1. Consider the heat equation $u_t = \kappa u_{xx}$ for $\kappa > 0$, $x \in [0,1]$, and Dirichlet boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$. Suppose we have initial condition $u(x,0) = sin(5\pi x)$.

a) Find an exact solution to this problem.

b) Implement the backward Euler (BE) method to solve this heat equation problem. Specifically, use diffusivity $\kappa = 1/20$ and final time $T = 0.1$. Note that you do not need to use Newton's method to solve the implicit equation, which is a linear system, but you should use sparse storage and an efficient linear solver (backslash in MATLAB will work).

c) Suppose the timestep k and the space step h are related by $k = 2h$. What do you expect for the convergence rate $O(h^p)$? Then measure it by using the exact solution from a), at the final time, and the infinity norm $||\cdot||_\infty$, and $h = 0.05, 0.02, 0.01, 0.005, 0.002, 0.001$. Make a log-log convergence plot of h versus the error.

d) Repeat parts b) and c) but with the trapezoidal rule instead of BE. (That is, implement and measure the convergence rate of Crank-Nicolson, with everything else the same.)

Solution, part a:
The exact solution is

$$u(x,t) = \exp(-\sqrt{5\kappa\pi t}) \sin(5\pi x)$$

Solution, part b:
See worksheet.

Solution, part c:
For Backwards Euler, the expected rate of convergence is $O(h)$. For Crank Nicolson the expected rate of convergence is $O(h^2)$. Log-log plots verifying these rates can be found in the worksheet.

2. Consider the PDE

$$u_t = \partial_x(p(x)u_x)$$

where $p(x)$ is a given function. We wish to solve the PDE on the region $0 \le x \le 1$, $0 \le t \le T$ with $u = 0$ at $x = 0, 1$ We will apply the following finite difference scheme to it:

$$u_{i,j+1} = u_{i,j} + \frac{k}{h^2}[(u_{i+1,j} - u_{i,j})p_{i+\frac{1}{2}} - (u_{i,j} - u_{i-1,j})p_{i-\frac{1}{2}}]$$

where $p_{i\pm\frac{1}{2}} = p(x_i \pm h/2)$.

a) Estimate the local truncation error in terms of powers of h and k and in terms of derivatives of u and derivatives of p. I'm looking for an answer akin to the estimate we derived for the heat equation of the form

$$|\tau| \le \max|u_{xxxx}|\left[\frac{k}{2} + \frac{h^2}{h}\right]$$

that we derived for the heat equation with no forcing term.

b) Show that the method is convergent, assuming $0 < p(x)k < h^2/2$. You will want to revist the proof from class that the explict method for the standard heat equation is convergent.

Solution, part a:

From Taylor's theorem we have the following:

$$\frac{u(x_i,t_{j+1}) - u(x_i,t_j)}{k} = u_t + O(k) \tag{1}$$
$$u(x_{i+1},t_j) - u(x_i,t_j) = u_x h + u_{xx}h^2/2 + u_{xxx}h^3/6 + O(h^4) \tag{2}$$
$$u(x_i,t_j) - u(x_{i-1},t_j) = u_x h - u_{xx}h^2/2 + u_{xxx}h^3/6 + O(h^4) \tag{3}$$
$$p(x_{i\pm 1/2}) = p \pm p_x \frac{h}{2} + O(h^2) \tag{4}$$

Hence

$$u(x_{i+1},t_j) - u(x_i,t_j)p(x_{i+1/2}) - u(x_i,t_j) - u(x_{i-1},t_j)p(x_{i-1/2}) = pu_{xx}h^2 + p_x u_x h^2 + O(h^4)$$

and we conclude the local truncation error is

$$u_t + O(k) - \frac{1}{h^2}(pu_{xx}h^2 + p_x u_x h^2 + O(h^4)) = u_t - \partial_x(pu_x) + O(k) + O(h^2) = O(k) + O(h^4)$$

Solution, part b:

If $U_{i,j}$ is the numerical solution and u_{ij} is the true solution evaluated at the grid points, then the error $E_{ikj} = U_{i,j} - u_{i,j}$ satisfies

$$E_{i,j+1} = E_{i,j} + \frac{k}{h^2}[(E_{i+1,j} - E_{i,j})p_{i+\frac{1}{2}} - (E_{i,j} - E_{i-1,j})p_{i-\frac{1}{2}}] + \tau_{i,j} \tag{5}$$
$$= (1 - \frac{k}{h^2}p_{i+1/2} - \frac{k}{h^2}p_{i-1/2})E_{i,j} + \frac{k}{h^2}E_{i+1,j}p_{i+1/2} + \frac{k}{h^2}E_{i-1,j}p_{i-1/2} + \tau_{ij} \tag{6}$$

where $\tau_{i,j}$ is a local truncation error. Let us suppose $0 < p(x)k < h^2/2$. Then $(1 - \frac{k}{h^2}p_{i+1/2} - \frac{k}{h^2}p_{i-1/2}) > 0$ and we compute

$$|E_{i,j+1}| \le (1 - \frac{k}{h^2}p_{i+1/2} - \frac{k}{h^2}p_{i-1/2})|E_{i,j}| + \frac{k}{h^2}p_{i+1/2}|E_{i+1,j}| + \frac{k}{h^2}p_{i-1/2}|E_{i-1,j}| + \tau_{i,j}.$$

Setting $E_j = \max_i |E_{i,j}|$ we conclude

$$E_{j+1} \le \left[(1 - \frac{k}{h^2}p_{i+1/2} - \frac{k}{h^2}p_{i-1/2}) + \frac{k}{h^2}p_{i+1/2}\frac{k}{h^2}p_{i-1/2}\right]E_j + \tau_{i,j} = E_j + \tau_{i,j}.$$

Having arrived at this inequality, the proof of convergence is now identical to that of the explicti method for the standard heat equation.

3.

a) Let

$$A = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

Compute $||A||_1$ and $||A||_\infty$.

b) Estimate $||A||_2$ as follows. Computer generate a figure containng the boundary of $A(B_1)$, where B_1 is the Euclidean ball of radius 1. Then use the figure to estimate the norm.

c) Suppose A is an $n \times n$ matrix, and choose $p \in [1, \infty]$. Show that $||A||_p = 0$ if and only if A is the 0 matrix.

d) For vectors in $\mathbb{R}^n$, it is known that $||x + y||_p \le ||x||_p + ||y||_p$ for any $p \in [1, \infty]$. This is the triangle inequality, and you need not prove it. But using this fact, show that the triangle inequality also holds for matrix norms $|| \cdot ||_p$ for p in the same range.

Solution, part a:

For a vector $x = (x_1, x_2)$, $Ax = (5x_1 + 6x_2, 7x_1 + 8x_2)$. Thus

$$||A_x||_1 = |5x_1 + 6x_2| + |7x_1 + 8x_2| \le 12|x_1| + 14|x_2| \le 14||x||_1.$$

Thus $||A||_1 \le 14$. But taking $x = (0, 1)$ we find

$$\frac{||Ax||_1}{||x||_1} = \frac{14}{1} = 14.$$

Thus$||A||_1 \ge 14$ as well and $||A||_1 = 14$.

Again, for a vector $x = (x_1, x_2)$, $Ax = (5x_1 + 6x_2, 7x_1 + 8x_2)$. Thus

$$\begin{aligned} ||A_x||_\infty &= \max |5x_1 + 6x_2|, |7x_1 + 8x_2| \\ &\le \max 5|x_1| + 6|x_2|, 7|x_1| + 8|x_2| \\ &\le \max(5||x||_\infty + 6||x_2||_\infty, 7||x||_\infty + 8||x||_\infty) \\ &\le 15||x||_\infty. \end{aligned} \tag{7}$$

Thus $||A||_\infty \le 15$. But taking $x = (1, 1)$ we have $||x||_\infty = 1$ and $Ax = (11, 15)$ so

$$\frac{||Ax||_\infty}{||x||_\infty} = \frac{15}{1} = 15.$$

Thus $||A||_\infty \ge 15$ and hence $||A||_\infty = 15$.

Solution, part b:

See worksheet.

Solution, part c:
Suppose A is the zero matrix. Then $||Ax||_p = 0$ for any vector x and

$$||A||_p = \sup_{x\neq 0} \frac{||Ax||_p}{||x||_p} = \sup_{x\neq 0} 0 = 0.$$

Now suppose A is not the zero matrix. Then one of its columns, column i, say, is not all zeros. Let e_i be the vector that is all zeros, except for a 1 in row i. Then $Ae_i = y$, where y is the i'th column of A. In particular $y \neq 0$ and $||y||_p > 0$. But then

$$||A||_p \geq \frac{||Ae_i||_p}{||e_i||_p} = \frac{||y||_p}{1} > 0.$$

Thus $A \neq 0$.

Thus $||A||_p = 0$ if and only if $A = 0$.

Solution, part d:
Suppose A and B are $n \times n$ matrices. Given a vector x,

$$||(A+B)x||_p = ||Ax + Bx||_p \leq ||Ax||_p + ||Bx||_p \leq ||A||_p||x||_p + ||B||_p||x||_p = (||A||_p + ||B||_p).$$

Note that we used the triangle inequality for vectors as well as the fundamental inequality for matrix norms:

$$||Ax||_p \leq ||A||_p||x||_p.$$

Assuming that $x \neq 0$ we find

$$\frac{||(A+B)x||_p}{||x||p} \leq ||A||_p + ||B||_p.$$

But the matrix norm is defined by

$$||A+B||_p = \sup_{x\neq 0} \frac{||(A+B)x||_p}{||x||p} \leq ||A||_p + ||B||_p$$

by the above.

4. Text, problem 3.7

See homework number 8.