1. Refresh your memory about the statement of Taylor's Theorem with remainder. (Appendix A.2 in your text will be helpful). Then use it to estimate the number of terms needed for the Taylor polynomial $p(x)$ for $\sin(x)$ such that

$$|p(x) - \sin(x)| < 10^{-4}$$

for all x in $[0, \pi]$. This will require estimating the size of the remainder term, and you are welcome to use a computer or calculator to assist in this computation. Then generate a graph showing the difference between $p(x)$ and $\sin(x)$ on the interval. What is the maximum error you actually observe?

Solution:
From Taylor's Theorem,

$$\sin(h) = \sin(0 + h) = p(h) + \frac{\sin^{(k+1)}(c)}{(k+1)!} h^{k+1}$$

where $p(x)$ is the kth order Taylor polynomial, and where c lies between 0 and h. For h in $[0, \pi]$ we estimate

$$\left| \frac{\sin^{(k+1)}(c)}{(k+1)!} h^{k+1} \right| \leq \frac{\pi^{k+1}}{(k+1)!}$$

Let $E_k = \pi^{k+1}/(k+1)!$. A numerical computation shows that $E_{13} \approx 10^{-3}$ but $E_{14} \approx 2 \times 10^{-5}$.

The Jupyter notebook shows a graph of the error in using the order 14 Taylor polynomial and confirms that the error is indeed less than 10^4.

2. Solve the initial value problem

$$y'' - y' - 6y = 0$$

for $y(t)$ subject to the initial condition $y(0) = 0$ and $y'(0) = 1$.

Then find the general solution of

$$y'' - y' - 6y = 1 + t.$$

Solution, part a:
The characteristic polynomial of the ODE is $\lambda^2 - \lambda - 6$ which has roots $\lambda_+ = 3$ and $\lambda_- = -2$. Thus the general solution is

$$y = Ae^{3t} + Be^{-2t}$$

for constants A and B. From the initial condition we determine $A + B = 0$ and $3A - 2B = 1$ so $A = 1/5$ and $B = -1/5$.

Solution, part b:
For the general solution, it is enough to find a single solution of the ODE, and then add on the general solution of the homogeneous version (which we already solved in the previous problem).

Let us guess a solution of the form $c_1 t + c0$. Substituting into the ODE we desire

$$-c_1 - 6(c_1 t + c0) = 1 + t$$

and therefore $c_1 = -1/6$ and $c_0 = -5/36$. The general solution is

$$y = Ae^{3t} + Be^{-2t} - \frac{t}{6} - \frac{5}{36}.$$

3. Implement the bisection algorithm for finding roots. (You can look at the Wikipedia entry for a reminder of how the algorithm works.) Your code should take as its arguments:

1. A function f. You will be solving $f(x) = 0$.
2. Two numbers a and b. The desired root should be in $[a, b]$.
3. A tolerance ϵ. The approximate root should be within distance ϵ of the true root.

It should return the approximate root.

Hand in both your code and a session showing its results in approximating $\sqrt{2}$ to within 10^{-8}.

Solution:

See Jupyter notebook.

4. Implement Newton's method. Your code should take as its arguments:

1. A function f. You will be solving $f(x) = 0$.
2. A function f'. This is the derivative of the function f.
3. A number x_0, which is the initial approximate root.
4. A tolerance ϵ. The approximate root should be be returned when two subsequent iterations of Newton's method yield approximations within ϵ of each other.

It should return the approximate value of the root and the number of iterations required to find it.

Newton's method can be finicky. Your code should handle gracefully error conditions that can occur.

Hand in both your code, and a session showing its results approximating $\sqrt{2}$ to within 10^{-12} starting from an initial approximation $x_0 = 2$.

Solution:

See Jupyter notebook.