

Linearization

Given a function $f(x)$, its linearization at $x = a$ is the function

$$L(x) = f(a) + f'(a)(x - a).$$

For example, if $f(x) = \sqrt{x}$ and $a = 4$ then $f(4) = 2$ and $f'(4) = 1/(2\sqrt{4}) = 1/4$. So

$$L(x) = 2 + \frac{1}{4}(x - 4).$$

The graph of the linearization is just the tangent line to the curve $y = \sqrt{x}$ at $x = 4$. So we expect that $L(x)$ is a good approximation for \sqrt{x} for x near 4. The point is that computing square roots is hard work (even if your calculator makes it look easy) but computing the value of a linear function like L is easy. In fact your calculator is doing a more sophisticated generalization of the linear approximation: stay tuned in Calculus II!

→ find the equation of the tangent line

at $x = 4$ $x = 4, y = f(4) = \sqrt{4} = 2$

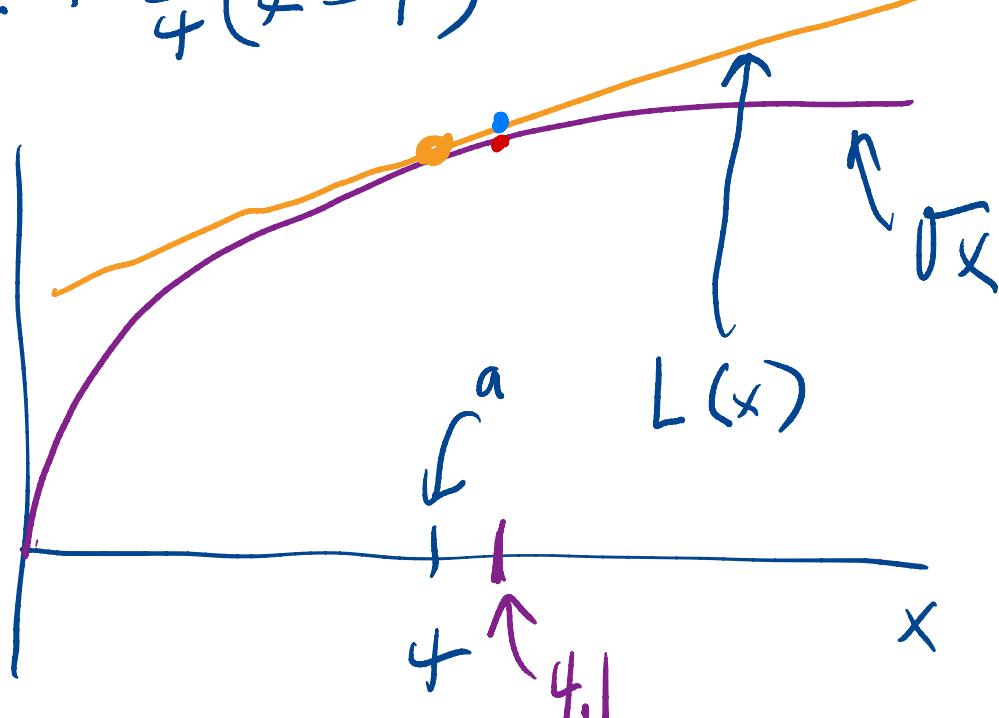
$$y - 2 = \frac{1}{4}(x - 4)$$

$$y = 2 + \frac{1}{4}(x - 4)$$

$$f'(4)$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$



1. Use the linear approximation of $f(x) = \sqrt{x}$ at $x = 4$ to approximate $\sqrt{4.1}$ and compare your result to its approximation computed by your calculator.

$$f(x) = \sqrt{x}$$

$$L(x) = \frac{f(a)}{\downarrow} + \underline{f'(a)}(x-a)$$

$$a=4 \quad f(a)=2 \quad f'(a)=\frac{1}{4}$$

$$L(x) = 2 + \frac{1}{4}(x-4)$$

$L(x) \approx f(x)$ if x is near 4.

$$L(4) = 2 + \frac{1}{4}(4-4) = 2 = \sqrt{4} = f(4)$$

$$L(4.1) \approx f(4.1) = \sqrt{4.1}$$

$$L(4.1) = 2 + \frac{1}{4}(4.1-4) = 2 + \frac{1}{4} \cdot \frac{1}{10}$$

$$\sqrt{4.1} = 2.0248\dots$$

2

$$= 2 + \frac{1}{40} = \boxed{2.025}$$

2. Use the linear approximation to approximate the cosine of $29^\circ = \frac{29}{30} \frac{\pi}{6}$ radians.

$$29^\circ \cdot \frac{2\pi}{360^\circ} = \frac{29}{30} \cdot \frac{\pi}{6}$$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \cos(x)$$

Want to approximate $f\left(\frac{29}{30} \frac{\pi}{6}\right)$

We'll use linearization.

We need: a close $a \rightarrow \frac{\pi}{6}$

1) a is close to $\frac{29}{30} \frac{\pi}{6}$

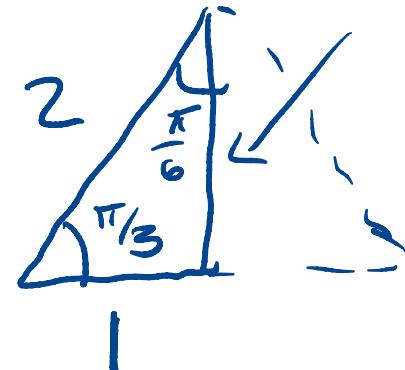
2) we can compute $f(a)$ and $f'(a)$
to obtain the linearization

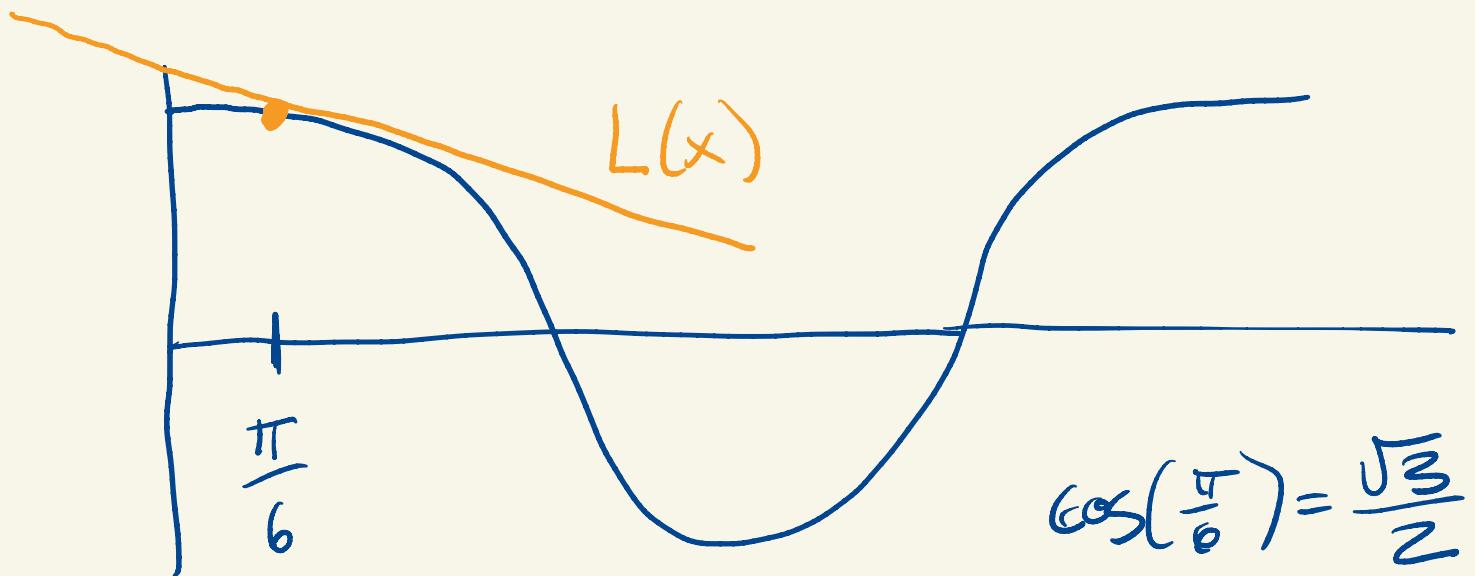
$$L(x) = f(a) + f'(a)(x-a)$$

$$f\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

$$f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$$

$$L(x) = \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(x - \frac{\pi}{6}\right)$$





Want: $\cos\left(\frac{29}{30}\frac{\pi}{6}\right)$

Use: $L\left(\frac{29}{30}\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2}\left(\frac{29}{30}\frac{\pi}{6} - \frac{\pi}{6}\right)$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{1}{30}\frac{\pi}{6}\right)$$

$$\frac{\sqrt{3}}{2} = 0.866\ldots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{60} \frac{\pi}{6}$$

$$= 0.87475\ldots$$

0.87461---

3. Find the linear approximation of $f(x) = \ln(x)$ at $a = 1$ and use it to approximate $\ln(0.5)$ and $\ln(0.9)$. Compare your approximation with your calculator's. Sketch both the curve $y = \ln(x)$ and $y = L(x)$ and label the points $A = (0.5, \ln(0.5))$ and $B = (0.5, L(0.5))$

$$L(x) = f(a) + f'(a)(x-a)$$

$$f(x) = \ln(x)$$

$$a = 1$$

$$\begin{aligned} f(a) &= \ln(a) \\ &= \ln(1) = 0 \end{aligned}$$

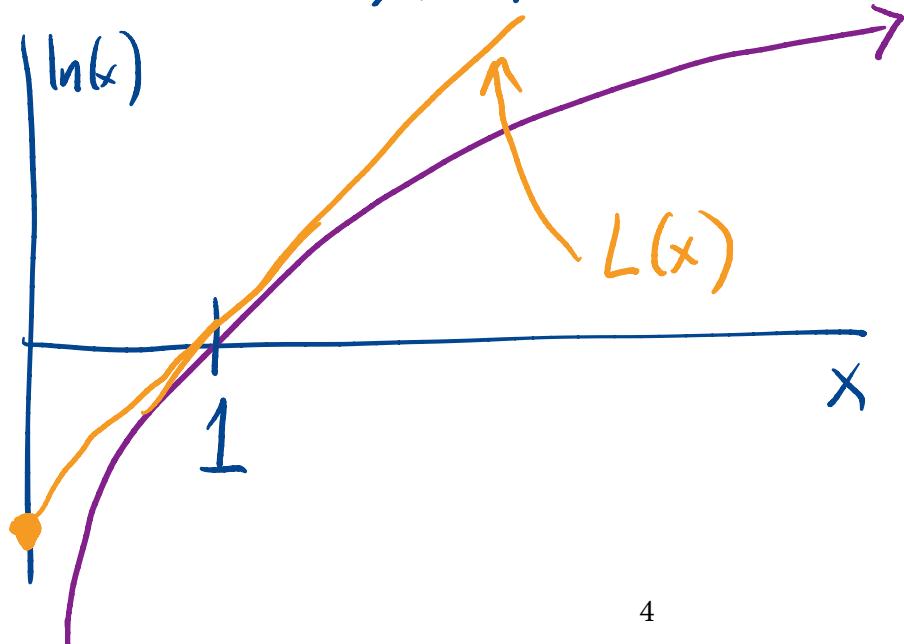
$$f'(a) = \ln'(a) = \frac{1}{a}$$

$$L(x) = x - 1$$

$$f'(1) = \frac{1}{1} = 1$$

$$L(x) = 0 + 1 \cdot (x-1)$$

$$= x - 1$$

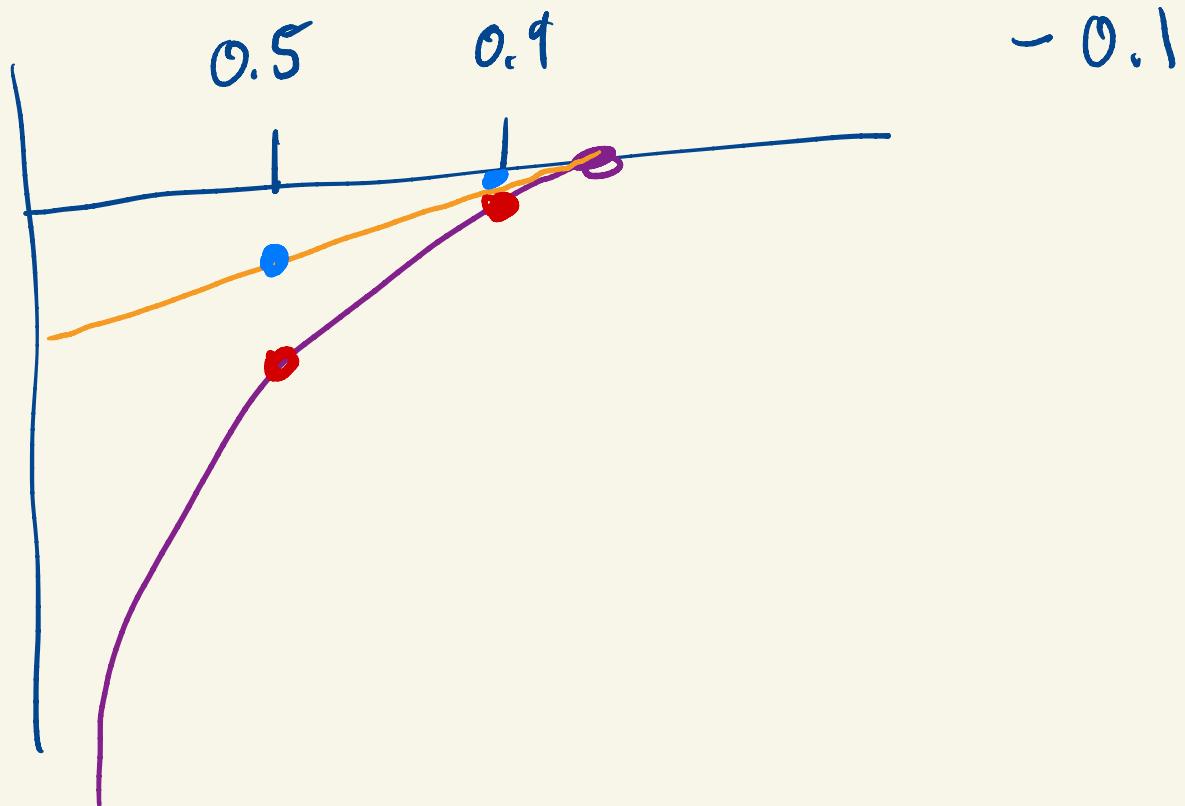


a) $\ln(0.5) \approx L(0.5) = L\left(\frac{1}{2}\right) = \frac{1}{2} - 1$
 \downarrow
 $= -\frac{1}{2} = -0.5$

~ -0.693

b) $\ln(0.9) \approx L(0.9) = L\left(\frac{9}{10}\right) = \frac{9}{10} - 1$
 \downarrow
 $= -\frac{1}{10}$

~ -0.105

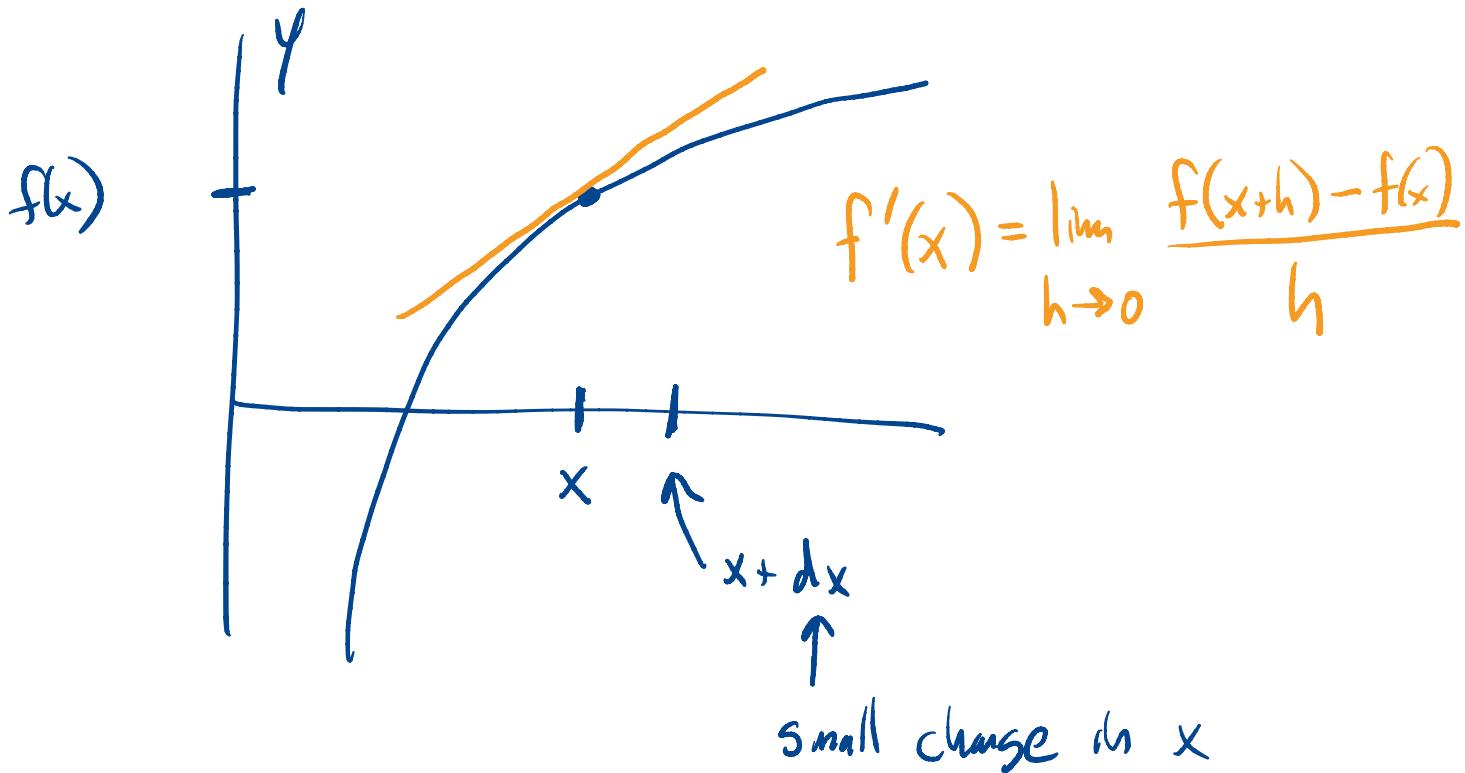


4. Find the linear approximation of $f(x) = e^x$ at $a = 0$ and use it to approximate $e^{0.05}$ and e^1 . Compare your approximations with your calculator's.

Differentials Suppose we have a variable $y = f(x)$. We define its differential to be

$$dy = f'(x)dx$$

where x and dx are thought of as variables you can control. What's the point? The value of dy is an estimate of how much y changes if we change x into $x + dx$. See the graph:



How much does $f(x)$ change when
I wiggle x to $x + dx$

If h is small

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

$$\underbrace{f(x+h) - f(x)}_{\Delta y} \approx f'(x) \cdot h$$

dx

$$\Delta y \approx f'(x) \cdot dx$$

↓
→ $dy \approx \Delta y$

$$dy = f'(x) \cdot dx$$

If $y = f(x)$ and we

change x to $x + dx$ then
 y changes to approximately $y + dy$

where $dy = f'(x) dx$

5. A tree is growing and the radius of its trunk in centimeters is $r(t) = 2\sqrt{t}$ where t is measured in years. Use the differential to estimate the change in radius of the tree from 4 years to 4 years and one month.

$$r(t) = 2\sqrt{t} \quad r'(t) = 2 \cdot \frac{1}{2\sqrt{t}}$$

$$dt = \frac{1}{12} \quad = \frac{1}{\sqrt{t}}$$

$$dr = r'(t) dt$$

$$dr = \frac{1}{\sqrt{t}} dt$$

$t=4$
 $dt = 1/12$

$$dr = \frac{1}{\sqrt{4}} \cdot \frac{1}{12} = \frac{1}{24} \quad 0.0416$$

$$\Delta r = 0.0414$$

- 6.** A coat of paint of thickness 0.05cm is being added to a hemispherical dome of radius 25m. Estimate the volume of paint needed to accomplish this task. [Challenge: will this be an underestimate or an overestimate? Thinking geometrically or thinking algebraically will both give you the same answer.]

7. The radius of a disc is 24cm with an error of $\pm 0.5\text{cm}$. Estimate the error in the area of the disc as an absolute and as a relative error.