Von Neumann Analysis

(fast, rule of thumb)

$\vec{v}$: $\quad v_k = e^{\sqrt{\ } r x_k} \qquad r \in \mathbb{R}$

Method: $\quad \vec{u}_{j+1} = (1 + \lambda D)\, \vec{u}_j + \vec{f}$ ignore

Von Neumann Analysis $\quad I + \lambda D$

(fast, rule of thumb)

$\vec{v}: \qquad v_k = e^{\mathcal{J} r x_k} \qquad r \in \mathbb{R}$

Method: $\quad \vec{u}_{j+1} = (I + \lambda D)\vec{u}_j$

$x_{i-1} = x_i - h$

Suppose $\quad \vec{u}_j = \vec{v}$

$$u_{i,j+1} = \lambda v_{i-1} + (1-2\lambda) v_i + \lambda v_{i+1}$$

$$= e^{Jrx_i}\left[\lambda e^{-Jrh} + \lambda e^{+Jrh} + (1-2\lambda)\right]$$

$$= e^{Jrx_i}\left[2\lambda\left[\cos(rh) - 1\right] + 1\right]$$

$$= e^{Jrx_i}\left[-4\lambda \sin^2\left(\frac{rh}{2}\right) + 1\right]$$

$$= v_i \underbrace{\left[1 - 4\lambda \sin^2\left(\frac{rh}{2}\right)\right]}_{\text{amplification factor}}$$

Upshot:

If $\vec{u}_j = v$ then

$$\vec{u}_{j+1} = \left[1 - 4\lambda \sin^2(rh)\right] \vec{u}_j$$

except at endpoints.

Want $\quad \left| 1 - 4\lambda \sin^2(rh) \right| \leq 1 \qquad \dfrac{k}{h^2}$

$$-2 \leq -4\lambda \sin^2(rh) \leq 2$$

$$-\frac{1}{2} \leq \lambda \sin^2(rh) \leq \frac{1}{2} \Rightarrow \lambda \leq \frac{1}{2}$$

Convergence:

Want $u_{i,j} \to u(x_i, t_j)$

as $h, k \to 0$. $\left(\max_{i,j} |u_{i,j} - u(x_i, t_j)| \to 0 \right)$

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We will only be able to do this assuming, simultaneously, $\frac{k}{h^2} < \frac{1}{2}$.

LTE (Carefully)

I $$\frac{u(x_i, t_j+k) - u(x_i, t_j)}{k} = u_t(x_i, t_j) + \frac{1}{2} u_{tt}(x_i, \tau_j) k$$

LTE (Carefully)

(I) $$\frac{u(x_i, t_j+k) - u(x_i, t_j)}{k} = u_t(x_i, t_j) + \frac{1}{2} u_{tt}(x_i, \tau_j) k$$

(II) $$\frac{u(x_i - h, t_j) - 2u(x_i, t_j) + u(x_i + h, t_j)}{h^2} = u''(x, t_j) + \frac{1}{6} u_{xxxx}(\xi_i, t_j)$$

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$$\text{I} - \text{II} - f(x_i, t_j) =$$

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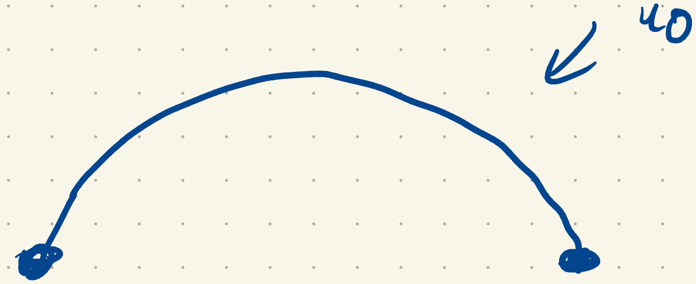
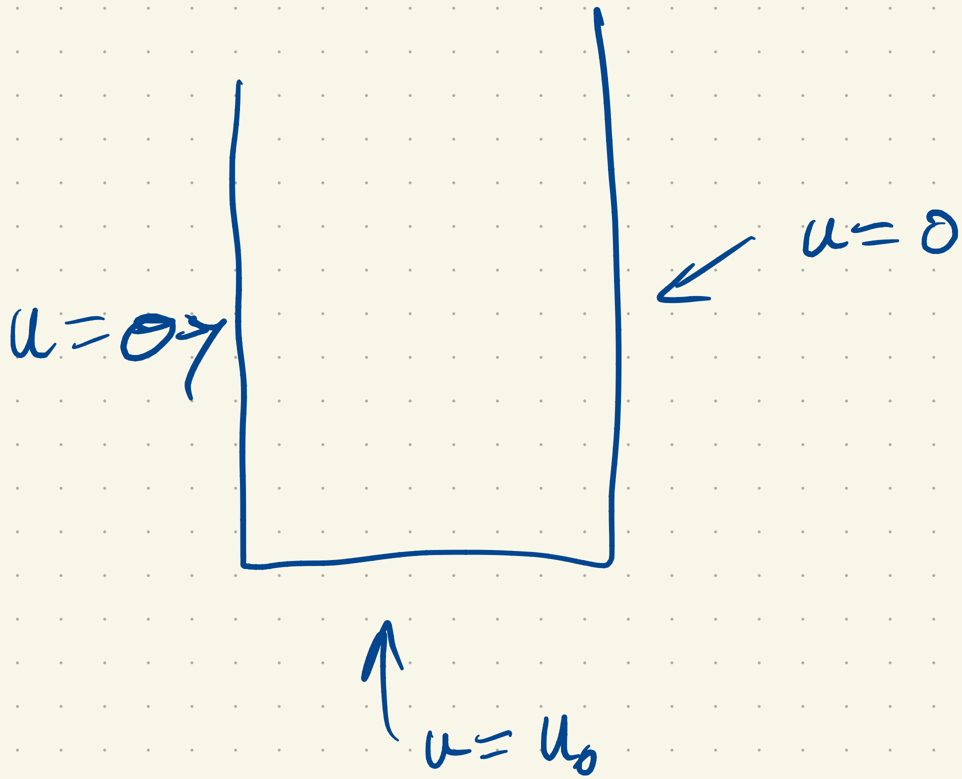
$$\text{I} + \text{II} - \underset{\substack{\uparrow \\ (u_t - u_{xx})(x_i, t_j)}}{f(x_i, t_j)} = \underbrace{\frac{1}{2} u_{tt}(x_i, \eta_j) k - \frac{1}{6} u_{xxxx}(\xi_i, t_j) h^2}_{\tau_{i,j}}$$

$u_{tt}\ k$ $\quad$ $u_{xxxx}\ h^2$





$$u_t = u_{xx}$$

$$u_{tt} = u_{xxt}$$

$$= u_{xxxx}$$

LTE (Carefully)

$$\frac{1}{2} u_{tt}(x_i, \eta_j) k - \frac{1}{6} u_{xxxx}(\xi_i, t_j)$$

$$u_t = u_{xx} + f$$

$$u_{tt} = u_{xxt} + f_t$$

$$= u_{xxxx} + f_t + f_{xx}$$

So: $$|\tau_{i,j}| \leq \left(\frac{1}{2}k + \frac{1}{6}h^2\right) \max \left(|u_{xxxx}| + \frac{1}{2}k \left(\max |f_t|\right)\right)$$

missing f_{xx} term!

LTE (Carefully)

$$\frac{1}{2} u_{tt}(x_i, \eta_j) k + \frac{1}{6} u_{xxxx}(\xi_i, t_j) \longleftarrow$$

$$u_t = u_{xx} + f$$

$$u_{tt} = u_{xxt} + f_t$$

$$= u_{xxxx} + f_t$$

u_{tt}, u_{xxxx} existing requires a compatability condition: $u_{tt} = 0$ on boundary so

$$u_{xxxx} + f_t = 0$$

is needed!

So:
$$|\tau_{i,j}| \leq \left(\frac{1}{2}k + \frac{1}{6}h^2\right) \max\left(|u_{xxxx}| + \frac{1}{2}k \left(\max |f_t|\right)\right)$$

Bound for LTE

$u_t = u_{xx} + f$

$u_{tt} = u_{xxt} + f_t$

$C = \max|u_{xxxx}| + \max|f_t| + \max|f_{xx}|$

$|\tau| \le \left(\frac{1}{2}k + \frac{1}{6}h^2\right) C$

Convergence of Direct Method

$U_{i,j} \rightarrow$ numerical solution

$$U_{i,j+1} = \lambda U_{i-1,j} + (1-2\lambda)U_{i,j} + \lambda U_{i+1,j} + k f_{i,j}$$

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$u_{i,j} \longrightarrow u(x_i, t_j)$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda) u_{i,j} + \lambda u_{i+1,j} + k f_{i,j} + k \tau_{i,j}$$

Convergence of Direct Method

$U_{i,j} \to$ numerical solution

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$u_{i,j} \longrightarrow u(x_i, t_j)$

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i+1,j} + kf_{i,j} + k\tau_{i,j}$$

Error: $E_{i,j} = U_{i,j} - u_{i,j}$

$E_j = \max_i |E_{i,j}|$ (error at time step j)

$$E_{i,j+1} = \lambda E_{i-1,j} + (1-2\lambda) E_{i,j} + \lambda E_{i+1,j} - k \tau_{i,j}$$

$$E_{i,j+1} = \underbrace{\lambda E_{i-1,j} + (1-2\lambda) E_{i,j} + \lambda E_{i+1,j}} - k\tau_{i,j}$$

If $0 < \lambda \leq \frac{1}{2}$ then is a weighted average

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$$|E_{i,j+1}| \leq \lambda |E_{i-1,j}| + |1-2\lambda| |E_{i,j}| + \lambda |E_{i+1,j}| + k|\tau_{i,j}|$$

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$$|E_{i,j+1}| \leq \lambda |E_{i-1,j}| + \underbrace{|1-2\lambda|}_{= 1-2\lambda \text{ if } 0<\lambda\leq 1/2} |E_{i,j}| + \lambda |E_{i+1,j}| + k|\tau_{i,j}|$$

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If $0 < \lambda \leq \frac{1}{2}$ then $\uparrow$ is a weighted average

$$|E_{i,j+1}| \leq \underbrace{\lambda |E_{i-1,j}|}_{\leq \lambda E_j} + \underbrace{|1-2\lambda|}_{= 1-2\lambda \text{ if } 0<\lambda\leq 1/2} |E_{i,j}| + \overbrace{\lambda |E_{i+1,j}|}^{\leq \lambda E_j} + k|\tau_{i,j}|$$

$\leq (1-2\lambda) E_j$

$$\boxed{E_{j+1} \leq E_j + k\,\tau}$$

Error simply compounds!

$$\tau = \max_{i,j} |\tau_{i,j}| \leq C\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

$$E_1 \leq E_0 + k\tau$$

Error simply compounds!

$$\tau = \max_{i,j} |\tau_{i,j}| \leq C\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

$$E_j = \max_i |E_{i,j}|$$

$$E_1 \leq E_0 + k\tau$$

$$\max_j E_j$$

$$E_2 \leq E_1 + k\tau \leq E_0 + 2k\tau$$

$$\underbrace{\max_j E_j}_{\to\ \max_{i,j}|E_{i,j}|} \le T\tau$$

Error simply compounds!

$$\tau = \max_{i,j} |\tau_{i,j}| \le C\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

$$E_j \le E_0 + T\tau$$

$$E_1 \le E_0 + k\tau$$

$$E_2 \le E_1 + \tau \le E_0 + 2k\tau$$

$$\vdots$$

$$E_M \le E_0 + Mk\tau = E_0 + T\tau.$$

$$\frac{k}{h^2} = \frac{1}{2} \qquad k = \frac{1}{2}h^2$$

$$h = k$$

$$E_j \le \cancel{E_0}^{\,0} + T\tau \qquad \frac{k}{2} \quad \frac{k^2}{6}$$

$$k \sim h^2$$

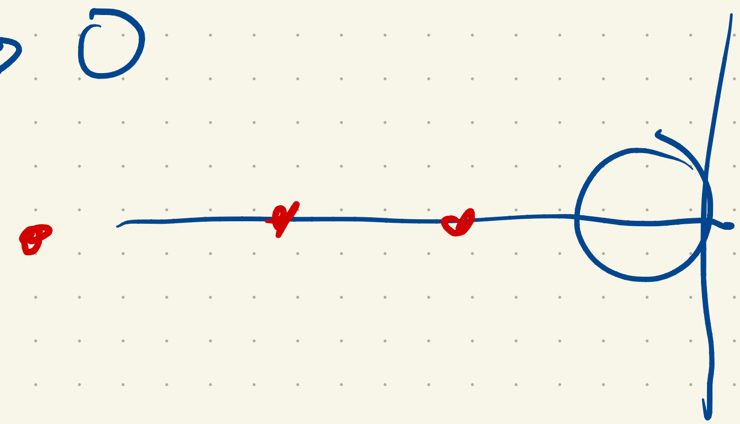
for all t $\quad \max_{i,j} \left| U_{i,j} - u(x_i, t_j) \right|$

$$\le CT\left(\frac{k}{2} + \frac{h^2}{6}\right)$$

Theorem: Suppose u is a smooth solution of heat equation (2 continuous time and 4 cts space derivatives).

Suppose $(h_n, k_n) \longrightarrow 0$

$$\frac{k_n}{h_n^2} \le \frac{1}{2}$$



$$(x_i^{(n)}, t_j^{(n)}) \to (x, t).$$

Then $U_{i,j}^{(n)} \longrightarrow u(x_i, t_j)$.

Implicit Method (Backwards Euler)

Replace $u_t(x_i, t_j) \approx \dfrac{u(x_i, t_j + k) - u(x_i, t_j)}{k}$

with $\dfrac{u(x_i, t_j) - u(x_i, t_j - k)}{k}$

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Replace $u_t(x_i, t_j) \approx \dfrac{u(x_i, t_j + k) - u(x_i, t_j)}{k}$

with $\dfrac{u(x_i, t_j) - u(x_i, t_j - k)}{k}$

$$\vec{u}_{j+1} = \vec{u}_j + \frac{k}{h^2} D \vec{u}_{j+1} + k \vec{f}_j$$

$$(I - \lambda D) u_{j+1} = \vec{u}_j + k \vec{f}_j$$

tridiagonal!

$O(N)$

$$u_{j+1} = \underbrace{(I - \lambda D)^{-1}}\left[\vec{u}_j + k\vec{f}_j\right]$$

Don't form this! Just solve:

$O(N)$.