Math F253 Midterm 2 Fall 2024

Name: ______________________ **Section: 901 (Maxwell)**

Student Id: ______________________

Rules:

You have 70 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

A scientific or graphing calculator is allowed.

A one page sheet of paper (8 1/2 in. x 11 in.) with handwritten notes on one side is allowed.

No other aids are permitted.

Include **units** in your answer whenever appropriate.

Place a box around your $\boxed{\text{FINAL ANSWER}}$ to each question where appropriate.

If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Turn off anything that might go beep during the exam.

Good luck!

Problem	Possible	Score
1	16	
2	12	
3	14	
4	12	
5	12	
6	10	
Extra Credit	5	
Total	76	

1. (16 points)

A depth map of a lake is being created. Depth D in meters is measured at locations (x, y) with x and y also in meters. At the point $P = (5, 7)$ m, the following is know about the depth:

$$D(5,7) = 8 \text{ m}$$
$$\frac{\partial D}{\partial x}(5,7) = \frac{1}{2}$$
$$\frac{\partial D}{\partial y}(5,7) = -\frac{1}{3}$$

a. The equations above leave off the units of $\partial D/\partial x$. What are those units?

b. Estimate the value of $D(5.1, 6.8)$ use either a differential or the linearization (your choice, it's the same thing!)

c. At the point $P = (5, 7)$, what direction would you travel in to see the lake get deeper fastest? Express your answer as a unit vector.

Continued....

Problem 1 continued....

Recall $P = (5,7)$ and

$$\begin{aligned} D(5,7) &= 8 \text{ m} \\ \frac{\partial D}{\partial x}(5,7) &= \frac{1}{2} \\ \frac{\partial D}{\partial y}(5,7) &= -\frac{1}{3} \end{aligned}$$

d. If your boat is at the point P and traveling with a velocity of $\langle 3,6\rangle$ m/s, what is the rate of change of depth of the lake that you observe? Units please.

e. If your boat is at the point P, what would be a non-zero velocity you could travel with and not see the depth change (for the moment)?

f. Find the equation of the tangent plane to the lake bottom[1] at the point P.

[1] We are using coordinates where z increases as we go down, not up. You should keep using those coordinates.

2. (12 points)

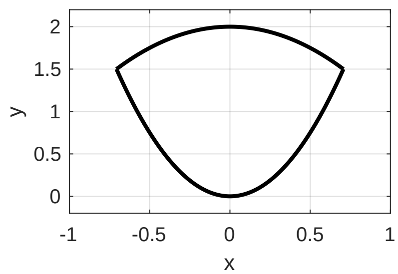
Consider the function $f(x, y) = x^3 + y^3 - 6xy + 17$.

a. Show that the points $(0, 0)$ and $(2, 2)$ are both critical points of this function. In fact, these are the only critical points. You need not show this, however.

b. Use the Hessian of f (i.e. the second derivative test) to classify each critical point as a local minimum, local maximum, or saddle point.

3. (14 points)

A thin metal plate is bounded by the curves $y = 3x^2$ and $y = 2 - x^2$ with x and y measured in centimeters. Its density is $\rho(x, y) = y$ g/cm^2.



a. Determine the x and y coordinates of the intersections of the two curves.

b. Compute the mass of the plate. You might want to simplify your task by observing that it suffices to compute the mass of the right half of the plate (why?).

c. Setup up the integral to compute M_x, the fist moment of inertia about the x axis. **Do not solve the integral.**

d. In fact, $M_x \approx 2.73$ cm. Use this figure to determine the y-coordinate of the centroid of the plate.

e. Without doing any computation, what do you expect M_y to be and why? Then mark the location of the centroid on the diagram above. Note that because the density is not constant, it won't be in the intuitive location for a constant density plate.

4. (12 points)

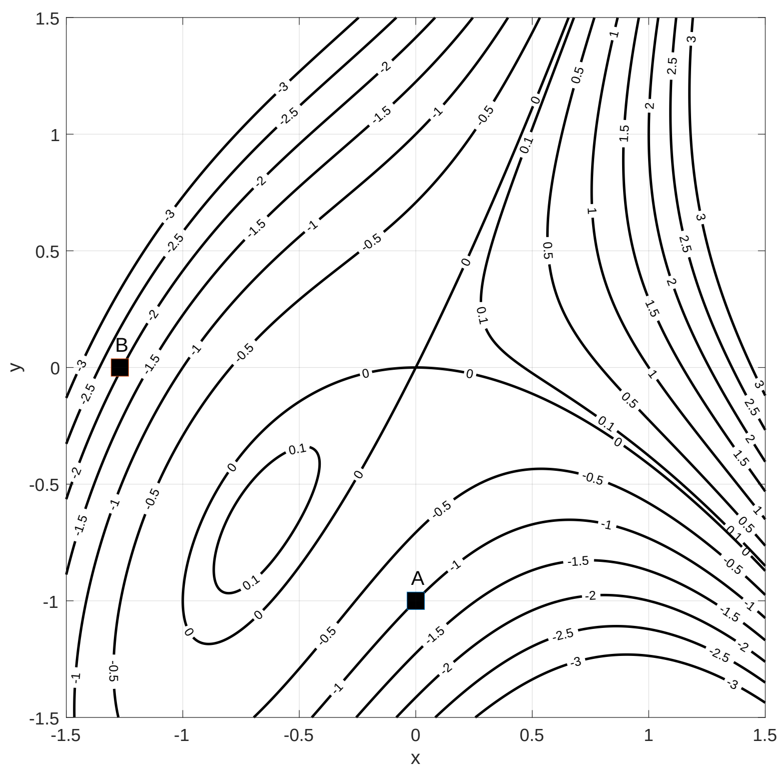
The figure below shows the level sets of a function $f(x,y)$.

a. Label each location in the figure where $\nabla f = \vec{0}$ with an asterisk: $*$

b. The points with asterisks are critical points. Classify each one as a local min, local max, or saddle. Just write the word next to the asterisk.

c. At each of locations A and B, draw an arrow that indicates the direction of the gradient.

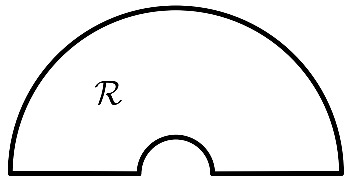
d. At which of location A and B is the gradient longer? Justify your answer briefly.



e. For the function $f(x,y) = y - x^3$ sketch the level sets $f(x,y) = c$ for $c = -1, 0, 1$.

5. (12 points)

Consider the saddle surface given by $z = f(x, y) = x^2 - y^2$ over the region $\mathcal{R}$ given by $1 \leq x^2 + y^2 \leq 9$ and $y \geq 0$.



a. Set up, but **do not compute**, an integral in polar coordinates that could be used to compute the volume of the solid with x and y in $\mathcal{R}$ and $0 \leq z \leq f(x, y)$.

b. Set up in polar coordinates an iterated integral you could use to computed the surface area of the surface $z = f(x, y)$ over the region. Don't compute the integral yet...

c. Now compute the surface area!

6. (10 points)

Consider the following iterated integral:

$$\int_0^2 \int_{2y}^4 y \sin(1 + x^3)\; dx\, dy.$$

a. This iterated integral corresponds to the double integral $\iint_{\mathcal{R}} y \sin(1 + x^3)\; dA$ for some region $\mathcal{R}$. Carefully sketch the region $\mathcal{R}$.

b. Now interchange the order of integration to compute the double integral.

7. (Extra Credit: 5 points)

In problem 2 you estimated a value of M_y without doing any computation. Set up the integral for M_y and then compute it to verify your suspicion. Hint: $\int_{-a}^{a} x^7 \, dx = ?$.