Exercise 1.3.9: (a) If $\sup A < \sup B$ then show that there exists an element $b \in B$ that is an upper bound for A.

(b) Give an example to show that this is not necessarily the case if we we only assume $\sup A \leq \sup B$.

Proof (a). Let $\epsilon = \sup B - \sup A$. By Lemma 1.3.7 there is a $b \in B$ such that $b > \sup B - \epsilon$. Notice that $\sup B - \epsilon = \sup A$. Hence $b > \sup A$. Since $\sup A$ is an upper bound for A, and since $b > \sup A$, b is an upper bound for A. □

Example for (b): We can take $A = B = (0, 1)$ Then $\sup A = \sup B = 1$. But if $b \in B$ then so is $(b + 1)/2$. Since $(b + 1)/2 > b$ and since $(b + 1)/2 \in B = A$, b is not an upper bound for A.

Exercise 1.3.11 : Decide if the following statements are true. Give a short proof for the true statements and a counterexample for the false statements.

(a) If A and B are nonempty, bounded, and satisfy $A \subseteq B$ then $\sup A \leq \sup B$.

(b) If $\sup A < \inf B$ for sets A and B, then there exists $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and $b \in B$.

(c) If there exists $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$ then $\sup A < \inf B$.

Solution:

(a) This is true. Indeed, every upper bound for B is evidently an upper bound for A. In particular $\sup B$ is an upper bound for A. Thus $\sup A \leq \sup B$ by definition of $\sup A$.

(b) This is true. Indeed, let $c = (\sup A + \inf B)/2$. Then $\sup A < c < \inf B$ by basic arithmetic. If $a \in A$ then $a \leq \sup A < c$ since $\sup A$ is an upper bound for A. Similarly, if $b \in B$ then $c < \inf B < b$. So for all $a \in A$ and $b \in B$, $a < c < b$.

(c) This is false. Let $A = (0, 1)$ and $B = (1, 2)$. Then for all $a \in A$ and $b \in B$, $a < 1 < b$. But $\sup A = \inf B$; we do not have a strict inequality.

Exercise 1.4.1: Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$ then ab and $a + b \in \mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $a \neq 0$ then $at \in \mathbb{I}$ as well.

(c) Part (a) says that the rational numbers are closed under multiplication and addition. What can be said about st and $s + t$ when $s, t \in \mathbb{I}$?

Solution:

(a) Suppose $a, t \in \mathbb{Q}$. Then $a = m/n$ and $b = p/q$ where $m, n, p, q \in \mathbb{Z}$ and $n, q \neq 0$. But then $a + b = (mq + np)/(nq)$ since $mq + np \in \mathbb{Z}$ and $nq \in \mathbb{Z}$ we conclude that $a + b \in \mathbb{Q}$. Similarly, $at = (mp)/(nq)$ so $at \in \mathbb{Q}$.

(b) Suppose $a + t = c$ where a and c are rational. Then $t = c - a$ which, by part a, is rational. Hence if a is rational and t is irrational, then c is irrational.

Suppose that $at = c$ where a and c are rational and $a \neq 0$. Then $t = c/a \in \mathbb{Q}$. So if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ with $a \neq 0$, then $at \in \mathbb{I}$

(c) Not much can be said about the product and sum of irrational numbers. For example, the sum $\sqrt{2} + \sqrt{2} = 2\sqrt{2} \in \mathbb{I}$ by part b). But $\sqrt{2} + (-\sqrt{2}) = 0 \in \mathbb{Q}$. Similarly, $\sqrt{2} \cdot (1/\sqrt{2}) = 1$. Note that $1/\sqrt{2}$ is irrational, for otherwise if $1/\sqrt{2} = c$ is rational, then $\sqrt{2} = 1/c$ and is also rational. So the product of two irrational number can be rational. On the other hand, it is known that $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{6}$ are all irrational (emulate the proof for $\sqrt{2}$). Since $\sqrt{6} = \sqrt{2}\sqrt{3}$ it is also possible for the product of irrational numbers to be irrational.

Exercise 1.4.2: Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + (1/n)$ is an upper bound for A but $s - (1/n)$ is not an upper bound for A. Show that $s = \sup A$.

Solution:

We first show that if $x < s$, then x is not an upper bound for A. Indeed, suppose $x < s$ and pick $n \in \mathbb{N}$ such that $1/n < s - x$. Observe that $x < s - (1/n)$. Since $s - (1/n)$ is not an upper bound for A there exists $a \in A$ such that $s - (1/n) < a$. But then $x < a$ as well, so x is not an upper bound. We conclude, therefore, that if x is an upper bound for A then $s \leq a$.

To show that $s = \sup A$ it remains to show that s is an upper bound for A. Indeed, suppose $x > s$ and pick $n \in \mathbb{N}$ such that $1/n < x - s$. Hence $s + (1/n) < x$. Now $s + (1/n)$ is an upper bound for A and therefore $x \notin A$. We have therefore shown that if $x > s$ then $x \notin A$. Equivalently, if $x \in A$ then $x \leq s$.

Exercise 1.4.3: Show that $\cap_{n=1}^{\infty}(0, 1/n) = \emptyset$.

Solution:
We need show that 0 is a lower bound, and that if b is any other lower bound, then $b \leq 0$. Clearly 0 is a lower bound since $1/n > 0$ for every $n \in \mathbb{N}$. Suppose b is any other lower bound. We cannot have $b > 0$, for otherwise (by Theorem 1.4.2) there exists $n \in \mathbb{N}$ such that $1/n < b$ and b is not a lower bound. Hence $b \leq 0$ as required.

Exercise 1.4.4: Let $a < b$ be real numbers and let $T = [a, b] \cap \mathbb{Q}$. Show that $\sup T = b$.

Solution:
We observe trivially that b is an upper bound for T. Thus, we need only show that if c is an

upper bound for T, then $b \leq c$. Proceeding via the contrapositive, suppose $c < b$; we will show that c is not an upper bound for T. Indeed, consider $d = \max(a, c)$. Since $d < b$, there is a rational number $q \in (d, b)$. Since $(d, b) \subseteq [c, d]$ we know that $q \in T$. Since $c \leq d < q$, c is not an upper bound for T.

Exercise 1.4.5: Use Exercise 1.4.1 to provide a proof of Corollary 1.4.4 by considering real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.

Solution:
Suppose $a < b$. We wish to find an irrational number c with $a < c < b$. Consider the interval $(a - \sqrt{2}, b - \sqrt{2})$. By Theorem 1.4.3 we can find a rational number r such that $a - \sqrt{2} < r < b - \sqrt{2}$. Hence $a < r + \sqrt{2} < b$. Let $c = r + \sqrt{2}$. By exercise 1.4.2, c is the sum of a rational number and an irrational number, and is therefore irrational. Thus c satisfies the desired properties.