In the first part of this worksheet we will get to know a method for computing an approximation of $\sqrt{2}$ to many digits of accuracy using only addition, subtraction, multiplication and division, and indeed using only a few such operations.

1. Consider the function

$$F(x) = x^2 - 2.$$

If we solve $F(a) = 0$ for some $a \geq 0$, what is the value of a?

$a = \sqrt{2}$

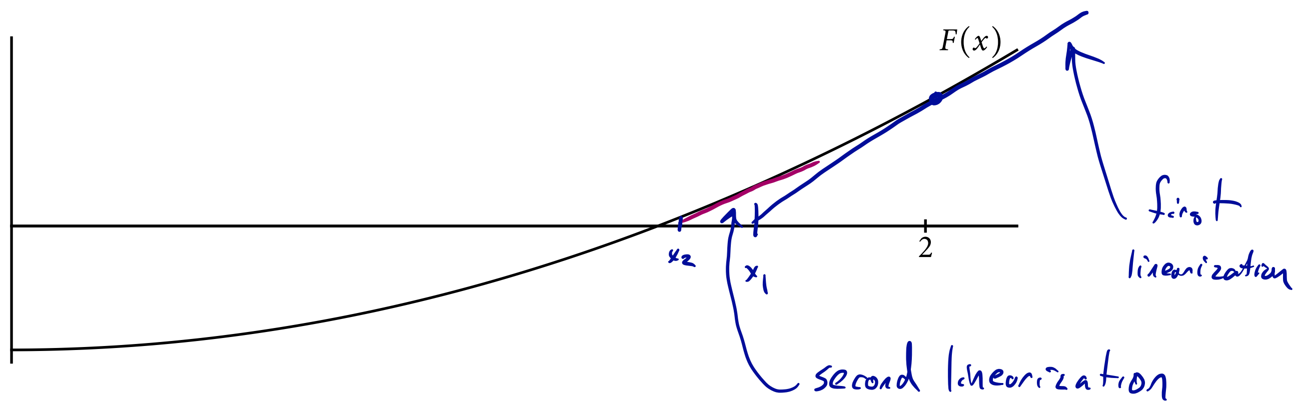
2. Find the linearization $L(x)$of $F(x)$ at $x = 2$. Leave your answer in point-slope form.

$F(2) = 2$

$F'(2) = 4$

$L(x) = 2 + 4(x-2)$

3. I've graphed $F(x)$ for you below. Add to this diagram the graph of $L(x)$.



4. Find the number x_1 such that $L(x_1) = 0$.

$4(x-2) = -2$

$x - 2 = -\frac{1}{2}$

$x = \frac{3}{2} \longrightarrow \boxed{x_1 = \frac{3}{2}}$

5. What good is the number x_1? Keep in mind that you want to solve $F(x) = 0$. You solved $L(x) = 0$ instead.

Since $F(x) \approx L(x)$ near $x=2$, if $L(3/2) = 0$ then $F(3/2) \approx 0$.

6. In the diagram above, label the point x_1 on the x-axis.

7. Let's do it again! Find the linearization $L(x)$ of $F(x)$ at $x = x_1$.

$F(3/2) = \frac{9}{4} - 2 = \frac{1}{4}$

$F'(3/2) = 3$

$L(x) = \frac{1}{4} + 3\left(x - \frac{3}{2}\right)$

8. Add the graph of this new linearization to your diagram on the first page.

9. Find the number x_2 such that $L(x_2) = 0$. Then label the point $x = x_2$ in the diagram.

$3\left(x - \frac{3}{2}\right) = -\frac{1}{4}$

$x = \frac{3}{2} - \frac{1/4}{3} = \frac{3}{2} - \frac{1}{12} = \frac{17}{12}$

10. To how many digits does x_2 agree with $\sqrt{2}$

$\frac{17}{12} = \underbrace{1.41}6\ldots \qquad \sqrt{2} = \underbrace{1.41}4\ldots$ → 3 digits

11. Let's be a little more systematic. Suppose we have an estimate x_k for $\sqrt{2}$.

- Compute $F(x_k)$. $\quad x_k^2 - 2$
- Compute $F'(x_k)$. $\quad 2x_k$
- Compute the linearization of $F(x)$ at $x = x_k$.

$L(x) = L(x) = (x_k^2 - 2) - 2x_k(x - x_k)$

- Find the number x_{k+1} such that $L(x_{k+1}) = 0$. You should try to find as simple an expression as you can.

$2x_k(x - x_k) = 2 - x_k^2$

$x = x_k + \frac{2 - x_k^2}{2x_k} = \frac{x_k}{2} + \frac{1}{x_k}$

$$\boxed{x_{k+1} = \frac{x_k}{2} + \frac{1}{x_k}}$$

12. Starting with $x_0 = 2$, compute x_1 and x_2 with your shiny new formula. Verify that they agree with your earlier expressions for x_1 and x_2.

$x_1 = 2/2 + \frac{1}{2} = \frac{3}{2}$ ✓ $\quad x_2 = \frac{3/2}{2} + \frac{1}{3/2}$

$= \frac{3}{4} + \frac{2}{3} = \frac{17}{12}$ ✓

13. Compute x_4. To how many digits does it agree with $\sqrt{2}$?

$x_3 = 1.4142156862745097$

$x_4 = 1.4142135623746899$

$\sqrt{2} = 1.4142135623730951$

16 digits of accuracy!

Newton's Method In General

We wish to solve $F(x) = 0$ for a differentiable function $F(x)$. We have an initial estimate x_0 for the solution.

Linearization! $L(x) = F(x_0) + F'(x_0)(x - x_0)$

Solve $L(x) = 0$: $\quad F'(x_0)(x - x_0) = -F(x_0)$

$$x - x_0 = -\frac{F(x_0)}{F'(x_0)}$$

$$x = \underbrace{x_0 - \frac{F(x_0)}{F'(x_0)}}_{x_1}$$

In general $\boxed{x_{k+1} = x_k - \frac{F(x_k)}{F'(x_k)}}$

14. Try to solve

$$e^{-x} - x = 0$$

by hand.

$$e^{-x} = x$$

$$\ln(e^{-x}) = \ln(x)$$

$$-x = \ln(x) \qquad \text{crud.}$$

15. Explain why there is a solution between $x = 0$ and $x = 1$.

Let $F(x) = e^{-x} - x$

$F(0) = 1 - 0 = 1$

$F(1) = e^{-1} - 1 < 0$

Intermediate value theorem!

16. Starting with $x_0 = 1$, find an approximation of the solution of $e^{-x} - x = 0$ to 6 decimal places. During your computation, keep track of each x_k to at least 10 decimal places of accuracy.

$F(x) = e^{-x} - x$

$F'(x) = -e^{-x} - 1$

$$x_{k+1} = x_k - \frac{e^{-x_k} - x_k}{-e^{-x_k} - 1} = x_k + \frac{1 - x_k e^{x_k}}{1 + e^{x_k}}$$

$$= \frac{x_k + 1}{e^{x_k} + 1}$$

$x_0 = 1$

$x_1 = 0.5378828427\ldots$

$x_2 = 0.5669869914\ldots$

$x_3 = 0.5671432859\ldots$

$x_4 = 0.\underbrace{567143}_{\text{looks like these are right}}2904\ldots$

(in fact, x_4 is correct to 16 digits of accuracy!)