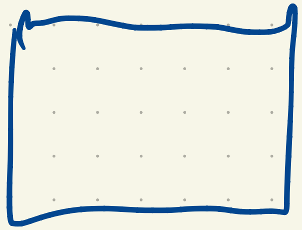


fix k, let $h \to 0$

$$\frac{k}{h^2} \to \infty$$

$$\lambda \le \frac{1}{2}$$

$$\lambda = \frac{k}{h^2}$$

$c \to \infty$

$k = \lambda h^2$

$c = 2, 4, 8, 10$

k, h

$c^2 M, cN$

h

k

$k \sim \frac{1}{M}$ $h \sim \frac{1}{N}$

error (c)

(h)

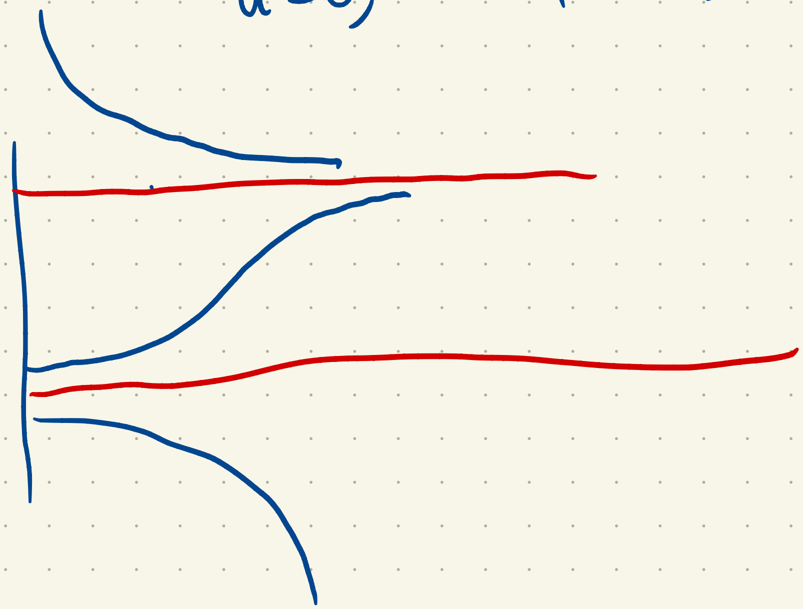
(k)

$$\frac{k}{h^2} = \frac{N^2}{M}$$

$$u' = 30\,u\,(1-u)$$

$\boxed{\text{Why is } \lambda = -30 \text{ key?}}$

$u=0$, $u=1$ solutions.



$u' = 30u(1-u)$

Why is $\lambda = -30$ key?

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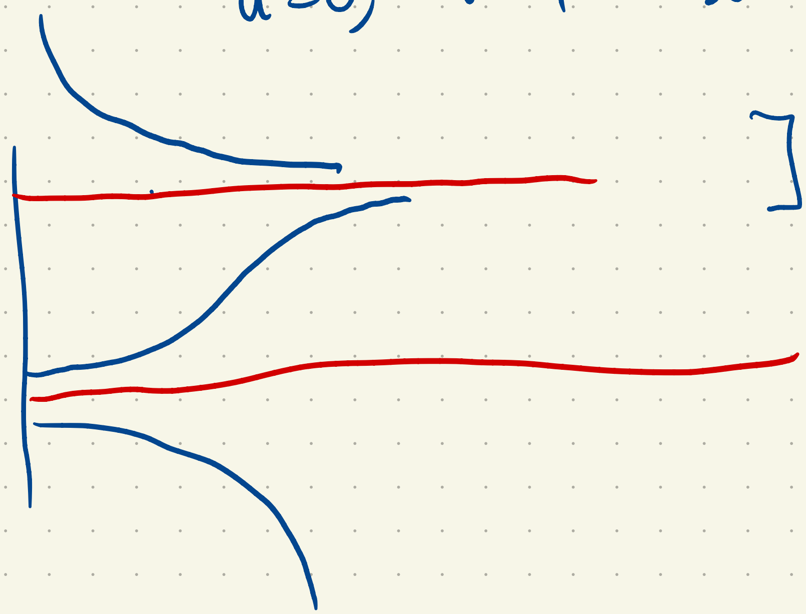

] stable equilibrium $\to$ transients!

] unstable equilibrium

$u' = 30u$ ~~$+ \ldots$~~

$u' = 30u(1-u)$

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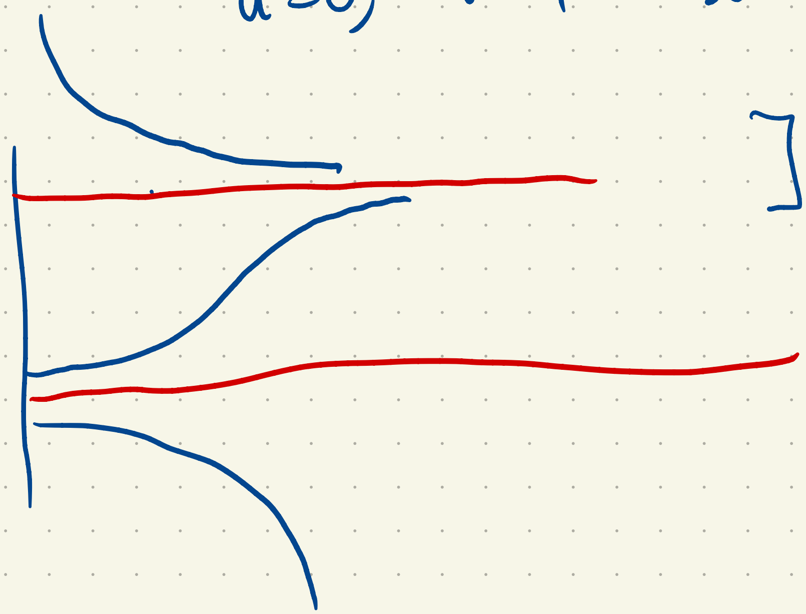

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If $u \sim 0$ $\quad u' = 30u - 30u^2 \approx 30u$ (exp growth!)

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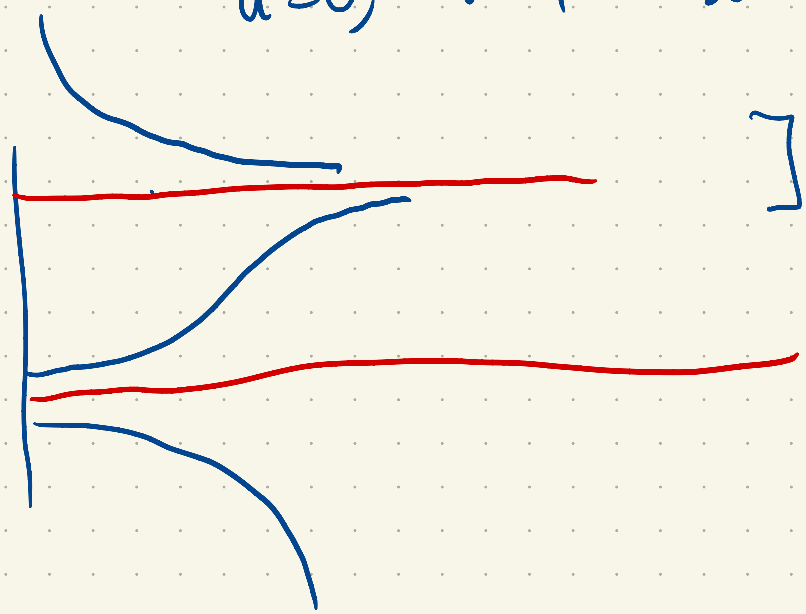


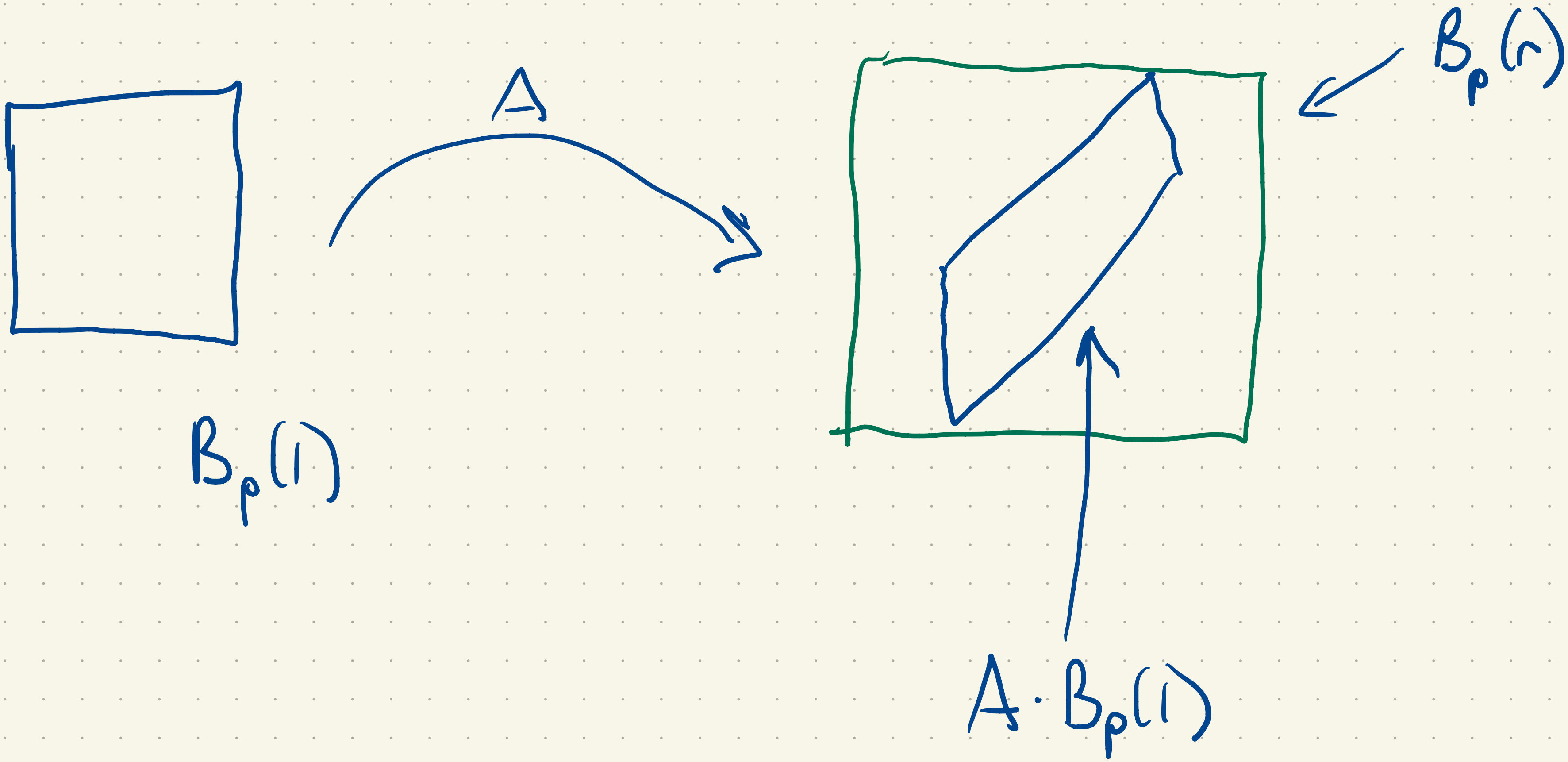
] stable equilibrium $\rightarrow$ transients!

] unstable equilibrium

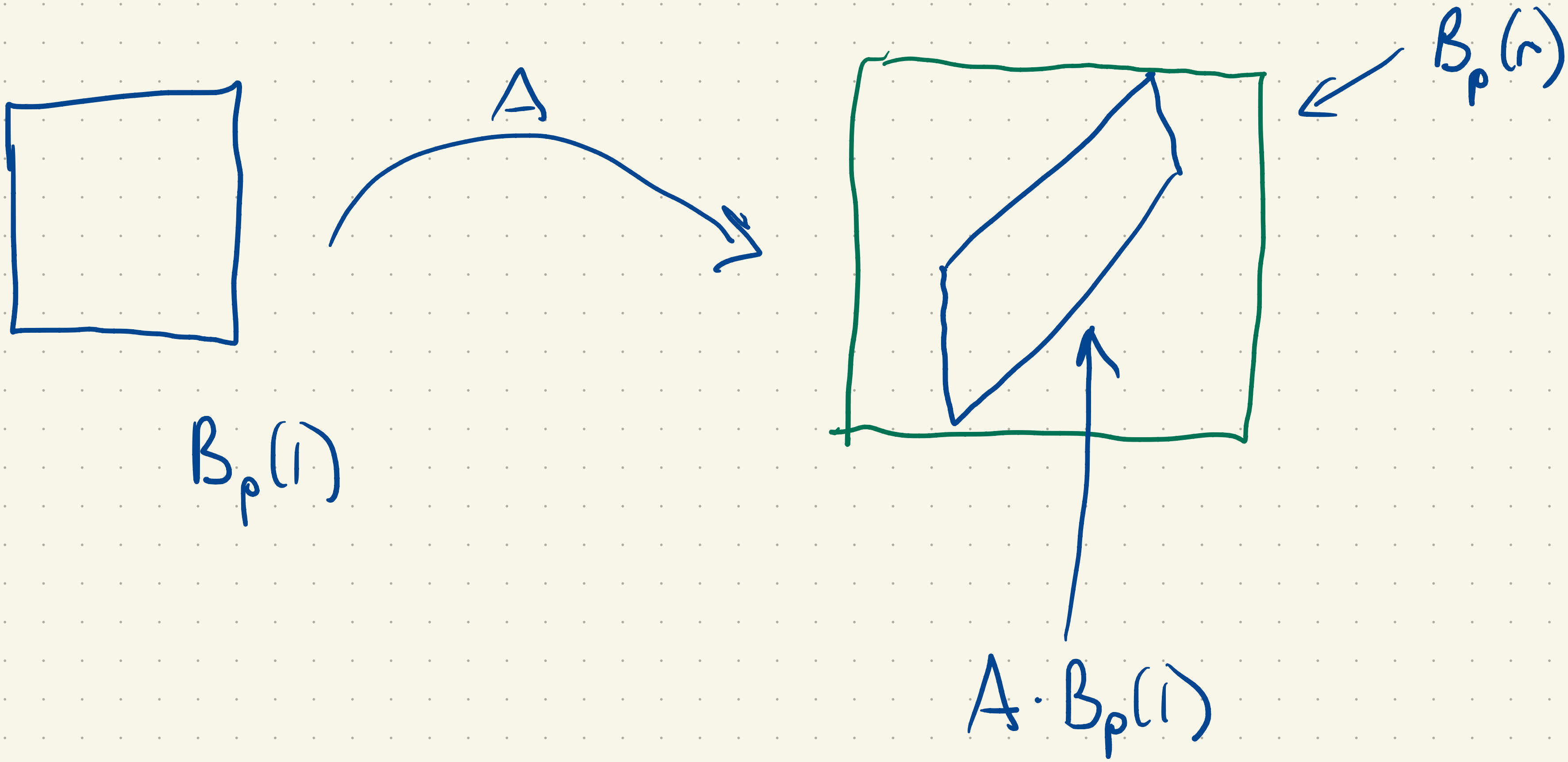
If $u \sim 0$ $\quad u' = 30u - 30u^2 \approx 30u$ (exp growth!)

$u = 1 + w$ $\quad w' = 30(1+w)(-w) = -30w - 30w^2$

Matrix Norms A: $n \times n$ matrix



Matrix Norms A: $n \times n$ matrix



$\|A\|_p$ is smallest r so $A\,B_p(1) \subseteq B_p(r)$

$$A \cdot B_p(1) = \{ Ax : x \in B_p(1) \}$$

$$\|A\|_p = \inf_r : A \cdot B_p(1) \subseteq B_p(r)$$

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Note: if $x \in B_p(1)$, $\|Ax\|_p \leq \|A\|_p$

smallest number that works for all x!

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$$\|A\|_p = \inf_r : A \cdot B_p(1) \subseteq B_p(r)$$

Note: if $x \in B_p(1)$, $\|Ax\|_p \leq \|A\|_p$

smallest number that works for all x!

Alt: $\|A\|_p = \sup_{\|x\|_p \leq 1} \|Ax\|_p.$

Exercise $\|A\|_p = \underset{\|x\|=1}{\overset{\max}{\sup}} \|Ax\|_p$

Moreover if $x \neq 0$

$$\frac{\|Ax\|_p}{\|x\|_p} = \left\|A \frac{x}{\|x\|_p}\right\|_p \leq \|A\|_p$$

$$\sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p$$

$$\geq \sup_{\|x\|_p = 1} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p = \|A\|_p$$

Exercise $\|A\|_p = \sup_{\|x\|=1} \|Ax\|_p$

Moreover if $x \neq 0$

$$\frac{\|Ax\|_p}{\|x\|_p} = \left\|A \frac{x}{\|x\|_p}\right\|_p \leq \|A\|_p$$

$$\sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \|A\|_p$$

So $\|A\|_p \leq \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p$

Three formulations of the matrix norm:

$$\|A\|_p = \sup_{\|x\|=1} \|Ax\|_p$$

$$= \sup_{\|x\|\le 1} \|Ax\|_p$$

$$= \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Triangle Inequality:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \qquad x, y \in \mathbb{R}^n$$

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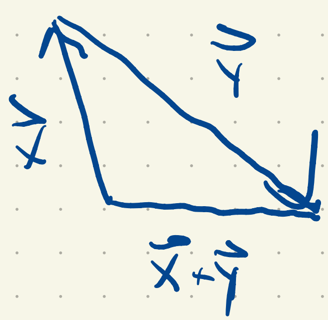


Exercise: prove this for

$p=1$, $p=\infty$

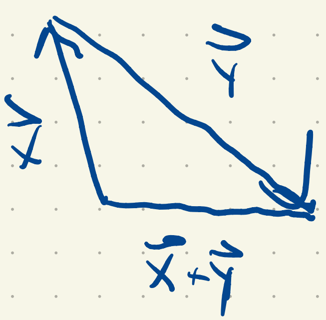
Triangle Inequality:

$$|x \cdot y| \leq \|x\|_2 \|y\|_2$$

Cauchy-Schwarz

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \qquad x, y \in \mathbb{R}^n$$

$$x \cdot y = |x||y| \cos\theta$$



Exercise: prove this for

$p=1, \quad p=\infty$

$$\|x\|_2^2 = x \cdot x$$

$$(x+y)\cdot(x+y)$$

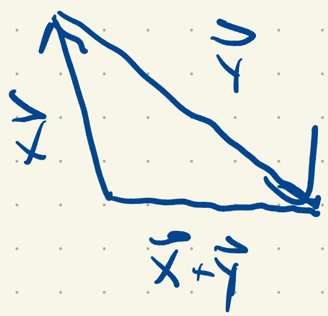
$$\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\, x \cdot y$$

$$\leq \|x\|_2^2 + \|y\|_2^2 + 2\|x\|_2\|y\|_2$$

$$= (\|x\|_2 + \|y\|_2)^2$$

Triangle Inequality:

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p \qquad x,y \in \mathbb{R}^n$$



Exercise: prove this for

$p=1, \quad p=\infty$

Cauchy Schwartz

$$\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 + 2\, \underline{x \cdot y}$$

$$\leq \|x\|_2^2 + \|y\|_2^2 + 2\, \underline{\|x\|_2 \|y\|_2}$$

$$= (\|x\|_2 + \|y\|_2)^2$$

Triangle Inequality

$$\|(A+B)\|_p \leq \|A\|_p + \|B\|_p$$

Triangle Inequality

$$\|(A+B)x\|_p \leq \|Ax\|_p + \|Bx\|_p$$

On your homework

Multiplicative Property: $\|AB\|_p$

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

Multiplicative Property:

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

$$\left(\frac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p \quad \text{if} \quad x \neq 0 \right)$$

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$$\|ABx\|_p \leq \|A\|_p \|Bx\|_p \leq \|A\|_p \|B\|_p \|x\|_p$$

Multiplicative Property:

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

$$\left(\frac{\|Ax\|_p}{\|x\|_p} \leq \|A\|_p \quad \text{if} \quad x \neq 0 \right)$$

$$\|ABx\|_p \leq \|A\|_p \|Bx\|_p \leq \|A\|_p \|B\|_p \|x\|_p$$

$$\frac{\|ABx\|_p}{\|x\|_p} \leq \|A\|_p \|B\|_p \Rightarrow \|AB\|_p \leq \|A\|_p \|B\|_p$$

Multiplicative Property:

$$\|AB\|_p \leq \|A\|_p \|B\|_p$$

For exercise:

$\|A\|_\infty$: maximum 1-norm of rows

$\|A\|_1$: maximum 1-norm of columns

For exercise:

$-1 \cdot x + 2y \quad 3\epsilon$

$3 \cdot x - 4y$

$\|A\|_\infty$: maximum 1-norm of rows

$\|A\|_1$: maximum 1-norm of colums

IOU

$$\left\| \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\infty$$

make big with

$|x| \le 1, \ |y| \le 1$

$x = 1, \ y = -1$

-3

$3 + 4$

For exercise:

$\|A\|_\infty$: maximum 1-norm of rows

$\|A\|_1$: maximum 1-norm of columns

Alas, we'll be more interested in the

2-norm and this is harder to compute.

stay tuned...

Theta Method:

$$\underbrace{\left(I - (1-\theta)\lambda D\right)}_{B} \vec{U}_{j+1} = \underbrace{\left(I + \lambda\theta D\right)}_{A} \vec{U}_j + k\vec{f}_j$$

Have convergence if $\theta\lambda \leq \frac{1}{2}$

$$\sup_j \| \vec{U}_j - \vec{u}_j \|_\infty \to 0$$

Theta Method:

$$\underbrace{\left(I - (1-\theta)\lambda D\right)}_{B} \vec{U}_{j+1} = \underbrace{\left(I + \lambda\theta D\right)}_{A} \vec{U}_j + k\vec{f}_j$$

$$\theta\lambda \leq \frac{1}{2} \Rightarrow \sup_j \|\vec{U}_j - \vec{u}_j\|_\infty \to 0$$

$$(2\theta - 1)\lambda \leq \frac{1}{2} \Rightarrow \sup_j \frac{1}{\sqrt{N}} \|\vec{U}_j - \vec{u}_j\|_2 \to 0$$

Theta Method:

$$\underbrace{\left(I - (1-\theta)\lambda D\right)}_{B} \vec{U}_{j+1} = \underbrace{\left(I + \lambda\theta D\right)}_{A} \vec{U}_j + k\vec{f}_j$$

We'll show

1) $\|B^{-1}\|_2 \le 1$

2) $\|B^{-1}A\|_2 \le 1$

Theta Method:

$$\underbrace{\left(I-(1-\theta)\lambda D\right)}_{B}\vec{U}_{j+1}=\underbrace{\left(I+\lambda\theta D\right)}_{A}\vec{U}_j+k\vec{f}_j$$

We'll show

1) $\|B^{-1}\|_2 \leq 1$

2) $\|B^{-1}A\|_2 \leq 1$

assuming $(2\theta-1)\lambda \leq \frac{1}{2}$

Assuming $\|B^{-1}\|_2 \leq 1$, $\|B^{-1}A\|_2 \leq 1$:

$$B\vec{U}_{j+1} = A\vec{U}_j + k\vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j + k\tau_j$$

Assuming $\|B^{-1}\|_2 \le 1$, $\|B^{-1}A\|_2 \le 1$:

$$B\vec{U}_{j+1} = A\vec{U}_j + k\vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j + k\tau_j$$

$$BE_{j+1} = AE_j - k\tau_j$$

Assuming $\|B^{-1}\|_2 \leq 1$, $\|B^{-1}A\|_2 \leq 1$:

$$B\vec{U}_{j+1} = A\vec{U}_j + k\vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j + k\tau_j$$

$$B E_{j+1} = A E_j - k\tau_j$$

$$\vec{E}_{j+1} = B^{-1}A\vec{E}_j - kB^{-1}\tau_j$$

$$\|\vec{E}_{j+1}\|_2 \leq \|B^{-1}A\|_2 \, \|\vec{E}_j\|_2 + k\|B^{-1}\|_2 \|\tau_j\|_2$$

$$\|\vec{E}_{j+1}\|_2 \leq \|\vec{E}_j\|_2 + k\|\tau_j\|_2$$

$$\|\vec{E}_{j+1}\|_2 \le \|B^{-1}A\|_2 \, \|\vec{E}_j\|_2 + k\|B^{-1}\|_2 \|\tau_j\|_2$$

$$\|\vec{E}_{j+1}\|_2 \le \|\vec{E}_j\|_2 + k\|\tau_j\|_2$$

Assuming $\theta\lambda \le \frac{1}{2}$ (a stronger assumption) we had

$$\|\vec{E}_{j+1}\|_\infty \le \|E_j\|_\infty + k\|\tau_j\|_\infty$$

(a stronger consequence)

$$\left(\sum_{i=1}^{N} (\tau_{i,j})^2 \right)^{1/2} \qquad \tau_{i,j} = O(k) + O(h^2)$$

$$\max_j \|\vec{E}_j\|_2 \leq \|\vec{E}_0\|_2 + T \max_j \|\tau_j\|_2$$

$$(2\theta - 1)\lambda \leq \frac{1}{2}$$

$$\|E\|_{2,\infty} \leq \|\vec{E}_0\|_2 + T \|\tau\|_{2,\infty} \frac{1}{\sqrt{N}}$$

$$\longrightarrow \max_j \|\vec{E}_j\|_2$$

vs

$$\|E\|_{\infty,\infty} \leq \|E_0\|_\infty + T\|\tau\|_{\infty,\infty} \qquad \theta\lambda \leq \frac{1}{2}$$

$$\downarrow$$

$$\max_{i,j} |E_{i,j}|$$

$$\|\tau_j\|_2 \leq \sqrt{N} \|\tau_j\|_\infty$$