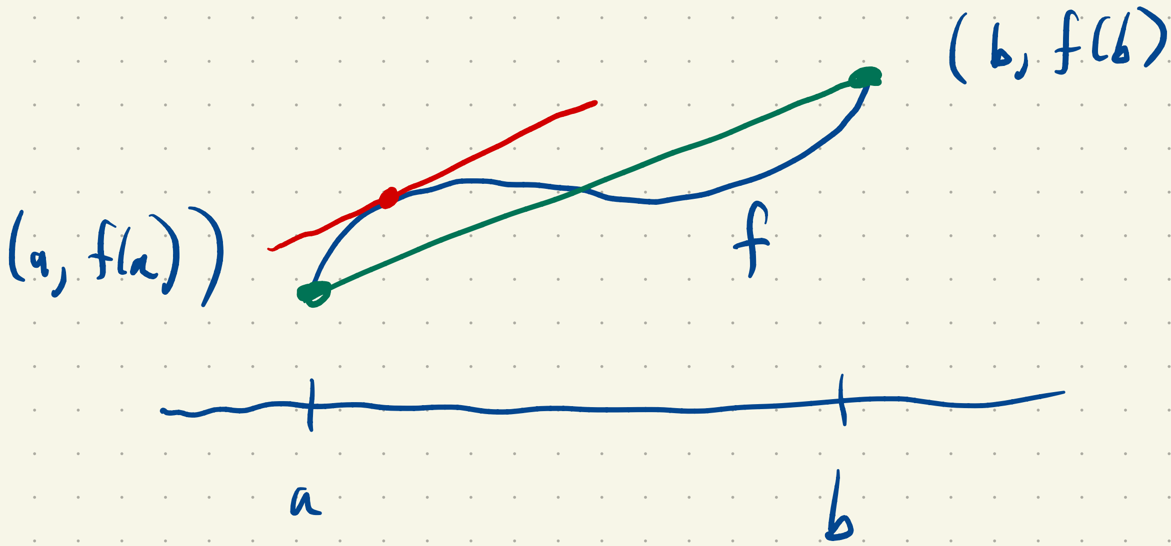
Taylor's Theorem

1) Mean Value Theorem



f is cts on $[a,b]$

f is diff on (a,b)

$\exists\, c$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

some $c \in (a,b)$

2) FTC

Given $f : [a,b] \to \mathbb{R}$, continuous with a continuous derivative

$$f(b) - f(a) = \int_a^b f'(s)\, ds$$

These are cousins:

$h = b - a \qquad a,\ a+h$

MVT: $f(a+h) = f(a) + f'(c) \cdot h$

$$f(a+h) = f(a) + \int_a^{a+h} f'(s)\,ds$$

$$f(a+h) = f(a) + h \cdot \left[\frac{1}{h} \int_a^{a+h} f'(s)\,ds \right]$$

$\to$ average of f' over $[a, a+h]$

between $\max f'$ and $\min f'$

(f' is continuous)

it equals $f'(c)$ for some c in the interval.

Next version:

1) $f(a+h) = f(a) + f'(a)h + \frac{1}{2} f''(c) h^2$

c is ~~between~~ in $(a, a+h)$

f' is cts on $[a,b]$

f'' exists on (a,b)

2) $f(a+h) = f(a) + f'(a) \cdot h + \int_a^{a+h} (a+h-s) f''(s)\, ds$

In general:

1) $$f(a+h) = \underbrace{f(a) + f'(a)\cdot h + \cdots + \frac{f^{(k)}(a)}{k!}h^k}_{} + \frac{f^{(k+1)}(c)\,h^{k+1}}{(k+1)!}$$

2) Integral version (wait a sec)

Taylor Polynomial $\quad b = a+h$

$$f(b) = f(a) + f'(a)(b-a) + \cdots + \frac{f^{(k)}(a)(b-a)^k}{k!} + R_k$$

$b \longrightarrow x$

$$f(x) = f(a) + f'(a)x - - \cdots - + \frac{f^{(k)}(a)}{k!} x^k$$

Example: Estimate the error in approximating $\log(x)$
$\ln(x)$

by its 10^{th} order Taylor polynomial on $[0.5, 1.5]$

centered at $a = 1$

Derivatives: $\log(x), \frac{+1}{x}, \frac{-1}{x^2}, + \frac{2!}{x^3}, \ldots, -\frac{9!}{x^{10}}$

$x = 1$ $\quad 0, 1, -1, 2!, -3!, \ldots, -9!$

$$\log(1+h) = 0 + 1\cdot h - \frac{1\cdot h^2}{2!} + \frac{2!\, h^3}{3!} \cdots\cdots - \frac{9!\, h^{10}}{10!} + R$$

$$= 0 + h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} \cdots\cdots - \frac{h^{10}}{10} + R$$

let's estimate $R = \dfrac{\log^{(11)}(c)\, h^{11}}{11!}$

$$= \frac{10! \left(\frac{1}{c}\right)^{11}}{11!} \cdot h^{11} = \frac{1}{11}\left(\frac{h}{c}\right)^{11}$$

$x = [1/2, 3/2]$

$h \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$c \in [\frac{1}{2}, \frac{3}{2}]$

estime $\frac{h}{c} \leq h$ for $h > 0$

$$R \leq \frac{1}{11} \cdot \left(\frac{1}{2}\right)^{11} \approx 0{,}00004$$