \lambda D \vec{u}_j + (1-\theta) \lambda D \vec{u}_{j+1} + \vec{f}$$

Θ - methods

$u' = f(t, u)$

$$\begin{array}{ccc} & (1-\theta) & \theta \end{array}$$

$$\vec{u}_{j+1} = \vec{u}_j + \theta \lambda D \vec{u}_j + (1-\theta) \lambda D \vec{u}_{j+1} + \vec{f}$$

$\Theta = 1$ Explicit

$\Theta = 0$ Backwards Euler

$\Theta = 1/2$ Crank-Nicholson

$$u_{j+1} = u_j + k \cdot \frac{1}{2}\left[f(t_j, u_j) + f(t_{j+1}, u_{j+1}) \right]$$

$0 \le \Theta \le 1$ generally

$$\vec{u}' = \frac{1}{h^2} D \vec{u}$$

Fourier Analysis

$$1 \quad -2 \quad 1$$

$$u_{i,j} = q^j e^{Jrx_i}$$

$$\vec{u}_{j+1} - (1-\theta)\lambda D\vec{u}_{j+1} = \vec{u}_j + \theta\lambda D\vec{u}_j \qquad (\text{drop } \vec{f})$$

$$\downarrow$$

$$\left[-(1-\theta)\lambda e^{-Jrh} + 1+2(1-\theta)\lambda - (1-\theta)\lambda e^{Jrh}\right]q =$$

$$\left[\theta\lambda e^{-Jrh} + (1-2\theta\lambda) + \theta\lambda e^{Jrh}\right]$$

Fourier Analysis

$$u_{i,j} = q^j e^{\sqrt{-1}\, r x_i}$$

$$\vec{u}_{j+1} - (1-\theta)\lambda D\,\vec{u}_{j+1} = \vec{u}_j + \theta\lambda D\,\vec{u}_j \qquad (\text{drop } \vec{f}\,)$$

Fourier Analysis

$$u_{i,j} = q^j e^{Jrx_i}$$

$$\vec{u}_{j+1} - (1-\theta)\lambda D\,\vec{u}_{j+1} = \vec{u}_j + \theta\lambda D\,\vec{u}_j \qquad (\text{drop } \vec{f}\,)$$

$$\downarrow$$

$$\left[-(1-\theta)\lambda e^{-Jrh} + 1+2(1-\theta)\lambda - (1-\theta)\lambda e^{Jrh}\right] q =$$

$$\left[\theta\lambda e^{-Jrh} + (1-2\theta\lambda) + \theta\lambda e^{Jrh}\right]$$

$$q = \frac{1 + 2\theta\lambda[\cos(rh) - 1]}{1 + 2(1-\theta)\lambda[1 - \cos(rh)]}$$

$$q = \frac{1 + 2\theta\lambda[\cos(\nu h) - 1]}{1 + 2(1-\theta)\lambda[1 - \cos(\nu h)]}$$

$$\left| 1 - 4\theta\lambda \sin^2 \right| \leq 1$$

$$q = \frac{1 - 4\theta\lambda \sin^2(\nu h/2)}{1 + 4(1-\theta)\lambda \sin^2(\nu h/2)}$$

$$|q| \leq 1 \qquad -1 \leq q \leq 1$$

$$q = \frac{1 + 2\theta\lambda[\cos(\nu h) - 1]}{1 + 2(1-\theta)\lambda[1 - \cos(\nu h)]}$$

$$q = \frac{1 - 4\theta\lambda \sin^2(\nu h/2)}{1 + 4(1-\theta)\lambda \sin^2(\nu h/2)}$$

Want $|q| \le 1$ to prevent mode from growing

$$-1 \le q \le 1$$

$$q = \frac{1 + 2\theta\lambda\left[\cos(\nu h) - 1\right]}{1 + 2(1-\theta)\lambda\left[1 - \cos(\nu h)\right]}$$

$$q = \frac{1 - 4\theta\lambda \sin^2(\nu h/2)}{1 + 4(1-\theta)\lambda \sin^2(\nu h/2)}$$

Want $|q| \le 1$ to prevent mode from growing

$$-1 \le \underbrace{q \le 1}_{\text{free}}$$

$$-1 \le q \Rightarrow$$

$$-1 - 4(1-\theta)\lambda \sin^2\left(\frac{rh}{2}\right) \le 1 - 4\theta\lambda \sin^2\left(\frac{rh}{2}\right)$$

$$-1 \leq q \Rightarrow$$

$$-1 - 4(1-\theta)\lambda \sin^2\left(\frac{\nu h}{2}\right) \leq 1 - 4\theta\lambda \sin^2\left(\frac{\nu h}{2}\right)$$

$$4\lambda \sin^2\left(\frac{\nu h}{2}\right)\left[\theta - (1-\theta)\right] \leq 2$$

$\theta = 0$ BE

$\theta = 1$ F

$\theta = 1/2$ CN

$-1 \leq q \Rightarrow$

$$-1 - 4(1-\theta)\lambda \sin^2\left(\frac{rh}{2}\right) \leq 1 - 4\theta\lambda \sin^2\left(\frac{rh}{2}\right)$$

$$4\lambda \sin^2\left(\frac{rh}{2}\right)\left[\theta - (1-\theta)\right] \leq 2$$

$\lambda \leq \frac{1}{2}$ F

$$4\lambda \sin^2\left(\frac{rh}{2}\right)\left[2\theta - 1\right] \leq 2$$

$$\lambda \left[2\theta - 1 \right] \leq \frac{1}{2}$$

$$\lambda\left[2\theta-1\right] \leq \frac{1}{2}$$

If $\theta \leq \frac{1}{2}$, this is always assured.

($\theta = 0$: BE

$\theta = 1/2$: CN)

$$\frac{k}{h^2} \leq \frac{1}{2} \frac{1}{2\theta-1}$$

$$k \leq \frac{h^2}{2} \cdot \frac{1}{2\theta-1}$$

If $\theta > \frac{1}{2}$, $\lambda \leq \frac{1}{2} \frac{1}{(2\theta-1)}$

$$\lambda\,[2\theta - 1] \le \frac{1}{2}$$

If $\theta \le \frac{1}{2}$, this is always assumed.

($\theta = 0$: BE

$\theta = 1/2$: CN)

If $\theta > \frac{1}{2}$, $\lambda \le \frac{1}{2} \dfrac{1}{(2\theta - 1)}$ → discuss!

LTE: $0 \leq \theta \leq \frac{1}{2}$

$\hookrightarrow$ expect no restrictions on time step.

Exercise: $\tau_{i,j} = O(k) + O(h^2)$ $\quad k \sim h^2$

<u>unless</u> $\theta = \frac{1}{2}$ $\quad \frac{k}{h^2} \sim 1$

$O(k^2) + O(h^2)$. $\quad k^2 \sim h^2$

$k \sim h$

Intuition: $\theta = \frac{1}{2} \sim$ trapezoid rule for ODEs

LTE:

Exercise: $\tau_{i,j} = O(k) + O(h^2)$

<u>unless</u> $\theta = \frac{1}{2}$

$$\underbrace{O(k^2) + O(h^2)}_{\text{Merits!}}.$$

Intuition: $\theta = \frac{1}{2} \sim$ trapezoid rule for ODEs

Convergence of θ - Methods (Round 1 !)

$$\left[1 + 2(1-\theta)\lambda\right] u_{i,j+1} = (1-\theta)\lambda \left[u_{i-1,j+1} + u_{i+1,j+1}\right]$$

$$+ \theta\lambda \left[u_{i-1,j} + u_{i+1,j}\right]$$

$$+ \left[1 - 2\theta\lambda\right] u_{i,j} \qquad \left(+ \vec{0}\right)$$

Discrete maximum principle:

$$\left[1 + 2(1-\theta)\lambda\right] u_{i,j+1} = (1-\theta)\lambda \left[u_{i-1,j+1} + u_{i+1,j+1}\right]$$
$$+ \theta\lambda \left[u_{i-1,j} + u_{i+1,j}\right]$$
$$+ \left[1-2\theta\lambda\right] u_{i,j}$$

$$2(1-\theta)\lambda + 2\theta\lambda + 1 - 2\theta\lambda = 1 + 2(1-\theta)\lambda$$

Discrete maximum principle:

$$\left[1+2(1-\theta)\lambda\right]u_{i,j+1} = (1-\theta)\lambda\left[u_{i-1,j+1}+u_{i+1,j+1}\right]$$
$$+\ \theta\lambda\left[u_{i-1,j}+u_{i+1,j}\right]$$
$$+\ \left[1-2\theta\lambda\right]u_{i,j}$$

$$\overbrace{2(1-\theta)\lambda}+\overbrace{2\theta\lambda}+\overbrace{1-2\theta\lambda} = 1+2(1-\theta)$$

Discrete maximum principle:

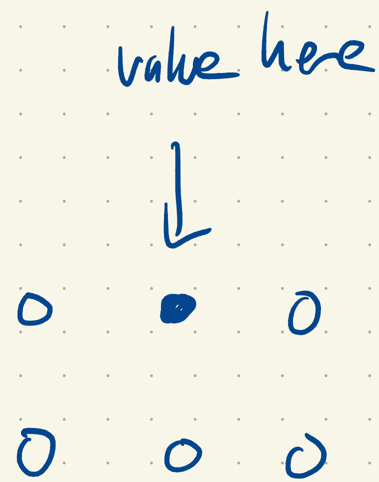
$$1 - 2\theta\lambda \geq 0$$

$$\frac{1}{2} \geq \theta\lambda$$

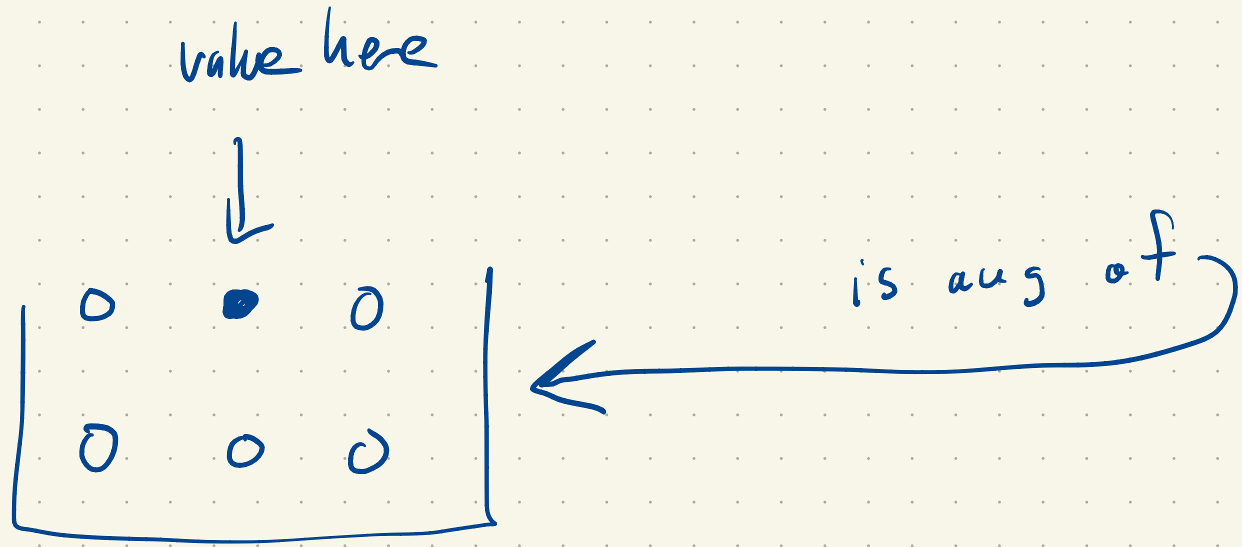
$$\begin{aligned}[1 + 2(1-\theta)\lambda]\, u_{i,j+1} = {} & (1-\theta)\lambda\, [u_{i-1,j+1} + u_{i+1,j+1}] \\ & + \theta\lambda\, [u_{i-1,j} + u_{i+1,j}] \\ & + [1 - 2\theta\lambda]\, u_{i,j}\end{aligned}$$

$$\underbrace{2(1-\theta)\lambda} + \underbrace{2\theta\lambda} + \underbrace{1 - 2\theta\lambda} = \underbrace{1 + 2(1-\theta)}$$

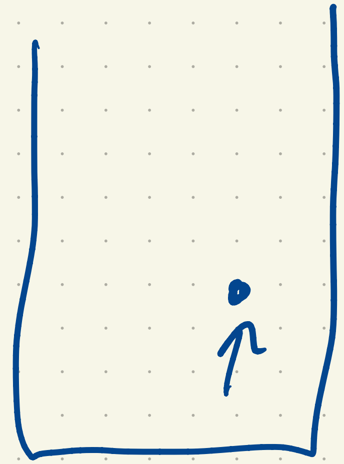
Discrete Max Principle



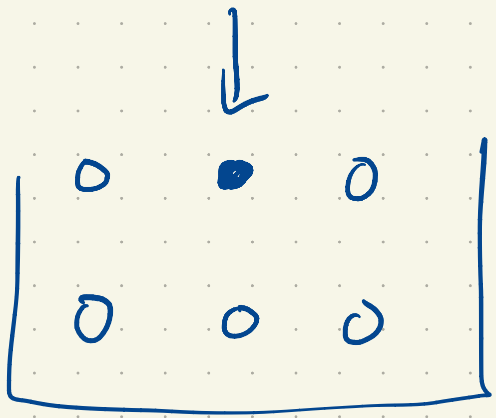
Discrete Max Principle



Discrete Max Principle



value here



is avg of

(honest average requires $1 - 2\theta\lambda > 0$

$$\boxed{\theta\lambda < \frac{1}{2}}$$

We'll prove convergence assuming $\theta\lambda < \frac{1}{2}$

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- $\theta = 0$, Backwards Euler

We'll prove convergence assuming $\theta\lambda < \frac{1}{2}$

- $\theta = 0$, Backwards Euler, no constraint

We'll prove convergence assuming $\theta\lambda < \frac{1}{2}$

- $\theta = 0$, Backwards Euler, no constraint

- $\theta = 1$, Explicit

We'll prove convergence assuming $\theta \lambda < \frac{1}{2}$

$$\frac{1}{2}\lambda < \frac{1}{2}$$

- $\theta = 0$, Backwards Euler, no constraint
- $\theta = 1$, Explicit, $\lambda < \frac{1}{2}$ as before

We'll prove convergence assuming $\theta\lambda \leq \frac{1}{2}$

- $\theta = 0$, Backwards Euler, no constraint
- $\theta = 1$, Explicit, $\lambda \leq \frac{1}{2}$ as before
- $\theta = \frac{1}{2}$, CN, $\lambda \leq 1$

$$\frac{k}{h^2} \leq 1$$

$$k \leq h^2$$

We'll prove convergence assuming $\theta\lambda < \frac{1}{2}$

- $\theta = 0$, Backwards Euler, no constraint
- $\theta = 1$, Explicit, $\lambda < \frac{1}{2}$ as before
- $\theta = \frac{1}{2}$, CN, $\lambda < 1$ discuss!

L^2 convergence $\qquad \theta \gtrless \frac{1}{2}$

Later: If we weaken our notion of closeness, can show convergence w/ $\lambda[2\theta-1] \leq \frac{1}{2}$

(so always if $\theta \leq \frac{1}{2}$)