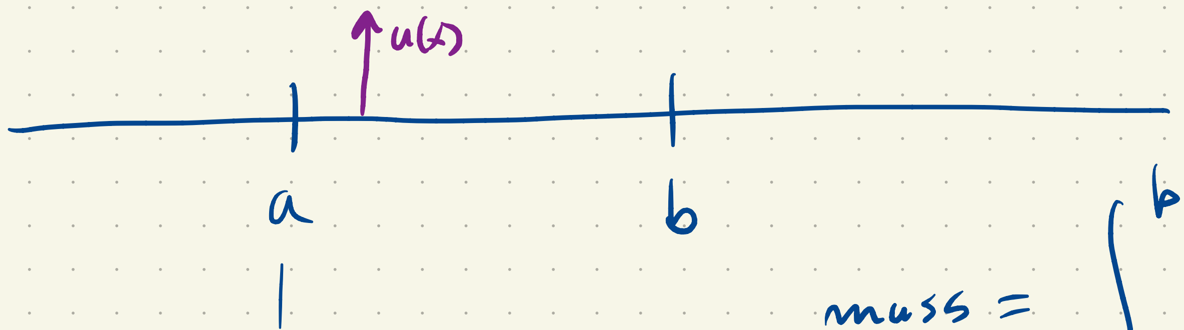
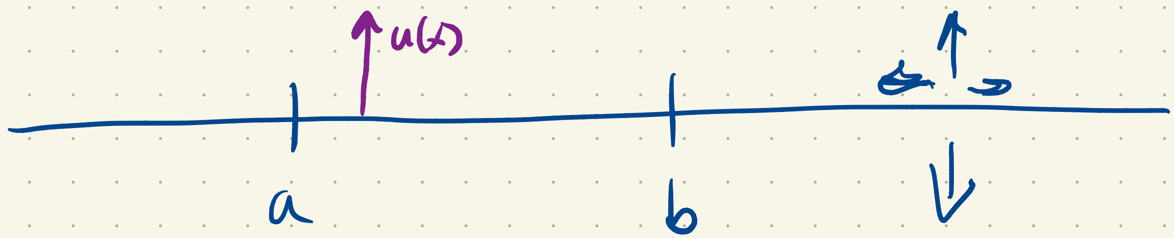
Wave Equation



mass = $\int_a^b \rho \, dx$

linear density: ρ

Wave Equation

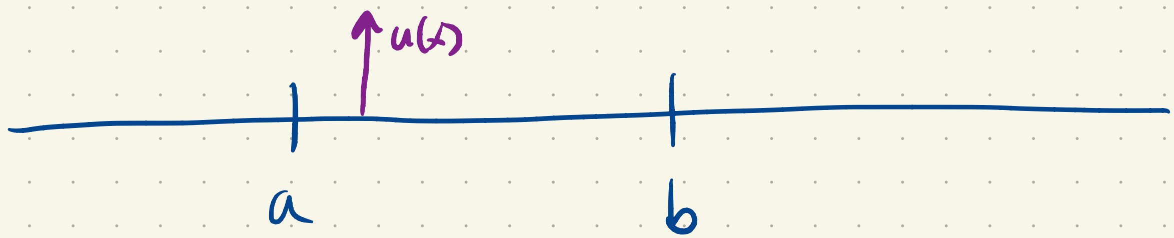


linear density: ρ

vertical displacement: u

vertical velocity: u_t

Wave Equation



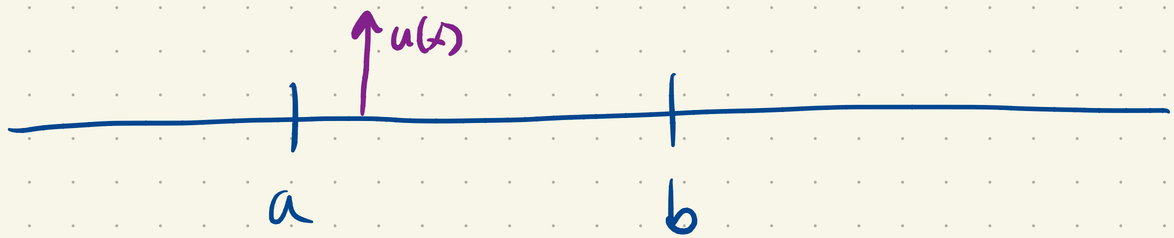
linear density: ρ

vertical displacement: u

vertical velocity: u_t

momentum: $$P = \int_a^b \rho\, u_t(s,t)\, ds$$

Wave Equation



linear density: ρ

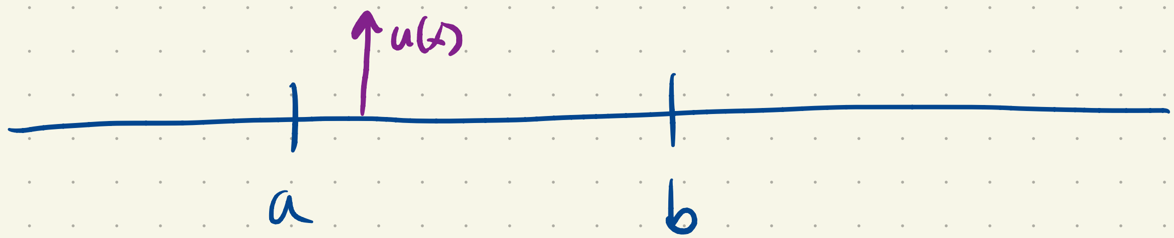
vertical displacement: u

vertical velocity: u_t

momentum: $P = \int_a^b \rho\, u_t(s,t)\, ds$

Newton 2: $F = ma$

Wave Equation



linear density: ρ

vertical displacement: u

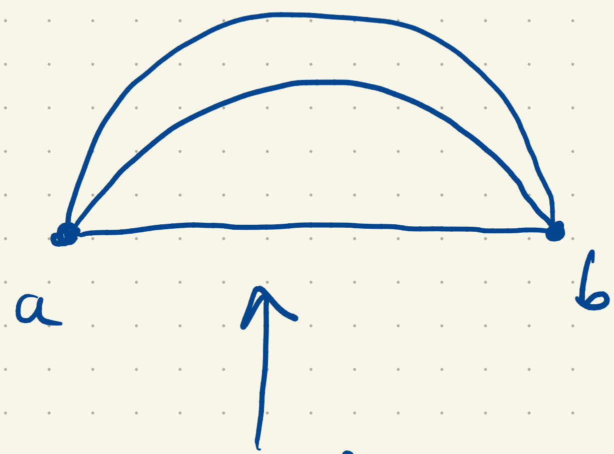
vertical velocity: u_t

momentum: $P = \int_a^b \rho\, u_t(s,t)\, ds$

Newton 2: $F = \cancel{ma}$

$$\frac{d}{dt} P = F$$

Restoring Force



internal forces
cancel

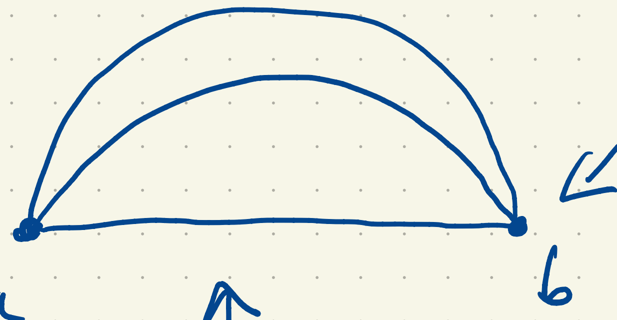
Restoring Force

force proportional to $+u_x$

(negative u_x $\Rightarrow$ force is down, restoring)

internal forces cancel

Restoring Force



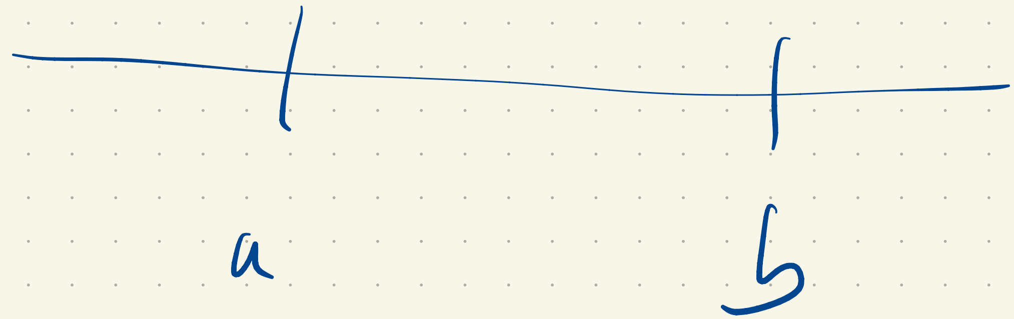
force proportional to $+u_x$

(negative u_x $\Rightarrow$ force is down, restoring)

internal forces cancel

force proportional to $-u_x$

Newton 2



$$\frac{d}{dt}\int_a^b \rho\, u_t(s,t)\, ds = K u_x(0,b) - K u_x(0,a)$$

Newton 2

$$\frac{d}{dt}\int_a^b \rho u_t(s,t)\,ds = Ku_x(0,b) - Ku_x(0,a)$$

$$= K\int_a^b u_{xx}(s,t)\,ds$$

Newton 2

$$\frac{d}{dt}\int_a^b \rho\, u_t(s,t)\,ds = K u_x(b,t) - K u_x(a,t)$$

$$= K\int_a^b u_{xx}(s,t)\,ds$$

$$\int_a^b \rho\, u_{tt}(s,t) - K u_{xx}(s,t)\,ds = 0$$

True for all a, b!

Wave Equation

f for external force

$$\rho u_{tt} - k u_{xx} = 0$$

Wave Equation

f for external force

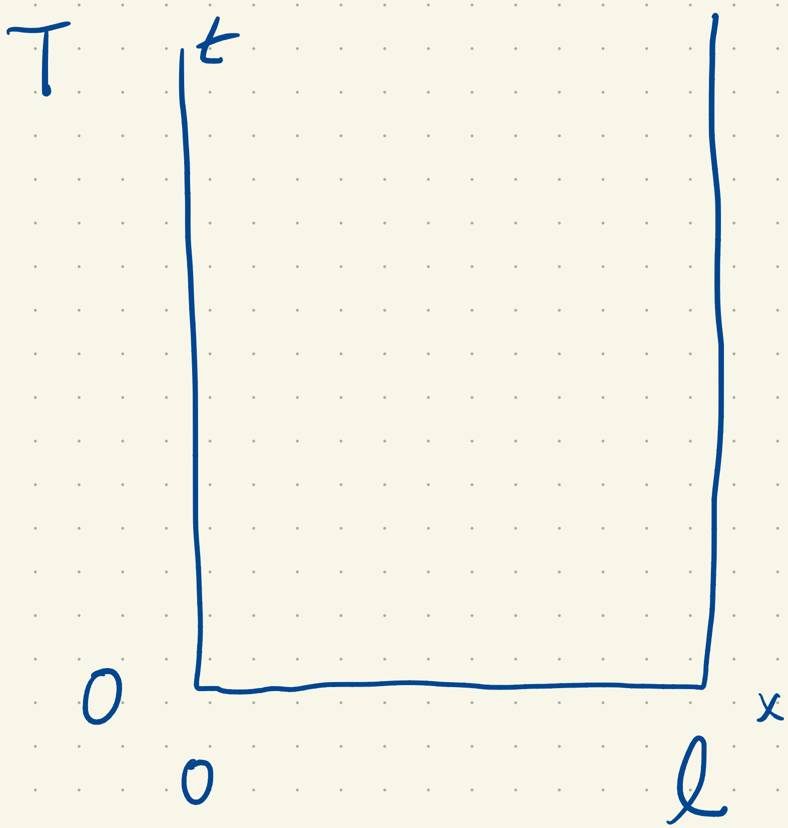
$$\rho u_{tt} - k u_{xx} = 0$$

$$c^2 = k/\rho$$

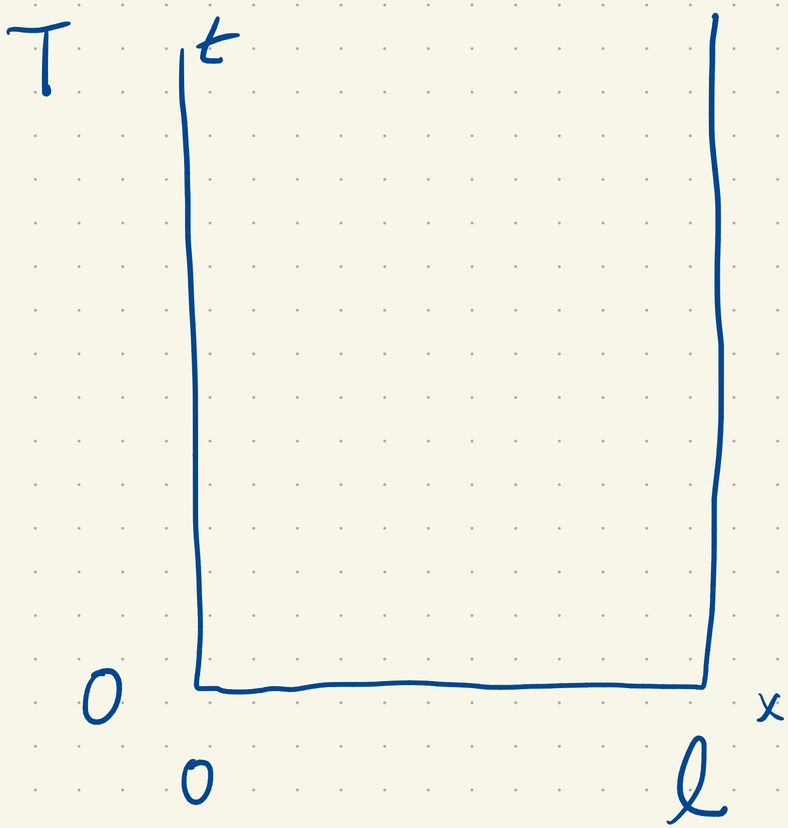
$$\boxed{u_{tt} - c^2 u_{xx} = 0}$$

velocity

Domain, BC's



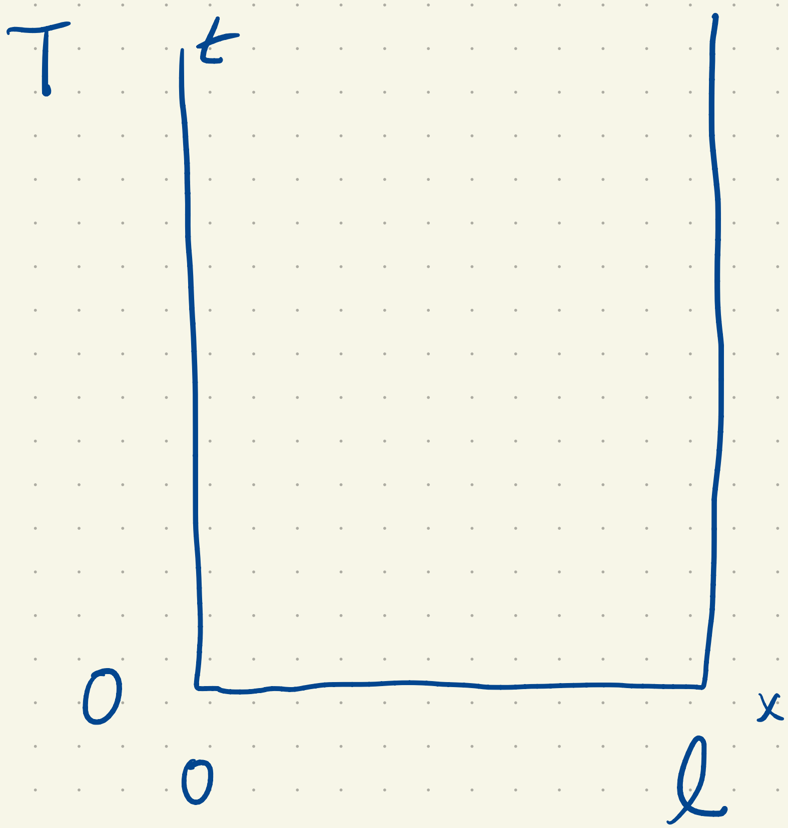
Domain, BC's



$$u(0,t)=0$$

$$u(\ell,t)=0$$

Domain, BC's



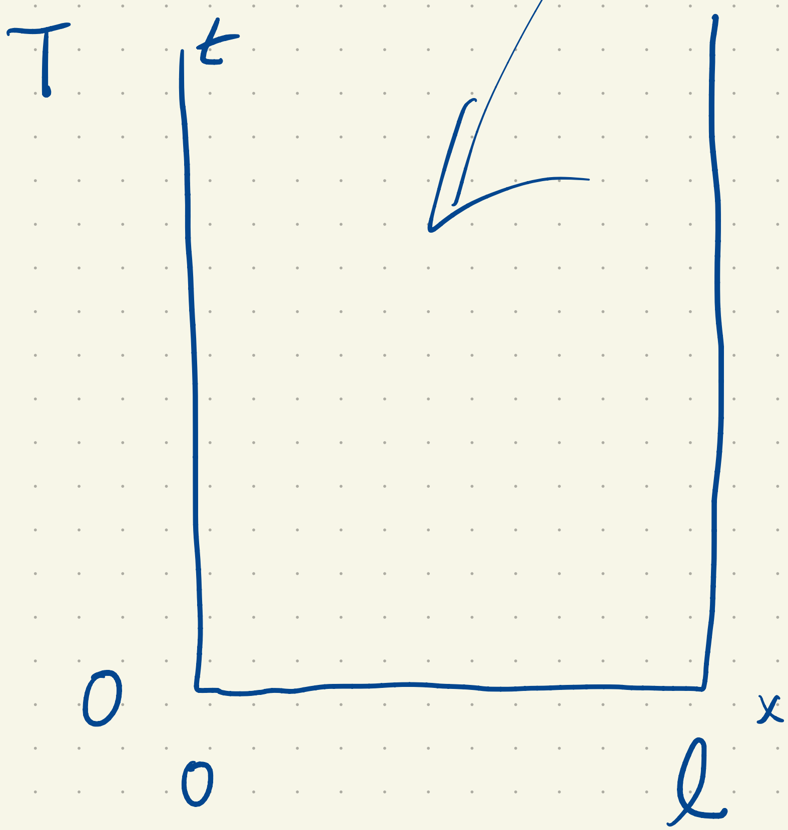
$u(0,t)=0$

$u(\ell,t)=0$

(position, momentum)

Domain, BC's

$$u_{tt} + c^2 u_{xx} = 0$$



$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = u_0(x)$$

$$u_t(x,0) = V(x)$$

initial velocity.

(position, momentum)

Sample solutions

$$u_{tt} - c^2 u_{xx} = 0$$

$$w(x) = \sin\left(\frac{k\pi}{\ell} x\right) \qquad w'' = -\left(\frac{k\pi}{\ell}\right)^2 w(x)$$

$$w(0) = 0$$

$$w(\ell) = \sin(k\pi) = 0$$

Sample solutions $\qquad e^{at}\sin\left(\frac{k\pi}{\ell}x\right)$

$$u_{tt} - c^2 u_{xx} = 0 \qquad a^2 - c^2\left(-\left(\frac{k\pi}{\ell}\right)^2\right)$$

$$w(x) = \sin\left(\frac{k\pi}{\ell}x\right) \qquad w'' = -\left(\frac{k\pi}{\ell}\right)^2 w(x)$$

$$w(0) = 0$$

$$w(\ell) = \sin(k\pi) = 0$$

$$h(t) = \cos(at) \qquad h'' = -a^2 h(t)$$

$$u(x,t) = w(x)h(t)$$

$$u_{tt} - c^2 u_{xx} = -\left[a^2 - c^2\left(\frac{k\pi}{l}\right)^2\right] w(x)h(t)$$

$$0 \qquad u$$

$$u(x,t) = w(x)\,h(t)$$

$$u_{tt} - c^2 u_{xx} = -\left[a^2 - c^2\left(\frac{k\pi}{\ell}\right)^2\right] w(x)\,h(t)$$

Solution if $a = \pm \dfrac{ck\pi}{\ell}$

irrelevant

$$u(x,t) = w(x)\,h(t) \qquad \underbrace{e^{I\omega t}}$$

$$u_{tt} - c^2 u_{xx} = -\left[a^2 - c^2\left(\frac{k\pi}{\ell}\right)^2\right] w(x)\,h(t)$$

Solution if $a = \pm \frac{ck\pi}{\ell}$ $\qquad e^{k\pi/\ell\, t}$

irrelevant

$$u(x,t) = \cos\left(\frac{ck\pi}{\ell} t\right) \sin\left(\frac{k\pi}{\ell} x\right)$$

$$u(x,t) = w(x)\,h(t)$$

$$u_{tt} - c^2 u_{xx} = -\left[a^2 - c^2\left(\frac{k\pi}{\ell}\right)^2\right] w(x)\,h(t)$$

Solution if $a = \pm \dfrac{ck\pi}{\ell}$

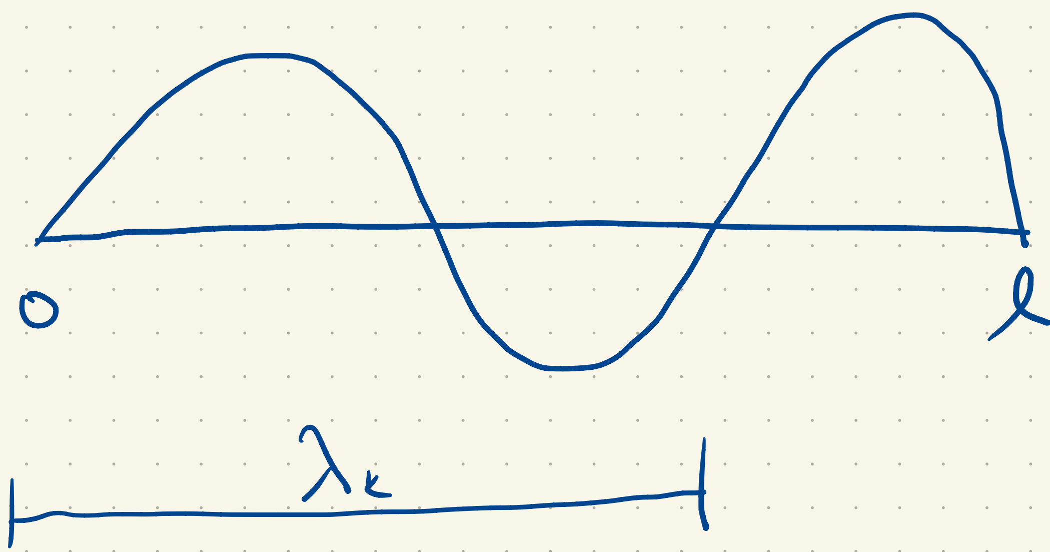
irrelevant

$$u(x,t) = \cos\left(\frac{ck\pi}{\ell}t\right)\sin\left(\frac{k\pi}{\ell}x\right)$$

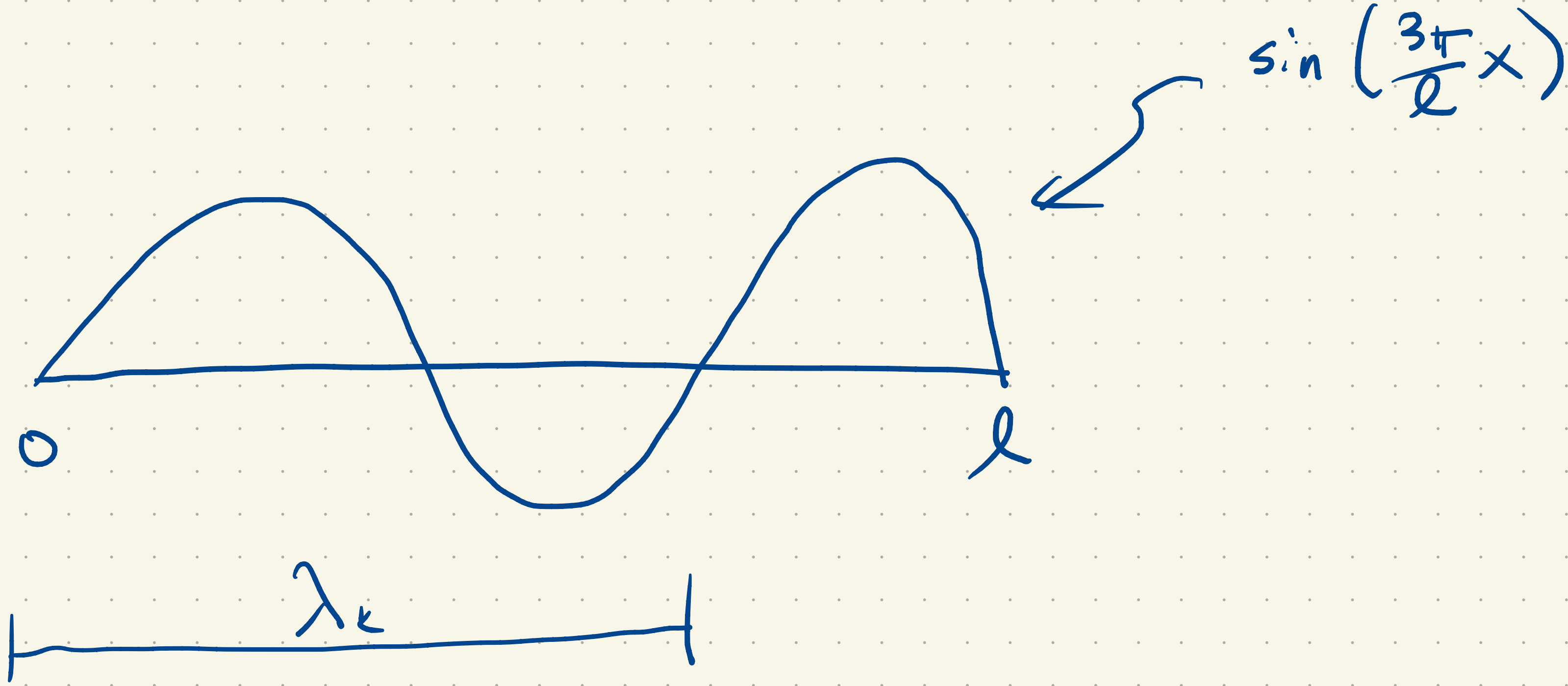
$$u(x,0) = \sin\left(\frac{k\pi}{\ell}t\right) \; ; \; u_t(x,0) = 0$$

Qualitative Behaviour

$\sin\left(\frac{3\pi}{\ell}x\right)$



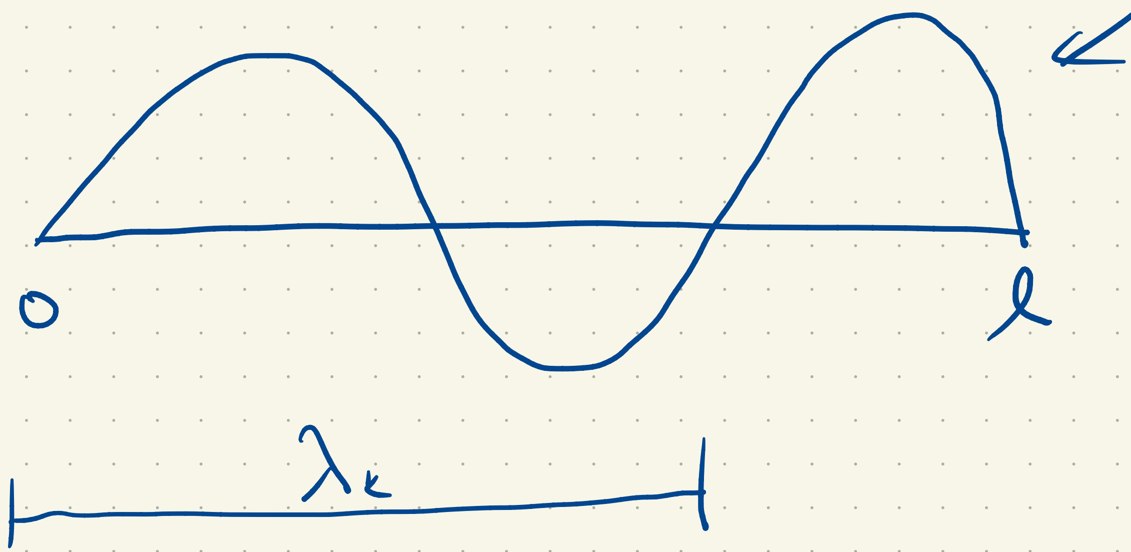
Qualitative Behaviour



wavelength: $\frac{3\pi}{\ell}\lambda_k = 2\pi \Rightarrow \lambda_k = \frac{2\ell}{3}$

Qualitative Behaviour

$\sin\left(\frac{3\pi}{\ell}x\right)$

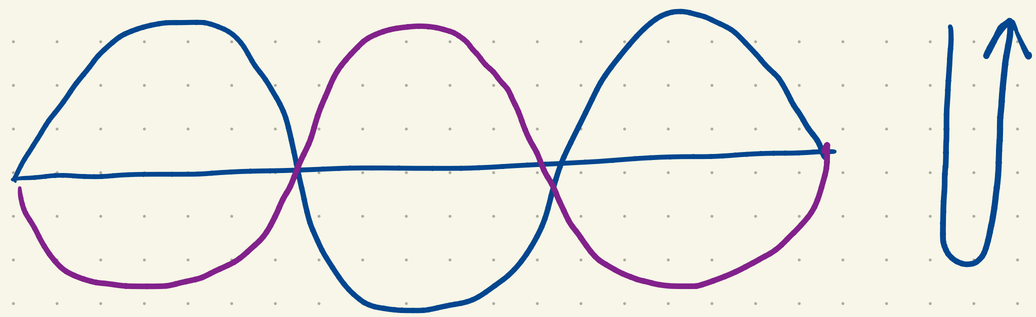


wavelength: $\frac{3\pi}{\ell}\lambda_k = 2\pi \Rightarrow \lambda_k = \frac{2\ell}{3}$

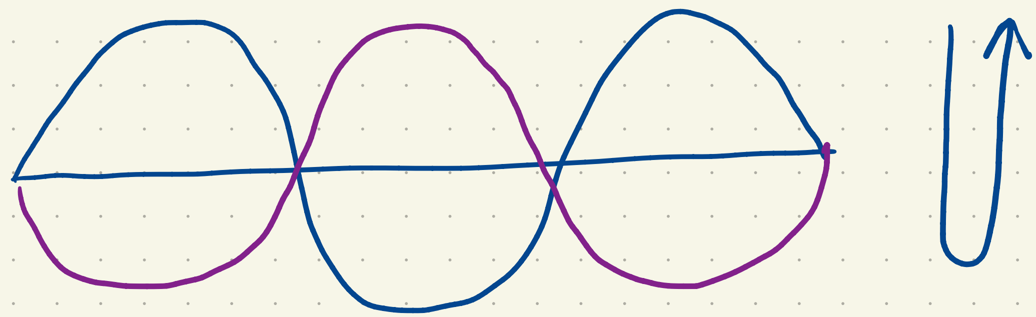
For k, $\lambda_k = \frac{2\ell}{k}$.

Period: τ_k

$\cos\left(c \frac{k\pi}{\ell} t\right)$

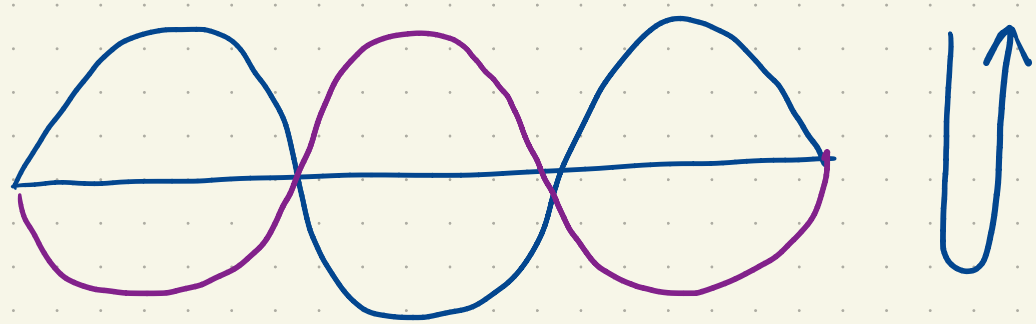


Period: τ_k



$\cos\left(c\frac{k\pi}{\ell}t\right)$ $\quad$ $\frac{ck\pi}{\ell}\tau_k = 2\pi$

Period: τ_k



$$\cos\left(c\frac{k\pi}{\ell}t\right) \qquad \frac{ck\pi}{\ell}\tau_k = 2\pi$$

$$\tau_k = \frac{2\ell}{kc} = \underbrace{\frac{1}{c}\lambda_k}$$

Time to travel dist λ_k at speed c.

Angular Frequency

$$\omega = \frac{2\pi}{\tau} = \frac{kc}{l}\pi$$

$\hookrightarrow$ radians/sec

$$f = \frac{\omega}{2\pi}, \quad \text{cycles/s}$$

Angular Frequency

$$\omega = \frac{2\pi}{\tau} = \frac{kc}{l}\pi \qquad f = \frac{\omega}{2\pi}, \quad \text{cycles/s}$$

$\hookrightarrow$ radians/sec

Upshot: Short wavelength oscillations

$\Longleftrightarrow$

High frequency oscillations.

Another family

$$u(x,t) = \sin\left(c\frac{k\pi}{\ell}t\right)\sin\left(\frac{k\pi}{\ell}x\right)$$

$$u(x,0) = 0$$

$$u_t(x,0) = \frac{ck\pi}{\ell}\underset{1}{\cancel{\cos\left(c\frac{k\pi}{\ell}t\right)}}\sin\left(\frac{k\pi}{\ell}x\right)$$

Finite sum solutions

$$u(x,t) = \sum_{k=1}^{N} \left[a_k \cos\left(c\frac{k\pi}{\ell} t\right) + b_k \sin\left(c\frac{k\pi}{\ell} t\right) \right] \sin\left(\frac{k\pi}{\ell} x\right)$$

$$u_{tt} - c^2 u_{xx} = 0$$

linear

$$u_1 \ , \ u_2$$

$$a u_1 + b u_2$$

Finite sum solutions

$$u(x,t) = \sum_{k=1}^{N} \left[a_k \cos\left(c\frac{k\pi}{\ell}t\right) + b_k \sin\left(c\frac{k\pi}{\ell}t\right) \right] \sin\left(\frac{k\pi}{\ell}x\right)$$

$$\sum_{k=1}^{N} a_k \sin\left(\frac{k\pi x}{\ell}\right)$$

$$\int_0^{\ell} u(x,0) \sin\left(\frac{j\pi}{\ell}x\right) dx = \frac{\ell}{2} a_j$$

$$\int_0^{\ell} u_t(x,0) \sin\left(\frac{j\pi}{\ell}x\right) dx = \frac{cj\pi}{\ell} \frac{\ell}{2} b_j$$

Finite sum solutions

$$u(x,t) = \sum_{k=1}^{N} \left[a_k \cos\left(c\frac{k\pi}{\ell}t\right) + b_k \sin\left(c\frac{k\pi}{\ell}t\right) \right] \sin\left(\frac{k\pi}{\ell}x\right)$$

$$u_0 = \sum_{k=1}^{N} a_k \sin\left(\frac{k\pi}{\ell}x\right)$$

$$\int_0^\ell u(x,0)\sin\left(\frac{j\pi}{\ell}x\right)dx = \frac{\ell}{2}\, a_j$$

$$v = \sum_{k=1}^{N} \frac{ck\pi}{\ell} b_k \sin\left(\frac{k\pi}{\ell}x\right)$$

$$\int_0^\ell u_t(x,0)\sin\left(\frac{j\pi}{\ell}x\right)dx = \frac{cj\pi}{\ell}\,\frac{\ell}{2}\, b_j$$

Fourier Series Solutions?

$$u(x,t) = \sum_{k=1}^{\infty} \left[a_k \cos\left(c\frac{k\pi}{\ell}t\right) + b_k \sin\left(c\frac{k\pi}{\ell}t\right) \right] \sin\left(\frac{k\pi}{\ell}x\right)$$

$$a_j = \frac{2}{\ell} \int_0^{\ell} u(x,0) \sin\left(\frac{j\pi}{\ell}x\right) dx$$

$$b_j = \frac{1}{cj} \frac{2}{\pi} \int_0^{\ell} u_t(x,0) \sin\left(\frac{j\pi}{\ell}x\right) dx$$

D'Alembert Solution $(-\infty < x < \infty)$

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u$$

D'Alembert Solution $(-\infty < x < \infty)$

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u$$

$$\begin{aligned} r &= x - ct \\ s &= x + ct \end{aligned} \implies \begin{aligned} x &= \frac{s+r}{2} \\ t &= \frac{s-r}{2c} \end{aligned}$$

D'Alembert Solution $(-\infty < x < \infty)$

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

$$\begin{aligned} r &= x - ct \\ s &= x + ct \end{aligned} \implies \begin{aligned} x &= \frac{s+r}{2} \\ t &= \frac{s-r}{2c} \end{aligned}$$

$$U(r,s) = u\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)$$

D'Alembert Solution $(-\infty < x < \infty)$

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u$$

$$r = x - ct \qquad\qquad x = \frac{s+r}{2}$$
$$\Rightarrow$$
$$s = x + ct \qquad\qquad t = \frac{s-r}{2c}$$

$$U(r,s) = u\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)$$

$$\partial_r U = \frac{1}{2}\left[u_x - \frac{1}{c}u_t\right]$$

D'Alembert Solution $(-\infty < x < \infty)$

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)\, u$$

$$\begin{aligned} r &= x - ct \\ s &= x + ct \end{aligned} \implies \begin{aligned} x &= \frac{s+r}{2} \\ t &= \frac{s-r}{2c} \end{aligned}$$

$$U(r,s) = u\left(\frac{s+r}{2}, \frac{s-r}{2c}\right)$$

$$\partial_r U = \frac{1}{2}\left[u_x - \frac{1}{c} u_t\right] = -\frac{1}{2c}\left[u_t - c u_x\right]$$

$$= -\frac{1}{2c}(\partial_t - c\partial_x)\, u$$

$$\partial_s U = \frac{1}{2}\left[u_x + \frac{1}{c}u_t\right] = \frac{1}{2c}\left[u_t + c\,u_x\right]$$

$$= \frac{1}{2c}\left(\partial_t + c\,\partial_x\right)u$$

$$\partial_s U = \frac{1}{2}\left[u_x + \frac{1}{c}u_t\right] = \frac{1}{2c}\left[u_t + c\,u_x\right]$$

$$= \frac{1}{2c}\left(\partial_t + c\,\partial_x\right)u$$

$$\partial_r \partial_s U = \frac{-1}{(2c)^2}\left(\partial_t + c\,\partial_x\right)\left(\partial_t - c\,\partial_x\right)u = 0$$

$$\partial_s \left[\partial_r U \right] = 0$$

$$\partial_r U = f(r)$$

$$\partial_s \left[\partial_r U \right] = 0$$

$$\partial_r U = f(r)$$

$$U = F(r) + G(s) \; ; \; F' = f$$

$$\partial_s \left[\partial_r U \right] = 0$$

$$\partial_r U = f(r)$$

$$U = F(r) + G(s) \; ; \; F' = f$$

$$u\big(x(r,s), t(r,s)\big) = F(r) + G(s)$$

$$\partial_s \left[\partial_r U \right] = 0$$

$$\partial_r U = f(r)$$

$$U = F(r) + G(s) \; ; \; F' = f$$

$$u\big(x(r,s), t(r,s)\big) = F(r) + G(s)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$\longrightarrow \qquad \longleftarrow$$

General Solution is a superposition of two waves, each traveling at speed c, one left, one right.

Initial Conditions

$$u(x,0) = u_0(x)$$
$$u_t(x,0) = v(x)$$

$$u(x,t) = F(x-ct) + G(x+ct)$$

Initial Conditions

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$u(x,0) = F(x) + G(x) = u_0(x)$$

$$u_t(x,0) = -cF'(x) + cG'(x) = v(x)$$

u_0

Initial Conditions

$$u(x,t) = F(x-ct) + G(x+ct)$$

$$u(x,0) = F(x) + G(x)$$

$$u_t(x,0) = -cF'(x) + cG'(x)$$

Case 1) u_0 given, $v \equiv 0$

2) $u_0 \equiv 0$, v given