Exercise Supplemental 1: Show that the sequence $(-1)^n$ does not converge.

Proof. Suppose to the contrary that $(-1)^n$ converges to some L. Then there exists N so that if $n \ge N$,

$$|(-1)^{n+1} - L| < 1.$$

By picking values of $n \ge N$ so tha $n + 1$ is even and odd we conclude

$$|-1 - L| < 1 \quad \text{and} \quad |1 - L| < 1.$$

But

$$2 = |2 - L + L| \le |1 + L| + |1 - L| = |-1 - L| + |1 - L| < 1 + 1.$$

Hence $2 < 2$, a contradiction. □

Exercise Supplemental 2:

(a) Show that for all $n \in \mathbb{N}$, $2^n \ge n$.

(b) Show that $\lim_{n\to\infty} 1/2^n = 0$.

Part (a). First, observe that $2^1 = 2 > 1$. Now suppose for some $n \ge \mathbb{N}$ that $2^n \ge n$. Then

$$2^{n+1} = 22^n \ge 2^n + 2^n \ge n + n \ge n + 1.$$

□

Part (b). Let $n \in \mathbb{N}$. Then $2^n > 0$, and hence $0 < 1/2^n$. By part (a), $2^n \ge n$ and hence $1/2^n \le 1/n$. That is, for all $n \in \mathbb{N}$,

$$0 \le 1/2^n \le 1/n.$$

Now let $\epsilon > 0$. Pick N so that $1/N < \epsilon$. Then if $n \ge N$,

$$-\epsilon < 0 \le \frac{1}{2^n} \le \frac{1}{n} \le \frac{1}{N} < \epsilon.$$

That is, if $n \ge N$ then

$$\left|0 - \frac{1}{2^n}\right| < \epsilon$$

and $1/2^n \to 0$. □

Exercise 2.2.2: From the definition, compute the given limits.

Part (a). Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $1/N < 25/3\epsilon$. Then if $n \ge N$

$$\left|\frac{2}{5} - \frac{2n+1}{5n+4}\right| = \frac{3}{25n+20} \le \frac{3}{25n} \le \frac{3}{25N} < \epsilon.$$

□

Part (b). Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $1/N < \epsilon/2$. If $n \geq N$ then

$$\left|0 - \frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} \leq \frac{2n^2}{n^3} = \frac{2}{n} \leq \frac{2}{N} < \epsilon.$$

□

Part (c). We begin with a lemma.

Lemma: If $x, y > 0$ then $x < y$ if and only if $x^{1/3} < y^{1/3}$.

Proof. Suppose $x^{1/3} < y^{1/3}$. Repeatedly using the fact that if $a \leq b$ and $c > 0$ then $ac < bc$ we find

$$x = (x^{1/3})^3 = (x^{1/3})^2 x^{1/3} < (x^{1/3})^2 y^{1/3} < x^{1/3}(y^{1/3})^2 < (y^{1/3})^3.$$

We prove the converse by the contrapositive. That is, we wish to show that if $x^{1/3} \geq y^{1/3}$ then $x \geq y$.

Suppose $x^{1/3} \geq y^{1/3}$. If the inequality is strict, the forward direction shows $x > y$. If equality holds, then cubing both sides of $x^{1/3} = y^{1/3}$ we find $x = y$. Either way, $x \geq y$. □

We now return to the main proof. Let $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that $1/N < \epsilon^3$. Then if $n \geq N$, using the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$, we find

$$\left|0 - \frac{\sin(n^2)}{\sqrt[3]{n}}\right| = \frac{|\sin(n^2)|}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}}.$$

Using the lemma we see that if $n \geq N$,

$$\frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} = \left(\frac{1}{N}\right)^{1/3} \leq (\epsilon^3)^{1/3} = \epsilon.$$

In conclusion, if $n \geq N$,

$$\left|0 - \frac{\sin(n^2)}{\sqrt[3]{n}}\right| < \epsilon$$

and the sequence converges to 0. □

Exercise 2.2.3: Describe what needs to be shown to disprove the given statements.

Solution:

(a) Find a single college where there are no students at least seven feet tall.

(b) Find a single college in the US where every professor gives at least one student a grade less than a B.

(c) Show that at every college in the US there is a student that is less than six feet tall.

Exercise 2.2.6: Prove that limits are unique.

Proof. Let (a_n) that converges to limits L_1 and L_2. Let $\epsilon > 0$ Pick $N_1 \in \mathbb{N}$ so that if $n \geq N_1$ then $|L_1 - a_n| < \epsilon/2$. Similarly, pick $N_2 \in \mathbb{N}$ so that if $n \geq N_2$ then $|L_2 - a_n| < \epsilon/2$. Setting $N = \max(N_1, N_2)$ we find

$$|L_1 - L_2| \leq |L_1 - a_N| + |a_N - L_2| < 2\epsilon/2 = \epsilon.$$

This inequality holds for all $\epsilon > 0$. But then $L_1 = L_2$, for otherwise we could pick $\epsilon = |L_1 - L_2|$ to produce a contradiction. □

Exercise 2.2.5(a): Determine, with a proof, $\lim_{n\to\infty}[[5/n]]$.

Solution:
Claim: The limit is 0.

Proof. let $\epsilon > 0$. Observe that if $n > 5$ then $0 \leq 5/n < 1$ and hence $[[5/n]] = 0$. That is, if $n \geq 6$,

$$|0 - [[5/n]]| = |0 - 0| = 0 < \epsilon.$$

□

Exercise 2.3.9(a)(c):

(a) If (a_n) is a bounded sequence and $b_n \to 0$, show $a_n b_n \to 0$.

(c) Prove Theorem 2.3.3(iii) for the case $a = 0$.

Solution:

(a) *Proof.* Let $M > 0$ be a bound for the sequence (a_n). So $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ so that if $n \geq N$ then

$$|b_n| < \frac{\epsilon}{M}.$$

Then, if $n \geq N$,

$$|0 - a_n b_n| = |a_n||b_n| \leq M|b_n| < M\frac{\epsilon}{M} = \epsilon.$$

□

(c) *Proof.* Suppose $a_n \to L$ and $b_n \to 0$. Then (a_n) is bounded (Theorem 2.3.2). Hence part (a) implies $a_n b_n \to 0$. □