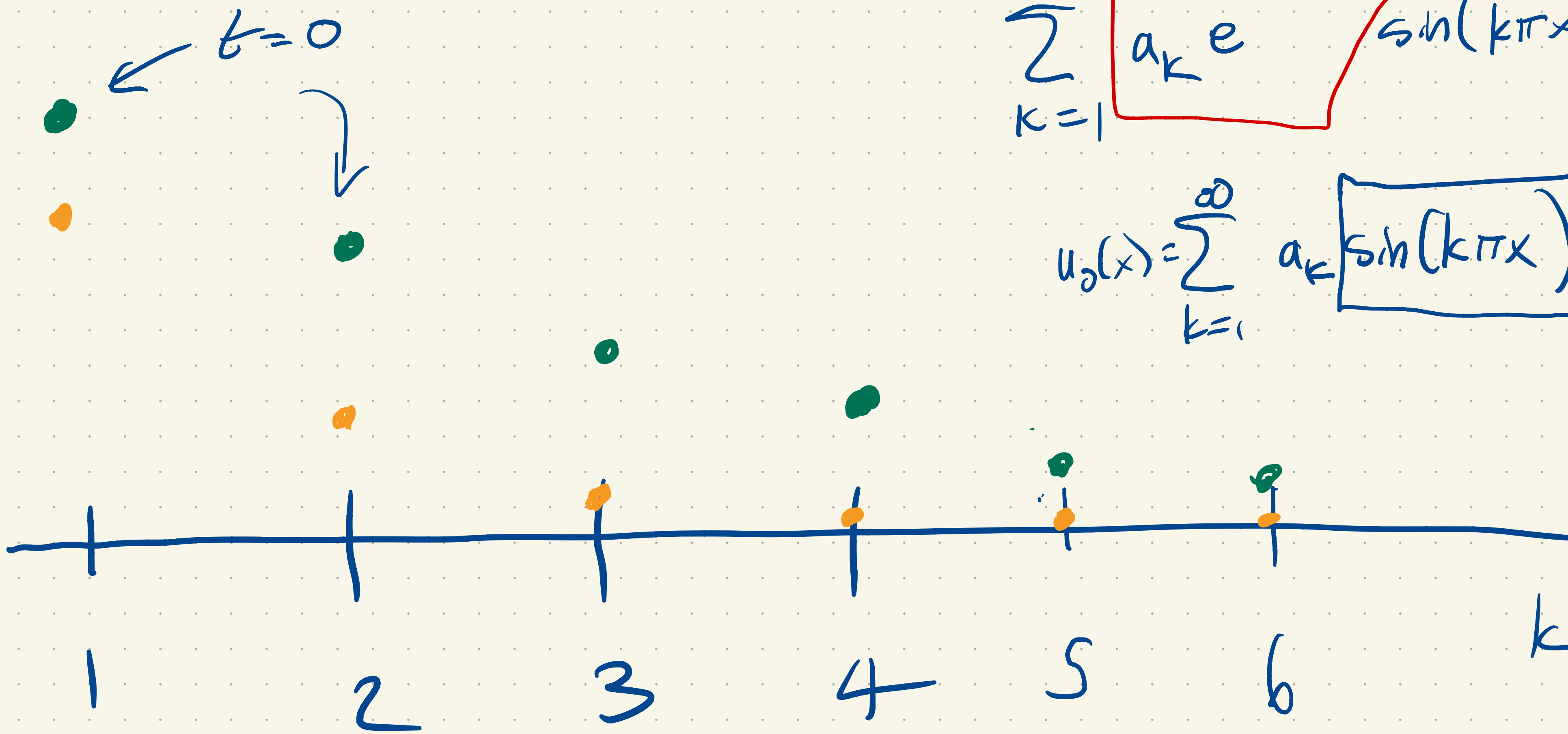
Initial and true solution a little later

$$\sum_{k=1}^{6} \boxed{a_k e^{-k^2\pi^2 t}} \sin(k\pi x)$$

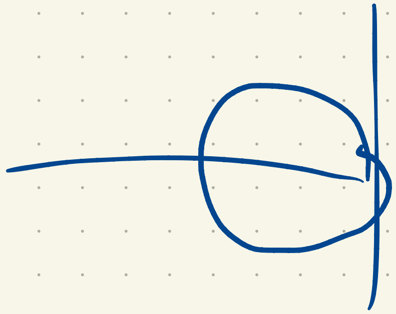
$$u_0(x) = \sum_{k=1}^{\infty} a_k \boxed{\sin(k\pi x)}$$



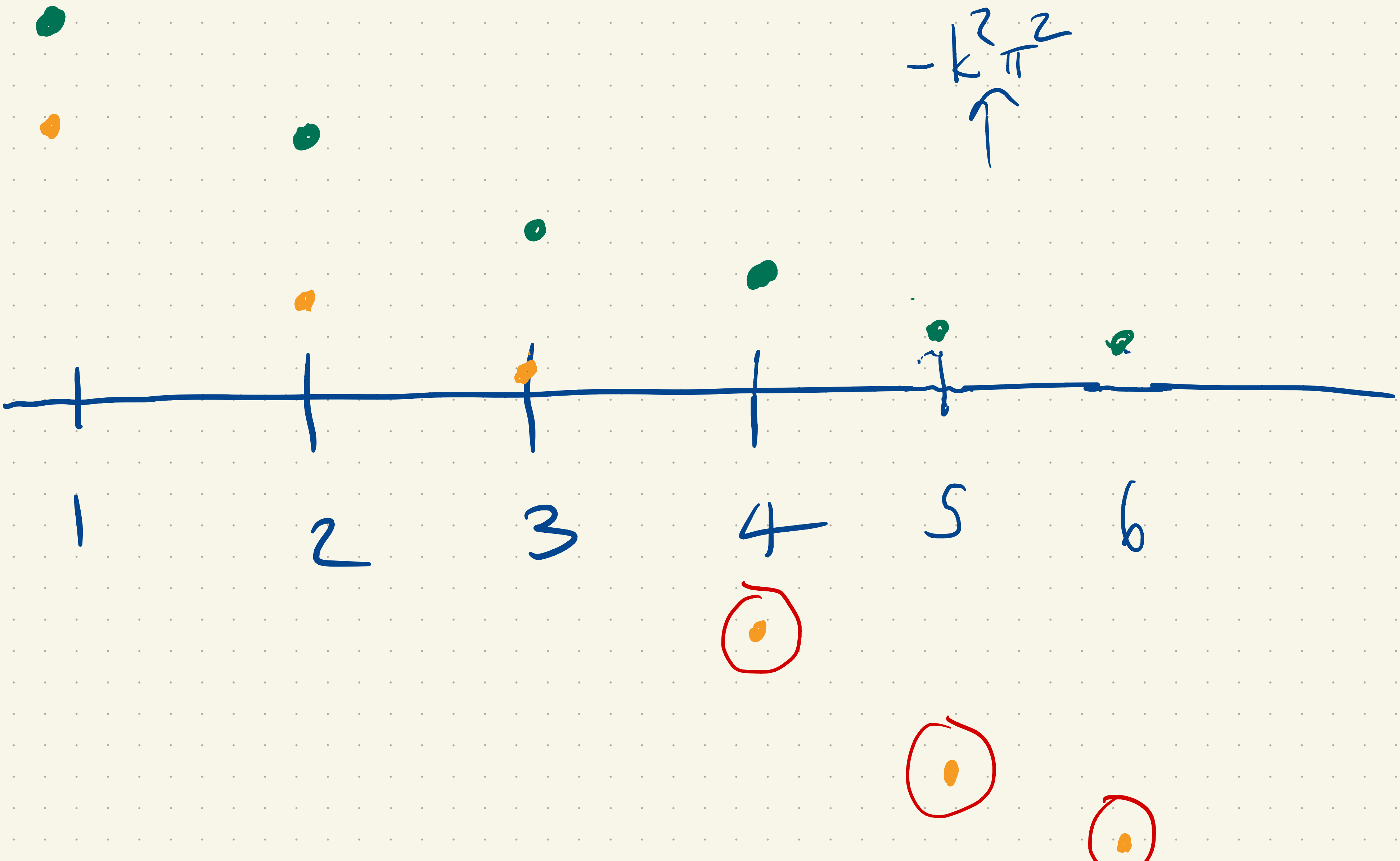
$$\sum_{k=1}^{\infty} 1 \qquad \sum_{k=1}^{\infty} a_k$$

$$O\left(k^{-l-1-\varepsilon}\right)$$

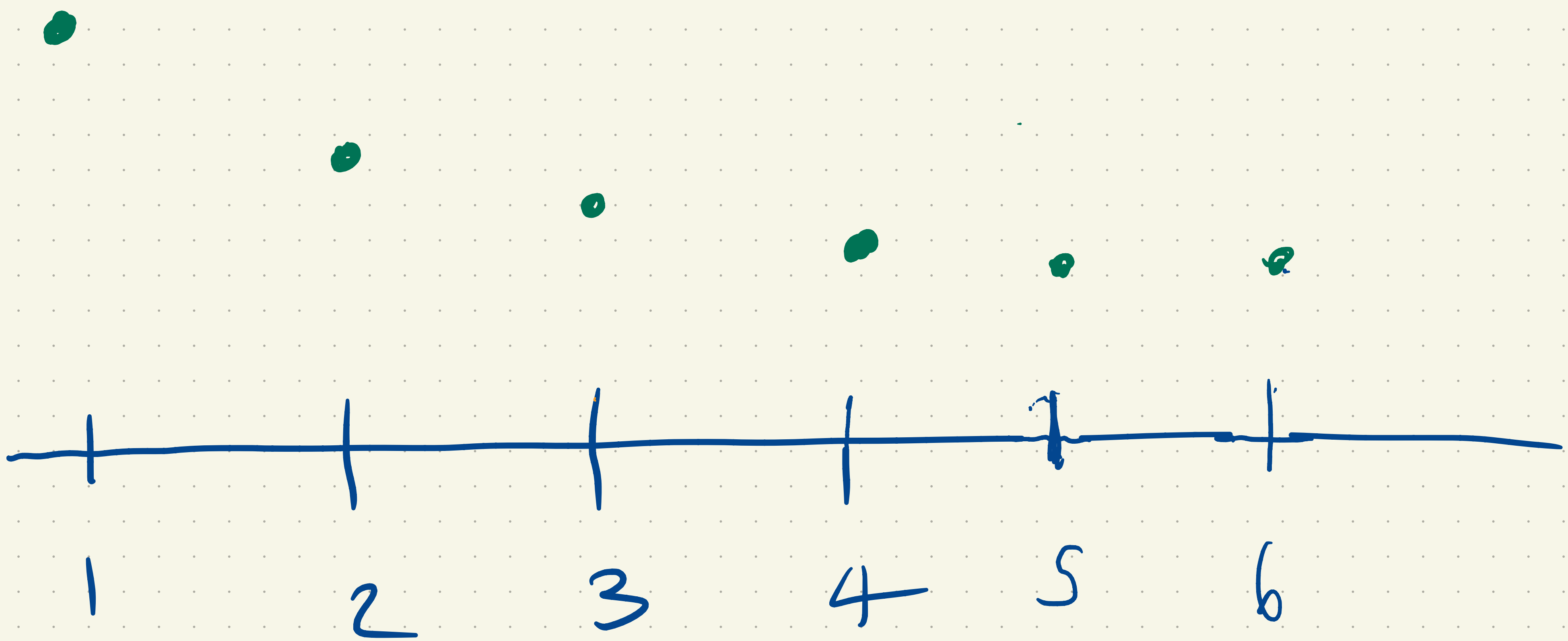
$$\Rightarrow C^{l}$$



Forward Euler violating $\lambda \leq \frac{1}{2}$

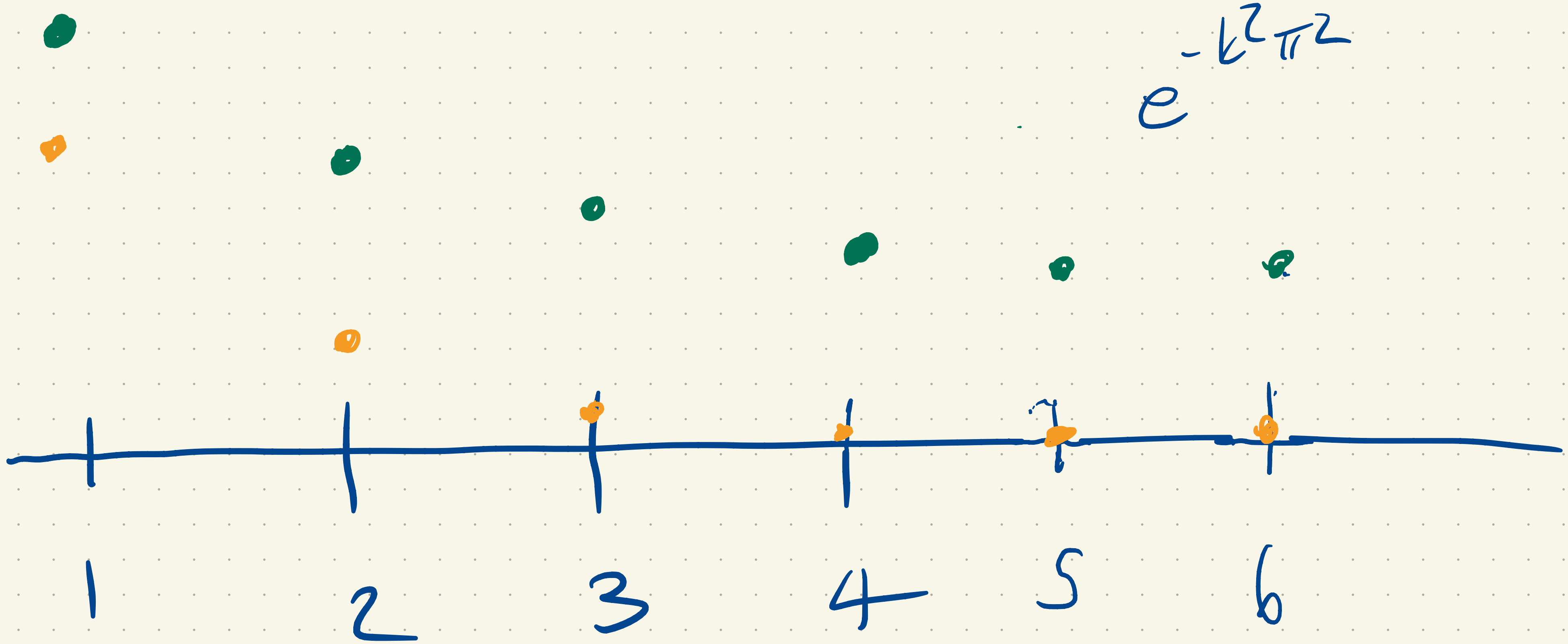


A heavy tail (many high frequency components; not so smooth)

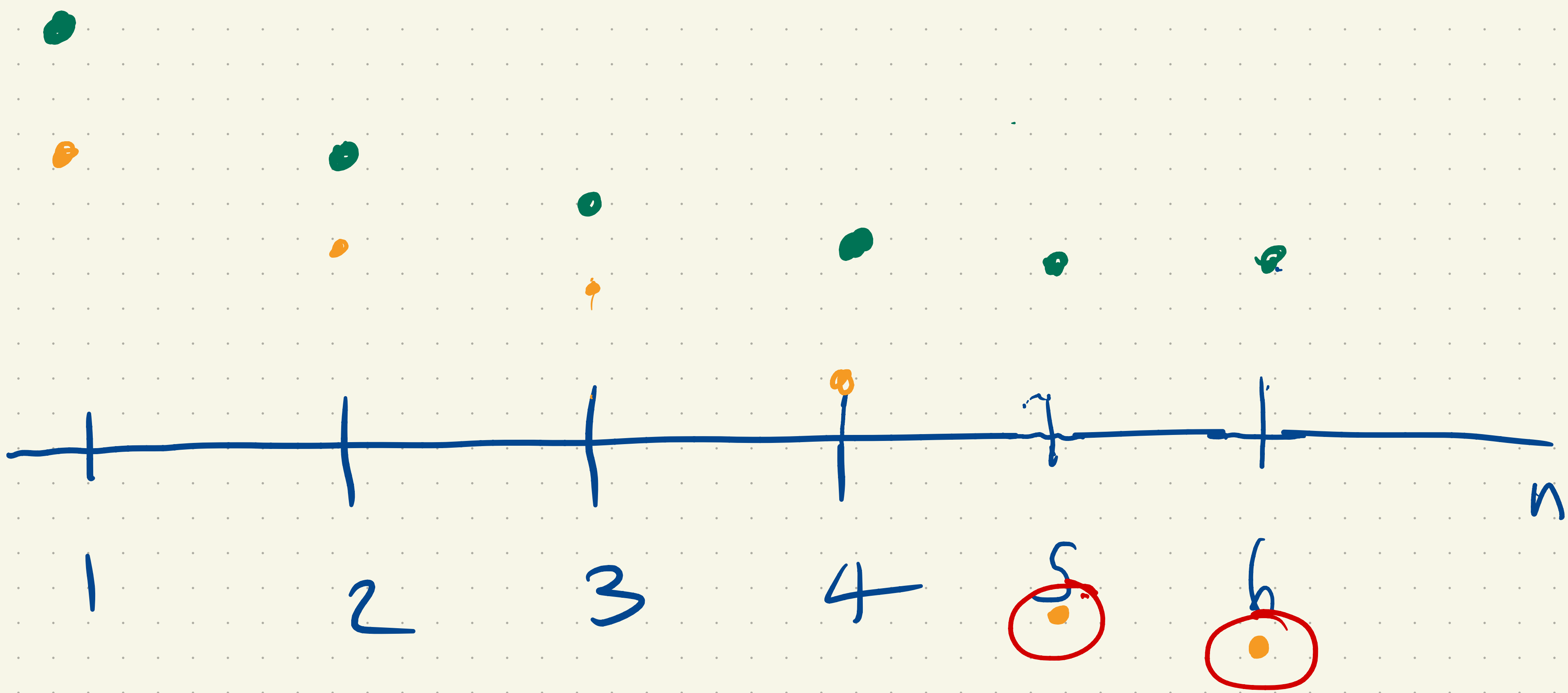


True solution some time later....

$$e^{-k^2\pi^2}$$

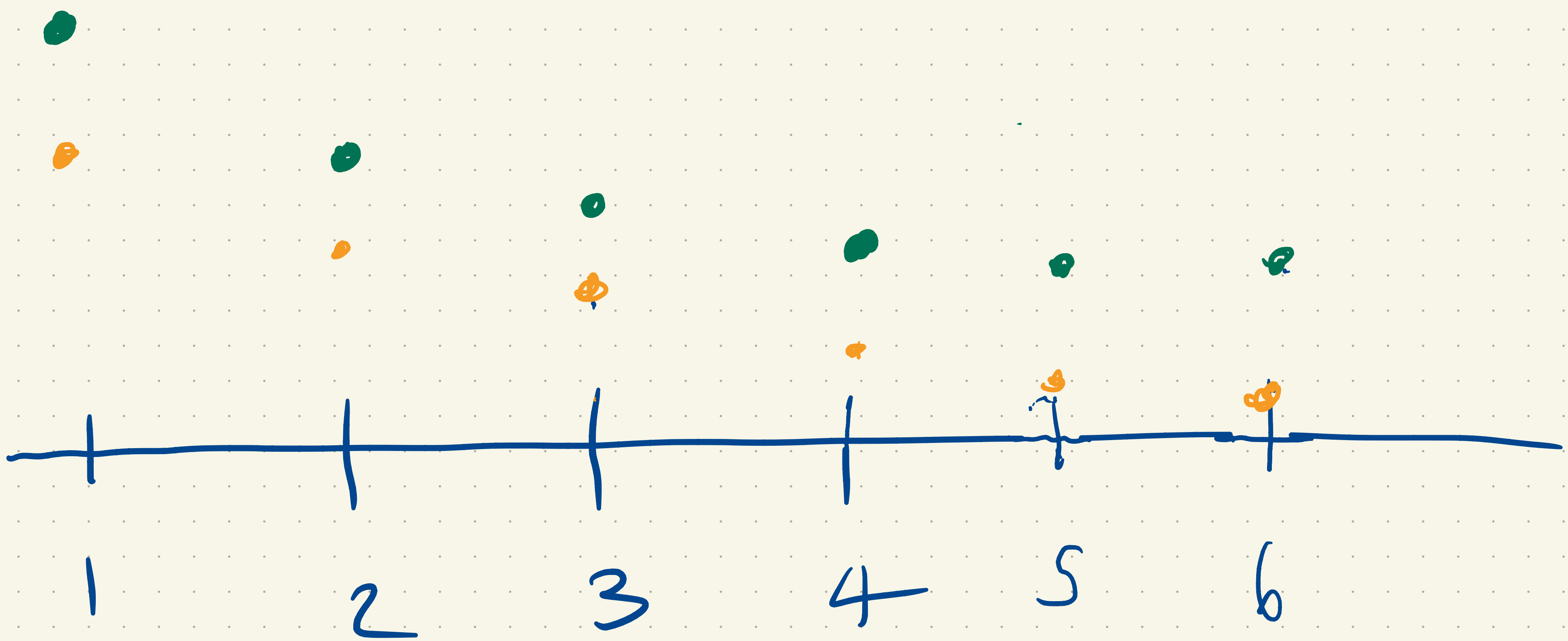


Crank - Nicholson



$$\lambda_n = -n^2\pi^2 \qquad \frac{2+z}{2-z} \qquad z = \lambda_n k$$

Backwards Euler



$\lambda_n = -n^2\pi^2$

$$\frac{1}{1-z}$$

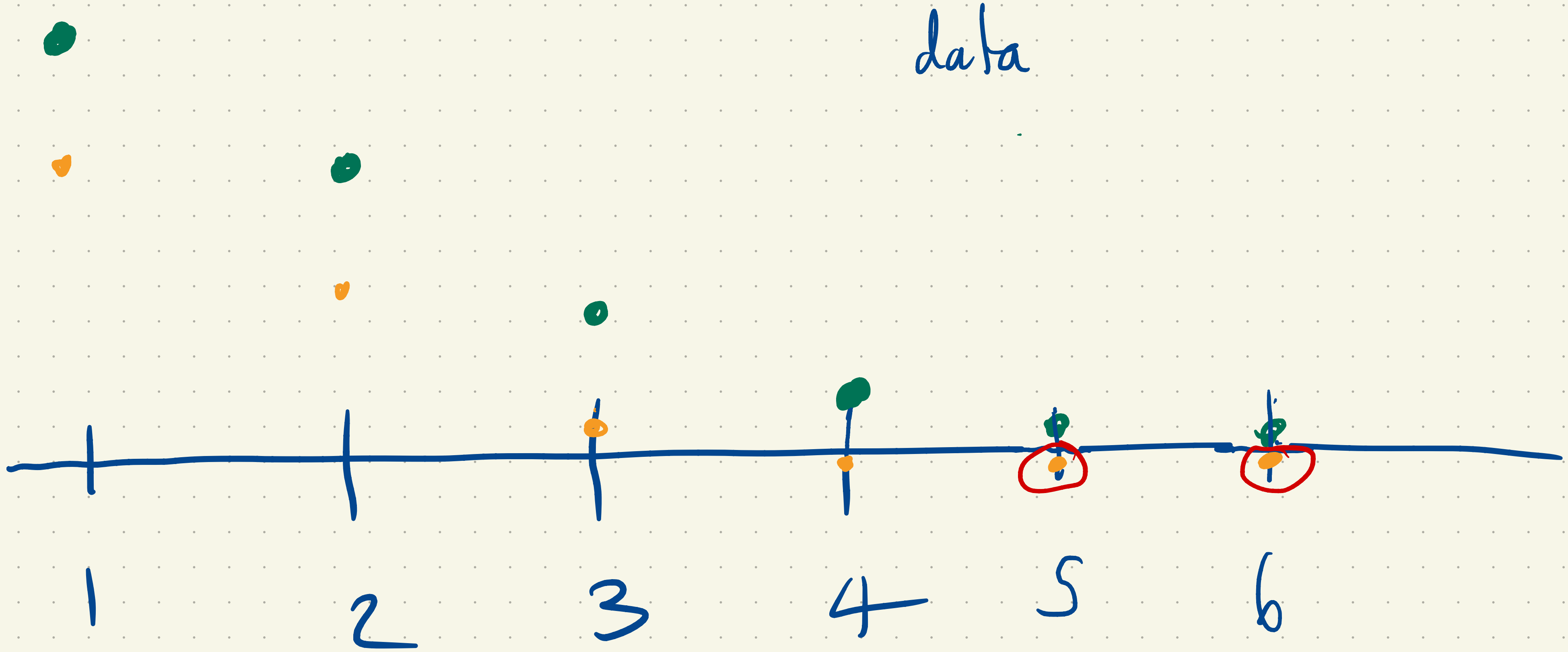
$z = \lambda_n k$

$O(h^2)$

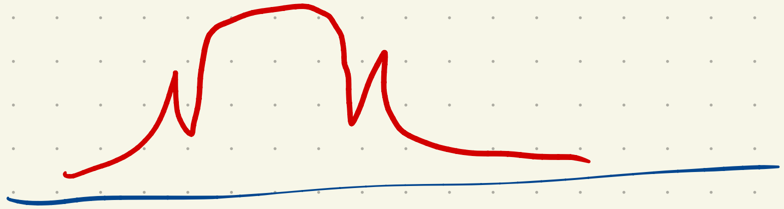
$+O(k)$

Crank - Nicholson

applied to smooth initial data



$$\lambda_n = -n^2\pi^2 \qquad \frac{2+z}{2-z} \qquad z = \lambda_n k$$



Def: A one-step numerical method for ODEs is <u>L-stable</u> if

1) It is A-stable
2) $|R(z)| \to 0$ as $|z| \to \infty$

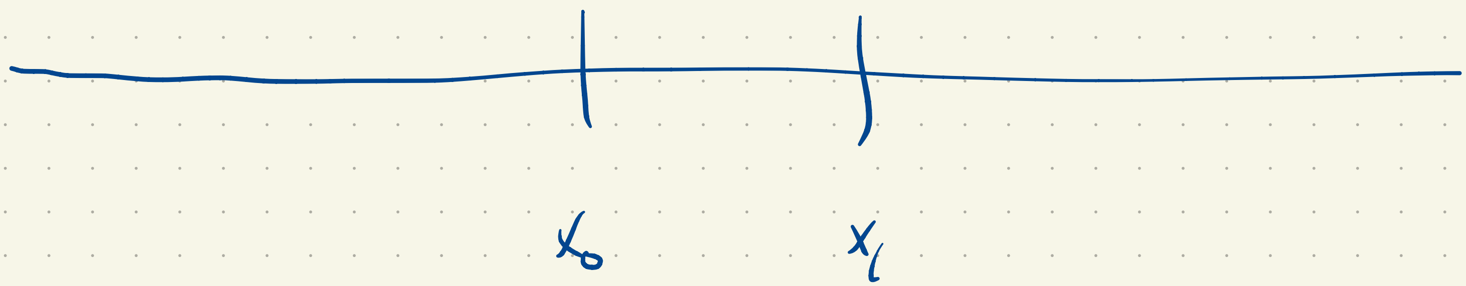
$$R(z)$$

where $u_{n+1} = R(\lambda h)\, u_n$ when applied to $u' = \lambda u$.

$$\frac{2+z}{2-z},$$

$$\frac{1}{1-z}$$

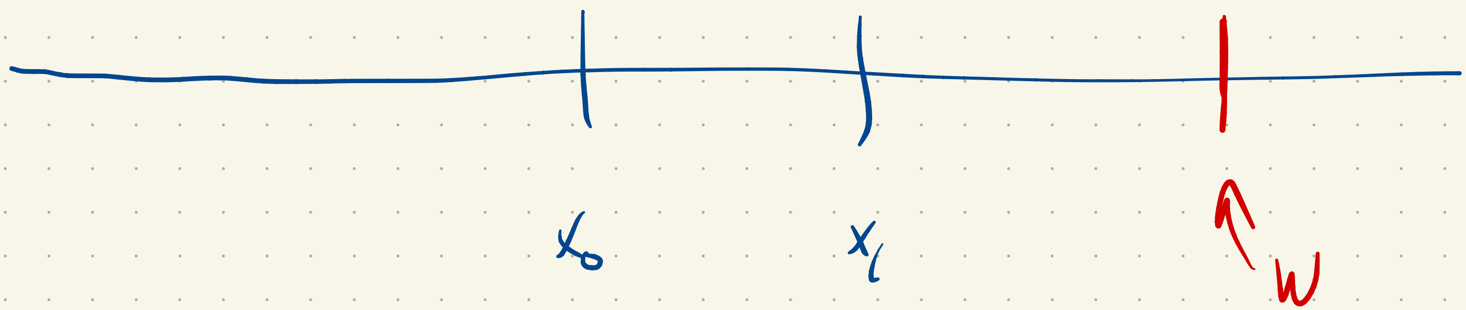
Hyperbolic PDEs



$u(x)$ a linear density

$$\int_{x_0}^{x_1} u(s)\,ds \to \text{stuff in } [x_0, x_1]$$

Hyperbolic PDEs



$u(x)$ a linear density

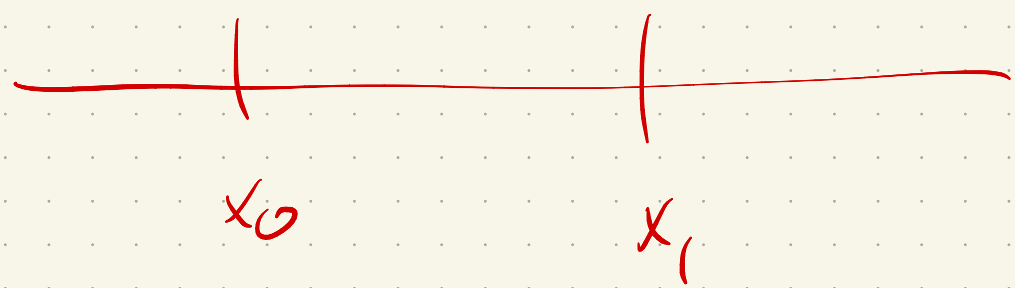
$$\int_{x_0}^{x_1} u(s)\,ds \to \text{stuff in } [x_0, x_1]$$

stuff/time

Flux: transport of u, to the right, at x, per unit time: w

$$\frac{d}{dt}\int_{x_0}^{x_1} u(s,t)\,ds = -w(x_1,t) + w(x_0,t)$$

$$= -\int_{x_0}^{x_1} w_x(s,t)\,ds$$



$$\frac{d}{dt}\int_{x_0}^{x_1} u(s,t)\,ds = -w(x_1,t) + w(x_0,t)$$

$$= -\int_{x_0}^{x_1} w_x(s,t)\,ds$$

$$\int_{x_0}^{x_1} u_t(s,t) + w_x(s,t)\,ds = 0$$

Hypothesis: There is a velocity v and

$$w(x,t) = v(x,t)\,u(x,t)$$

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$$w(x,t) = v(x,t)\,u(x,t)$$

$$\uparrow \qquad \uparrow$$

$$\frac{\text{dist}}{\text{time}} \quad \frac{\text{stuff}}{\text{dist}} \rightarrow \frac{\text{stuff}}{\text{time}} \checkmark$$

Hypothesis: There is a velocity v and

$$w(x,t) = v(x,t)\,u(x,t)$$

$$\underbrace{v(x,t)}_{\frac{\text{dist}}{\text{time}}}\ \underbrace{u(x,t)}_{\frac{\text{stuff}}{\text{dist}}} \Rightarrow \frac{\text{stuff}}{\text{time}} \checkmark$$

$$\int_{x_0}^{x_1} u_t(x,t) + \partial_x\big(v(x,t)\,u(x,t)\big)\,dx = 0$$

$\forall\ x_0, x_1$

Transport Equation:

$$\boxed{u_t + \partial_x (v u) = 0}$$

$v(x,t)$

known in advance

solve for $u(x,t)$

Transport Equation:

$$\boxed{u_t + \partial_x (vu) = 0}$$

$$\int_{x_0}^{x_1} u_t + \partial_x (vu) = \int_{x_0}^{x_1} f$$

what does this mean?

$$\frac{d}{dt} \int_{x_0}^{x_1} u = \text{flux} + \int_{x_0}^{x} f(s,t)\, ds$$

Easy case: $v(x,t) = a$, constant

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Super easy: $a = 0$

$u_t = 0$

$\partial_t u(x,t) = 0$

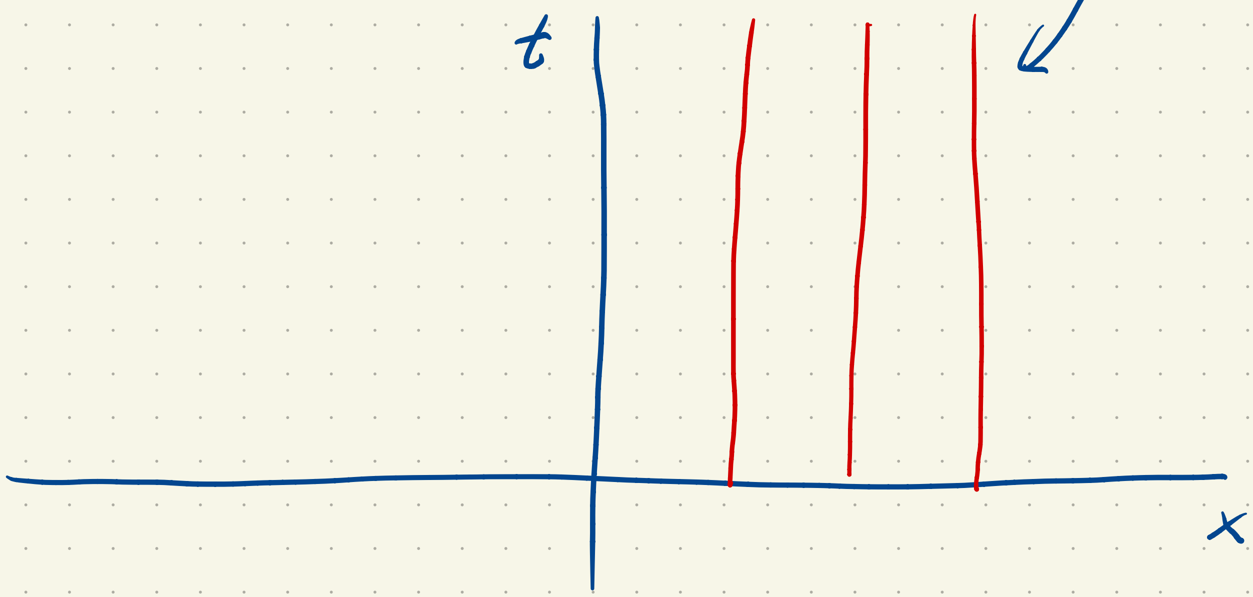
$\partial_t u_0(x) = 0$

Easy case: $v(x,t) = a$, constant

Super easy: $a = 0$

$$u_t = 0$$

u constant here



$$u(x,t) = u_0(x)$$

$u_t + a u_x = 0$ $\quad$ a constant.

Are there curves along which u is constant?

$$\gamma(s) = (x(s), t(s))$$

$$\frac{d}{ds} u(x(s), t(s)) = u_t \dot{t} + u_x \dot{x}$$

$\frac{d}{ds}$

So if $\dot{t} = 1$, $\dot{x} = a$ then yes!

$u_t + a\, u_x = 0$ $\qquad a$ constant.

Are there curves along which u is constant?

$u_t + a\,u_x = 0 \qquad a$ constant.

Are there curves along which u is constant?

$$\gamma(s) = (x(s), t(s))$$

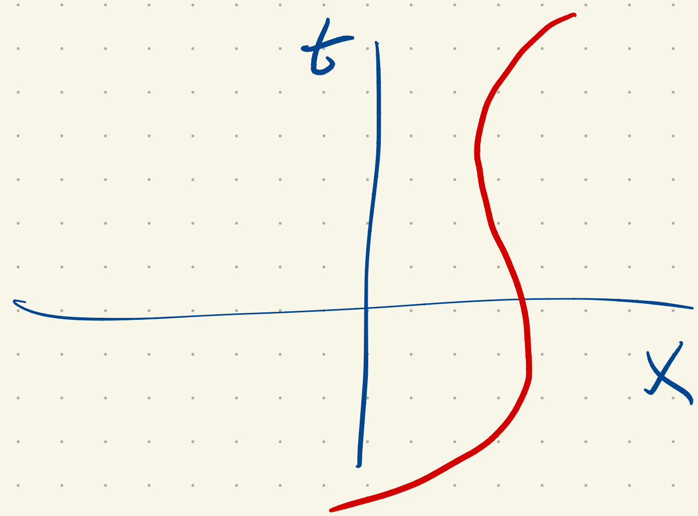
$$\frac{d}{ds}$$

$$\frac{d}{ds}\, u(x(s), t(s)) = u_t \dot{t} + u_x \dot{x}$$

$\dot{t} = 2$

$\dot{x} = 2d$

want $= 0$



$$u_t + a\,u_x = 0 \qquad a \text{ constant.}$$

Are there curves along which u is constant?

$$\gamma(s) = (x(s), t(s))$$

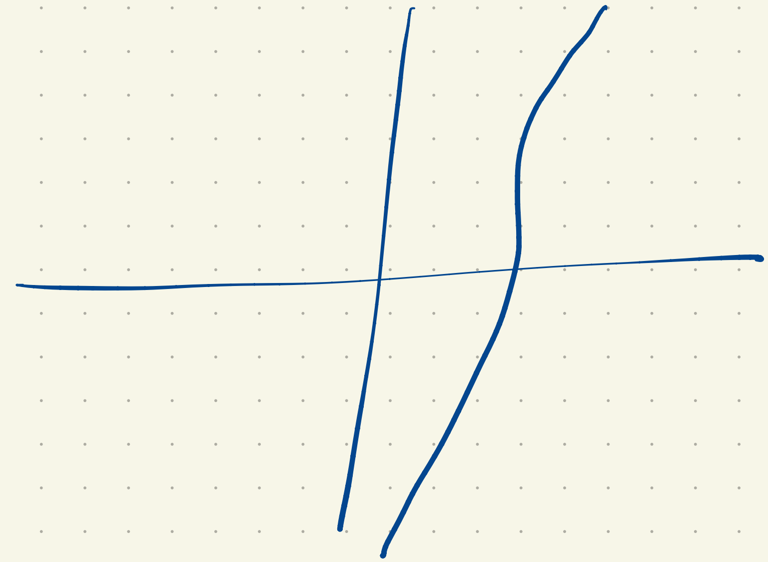
$$\frac{d}{ds} u(x(s), t(s)) = u_t \dot{t} + u_x \dot{x} \qquad \dot{\ } \to \frac{d}{ds}$$

So if $\dot{t} = 1$, $\dot{x} = a$ then yes!

$x(0) = r$

$t(0) = 0$

$\dot{x} = a \Rightarrow x = as + r$

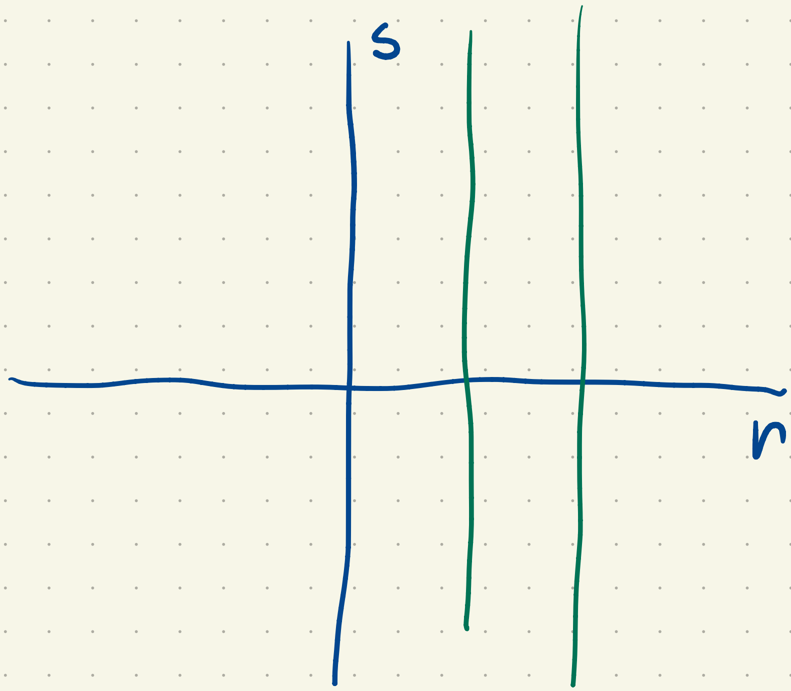


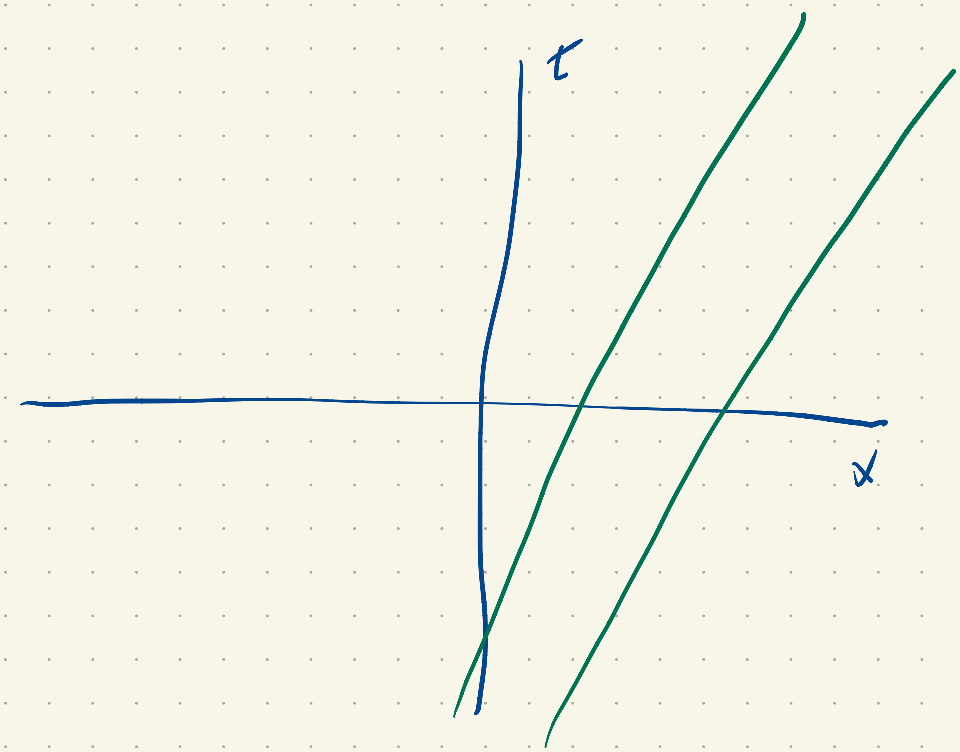
$\dot{t} = 1 \Rightarrow t = s + w$

One point on the curve: $s = 0 \to (r, w)$

Lets ensure the curve passes through

$$(\underset{\underset{x_0}{\uparrow}}{r}, 0)$$ by taking $w = 0$.





$x(r,s) = as + r$

$t(r,s) = s$

$at + r$