There are problems and a total of 55 points on this exam.

1. [10 points]

Compute the second-order Taylor polynomial $p(x)$ for $f(x) = 1/(1+x)$ centered at $x = 0$. Then use the remainder term to estimate

$$\max_{0\le x\le \frac{1}{2}} |f(x) - p(x)|.$$

2. [10 points]

Compute the local truncation error for the approximation of $f'(x)$ given by

$$\frac{f(x+h) - f(x-h)}{2h}.$$

For full credit you should include an estimate for the size of the error in terms of derivatives of f.

3. [15 points]

Consider the Adams-Moulton method for solving the ODE $u' = f(t,u)$ given by

$$u_{n+2} = u_{n+1} + \frac{h}{12}\left[-f_n + 8f_{n+1} + 5f_{n+2}\right]$$

where f_n is shorthand for $f(t_n, u_n)$.

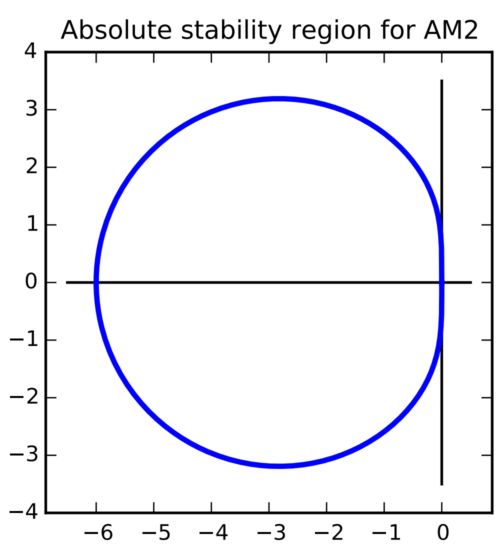
a) Show that this method is consistent.

b) Show that the method is zero stable.

c) What does the zero stability of this method imply?

d) Compute the absolute stability polynomial for this method.

e) The figure below exhibits the boundary of the absolute stability region for this method. Explain a technique for generating a diagram such as this.



4. [10 points]

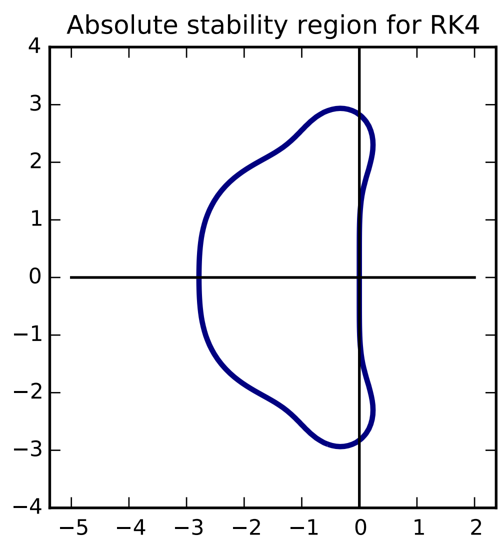
Consider the heat equation $u_t = u_{xx}$ for $0 \le x \le 1$ with boundary conditions $u|_{x=1,0} = 0$. Suppose we discritize in space using the standard centered difference approximation for the second derivative. The system then becomes an ODE

$$u_t = \frac{1}{h^2} Du$$

where h is the space step size and where D is a familiar tri-diagonal matrix. Suppose we now discretize the time variable using the fourth-order RK4 Runge-Kutta method.

a) What order of accuracy do you expect for the local truncation error? [Do **not** prove that the method has this order of accuracy]. Compare it to the order of accuracy of the Crank Nicolson method (i.e. the θ-method with $\theta = 1/2$).

b) Do you expect this method will perform better or worse than Crank Nicolson? While a proof is not needed, your answer should give concrete reasons for your expectations. A disussion of time step size is required. You may find the diagram below helpful.



5. [10 points]

Consider the PDE

$$u_t = u_{xx} + u_x + f$$

for $0 \le x \le 1$ with boundary conditions $u|_{x=0,1} = 0$. Formulate an explict finite difference scheme for approximating the solution of this equation that is first order in time and second order in space. You should use the following notation:

- There are $N+1$ space steps and M time steps.
- k is the time step size and h is the space step size.
- $U_{i,j}$ is the approximation of $u(x_i, t_j)$ where $x_i = ih$ and $t_j = jk$.
- $F_{i,j}$ is the known value of $f(x_i, t_j)$

You do **not** need to show that your method has the requested order of accuracy; you merely need to exhibit the scheme.