Exercise Supplemental 1: Suppose $(a_n) \to a$ and $a \neq 0$. Show that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \neq 0$.

Proof. Since $a \neq 0$ there exists N so that if $n \geq N$ then

$$|a_n - a| < |a|.$$

We claim that if $n \geq N$ then $a_n \neq 0$. Indeed, if for some $n \geq N$ we had $a_n = 0$ then $|a_n - a| = |0 - a| = |a|$. But this contradicts the fact that for this n, $|a_n - a| < |a|$. □

Exercise Supplemental 2:

1. Show that if $a, b \geq 0$ and $a > b$, then $\sqrt{a} > \sqrt{b}$.

   Proof. Suppose $\sqrt{a} > \sqrt{b}$. Then using the fact that if $x < y$ and $c > 0$ we have $xc < yc$ we find
   $$b = \sqrt{b}\sqrt{b} < \sqrt{b}\sqrt{a} < \sqrt{a}\sqrt{a} = a.$$
   That is $b < a$.

   The converse is proved via the contrapositive: if $\sqrt{a} \leq \sqrt{b}$ then $a \leq b$. Indeed, if $\sqrt{a} = \sqrt{b}$ then squaring both sides we have $a = b$. Otherwise, $\sqrt{a} < \sqrt{b}$ and the forward direction implies $a < b$. Either way, $a \leq b$. □

2. Exercise 2.3.1(a)

   Proof. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that if $n \geq N$
   $$|0 - x_n| < \epsilon^2.$$
   As a consqeunce, via part (a), if $n \geq N$ then
   $$\sqrt{x_n} < \epsilon.$$
   That is, if $n \geq N$,
   $$\left|0 - \sqrt{x_n}\right| = \sqrt{x_n} < \epsilon$$
   and $\sqrt{x_n} \to \epsilon$. □

Exercise 2.3.3:

We need to show that $\lim y_n$ exists and that it equals l. To do this, let $\epsilon > 0$. There exists N_1 so that if $n \geq N_1$, then $l - \epsilon < x_n < l + \epsilon$, and there exists N_2 so that if $n \geq N_2$, then $l - \epsilon < z_n < l + \epsilon$. Let $N = \max(N_1, N_2)$. Then if $n \geq N$ we have

$$l - \epsilon < x_n \leq y_n \leq z_n < l + \epsilon.$$

Hence, if $n \geq N$, then

$$|y_n - l| < \epsilon.$$

In conclusion, given $\epsilon > 0$, there exists N so that if $n \geq N$, then $|y_n - l| < \epsilon$. In other words, $\lim y_n = l$.

Exercise 2.3.10: For full credit, all arguments should be short!

(a) False. This is true only if one the limits of (a_n) or (b_n) exist (in which case they both do). As a counterexample, consider $a_n = (-1)^n$ and $b_n = -a_n$.

(b) This is true. Let $\epsilon > 0$. Pick N so that if $n \geq N$then $|b - b_n| < \epsilon$. Exercise 1.2.6(d) then implies that if $n \geq N$, then

$$||b| - |b_n|| \leq |b - b_n| < \epsilon.$$

(c) This is true and simply a consequence of the Algebraic Limit Theorem: $b_n = a_n + (b_n - a_n)$.

(d) This is true. Indeed,

$$-a_n \leq b - b_n \leq a_n$$

for every n. Now apply the squeeze theorem to conclude $b_n - b \to 0$.

Exercise Supplemental 3: Show that if $|b_n| \to 0$, then $b_n \to 0$. Then show that this statement is false if we replace 0 with any other real number.

Proof. Suppose $|b_n| \to 0$. Let $\epsilon > 0$. Pick N so that if $n \geq N$ then $|0 - |b_n|| < \epsilon$. So, if $n \geq N$,

$$|0 - b_n| = |b_n| = |0 - |b_n|| < \epsilon.$$

□

As for the counterexamples, suppose $c > 0$. Let $b_n = -c$ for all n. Then $|b_n| \to c$ but $b_n \to -c \neq c$.

Exercise Supplemental 4: Consider the series $\sum_{n=1}^{\infty} 1/n^2$. Give a careful proof by induction that the partial sums

$$s_k = \sum_{n=1}^{k} 1/n^2$$

satisfy $s_k < 2 - 1/k$.

Proof. We show that the desired inequality holds for $k \geq 2$; it is false when $k = 1$. First, observe that when $k = 2$

$$s_2 = \frac{5}{4} < \frac{3}{2} = 2 - \frac{1}{2}.$$

Suppose for some $k \in \mathbb{N}$ that $s_k < 2 - 1/k$. Then

$$\begin{aligned} s_{k+1} &= s_k + \frac{1}{(k+1)^2} \\ &< 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\ &< 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\ &= 2 - \frac{1}{k} + \left[\frac{1}{k} - \frac{1}{k+1}\right] \\ &= 2 - \frac{1}{k+1}. \end{aligned}$$

□

Exercise 2.4.3(a): Hint: Use the Monotone Convergence Theorem!

Let $a_1 = \sqrt{2}$ and define $a_{k+1} = \sqrt{2 + a_k}$ for every k. We claim the sequence is monotone increasing and bounded above by 2.

Certainly $a_1 \leq 2$. Suppose some $a_k \leq 2$. Then $a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2} = 2$. Hence we have shown by induction that $a_k \leq 2$ for every k.

We now show that the sequence is monotone increasing. Indeed for any k,

$$a_{k+1} = \sqrt{2 + a_k} \geq \sqrt{a_k + a_k} = \sqrt{2a_k} \geq \sqrt{a_k \cdot a_k} = a_k.$$

The Monotone Convergence Theorem implies (a_k) converges to a limit L. Taking the limit of the recursive equation we conclude

$$L = \sqrt{2 + L}$$

and hence $L = 2$ or $L = -1$. But the sequence is increasing from $\sqrt{2}$, which rules out $L = -1$ Hence $L = 2$.