CFL $\Rightarrow$ stability $\quad \frac{kc}{h} \leq 1$

Von Neumann

$$u_{i,j} = c_j e^{I r x_i} \qquad (c_j = K^j \text{ shortly})$$

CFL $\Rightarrow$ stability

$$\underbrace{\frac{kc}{h}}_{\lambda} \le 1$$

$$|\lambda| \le 1$$

$$0 \le \lambda$$

Von Neumann

$$u_{i,j} = c_j \, e^{I r x_i} \qquad (c_j = K^j \text{ shortly})$$

$$\left[-I\omega k = \ln(K) \Rightarrow u_{i,j} = e^{I(r x_i - \omega t_j)} \right]$$

plane waves!

Substitute $c_j e^{Irx_i}$ into

$$u_{i,j+2} = 2u_{i,j+1} + \lambda^2 \left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} \right] - u_{i,j}$$

Substitute $c_j e^{Irx_i}$ into

$$u_{i,j+2} = 2u_{i,j+1} + \lambda^2\left[u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}\right] - u_{i,j}$$

$$c_{j+2} - 2\left[(1-\lambda^2) + \lambda^2\cos(rh)\right]c_{j+1} + c_j = 0$$

Recurrence! Try $c_j = k^j$, stability: $|k| \leq 1$

$$K^2 - 2\underbrace{\left[(1-\lambda^2) + \lambda^2\cos(wh)\right]}_{s}K + 1 = 0$$

$$K^2 - 2sK + 1 = 0$$

$$K = \underbrace{s \pm \sqrt{s^2 - 1}}$$

$$s \geq 1$$

$$K = s \pm \sqrt{s^2 - 1}$$

1) $|s| > 1 \Rightarrow$ unstable

$s > 1$ $\quad s +$ stuff

$s < 1$ $\quad s -$ stuff

$$K = s \pm \sqrt{s^2 - 1}$$

K^j

1) $|s| \geq 1 \Rightarrow$ unstable

$s > 0$ $\quad s+$ stuff

$s < 0$ $\quad s -$ stuff

2) $|s| < 1 \Rightarrow$ complex conjugates $K = r, r^*$

$$K^2 - 2K + 1 = (K - r)(K - r^*)$$

$$= K^2 - 2\mathrm{Re}(r)K + |r|^2$$

$\Rightarrow$ stable, $|K| = 1$

3) $|s| = 1$

$$j+2 - 2(j+1) + j = 0$$

$s = 1$:

$$c_{j+2} - 2c_{j+1} + c_j = 0$$

$c_j = j$ solves.

$$u_{r,j} = j e^{Irx_i}$$

$c_j = 1^j$ always

$$K^j$$

3) $|s| = 1$

$s = 1$:

$$c_{j+2} - 2c_{j+1} + c_j = 0$$

$c_j = j$ solves.

$s = -1$

$$c_{j+2} + 2c_{j+1} + c_j = 0$$

$c_j = (-1)^j j$ solves

$c_j = (-1)^j$

3) $|s| = 1$

$s = 1$:

$$c_{j+2} - 2c_{j+1} + c_j = 0$$

$c_j = j$ solves.

$s = -1$

$$c_{j+2} + 2c_{j+1} + c_j = 0$$

$c_j = (-1)^j j$ solves

still unstable.

($\bar{K}$ a root, $\bar{K}^j j$ solves)

Stability: $|s| < 1$

$$s = (1 - \lambda^2) + \lambda^2 \cos(rh)$$

Stability: $|s| < 1$

$$s = (1-\lambda^2) + \lambda^2 \cos(rh)$$

$$= 1 + \lambda^2 [\cos(rh) - 1]$$

$$= 1 - 2\lambda^2 \sin^2\left(\frac{rh}{2}\right)$$

$$-1 < 1 - 2\lambda^2 \sin^2\left(\frac{rh}{2}\right) < 1$$

$$-2 < -2\lambda^2 \sin^2\left(\frac{rh}{2}\right) < 0$$

$$\boxed{0 < \lambda^2 \sin^2\left(\frac{rh}{2}\right) < 1}$$

$|\lambda| \le 1$

Can $\sin^2\left(\frac{rh}{2}\right) = 0$?

$|s| < 1$

Can $\sin^2\left(\frac{rh}{2}\right) = 1$?

needed for stability

$\frac{rh}{2} = k\pi$

$$u_{i,j} = \operatorname{Im}\left[c_j e^{I r x_i}\right]$$

$$\frac{n}{N+1}\frac{\pi}{2}$$

$$1 \le n \le N$$

$$u_{i,j} = c_j \sin(r x_i)$$

$$u_{0,j} = 0 \quad \text{free}$$

$$1 \le n \le N$$

$$\text{but } u_{N+1,j} = c_j \sin(r\ell)$$

$$r = \frac{\pi}{\ell} n$$

$$\frac{rh}{2} = \frac{r\ell}{2N+1} = \underbrace{\frac{n}{(N+1)}\frac{\pi}{2}}$$

$$0 < \sin\left(\frac{rh}{2}\right) < 1$$

DEMO

Initial Condition Trouble

$$u_{i,1} = u_0(x_i) + v(x_i)k$$

$$\frac{u_{i,2} - 2u_{i,1} + u_{i,0}}{k^2} = \frac{u_{i,2} - 2\left(u_0(x_i) + u_t(x_i)k\right) + u_{i,0}}{k^2}$$

LTE component

$$u_i(x_i, 2k) = u(x_i, 0) + u_t(x_i, 0)2k + u_{tt}(x_i, 0)\frac{(2k)^2}{2} + O(k^3)$$

$$u_i(x_i, 2k) = u(x_i, 0) + u_t(x_i, 0)2k + u_{tt}(x_i, 0)\frac{(2k)^2}{2} + O(k^3)$$

$$u(x_i, 2k) - 2\left[u(x_i, 0) + u_t(x_i, 0)k\right] + u(x_i, 0)$$

$$u_i(x_i, 2k) = u(x_i, 0) + u_t(x_i, 0)\,2k + u_{tt}(x_i, 0)\frac{(2k)^2}{2} + O(k^3)$$

$$u(x_i, 2k) - 2\left[u(x_i, 0) + u_t(x_i, 0)\,k\right] - u(x_i, 0)$$

$$= 2\,u_{tt}\,k^2 + O(k^3)$$

u_{tt}

$$u_i(x_i, 2k) = u(x_i, 0) + u_t(x_i, 0)2k + u_{tt}(x_i, 0)\frac{(2k)^2}{2} + O(k^3)$$

$$u(x_i, 2k) - 2[u(x_i, 0) + u_t(x_i, 0)k] - u(x_i, 0)$$

$$= 2u_{tt}k^2 + O(k^3)$$

$$u_{tt} + [u_{tt} + O(k)]$$

First LTE is $u_{tt} + O(k)$, first error

contributes: $ku_{tt} + O(k^2) = O(k)$ error on step 1.

$$u_i(x_i, 2k) = u(x_i, 0) + u_t(x_i, 0)\,2k + u_{tt}(x_i, 0)\frac{(2k)^2}{2} + O(k^3)$$

$$u(x_i, 2k) - 2\overbrace{\left[u(x_i, 0) + u_t(x_i, 0)\,k\right]}^{u(x_i,k)} - u(x_i, 0)$$

$\frac{1}{2}u_{tt}(x_i, 0)k^2$

$$= 2\,u_{tt}\,k^2 + O(k^3)$$

First LTE is $u_{tt} + O(k)$, first error is

Contributes: $k u_{tt} + O(k^2) = O(k)$ error on step 1.

Correct the missing term!

$$u_{i,1} = u_0(x_i) + v(x_i)\,k + u_{tt}(x_i, 0)\,\frac{k^2}{2} + O(k^3)$$

Correct the missing term!

$$u_{i,1} = u_0(x_i) + v(x_i)k + u_{tt}(x_i,0)\frac{k^2}{2} + O(k^3)$$

$$= u_0(x_i) + v(x_i)k + \frac{c^2k^2}{2} u_{xx}(x_i,0) + O(k^3)$$

$$= u_0(x_i) + v(x_i)k + \frac{c^2k^2}{2h^2}\left[u_0(x_{i+1}) - 2u_0(x_i) + u_0(x_{i-1})\right]$$

+ terms

$$= u_0(x_i) + v(x_i)k + \frac{1}{2}\lambda^2\left[u_0(x_{i+1}) - 2u_0(x_i) + u_0(x_{i-1})\right] + O(k^3)$$

Alternative: Ghost Points (on HW)

Introduce $u_{i,-1}$

Use centered difference $\dfrac{u(x_i,-k) - u(x_i,k)}{2k} = u_t(x_i) + O(k^2)$

$$u(x_i,-k) = \underbrace{u(x_i,k) - 2k\,v(x_i)} + O(k^3)$$

substitute $u_{i,-1} = u_{i,1} - 2k\,v(x_i)$

at first iteration.

DEMO

(Are more timesteps better?)

$$u(2k) = u + u_t 2k + \frac{1}{2} u_{tt} (2k)^2 + O(k^3)$$

$$u(k) = u + u_t k + \frac{1}{2} u_{tt} k^2 + O(k^3)$$

$$u(2k) - 4u(k) + 3u = -2u_t k + O(k^3)$$

$$\frac{-u_2 + 4u_1 - 3u_0}{2k} = u_t + O(k^2)$$

$$4u_1 = 3u_0 + 2kv + u_2$$

$$u_{i,2} = 2u_{i,1} + \lambda^2 \left[u_{i-1,1} - 2u_{i,1} + u_{i+1,1} \right] - u_{i,0}$$