Convergence

$$\|L^h E^h\|_p \leq \|L^h\|_p \|E^h\|_p$$

[consistency + stability => convergence]

Consistency: $\tau = O(h^2)$ ✓ $\qquad L^h E^h = \tau$

Stability?

$$L^h U^h = f^h$$

$$L^h u^h = f^h + \tau$$

$$L^h \underbrace{(u^h - U^h)}_{E^h} = \tau$$

Stability

$$E^h = (L^h)^{-1}\tau$$

$$\|E^h\|_p \le \|(L^h)^{-1}\|_p \, \|\tau\|_p$$

$$L^h E^h = \tau$$

- Enemy: does $(L^h)^{-1}$ scale up the small errors τ?

$$\boxed{\|(L^h)^{-1}\|_p \le C \quad \text{as } h \to 0}$$

$$\frac{1}{h^2}\begin{bmatrix} a & c & & \\ b & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}$$

Stability

$$E^h = (L^h)^{-1}\tau \qquad u_{xx} = f$$

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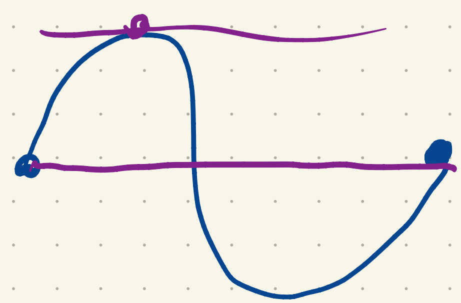
$$e'(a) = 0$$

$$e' = \int_0^x \tau \, ds$$



Stability

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$$e = \int_a^x \int_0^\sigma \tau \, ds \, d\sigma \qquad |e| \le \ell^2 \max |\tau|$$

Stability:

$$\|(L^h)^{-1}\|_2 \leq c\,\ell^2 \quad \text{as } h \to 0$$

If $$\|(L^h)^{-1}\|_p \leq C \quad \text{as } h \to 0,$$

stability in p-norm.

$$\|E^h\|_\infty \leq C_\infty \|\tau\|_\infty = O(h^2)$$

Stability:

If $\|(L^h)^{-1}\|_p \leq C$ as $h \to 0$,

stability in p-norm.

$$\|E^h\|_2 \leq C_2 \underbrace{\|\tau\|_2}_{\left[\sum_{k=1}^{N} (\tau_k)^2\right]^{1/2}} \leq C_2 \sqrt{N} \|\tau\|_\infty$$

Stability:

If $\|(L^h)^{-1}\|_p \leq C$ as $h \to 0$,

stability in p-norm.

$$\|E^h\|_2 \leq C_2 \|\tau\|_2 \leq C_2 \sqrt{N} \|\tau\|_\infty$$

$$\left[\int e^2\right]^{1/2} \quad \frac{1}{\sqrt{N}} \|E_2^h\|_2 \leq C_2 \|\tau\|_\infty = O(h^2)$$

Stability in 2-norm (special case)

$$L^h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \end{bmatrix}$$

$u_{xx} = f$

$L^h = L$ is symmetric $\Rightarrow$ L^{-1} is symmetric

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1

$1/100$

$$L = \begin{bmatrix} -10 & 0 \\ 0 & 0.01 \end{bmatrix} \qquad \|L\| = 10$$

$$L^{-1} = \begin{bmatrix} -1/10 & 0 \\ 0 & 100 \end{bmatrix}$$

$L^h = L$ is symmetric $\Rightarrow$ L^{-1} is symmetric

$$\|L^{-1}\| = 100$$

For a symmetric matrix, $\|B\|_2$ is $|\lambda|$ ↳ largest eigenvalue

$$\|L^{-1}\|_2 \qquad \frac{1}{|\lambda|}$$

smallest eigenvalue

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Eigenvectors: $w_i = \sin(r_k x_i)$ $\quad r_k = \frac{\pi}{\ell} k \quad 1 \leq k \leq N$

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$\sin(x) \sim x$

Eigenvectors: $w_i = \sin(r_k x_i)$ $\quad r_k = \frac{\pi}{\ell} k \quad 1 \leq k \leq N$

$$\lambda_k = \frac{1}{h^2}\left[-2 + 2\cos(r_k h)\right]$$

$$= -\frac{4}{h^2}\sin^2\left(\frac{r_k h}{2}\right)$$

$-1 + \cos(\theta)$

$h = \frac{\ell}{N+1}$

$r_1 = \frac{\pi}{N+1}$

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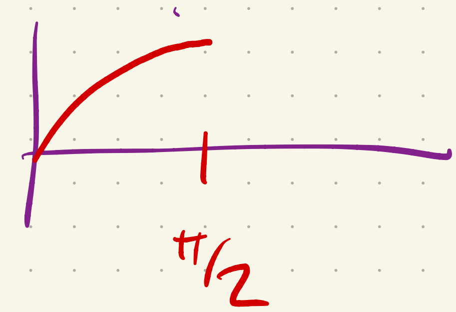

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$$0 < \frac{r_k h}{2} < \frac{\pi}{2}$$

$$\frac{1}{2}\,\frac{\pi}{\ell}\,\frac{k}{N+1}\,\ell$$

$$\frac{k}{N+1}\,\frac{\pi}{2}$$

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Smallest eigenvalue, h small,

$$\lambda_1 \approx -\frac{4}{h^2}\left(\frac{r_1^2 h^2}{4}\right) = -r_1^2 = -\frac{\pi^2}{\ell^2}$$

$$\frac{1}{|\lambda|} \approx \frac{\ell^2}{\pi^2}$$

$e'' = \tau$

$$\|(L^h)^{-1}\|_2 \longrightarrow \frac{\ell^2}{\pi^2} \quad \text{as} \quad h \to 0$$

Stability in 2-norm!

$$\|(L^h)^{-1}\|_2 \ \dot{\approx}\ \frac{\ell^2}{\pi^2} \longleftarrow \text{as predicted.}$$

$$\partial_x^2 \sin(k\pi x) \qquad [0,l]$$

$$-(k\pi)^2 \sin(k\pi x)$$

$$\|(L^h)^{-1}\|_2 \approx \frac{l^2}{\pi^2} \leftarrow \text{as predicted.}$$

Biggest eigenvalues of (L^h) ? $\quad \frac{\pi}{2}\frac{N}{N+1}$

$-\frac{4}{h^2}\sin^2\left(\frac{rh}{2}\right)$ controlled by $\frac{1}{h^2} = O(N^2)$

$$\sin''(kx) = -k^2 \sin(kx)$$

Direct Proof of L^∞ Convergence

$$a_k E_k = -b_k E_{k-1} - c_k E_{k+1} + \tau_k h^2$$

$$b_k = 1 - \frac{p_k h}{2}, \quad c_k = 1 + \frac{p_k h}{2}, \quad a_k = -2 - q_k h^2$$

$$\frac{1}{h^2}\begin{bmatrix} a & c & & 0 \\ b & \ddots & \ddots & \\ & \ddots & \ddots & c \\ 0 & & b & a \end{bmatrix}$$

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Assuming $q_k \leq \gamma < 0$, $\left|\frac{p_k h}{2}\right| \leq \frac{1}{2}$

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Assuming $q_- \leq \gamma < 0$, $\left|\frac{p_k h}{2}\right| < 1$

$$(2 - \gamma h^2)|E_k|$$

$$\leq 2\left[\left(\frac{1}{2} - \frac{p_k h}{4}\right)|E_{k-1}| + \left(\frac{1}{2} + \frac{p_k h}{4}\right)|E_{k+1}|\right] + h^2|\tau_k|$$

$$2|E_k| \leq \left(1 - \frac{p_k h}{2}\right)|E_{k-1}| + \left(1 + \frac{p_k h}{2}\right)|E_{k+1}| + h^2|\tau_k|$$

$$\hat{E} = \max_{1 \leq k \leq N} |E_k|, \qquad \hat{\tau} = \max_{1 \leq k \leq N}$$

$$(2 + \gamma h^2)\hat{E} \leq 2 \max(\hat{E}, |E_0|, |E_{N+1}|) + h^2|\hat{\tau}|$$

$$\leq 2\hat{E} + 2\max(|E_0|, |E_{N+1}|) + h^2|\hat{\tau}|$$

$$\gamma h^2 \hat{E} \leq 2\max(|E_0|, |E_{N+1}|) + h^2|\hat{\tau}|$$

$$(2+\gamma h^2)|E_k| \qquad \boxed{E_0,\ E_{N+1} = 0}$$

$$\leq 2\left[\left(\frac{1}{2} - \frac{p_k h}{4}\right)|E_{k-1}| + \left(\frac{1}{2} + \frac{p_k h}{4}\right)|E_{k+1}|\right] + h^2|\tau_k|$$

$$\hat{E} = \max_{1\leq k\leq N} |E_k| \ ; \quad \hat{\tau} = \max |\tau_k|$$

$$(2+\gamma h^2)|E_k| \leq 2\hat{E} + h^2\hat{\tau}$$

$$(2 \leq k \leq N-1)$$

almost: $\hat{E} \leq \gamma^{-1}\hat{\tau}$

$$\max_k |E_k|$$

$$(2-\gamma h^2)|E_k| \leq 2\hat{E} + h^2 \hat{\tau}$$

$$\boxed{2|E_k| - \gamma h^2 |E_k| \leq 2\hat{E} + h^2 \hat{\tau}}$$

$$2\hat{E} - \gamma h^2 \hat{E} \leq 2\hat{E} + h^2 \hat{\tau}$$

$$-\gamma h^2 \hat{E} \leq h^2 \hat{\tau} \Rightarrow -\gamma \hat{f} \leq \hat{\tau}$$

$$(2+\gamma h^2)|E_k|$$

$$\leq 2\left[\left(\frac{1}{2}-\frac{p_k h}{4}\right)|E_{k-1}| + \left(\frac{1}{2}+\frac{p_k h}{4}\right)|E_{k+1}|\right] + h^2|\tau_k|$$

$$(2+\gamma h^2)|E_1| \leq 2\left[\frac{3}{4}|E_2| + \frac{3}{4}|E_0|\right] + h^2|\tau_k|$$

$$(2+\gamma h^2)|E_1| \leq \frac{3}{2}\hat{E} + \frac{3}{2}|E_0| + h^2|\tau_k|$$

$$|E_1| \leq \frac{3}{4}\hat{E} + \frac{3}{4}|E_0| + |\tau_k| \qquad (h \leq 1)$$

Upshot

$$\hat{E} \leq C \left[\max(|E_0|, |E_{N+1}|) + \hat{\tau} \right]$$

Related Issue: Condition Number.

We want to solve $Ax = b$ but b isn't exact.

We solve $A\hat{x} = \hat{b}$ instead.

What's the relative error in x?

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We solve $A\hat{x} = \hat{b}$ instead.

What's the relative error in x?

$Ax = b \quad x = A^{-1}b$

$A\hat{x} = \hat{b} \quad \hat{x} = A^{-1}\hat{b}$

What's the relative error in x?

$$\frac{\|\hat{x} - x\|}{\|x\|} = \frac{\|A^{-1}(\hat{b} - b)\|}{\|x\|^2}$$

What's the relative error in x?

$$\frac{\|\hat{x}-x\|^2}{\|x\|^2} = \frac{\|A^{-1}(\hat{b}-b)\|}{\|x\|} \leq \frac{\|A^{-1}\| \, \|\hat{b}-b\|}{\|x\|}$$

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$$\leq \underbrace{\|A^{-1}\| \, \|A\|}_{\kappa} \frac{\|\hat{b} - b\|}{\|b\|}$$

$\kappa \to$ condition number

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$$\le \|A^{-1}\| \, \|A\| \, \frac{\|\hat{b} - b\|}{\|b\|}$$

relative error in b

Condition Number

$$\frac{\|\hat{x} - x\|^2}{\|x\|^2} \le \underbrace{\|A^{-1}\| \, \|A\|}_{\text{you can't, in general, beat this}} \frac{\|\hat{b} - b\|}{\|b\|}$$

relative error in r.h.s (double precision: 10^{-15})

$$K(A) = \|A\| \, \|A^{-1}\|$$ in some norms

Condition Number of L^h

$$L^h = \frac{1}{h^2}\begin{bmatrix} -2 & 1 & & 0 \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & -2 \end{bmatrix}$$

$$\|(L^h)^{-1}\|_2 \sim 1 \qquad \kappa \sim 1 \cdot N^2$$

$$\|L^h\|_2 = O(N^2)$$

$\hookrightarrow |\lambda| \leftarrow$ biggest eigenvalue

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$\|A\|_2 = \max(|\lambda|)$ if A is symmetric.

$$\|A^{-1}\|_2 = \max(|\lambda|^{-1}) = \left[\min(|\lambda|)\right]^2$$

$$\min(|\lambda|) \longrightarrow \frac{\pi^2}{\ell^2}, \quad \max(|\lambda|) = \sim \frac{1}{h^2} \sim \frac{N^2}{\ell^2}$$

$$K \sim \frac{1}{\pi^2} N^2$$

$N = 1000 \Rightarrow K \sim 10^6$

$\Rightarrow$ if r.h.s. has a relative error 10^{-15},

solution has a relative error 10^{-9}

$$10^6 \cdot 10^{-15} = 10^{-9}$$