1. Consider the matrix

$$A = \begin{pmatrix} 23 & -8 & 4 \\ 21 & -8 & 5 \\ -126 & 42 & -19 \end{pmatrix}.$$

a) Show that $v_1 = [-1, -2, 3]$, $v_2 = [1, 3, 0]$ and $v_3 = [0, 1, 2]$ are eigenvectors of A, and determine their associated eigenvalues.

b) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

c) Compute the solution of

$$u' = Au$$

with initial condition $u(0) = v_2 + v_3$. Show, by plugging your solution into the ODE, that your solution really is a solution.

d) Determine the exact solution of

$$u' = Au$$

with initial condition $u(0) = [1, 5, 5]$.

Solution, part a:
To show that x is an eigenvector, it is enough to show that $Ax = \lambda x$ for some number λ. A routine computation shows that $Av_1 = -5v_1$ $Av_2 = -v2$ and $Av_3 = 2v_3$ Thus $\lambda_1 = -5$, $\lambda_2 = -1$ and $\lambda_3 = 2$.

Solution, part b:
The solution is $u(t) = e^{2t}v_3$. Observe that $u(0) = e^0 v_3 = v_3$. Moreover, $u'(t) = 2e^{2t}v_3$ and $Au = e^{2t}Av_3 = e^{2t}2v_3 = 2u$. Comparing we find $u' = 2u = Au$.

Solution, part c:
The solution is $u(t) = e^{-t}v_2 + e^{2t}v_3$. Indeed,

$$u(0) = e^0 v_2 + e^0 v_3 = v_2 + v_3.$$

Moreover,

$$u'(t) = -e^{-t}v_2 + 2e^{2t}v_3$$

and

$$Au(t) = e^{-t}Av_2 + e^{2t}Av_3 = -e^{-t}Av_2 + 2e^{2t}Av_3.$$

So $u' = Au$.

Solution, part d:
We can write

$$[1, 5, 5] = v_1 + 2v_2 + v_3.$$

The solution of the IVP is therefore

$$u(t) = e^{-5t}v_1 + 2e^{-t}v_2 + e^{2t}v_3.$$

2. Suppose you wish to apply the RK4 method to solve the ODE of the previous problem. What is the largest time step you can use before issues concerning absolute stability arise in your solution?

Solution:
From the last homework assignment we know that the region of absolute stability for RK4 extends along the negative real axis up to about -2.9. For and ODE of the form $u' = \lambda u$ with $\lambda < 0$, we will encounter difficulties due to stability unless $z = \lambda h$ lies in the region of absolute stability. The ODE of the previous problem breaks into three decoupled ODEs of the form $u' = \lambda u$ along the eigenvectors. There are two negative eigenvalues, -1 and -5. Thus we need $-5h > -2.9$ and $-1h > -2.9$. Clearly the first condition is more restrictive and we need

$$h < \frac{2.9}{5} \approx 0.58.$$

3.

a) Use your Newton solver from last week's homework to implement the trapezoidal rule for solving systems of ODEs.

b) Determine the exact solution to the problem

$$\begin{aligned} u' &= 1 \\ v' &= v - u^2 \end{aligned} \tag{1}$$

with initial condition $u(0) = 0$ and $v(0) = 1$.

c) Test your solver against the previous exact solution and confirm that it has the predicted order of accuracy.

Solution, part a:
See worksheet for the implementation.

Solution, part b:
The system is what is called semi-decoupled. We can solve the system for u, and then knowing u, solve for v. Indeed, $u' = 1$ together with $u(0) = 0$ implies $u(t) = t$. Thus $v' = v - t^2$. A particular solution is $v(t) = t^2 + 2t + 2$ (just guess a solution of the form of a quadratic and solve for the coefficients), so the general solution is

$$v(t) = ce^t + t^2 + 2t + 2$$

The initial condition $v(0) = 1$ implies $c = -1$.

Solution, part c:

4. Implement the explict method for solving the heat equation with right-hand side function

$$u_t = u_{xx} + f$$

on $0 \le x \le 1$ and $0 \le t \le T$. You function should have the following signature:

forcedheat(f,u0,N,M)

where

- $f(x,t)$ is a function and provedes the desired forcing term
- $u0(x)$ is a function and provides the desired initial condition.
- $N+1$ is the number of interior spatial steps
- M is the number of time steps

It should return (x,t,u) where x is an array of grid coordinates that includes 0 and 1, t is a vector of t coordinates that includes 0 and T, and where u is an $(N+2) \times (M+1)$ matrix where column j encodes the solution at time t_j.

Test your code as follows

- Compute what f is if the solution is $u(t,x) = \sin(t)x(1-x)$.
- Now, working on $0 \le x \le 1$ and $0 \le t \le 2\pi$ compute solutions with this forcing term and compare your solution with the exact solution. By working with various grid sizes, confirm that your code has the expected order of convergence.

Solution:
If $u = \sin(t)x(1-x)$ then

$$u_t = \cos(t)x(1-x)$$

and

$$u_{xx} = -2\sin(t).$$

Thus

$$f = u_t - u_{xx} = \cos(t)x(1-x) + 2\sin(t).$$

For convergence analysis, see the worksheet.