PDE:

Heat equation (rep of class of parabolic PDE
that model diffusion processes)

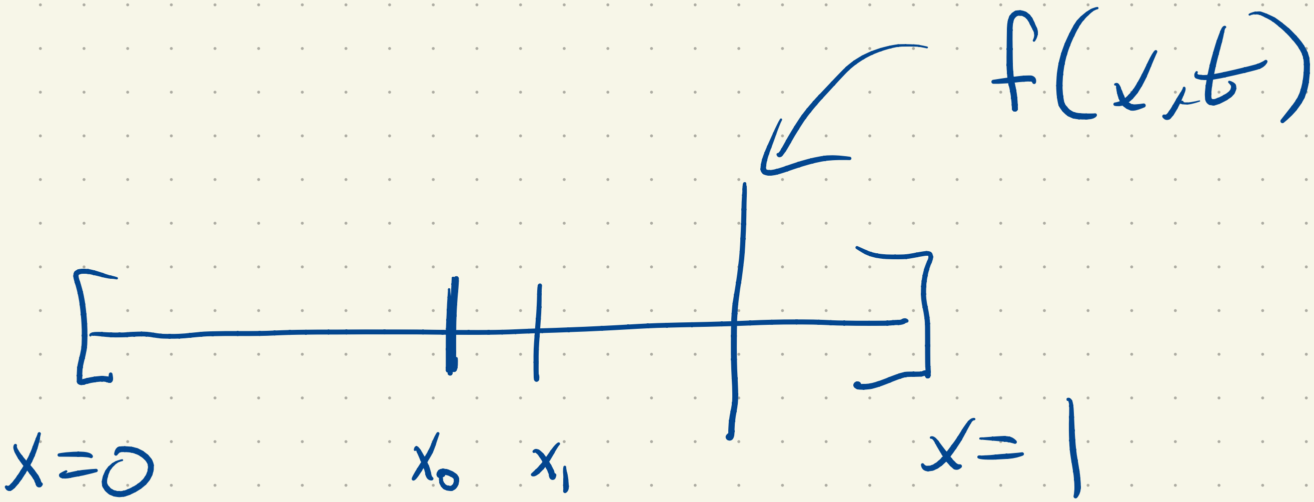
Space domain: $[0,1]$ (e.g. a rod)

u: a linear density of some kind

$[u] = \frac{\text{stuff}}{\text{length}}$ (energy, particles, heat $\approx$ temp)

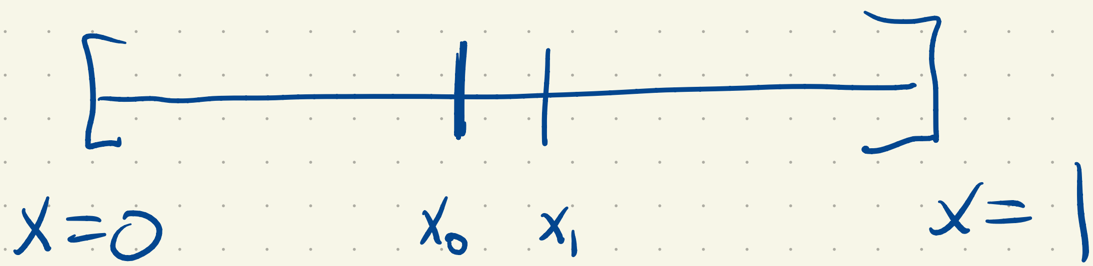
flux: $f(x,t)$: at time t, at position x,
rate at which stuff is passing, $\rightarrow$

units: $[u][x]/[t]$



$$\int_{x_0}^{x_1} u(s,t)\,ds$$

units: $[u][x]/[t]$



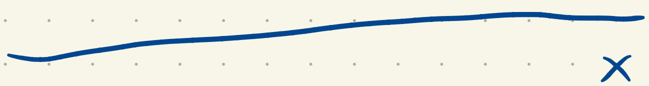
$\int_{x_0}^{x_1} u(t,s)\,ds$: <u>total stuff in $[x_0, x_1]$</u>

$$\frac{d}{dt}\int_{x_0}^{x_1} u(s,t)\,ds = f(x_1,t) - f(x_0,t)$$

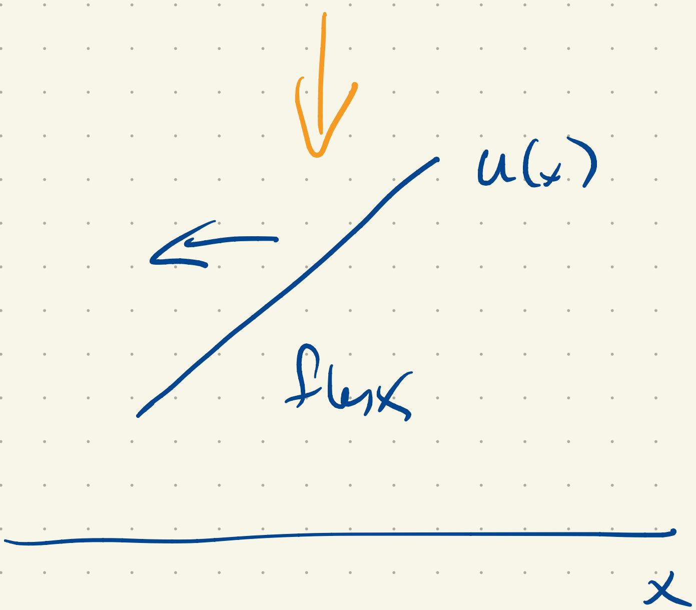
[conservation of stuff]

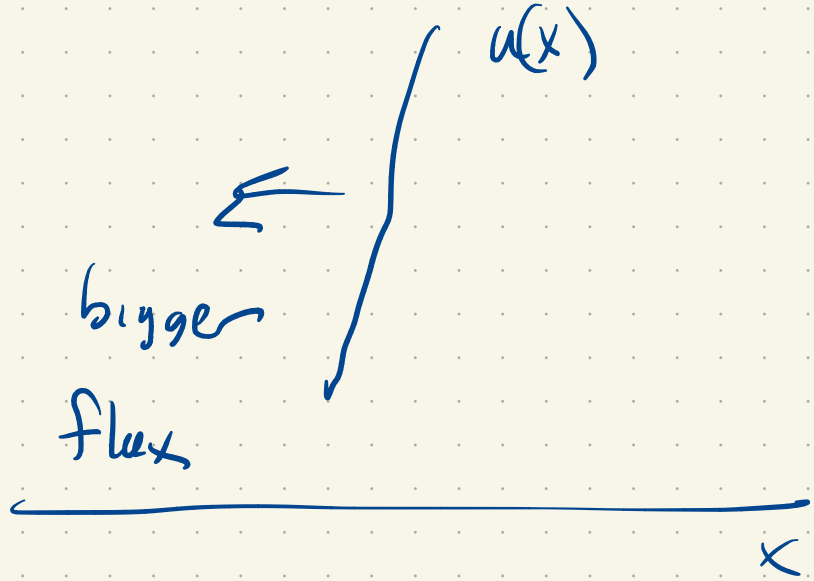
Leveling hypothesis:

$$f(x,t) \sim u_x$$



→ no flux





$$f(x,t) = k\,u_x \qquad k > 0 \text{ a constant.}$$

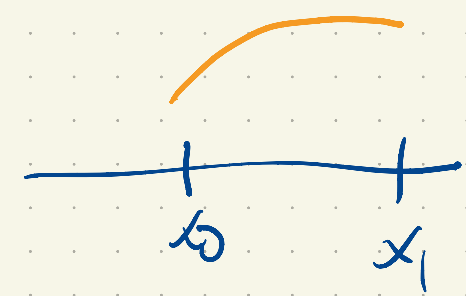
$$\frac{d}{dt}\int_{x_0}^{x_1} u(s,t)\,ds = k\left[u_x(x_1,t) - u_x(x_0,t)\right]$$

$$f(x,t) = k\,u_x \qquad k \text{ a constant.}$$

$$\int_a^b \partial_x f(s)\,ds = f(b) - f(a)$$

$$\frac{d}{dt}\int_{x_0}^{x_1} u(s,t)\,ds = k\left[u_x(x_1,t) - u_x(x_0,t)\right]$$

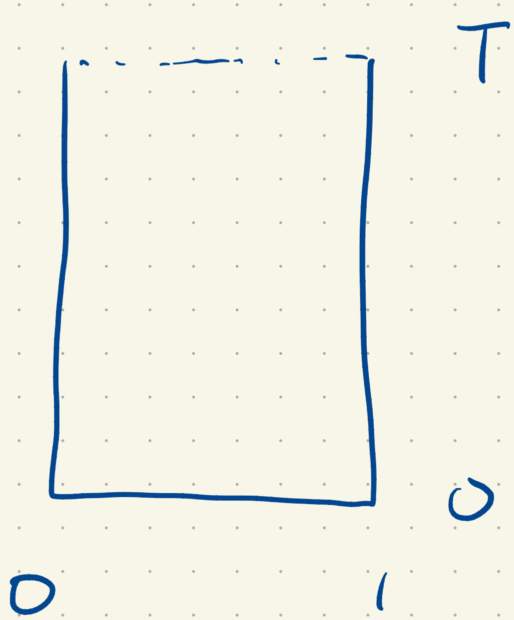
$$= k\int_{x_0}^{x_1} u_{xx}(t,s)\,ds$$



$$\int_{x_0}^{x_1}\left[u_t(t,s) - k\,u_{xx}(t,s)\right] ds = 0$$

True on all subintervals $\Rightarrow$ $\underbrace{u_t - k u_{xx} = 0}_{\text{heat equation.}}$

Domain:

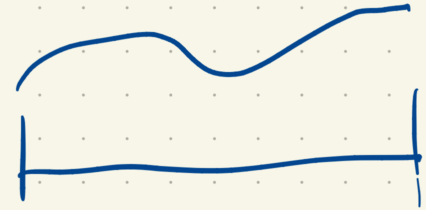


$\Omega = [0,1] \times [0,T]$

Exercise:

If $u_t - k u_{xx} = g(x,t)$ what does g mean?

Exercise: If $k = k(x,t)$



$$u_t - \partial_x\big(k(x,t)\, u_x\big) = 0$$

By choice of units time we can take $k=1$ if k is constant.

Boundary conditions (unique to each PDE)

1) Initial condition $u(x,t_0) = u_0(x)$

(initial heat density) $\to t_0 = 0$

2) End conditions at $x=0$, $x=1$

- Dirichlet: u specified at ends.
- Neumann: flux specified at ends

$k u_x$

↳ usually just u_x

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- Dirichlet: u specified at ends.
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$$-ku_x$$

↳ usually just u_x

Less common: Robin

flux is proportional to u

$$-ku_x = \alpha u$$

$$u_x + cu = 0$$

Homogeneous: $u(0) = u(1) = 0$

$u_x(0) = u_x(1) = 0.$

$u_t = u_{xx}$

$u(0,t) = u(1,t) = 0$

Homogeneous: $u(0) = u(1) = 0$

$$u_x(0) = u_x(1) = 0.$$

For simplicity, we'll focus for now on homogeneous Dirichlet.

Structure:

$$u_t = u_{xx}$$

$$\downarrow$$

ODE analog: $$\vec{w}_t = A\vec{w}$$

(think of u as a vector with one degree of freedom per x!)

Suppose $A\vec{v} = \lambda \vec{v}$

$\vec{w}(t) = e^{\lambda t}\vec{v}$ is a solution

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Suppose $A\vec{v} = \lambda \vec{v}$

$u' = \lambda u$

$c e^{\lambda t}$

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$\vec{w}_t = \lambda e^{\lambda t}\vec{v}$ ✓

$$\begin{aligned} A\vec{w} &= e^{\lambda t} A\vec{v} \\ &= \lambda e^{\lambda t}\vec{v} \quad ✓ \end{aligned}$$

If $\vec{v}_1, \dots, \vec{v}_n$ is a basis of eigenvectors

$\lambda_1, \dots, \lambda_n$

$$\vec{w} = c_1 e^{\lambda_1 t} \vec{v}_1 + \dots + c_n e^{\lambda_n t} \vec{v}_n \quad \text{solves ODE}$$

Every solution is a linear combo.

$$\vec{w}\,' = A w$$

$$\vec{w}(0) = \vec{w}_0 \longrightarrow \vec{w}_0 = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$\vec{w}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + \dots + c_n e^{\lambda_n t} \vec{v}_n$$

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Every solution is a linear combo.

[Warning: $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ does not have a basis

of eigenvectors. Only eigenvalue is $\lambda = 1$

$$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$A\vec{v} = \lambda \vec{v}$

For $\quad u_t = u_{xx}$

eigenvector analog: $\left.\begin{array}{l} v_{xx} = \lambda v \\ v(0) = 0 \\ v(1) = 0 \end{array}\right]$ eigenfunctions

$\lambda > 0: \quad v(x) = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$

$\lambda < 0 \quad v(x) = c_1 \cos(\sqrt{-\lambda} x) + c_2 \sin(\sqrt{-\lambda}\, x)$

$\lambda = 0 \qquad v_{xx} = 0 \longrightarrow$ linear

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$\sin(\sqrt{-\lambda}) = 0$

To get $v(0) = 0$, $v(1) = 0 \Rightarrow$

a) $\lambda < 0$

b) $\sqrt{-\lambda} = k\pi$ $k = 1, 2, 3, \ldots$

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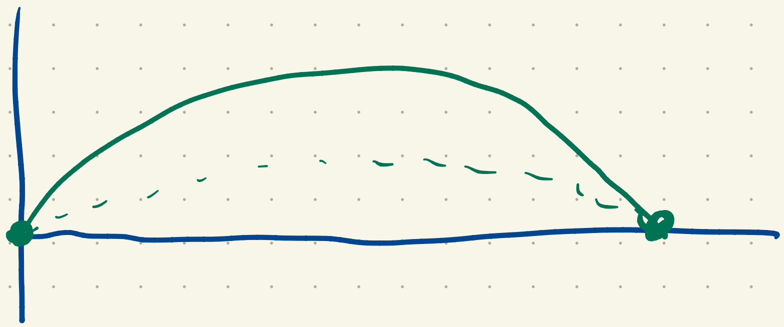
To get $v(0) = 0,\ v(1) = 0 \Rightarrow$

a) $\lambda < 0$

b) $\sqrt{-\lambda} = k\pi \Rightarrow \lambda = -k^2\pi^2$

$v_k(x) = \sin(k\pi x)$, $\lambda_k = -k^2\pi^2$

solution: $u(x,t) = e^{-k^2\pi^2 t} \sin(k\pi x)$



$$v_k(x) = \sin(k\pi x), \qquad \lambda = -k^2\pi^2$$

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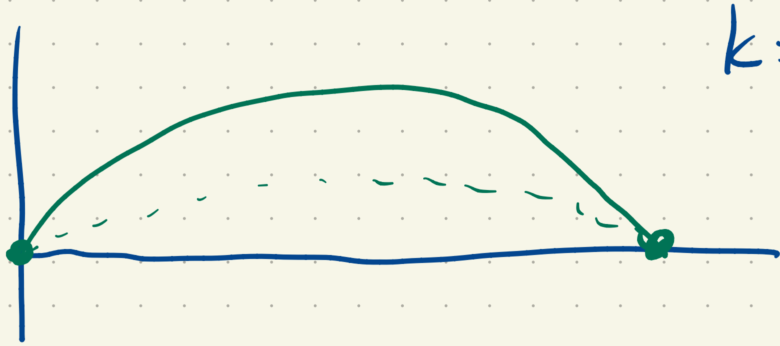
$k=1$



$k=2$