Closed Map Lemma:

Lemma. A continuous map from a compact space to a Hausdorff space is a closed map.

Pf: Suppose $f: X \to Y$ is continuous, X is compact and Y is Hausdorff. Suppose $A \subseteq X$ is closed. Since X is compact and A is closed, A is compact. Since f is continuous $f(A)$ is compact. Since Y is Hausdorff, $f(A)$ is closed.

Cor: If X is compact and Y is Hausdorff and $f: X \to Y$ is continuous and surjective then f is a quotient map.

Pf: f is continuous, surjective and closed (and hence takes saturated closed sets to closed sets.)

Cor: If X is compact and Y is Hausdorff and $f: X \to Y$ is continuous and bijective then f is a homeomorphism.

Pf: We need only establish that f^{-1} is continuous.

If $A \subseteq X$ is closed then $(f^{-1})^{-1}(A) = f(A)$ which is closed.

Cor: If X is compact and Y is Hausdorff and $f: X \to Y$ is continuous and injective then f is a top. embedding.

Pf: Note $f(X) \subseteq Y$ is Hausdorff. So $f: X \to f(X)$ is continuous and bijective from a compact space to a Hausdorff space and is hence a homeomorphism.

E.g. $[0,1]/\sim$ is homeomorphic to S^1

$0 \sim 1$.

$f : [0,1] \to S^1$

$f(x) = e^{2\pi i x}$

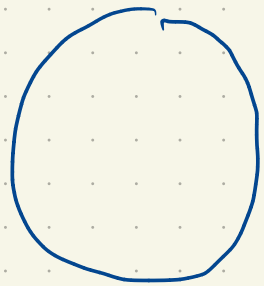
f is cts, surjective.
$[0,1]$ is compact
S^1 is Hausdorff.

$\Rightarrow f$ is a quotient map.

$$\begin{tikzcd}
& {[0,1]} \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\
{[0,1]/\sim} & & S^1
\end{tikzcd}$$

f and π make the same identifications

So the quotients are homeomorphic.



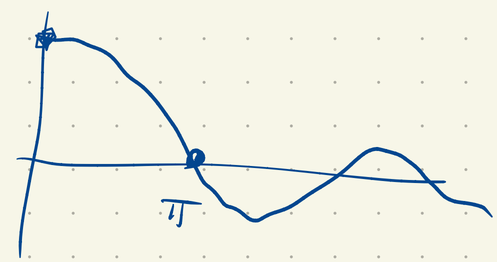
$D^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$

$D^2 / \partial D^2$ $\quad (a \sim b$ if $a, b \in \partial D^2)$

$\dfrac{\sin(x)}{x} \to 1$ as $x \to 0$.

Claim is $D^2/\partial D^2 \sim S^2$.

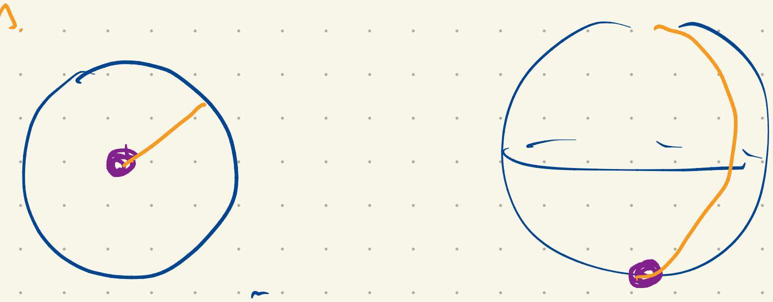
$$f(x,y) = \cos\left(\sqrt{x^2+y^2}\cdot\pi\right)\cdot(0,0,-1) + \frac{\sin\left(\sqrt{x^2+y^2}\cdot\pi\right)}{\pi\sqrt{x^2+y^2}}\cdot(x,y,0).$$



f is cts, surjective, domain is compact, codomain is Hausdorff and f makes the same identifications.

quotient map.

$$\begin{tikzcd} & D^2 \arrow[dl, "\pi"'] \arrow[dr, "f"] & \\ D^2/\partial D^2 & & S^2 \end{tikzcd}$$



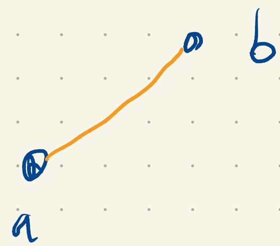
Prop: Suppose $K \subseteq \mathbb{R}^n$

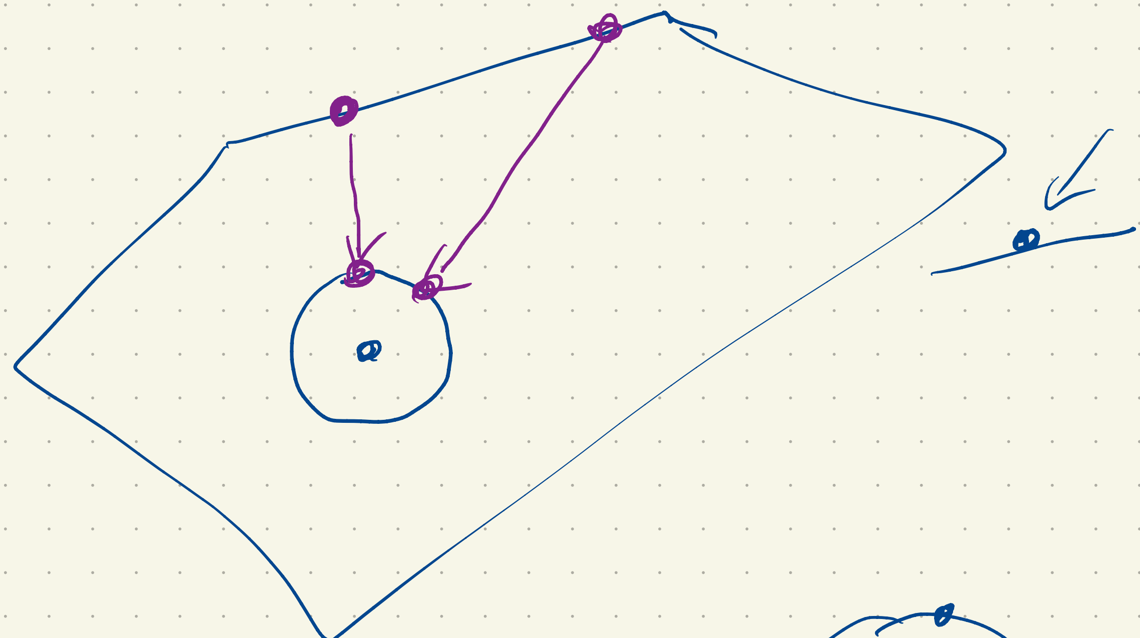
- is compact
- is convex
- has nonempty interior

Then K is homeomorphic to $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ by a homeomorphism taking ∂K to S^{n-1}.

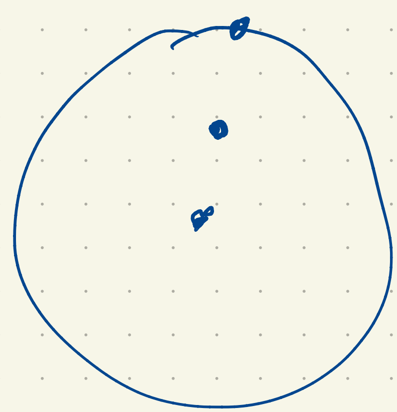
$A \subseteq \mathbb{R}^n$ is convex if whenever $a, b \in A$,

$\lambda a + (1-\lambda) b \in A \quad 0 \leq \lambda \leq 1.$





$$z \longmapsto \frac{z}{|z|}$$



Other notions of compactness

1) X is limit point compact if every infinite set in X has a limit point.

2) X is sequentially compact if every sequence in X has a convergent subsequence.

In general neither 1) nor 2) are equivalent to compactness.

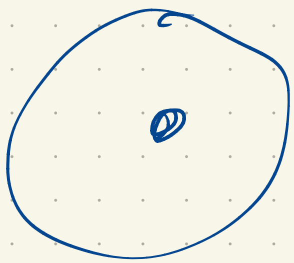
$X = \mathbb{Z}$ but $U \subseteq \mathbb{Z}$ is open if $U = -U$.

Claim: X is not compact but every nonempty set in X has a limit point.

$U_n = \{n, -n\}$ are open, no finite subcover.

If $A \subseteq \mathbb{Z}$ is nonempty then $n \in A$ for some n. Claim: $-n$ is a limit point of A.

Let U be open and containing $-n$. Then $n \in U$ as well so $U \cap A \neq \phi$.



Relations

1) compact $\Rightarrow$ limit point compact (always)

2) limit point cpct $\Rightarrow$ sequentially cpct (1st countable, Hausdorff)

3) sequentially compact $\Rightarrow$ compact (2nd countable <u>or</u> metrizable)

2nd countable + Hausdorff $\Rightarrow$ all 3 are equivalent

metrizable $\Rightarrow$

Prop: Compact spaces are limit point compact.

Pf: Suppose X is compact and $A \subseteq X$ has no limit points. We'll show that A is finite. Notice, since A has no limit points it is closed. Since X is compact and A is closed, A is compact. Let $a \in A$. Then a is not a limit point of A. Hence there is an open set U with $U \cap A = \{a\}$. Hence singletons are open in A. The open cover of A by singletons admits a finite subcover and hence A is finite.