Last class: Runge Kutta

Project: first contact by Monday

$$Y_1 = u_k + h\left[a_{11} f(t_k + c_1 h, Y_1) + a_{12} f(t_k + c_2 h, Y_2)\right]$$

$$Y_2 = u_k + h\left[a_{21} f(t_k + c_1 h, Y_1) + a_{22} f(t_k + c_2 h, Y_2)\right]$$

$$u_{k+1} = u_k + h\left[b_1 f(t_k + c_1 h, Y_1) + b_2 f(t_k + c_2 h, Y_2)\right]$$

$$Y_i \approx u(t_k + c_i h)$$

$$\begin{array}{c|cc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ \hline & b_1 & b_2 \end{array}$$

$$c_1 = a_{11} + a_{12}$$

$$c_2 = a_{21} + a_{22}$$

$$1 = b_1 + b_2$$

4 stage

$$
\begin{array}{ccccc}
c_1 & a_{11} & a_{12} & a_{13} & a_{14} \\
c_2 & a_{21} & a_{22} & a_{23} & a_{24} \\
c_3 & a_{31} & a_{32} & a_{33} & a_{34} \\
c_4 & a_{41} & a_{42} & a_{43} & a_{44} \\
 & b_1 & b_2 & b_3 & b_4
\end{array}
$$

4 stage

$$
\begin{array}{cc|cccc}
V_{i1} & c_1 & a_{11} & a_{12} & a_{13} & a_{14} \\
V_{i2} & c_2 & a_{21} & a_{22} & a_{23} & a_{24} \\
V_{i3} & c_3 & a_{31} & a_{32} & a_{33} & a_{34} \\
V_{i4} & c_4 & a_{41} & a_{42} & a_{43} & a_{44} \\
\hline
 & & b_1 & b_2 & b_3 & b_4
\end{array}
$$

explicit,

these vanish

4 stage

$$\begin{array}{c|cccc} c_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ c_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ c_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ c_4 & a_{41} & a_{42} & a_{43} & a_{44} \\ \hline & b_1 & b_2 & b_3 & b_4 \end{array}$$

explicit, these vanish

---

2 stage, explicit

$$\begin{array}{c|cc} c_1 & 0 & 0 \\ c_2 & a & 0 \\ \hline & b_1 & b_2 \end{array}$$

$c_1 = 0$

$c_2 = a$

$b_1 = 1 - b_2$

1st order except $ab_2 = \frac{1}{2} \Rightarrow$ 2nd order

Absolute stability: $u' = \lambda u$

$$f(t,u) = \lambda u$$

$$Y_1 = u_k$$

$$Y_2 = u_k + h a \lambda Y_1$$

$$u_{k+1} = u_k + h(b_1 \lambda Y_1 + b_2 \lambda Y_2)$$

$$= u_k + z(b_1 u_k + b_2(u_k + z a u_k))$$

$$= \left[1 + (b_1 + b_2) z + b_2 a z^2\right] u_k$$

$$= \underbrace{(1 + z + b_2 a z^2)}_{R(z),\ \text{stability function.}} u_k$$

abs. stab.

$$\boxed{|R(z)| \le 1}$$

For all the 2^{nd} order methods, $b_2 a = \frac{1}{2}$!

$$u_{k+1} = \underbrace{\left(1 + z + \frac{1}{2} z^2\right)}_{} u_k$$

$$|1+z+\frac{1}{2}z^2| = 1$$

need $\left|1 + z + \frac{1}{2} z^2\right| \leq 1$

For all the 2^{nd} order methods, $b_2 a = \frac{1}{2}$!

$$u_{k+1} = \underbrace{\left(1 + z + \frac{1}{2} z^2\right)}_{} u_k$$

need $\left| 1 + z + \frac{1}{2} z^2 \right| \leq 1$

Show worksheet.

The R-K method

Simpson's Rule:

$$\int_a^{a+h} f(s)\,ds = \frac{h}{6}\left[f(a) + 4f\left(a+\frac{h}{2}\right) + f(a+h)\right] + O(h^5)$$

The R-K method

Simpson's Rule:

$$\int_a^{a+h} f(s)\, ds = \frac{h}{6}\left[f(a) + 4 f\left(a+\frac{h}{2}\right) + f(a+h) \right] + O(h^5)$$

$$Y_1 = u_k$$

$$Y_2 = u_k + \tfrac{1}{2} h\, f\left(t_k + \tfrac{1}{2}h,\ Y_1\right)$$

$$Y_3 = u_k + \tfrac{1}{2} h\, f\left(t_k + \tfrac{1}{2}h,\ Y_2\right)$$

$$Y_4 = u_k + h\, f\left(t_k + h,\ Y_3\right)$$

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
\hline
 & 1 & 2 & 2 & 1
\end{array}$$

$$u_{k+1} = u_k + \frac{h}{6}\left[f(t_k, Y_1) + 2f(t_k+\tfrac{h}{2}, Y_2) + 2f(t_k+\tfrac{h}{2}, Y_3) + f(t_k+h, Y_4) \right]$$

- explicit
- 4^{th} order
- single stage
- You'll plot region of abs stability.

Summary of ODEs:

Solving $u' = f(t,u)$

LMMs

$$u_n \approx u(t_n) \qquad f_{n+j} = f(t_{n+j}, u_{n+j})$$

$$\alpha_k u_{n+k} + \cdots + \alpha_1 u_{n+1} + \alpha_0 u_n = h\left(\beta_k f_{n+k} + \cdots + \beta_1 f_{n+1} + \beta_0 f_n\right)$$

LTE:

$$t_n + jh$$

$$\frac{\sum_{j=0}^{k} \alpha_j u(t_{n+j})}{h} - \sum_{j=0}^{k} \beta_j f(t_{n+j}, u(t_{n+j})) = \tau_n$$

Consistent: $\tau_n \to 0$ as $h \to 0$

$$\sum_{j=0}^{k} \alpha_j = 0$$

$$\sum_{j=0}^{k} (j\alpha_j - \beta_j) = 0$$

Zero stable: Apply to $u' = 0$.

Want solutions to stay bounded as $h \to 0$

$$\sum_{j=0}^{k} \alpha_j \rho^j \leftarrow \text{char. poly}$$

ρ^n solves LMM w RHS $= 0$

Need roots $|\rho| \le 1$ (<1 sometimes)

Dahlquist:

if consistent: convergent $\iff$ zero stable

$\hookrightarrow \|e\|_\infty \to 0$

$e_i = u(t_i) - u_i$

$\to h\lambda \to z$

Absolute stability

Apply to $u' = \lambda u$

Ideally, if $\lambda < 0$, solutions decay.

Important if your system has a fast decaying component (transient) λu

Don't want failure to resolve the transient to spoil the numerical solution.

$$\underbrace{\overbrace{\alpha_k \rho^k + \cdots + \alpha_0 \rho}^{\sigma(\rho)} - z \overbrace{\left(\beta_k \rho^k + \cdots + \beta_0\right)}^{\eta(\rho)}}_{} = 0$$

want roots $|\rho| \leq 1$ (< 1)

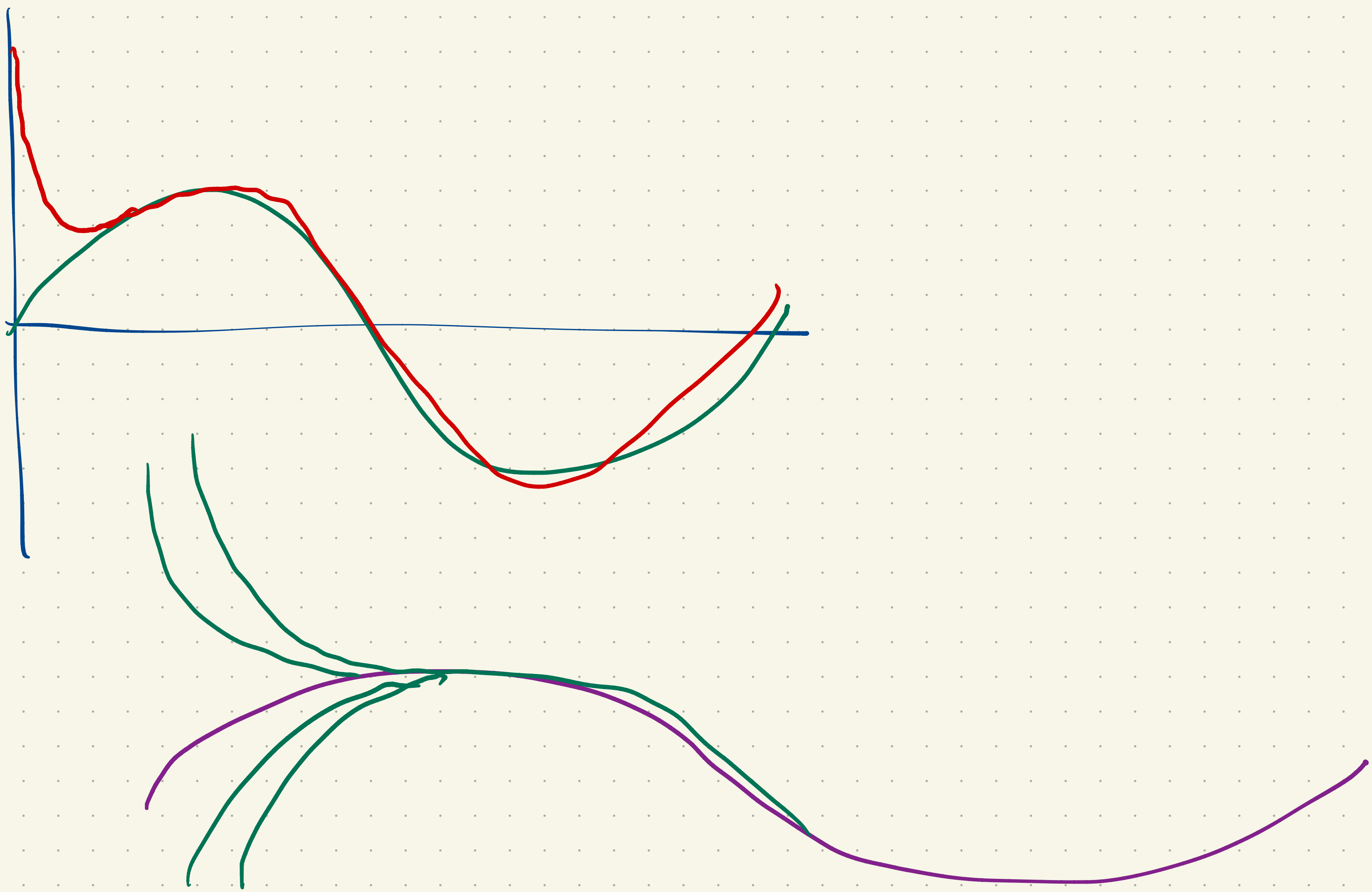
abs stability polynomial

ρ a root $\Rightarrow$

$$z = \frac{\sigma(\rho)}{\eta(\rho)}$$

look at $|\rho| = 1$ to set boundary.

$$u' = \lambda \left(u - \sin(t) \right) + \cos(t)$$



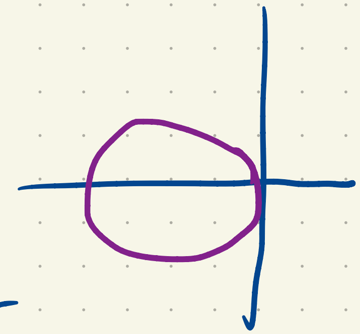
Euler:

$$\rho - 1 - z = 0$$

$$z = -1 + \rho \qquad |\rho| = 1$$

$$\hookrightarrow \rho = e^{i\theta}$$

$$z = -1 + e^{i\theta}$$

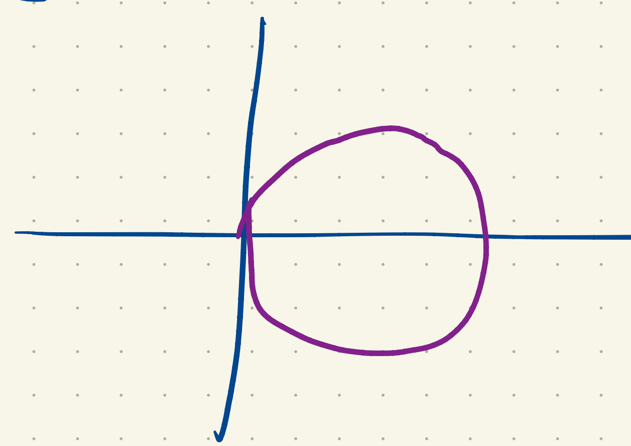


Backwards Euler:

$$\rho - 1 - z\rho = 0$$

$$z = 1 - 1/\rho \qquad \rho = e^{i\theta}$$

$$z = 1 - e^{-i\theta}$$



Abs stability for R-K

$$Y_1 = u_k + h \sum_{j=1}^{k} a_{1j} \overbrace{f(t_k + c_j h, Y_j)}^{\lambda Y_j}$$

$$\vdots$$

$$Y_n = u_k + h \sum_{j=1}^{k} a_{nj} \overbrace{f(t_k + c_j h, Y_j)}^{\lambda Y_j}$$

$$u_{k+1} = u_k + h \sum_{j=1}^{k} b_j \underbrace{f(t_k + c_j h, Y_j)}_{\lambda Y_j}$$

$$f(t, u) = \lambda u$$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = u_k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \lambda h \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}}_{A} \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$(I - zA)\vec{Y} = u_k \vec{W}$$

$$\vec{Y} = (I - zA)^{-1} \vec{W} \, u_k$$

$$u_{k+1} = u_k + h\lambda \, \vec{b} \cdot \vec{Y}$$

$$= u_k + z\, \vec{b}^T (I - zA)^{-1} \vec{w}\, u_k$$

$$= u_k \underbrace{\left[1 + z\, \vec{b}^T (I - zA)^{-1} \vec{w} \right]}$$

$R(z)$, stability function

$|R(z)| < 1$ desired.

$1 + z + \frac{1}{2} z^2$

Always a polynomial if method is explicit

$1 + z + \frac{1}{2} z^2$, e.g.