There are **6** problems and a total of 28 points on this exam.

1. [6 points]

Define a map from $\phi : \mathbb{C} \to \mathbb{R}^{2\times 2}$ by

$$\phi(a+ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{1}$$

Show that the restriction of ϕ to $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ is a group homomorphism into the group of 2×2 invertible real matrices. Compute the kernel of ϕ.

2. [6 points]

On you homework you showed that the matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $a \neq 0$ form a group, and that the inverse of such a matrix is

$$\begin{pmatrix} 1/a & -b/a \\ 0 & 1 \end{pmatrix}.$$

a) Show that the matrices of the form

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

are a subgroup.

b) Show (using conjugation) that this subgroup is, in fact, a normal subgroup.

3. [4 points]

We defined the length of a quaternion $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ to be $|q| = \sqrt{a^2+b^2+c^2+d^2}$. Use the multiplicative property of the determinant to show that $|q_1 q_2| = |q_1||q_2|$ for all quaternions. You may use the fact that quaternions can be represented as a matrix, but you need to be specific about the form of this representation.

4. [4 points]

Define an isometry of $\mathbb{R}^n$. Then show that multiplication by a unit quaternion is an isometry of $\mathbb{R}^4$

5. [4 points]

Represent the reflection through the plane orthogonal to $(1,1,1,1)$ in $\mathbb{R}^4$ as an operation involving quaternions.

6. [8 points]

a) Exhibit a surjective group homomorphism from the unit quaternions (i.e. $SU(2)$) to the rotations of $\mathbb{R}^3$ (i.e. $SO(3)$) in terms of quaternion multiplication. [You need only state the map; you do not need to prove it is a homomorphism or that it is surjective.]

b) State the kernel of the map in part a; you need not prove that it is the kernel.

7. [Extra Credit (4 points)]
In problem 2b, an alternative approach would be to show that the subgroup is the kernel of some group homomorphism. Exhibit the homomorphism.