$$\partial_s \left[\partial_r U \right] = 0$$

$$\partial_r U = f(r)$$

$$U = F(r) + G(s) \; ; \quad F' = f$$

$$u\left(x(r,s), t(r,s)\right) = F(r) + G(s)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$\longrightarrow \qquad \qquad \longleftarrow$$

Initial Conditions

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$u(x,0) = F(x) + G(x)$$

$$u_t(x,0) = -cF'(x) + cG'(x)$$

Case 1) u_0 given, $v \equiv 0$

2) $u_0 \equiv 0$, v given

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,0) = u_0$$

$$u_t(x,0) = V$$

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = u_0 \\ w_t(x,0) = 0 \end{cases}$$

$$\begin{cases} z_{tt} - c^2 z_{xx} = 0 \\ z(x,0) = 0 \\ z_t(x,0) = V \end{cases}$$

Case 1) $\quad F' = G' \qquad (-F' + G' \equiv 0)$

$$G = F + k$$

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$$F' = G'$$

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$$F + G = u_0$$

$$2F + k = u_0$$

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$$F' = G'$$

$$G = F + k$$

$$F + G = u_0$$

$$2F + k = u_0$$

$$F(x) = \frac{u_0(x) - k}{2}$$

$$G(x) = \frac{u_0(x) + k}{2}$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$= \frac{1}{2} u_0(x-ct) + \frac{1}{2} u_0(x+ct)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

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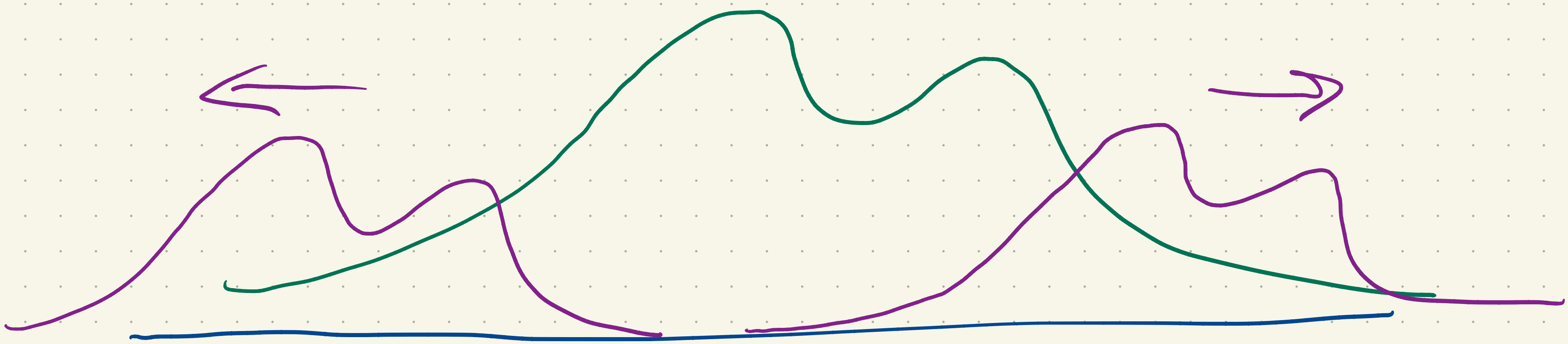
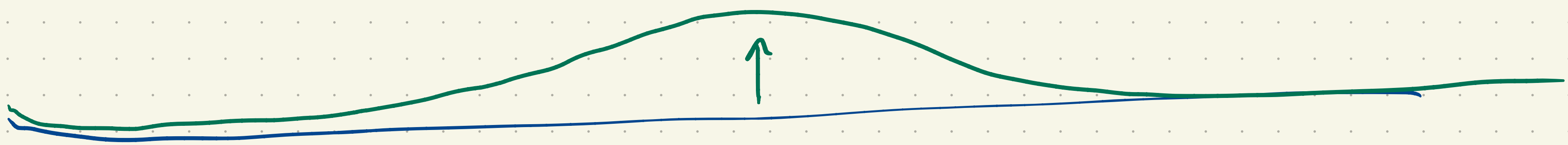




Case 2) $u_0 \equiv 0$

$$F + G = 0$$

$$G = -F$$

Case 2) $u_0 \equiv 0$

$$F + G = 0$$

$$G = -F$$

$$-cF' + cG' = V$$

Case 2) $u_0 \equiv 0$

$$F + G = 0$$

$$G = -F$$

$$-cF' + cG' = v$$

$$2c\,G' = v$$

$$G'(x) = \frac{1}{2c} v(x)$$

$$G'(x) = \frac{1}{2c} v(x)$$

$$G(x) = \frac{1}{2c} \int_a^x v(s)\, ds$$

$$G'(x) = \frac{1}{2c} v(x)$$

$$G(x) = \frac{1}{2c} \int_a^x v(s)\, dx$$

$$F(x) = -\frac{1}{2c} \int_a^x v(s)\, ds$$

$$= \frac{1}{2c} \int_x^a v(s)$$

$$G(x) = \frac{1}{2c} \int_a^x V(s)\,dx$$

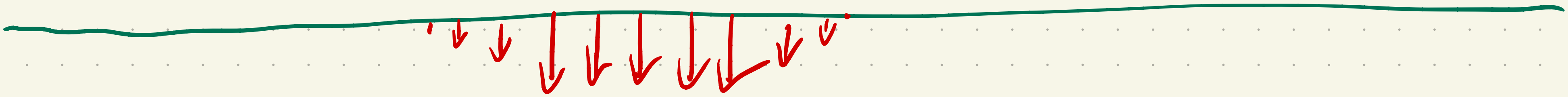
$$F(x) = \frac{1}{2c} \int_x^a V(s)\,dx$$

$$G(x) = \frac{1}{2c} \int_a^x V(s)\, dx$$

$$F(x) = \frac{1}{2c} \int_x^a V(s)\, dx$$

$$F(x-ct) + G(x+ct) = \frac{1}{2c} \left[\int_{x-ct}^{a} V(s)\, ds + \int_a^{x+ct} V(s)\, ds \right]$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} V(s)\, ds$$



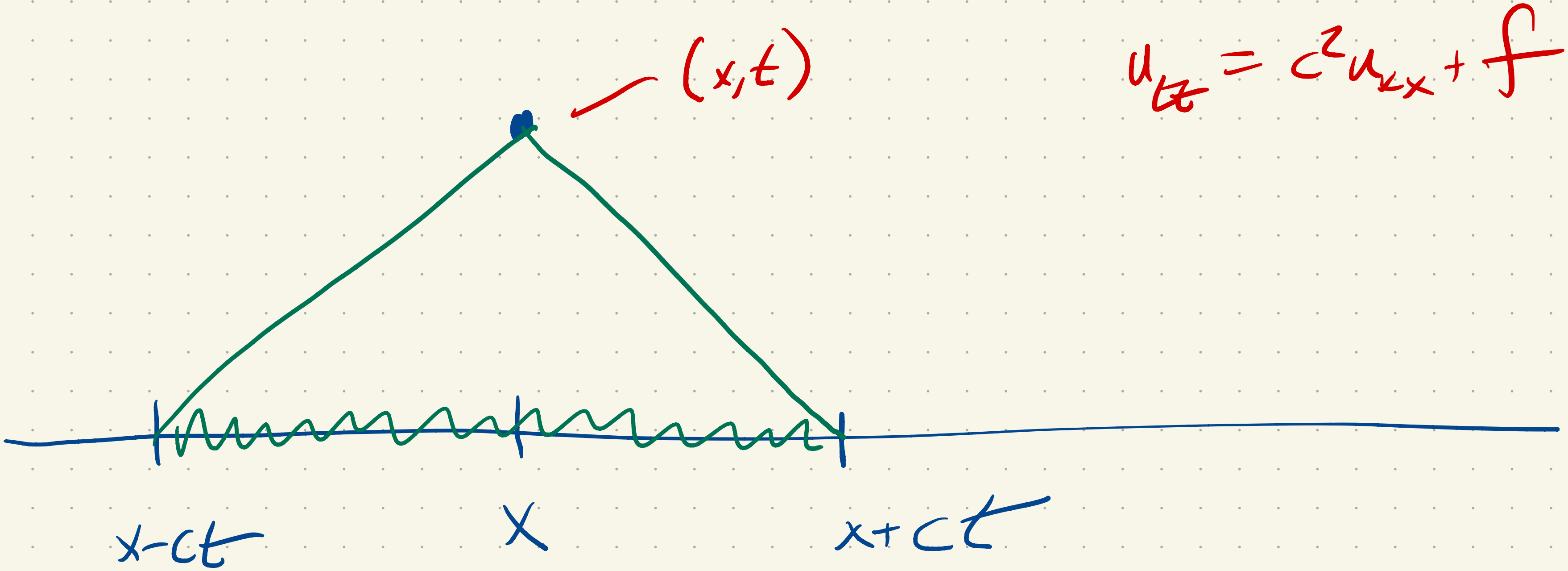
General Solution

$$\boxed{u(x,t) = \frac{1}{2}u_0(x-ct) + \frac{1}{2}u_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,ds}$$

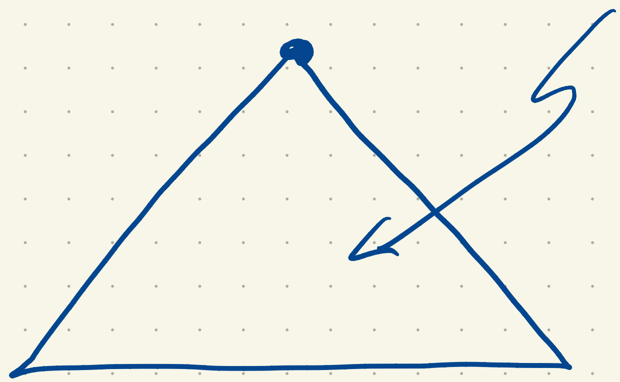
General Solution

$$\boxed{u(x,t) = \frac{1}{2}u_0(x-ct) + \frac{1}{2}u_0(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,ds}$$

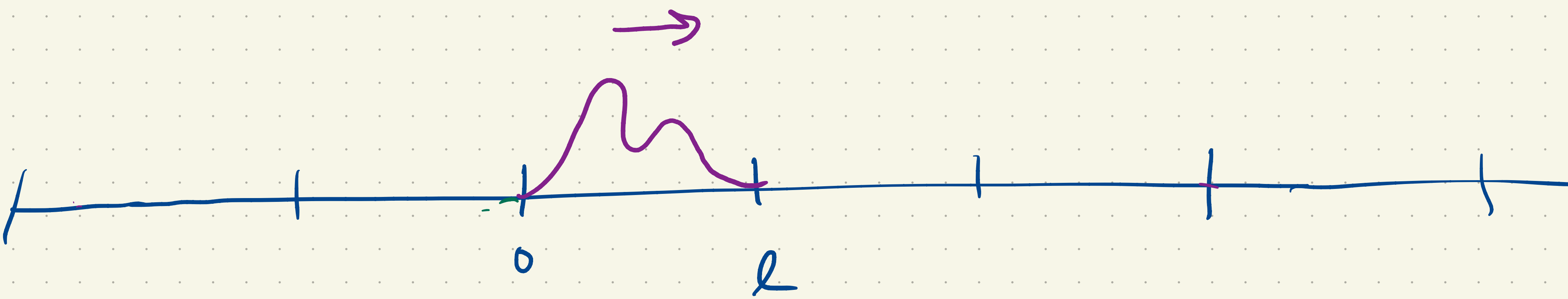
$u_{tt} = c^2 u_{xx} + f$



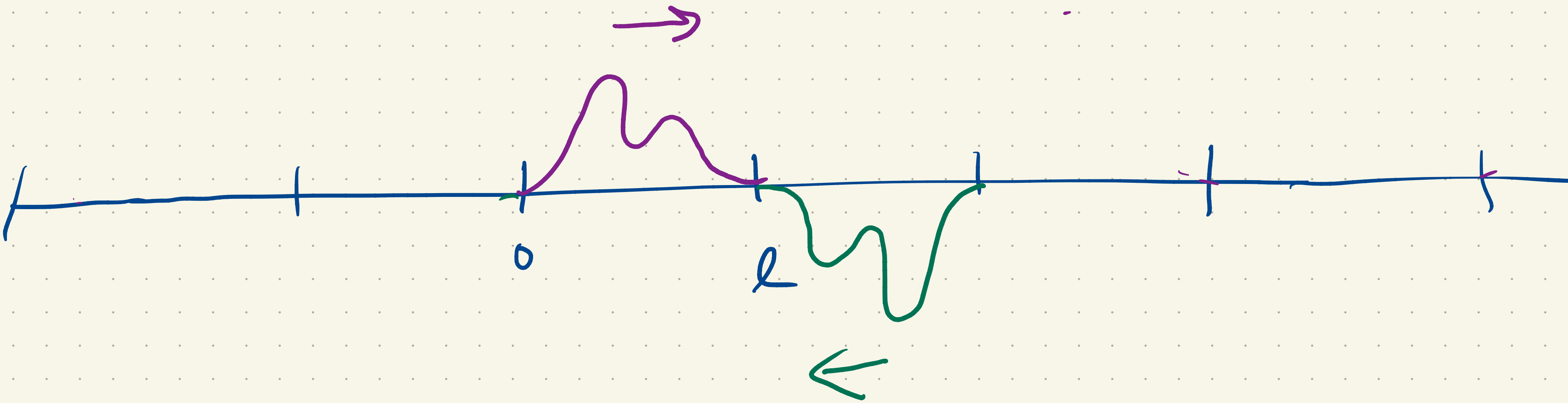
domain of dependence



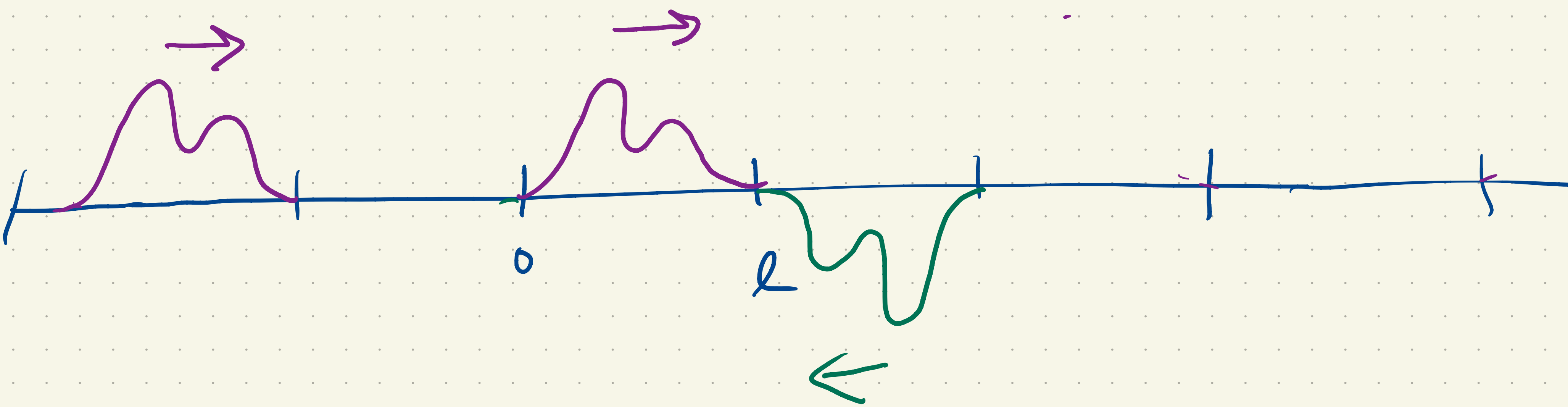
Boundary Condition Games



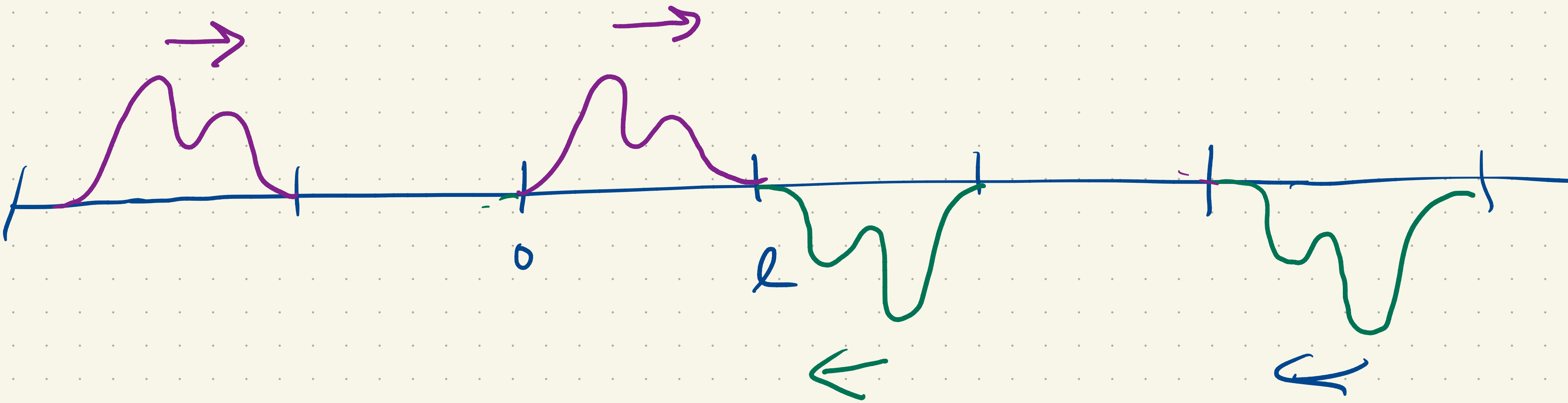
Boundary Condition Games



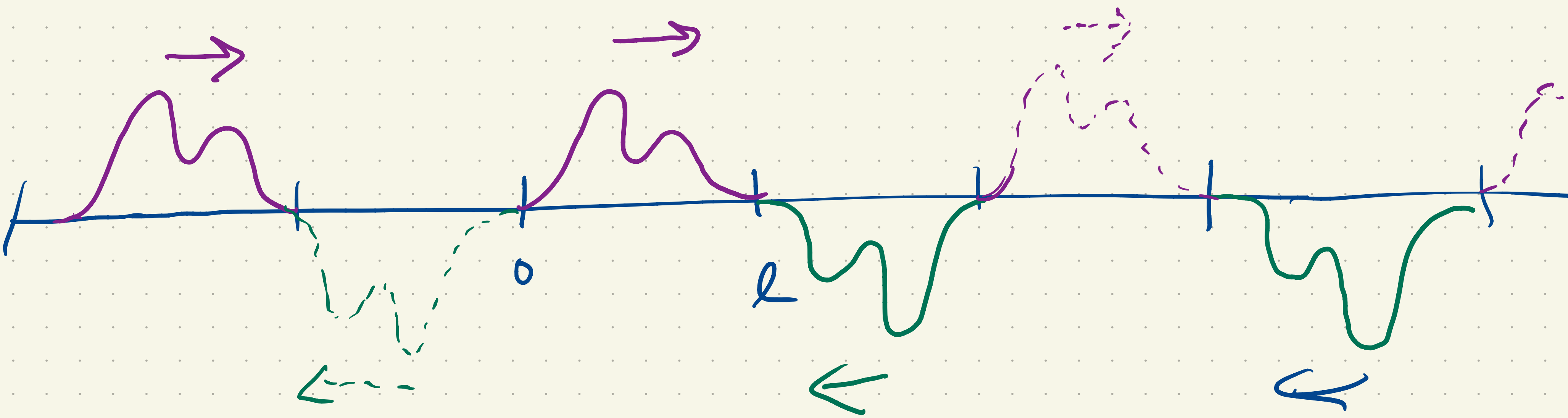
Boundary Condition Games



Boundary Condition Games



Boundary Condition Games



Dispersion Relations

Plane waves $\qquad e^{I(kx-\omega t)}$

k: real

ω: complex

wave number

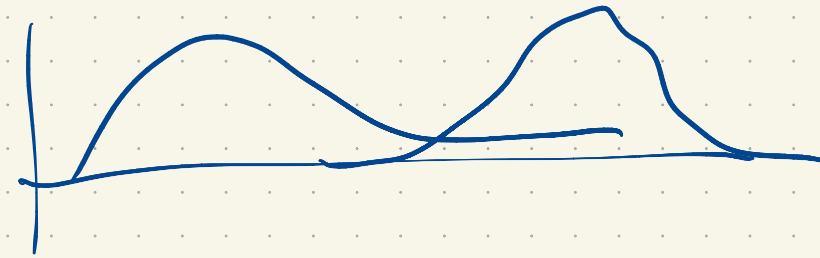
$$e^{Ikx}\,e^{-I\omega t}$$

complex amplitude in time

periodic in space, always

$$e^{I(kx-\omega t)}$$

wavenumber. $\lambda = \frac{2\pi}{k}$ is wavelength



$$\omega = \omega_r + I\,\omega_i$$

"complex frequency"

$$\underbrace{e^{\omega_i t}}_{\text{controls growth}} \cdot e^{Ik\left(x - \frac{\omega_r}{k} t\right)}$$

wave with velocity $\frac{\omega_r}{k}$

$$f(x-at)$$

$$f(z) = e^{Ikz}$$

Phase velocity

$$e^{I(kx-\omega t)}$$

$$v_p = \frac{\omega_r}{k}$$

Mantra: Linear hyperbolic const. coeff PDEs determine a relationship between wave number and complex frequency of plane wave solutions, and hence determine phase velocity from wave #.

Dispersion relation

E.g.

$$u_{tt} = c^2 u_{xx}$$

$$u = e^{I(kx-\omega t)}$$

$$-\omega^2 u = c^2 \cdot (-1) k^2 u$$

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$$\omega^2 = c^2 k^2$$

$$\omega = \pm ck$$

$$v_p = \frac{\omega}{k} = \pm c$$

Just one phase vel.

E.g.

$$u_{tt} = c^2 u_{xx}$$

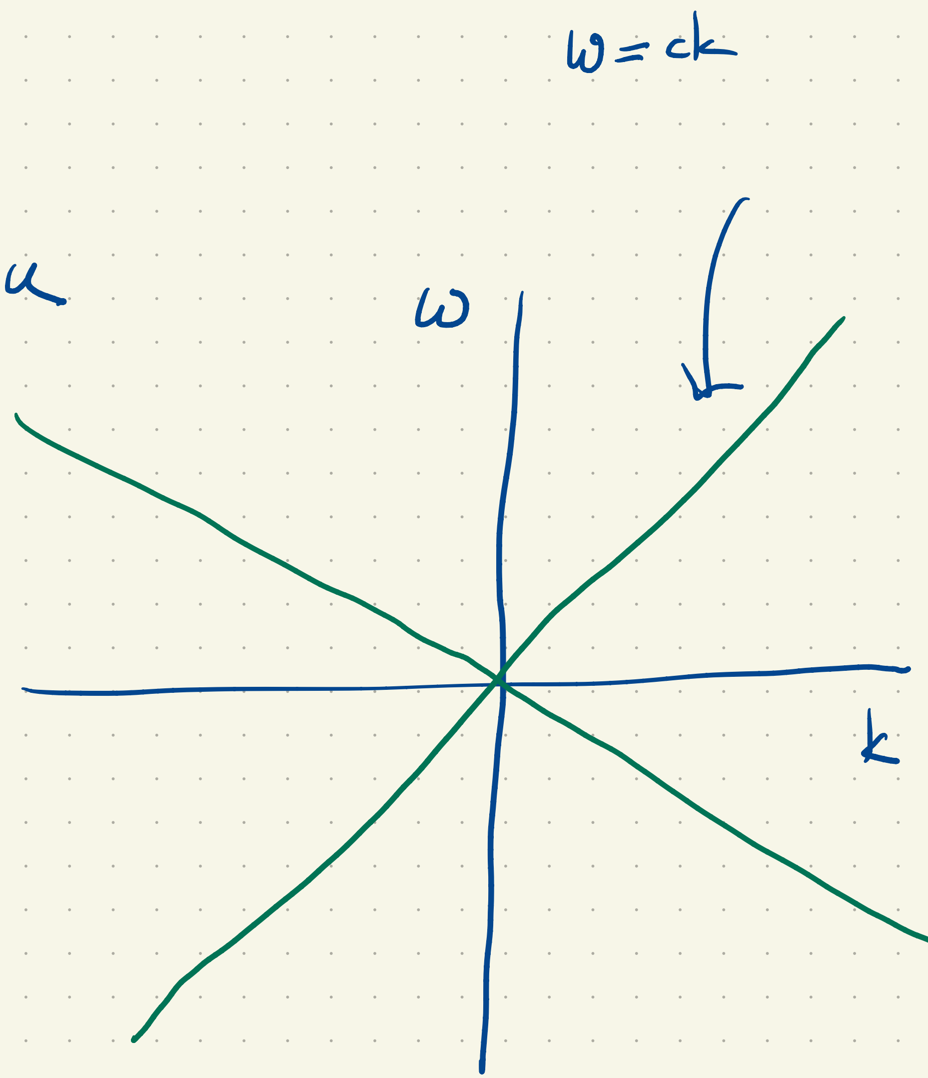
$$u = e^{I(kx-\omega t)}$$

$$-\omega^2 u = c^2 \cdot (-1) k^2 u$$

dispersion relation

$$\omega^2 = c^2 k^2$$

$$\omega = \pm ck$$



$$\frac{\omega}{k} = \pm c$$

Just one phase vel.

E.g. Klein-Gordon

$b>0$

$$u_{tt} = c^2 u_{xx} - bu$$

E.g. Klein-Gordon $\quad b>0$

$$u_{tt} = c^2 u_{xx} - bu \qquad e^{I(kx-\omega t)}$$

$$-\omega^2 u = -c^2 k^2 u - bu$$

$$\omega^2 = c^2 k^2 + b$$

$$\omega = \pm\sqrt{c^2 k^2 + b}$$

$$= \pm k\sqrt{c^2 + b/k^2}$$

E.g. Klein-Gordon $\quad b>0$

$$u_{tt} = c^2 u_{xx} - bu$$

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$$v_p = \frac{\omega_r}{k} = \pm c\sqrt{1 + \frac{b}{(ck)^2}}$$

E.g. Klein-Gordon

$b>0$

$$\boxed{x^2 - y^2 = 7}$$

$$u_{tt} = c^2 u_{xx} - bu$$

$$-\omega^2 u = -c^2 k^2 u - bu$$

$$\omega^2 = c^2k^2 + b$$

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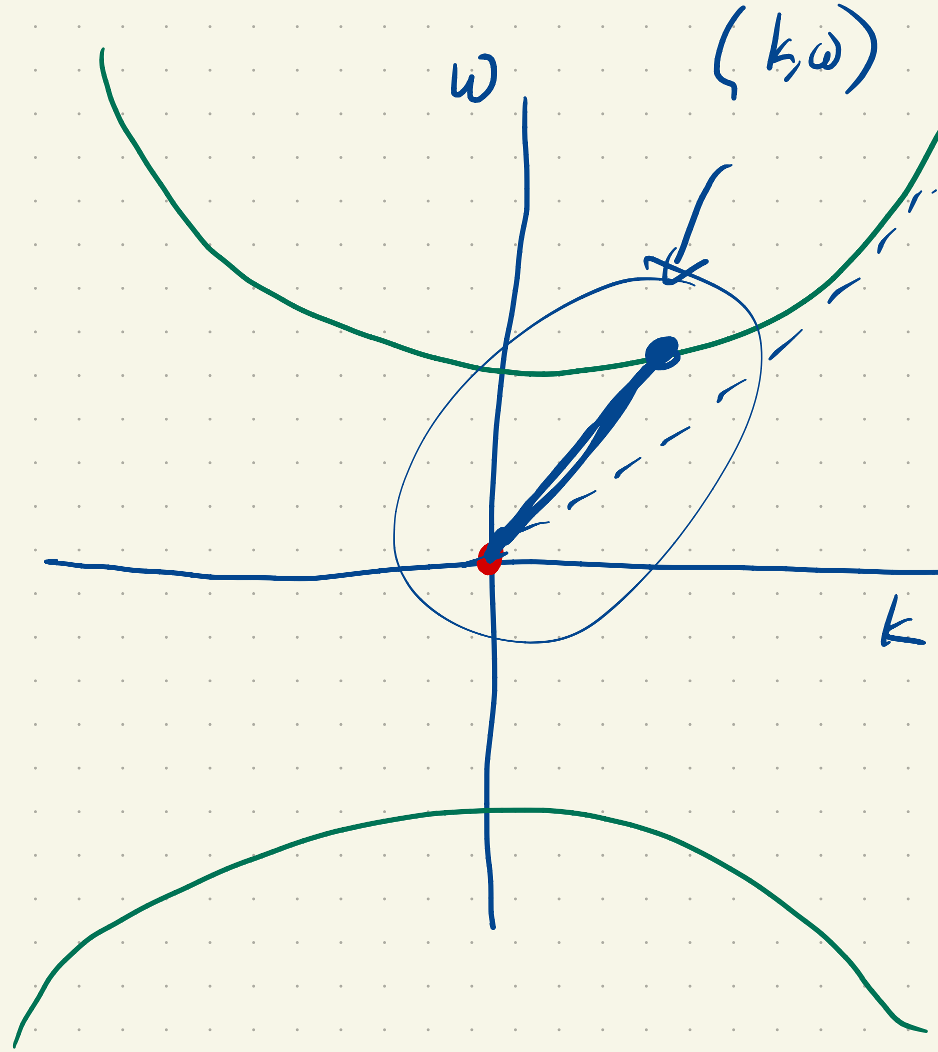

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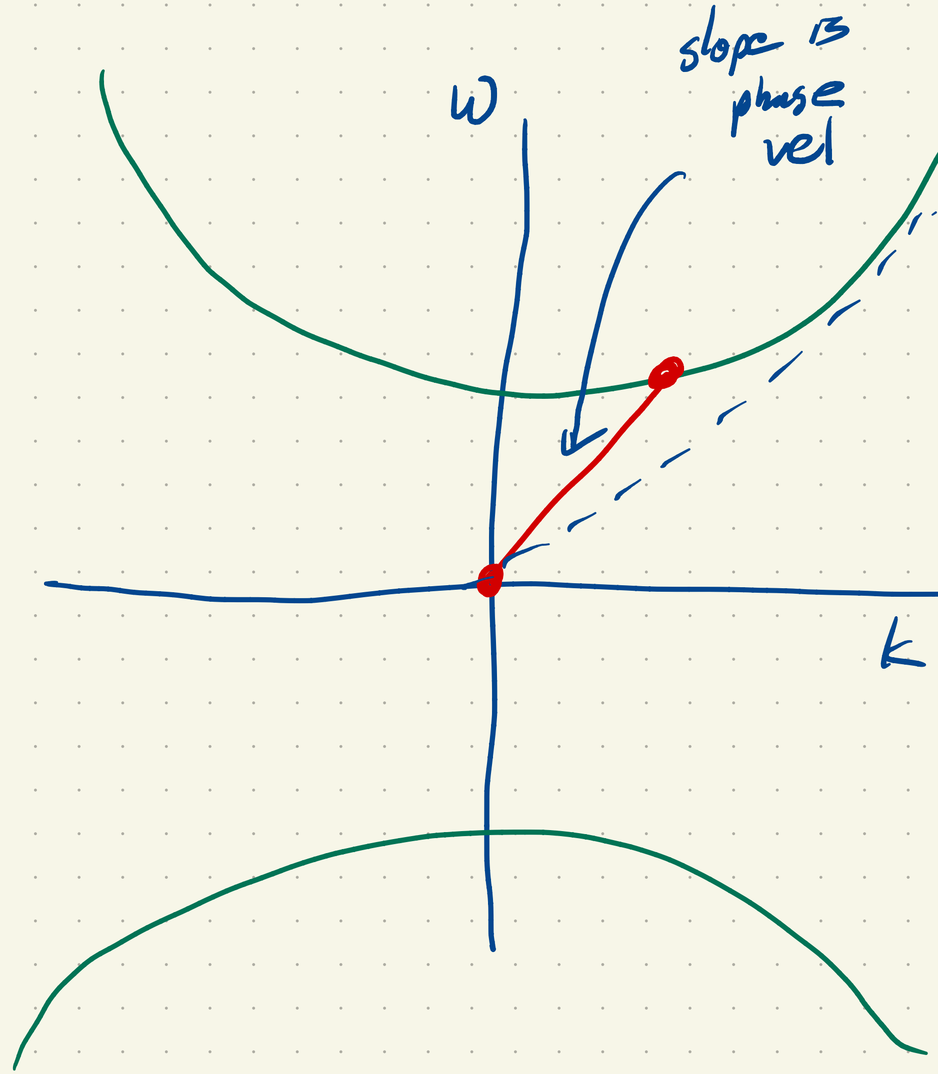


$$\lambda = \frac{2\pi}{k}$$

$$v_p = \frac{\omega}{k} = \pm c\sqrt{1 + \frac{b}{(ck)^2}}$$

$v_p(k)$

$$\lim_{k\to\infty} v_p(k) = c$$

$$\lim_{k\to 0} v_p(k)$$

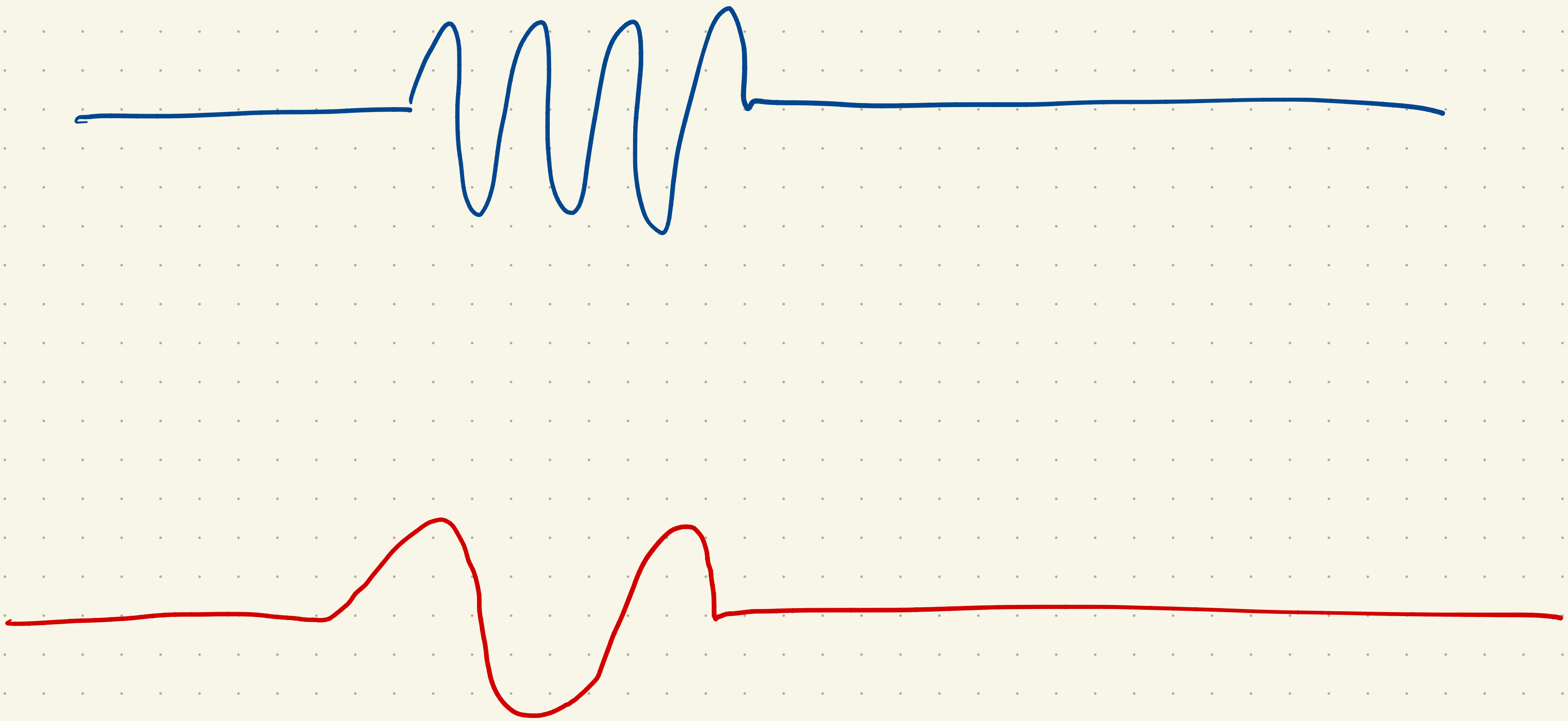
Large wave number $\Longleftrightarrow$ small wave length

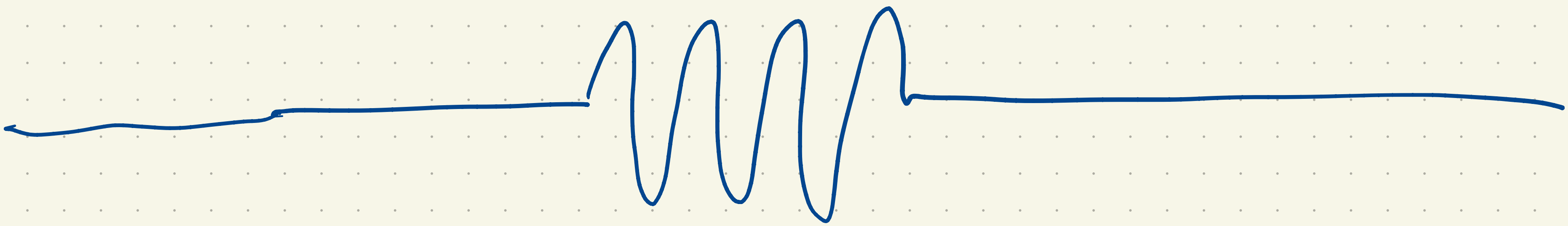
Phase velocity $\approx \pm c$

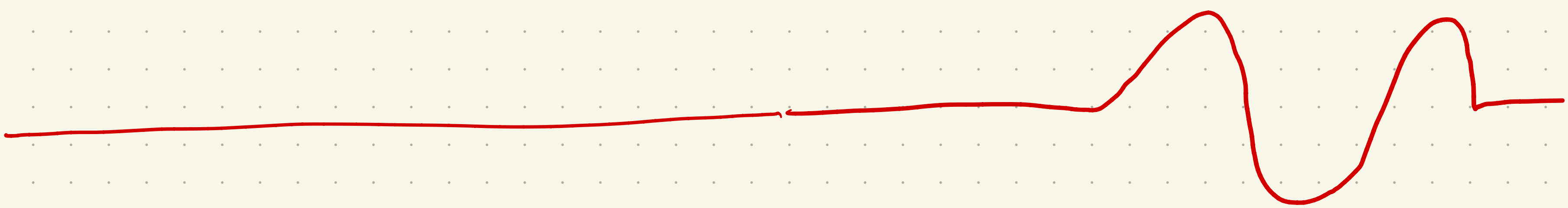
Small wave numbers travel with speed $> c$.

$\boxed{\text{Different speeds for different wave \#s}}$

$\hookrightarrow$ Dispersion.







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