Exercise 1.2.6 [Modified]: Use the triangle inequality to establish the following inequalities:

(a) $|a - b| \le |a| + |b|$;

(b) $||a| - |b|| \le |a - b|$.

Solution:

(a) *Proof.* By the triangle inequality and the fact that $|x| = |-x|$ for every $x \in \mathbb{R}$ we have

$$|a - b| \le |a| + |-b| = |a| + |b|.$$

□

(b) We start with a handy lemma:

Lemma: If $a \in \mathbb{R}$ and $M \ge 0$, and if

$$-M \le a \le M,$$

then $|a| \le M$.

Proof. We have two cases. If $a \ge 0$, then $|a| = a \le M$ as claimed. Otherwise, we have $|a| = -a$ and $-M \le a$. So $|a| = -a \le M$ as claimed. □

Now for the main result.

Proof. Notice that

$$|a| = |a - b + b| \le |a - b| + |b|.$$

Hence

$$|a| - |b| \le |a - b|.$$

But also,

$$|b| = |b - a + a| \le |b - a| + |a| = |a - b| + |a|.$$

Hence

$$-|a - b| \le |a| - |b|.$$

We have therefore shown that

$$-|a - b| \le |a| - |b| \le |a - b|.$$

By the Lemma we can conclude

$$||a| - |b|| \le |a - b|.$$

□

Exercise 1.2.7(b), (d): Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) : x \in A\}$.

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(d) Form and prove a conjecture concerning $f(A \cup B)$ and $f(A) \cup f(B)$.

Solution:

(b) Consider $f(x) = x^2$. Let $A = (-1, 0)$ and $B = (0, 1)$. Then $f(A) = f(B) = (0, 1)$, and $f(A) \cap f(B) = (0, 1)$. But $A \cap B = \emptyset$ and $f(A \cap B) = \emptyset$ as well.

(d) We claim that $f(A \cup B) = f(A) \cup f(B)$.

Proof. Suppose $y \in f(A \cup B)$. Then there exists $x \in A \cup B$ such that $y = f(x)$. If $f \in A$, then $y \in f(A)$. Otherwise $x \in B$ and $y \in f(B)$. Either way, $y \in f(A) \cup f(B)$. Hence $f(A \cup B) \subseteq f(A) \cup f(B)$.

Conversely, suppose $y \in f(A) \cup f(B)$. If $y \in f(A)$ then there exists $x \in A$ such that $y = f(x)$. This same $x \in A \cup B$ and hence $y \in f(A \cup B)$. A similar argument works if $y \in f(B)$ and we conclude that $f(A) \cup g(B) \subseteq g(A \cup B)$.

Since we have proven both set inclusions, $f(A \cup B) = f(A) \cup f(B)$. □

Exercise 1.2.11: Form the logical negation of each claim. Do not use the easy way out: "It is not the case that..." is not permitted

(a) For all real numbers satisfying $a < b$, there exists $n \in \mathbb{N}$ such that $a + (1/n) < b$.

(b) Between every two distinct real numbers there is a rational number.

(c) For all natural numbers $n \in \mathbb{N}$, $\sqrt{n}$ is either a natural number or is an irrational number.

(d) Given any real number $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.

Solution:

(a) There exists a pair of real numbers satisfying $a < b$ such that for any $n \in \mathbb{N}$, $a+1/n \geq b$.

(b) There exists a pair of distinct real numbers such that there is no rational number between them.

(c) For all natural number n, $\sqrt{n}$ is either a natural number or an irrational number

(d) There exists a natural number n such that $\sqrt{n}$ is rational, but not natural.

Exercise 1.2 Supplement: Show that the sequence $(x_1, x_2, x_3, \ldots)$ defined in Example 1.2.7 is bounded above by 2. That is, show that for every $i \in \mathbb{N}$, $x_i \leq 2$.

Proof. Recall that the sequence is defined by $x_1 = 1$ and $x_{n+1} = \frac{1}{2}x_n + 1$. We use induction to prove that the sequence is bounded above by 2. First, observe that $x_1 = 1$ by definition. Since $1 < 2$, this establishes the base case. Now suppose that $x_n < 2$ for some n. Then

$$x_{n+1} = \frac{1}{2}x_n + 1 < \frac{1}{2} \cdot 2 + 1 = 1 + 1 = 2.$$

Thus $x_{n+1} < 2$, as desired. □

Exercise 1.3.5: Let A be bounded above and let $c \in \mathbb{R}$. Define the sets $c + A = \{a + c : a \in A\}$ and $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup(A)$.

(b) Postulate a similar statment for $\sup(cA)$ when $c < 0$.

Proof (a). The case where $c = 0$ is trivial; $0A$ consists only of the zero element, and the supremum of this set is $0 = 0 \sup A$. So we may assume $c > 0$.

We start by showing that if b is an upper bound for A, then cb is an upper bound for cA. Indeed, if $x \in cA$ then $x/c \in A$ and hence $x/c \leq b$. But then since $c > 0$ we conclude that $x \leq cb$. Hence $c + b$ is an upper bound for $c + A$.

The remainder of the proof follows similarly to the previous one. Now let $\alpha = \sup A$. Then $c\alpha$ is an upper bound for cA, since α is an upper bound for A. Let β be any other upper bound for cA. Since $A = (1/c) \cdot (cA)$, we have $(1/c)\beta$ is an upper bound for A and hence $\alpha \leq 1/c\beta$. But then we have $c\alpha \leq \beta$. In summary, we have shown that $c\alpha$ is an upper bound for A and that if β is any upper bound for cA, then $c\alpha \leq \beta$. Hence $c\alpha$ is the least upper bound of cA. □

Statement for part (c): If $c < 0$ then $\sup(cA) = c \inf(A)$.

Exercise 1.3.7: Prove that if a is an upper bound for A an dif a is also an element of A, then $a = \sup A$.

Proof. Since a is an upper bound for A it suffices to show that it is a *least* upper bound. That is, we need to show that if b is any other upper bound for A, then $a \leq b$. But this is clear, for if b is an upper bound for A, then $a \leq b$ since $a \in A$. □

Exercise 1.3.8: Compute, without proof, the suprema and infima of the following sets.

(a) $\{m/n : m, n \in \mathbb{N} \text{ with } m < n\}$.

(b) $\{(-1)^m/n : n, m \in \mathbb{N}\}$.

(c) $\{n/(3n + 1) : n \in \mathbb{N}\}$.

(d) $\{m/(m + n) : m, n \in \mathbb{N}\}$.

(e) $\{n \in \mathbb{N} : n^2 < 10\}$.

(f) $\{n/(n+m) : n, m \in \mathbb{N}\}$.

(g) $\{n/(2n+1) : n \in \mathbb{N}\}$.

(h) $\{n/m : m, n \in \mathbb{N} \text{ with } m + n \leq 10\}$.

Solution:

(a) Supremum: 1. Infimum: 0.

(b) Supremum: 1/2. Infimum: -1.

(c) Supremum: 1/3. Infimum: 1/4.

(d) Supremum: 1. Infimum: 0.