Lengths of subsets of $\mathbb{R}$

We would like to find a function that determines the length of an arbitrary subset of $\mathbb{R}$. This would be a function $\ell : \mathcal{P}(\mathbb{R}) \to [0, \infty]$. It is not obvious what the length of a generic set should be. Nevertheless, it seems clear that such a function should satisfy the following properties.

1. $\ell([a,b]) = b - a$ if $a \le b$.
2. $\ell(A + c) = \ell(A)$ for every $c \in \mathbb{R}$.
3. $\ell(rA) = |r|\,\ell(A)$ for every $r \in \mathbb{R}$.
4. Monotonicity: if $A \subseteq B$, then $\ell(A) \le \ell(B)$.
5. Finite additivity: If A and B are disjoint, then $\ell(A \cup B) = \ell(A) + \ell(B)$.

Exercise 1: Show that monotonicity is a consequence of finite additivity.

The first three properties express that ℓ generalizes our notion of the lengths of intervals and that it is compatible with our geometric understanding of translations and scalings. Property 4) indicates that length can only grow as a set increases. Property 5) allows us to compute the length of a set by breaking it into two disjoint pieces and adding the length of each piece.

We will also want to be able to add the length of countably many sets, and this poses a challenge to our intuition of length. How long is $\mathbb{Z}$, and how long is $\mathbb{Q}$? It seems clear that $\mathbb{Z}$ should have length 0. One way to argue this is to note that we can break $\mathbb{Z}$ up into countably many disjoint pieces, and that each piece has length 0. Perhaps we can estimate the length of $\mathbb{Z}$ by adding the lengths of the individual pieces, and arrive at the estimate that $\ell(\mathbb{Z}) = 0$. However, this same argument would apply to $\mathbb{Q}$, and we would be forced to conclude that $\mathbb{Q}$ has length 0 as well! This is perhaps surprising, since one might want to make the case that $\mathbb{Q}$ has infinite length. The underlying question seems to be whether the length of a union of disjoint pieces can be different depending on the way in which the pieces are (disjointly) arranged in the line. Our answer to this question will be that length is independent of order, and we express this answer by one of the following two properties.

6. Countable subadditivity: if $\{A_k\}_{k=1}^\infty$ is a countable collection of sets, then
$$\ell(\cup_k A_k) \le \sum_k \ell(A_k).$$
7. Countable additivity: if $\{A_k\}_{k=1}^\infty$ is a countable collection of *disjoint* sets, then
$$\ell(\cup_k A_k) = \sum_k \ell(A_k).$$

Of these, property 6) seems to be the weaker one. It says that the length of a union of pieces can be estimated (from above) by summing the lengths of the pieces. This is a very mild condition to assume. Property 7) appears to be somewhat stronger, stating that the length of a disjoint union of pieces is exactly equal to the sum of the lengths of the pieces, and perhaps one would not want to make such a strong requirement about a length function. Fortunately, we do not have to decide which of these properties is appropriate – they are equivalent.

Exercise 2: Suppose $\ell : \mathcal{P}(\mathbb{R}) \to [0,\infty]$. Show that ℓ satisfies property 7) if and only if it satisfies properties 5) and 6). That is, ℓ is countably additive if and only if it is finitely additive and countably subadditive. The following exercise will be helpful.

Exercise 3: Suppose $\{A_k\}$ is a countable collection of sets. Show that there exist disjoint sets B_k such that $B_k \subseteq A_k$ and $\cup B_k = \cup A_k$. *Hint:* Let $B_k = A_k \setminus \left[\cup_{j=1}^{k-1} A_k\right]$ for $k > 1$.

Unfortunately, we have already asked for too much from a length function. We will show later in the course that there does not exist a length function satisfying properties 1), 2), and 7) (or equivalently properties 1), 2), 5), and 6)). One approach to this problem is to restrict our attention to length functions only satisfying properties 1)-5). But this severely restricts our ability to do limiting operations. Moreover, it turns out that the generalizations of properties 1)-5) to $\mathbb{R}^3$ are again too strong. That is, there does not exist a volume function on $\mathcal{P}(\mathbb{R}^3)$ that gives volume 1 to the unit cube, is invariant under rigid motions, and is finitely additive. This stunning fact is the content of the Banach-Tarski paradox – a special case of the paradox states the unit sphere can be broken into finitely many pieces which can then be reassembled into the sphere of radius 100 (leaving no gaps and making no overlaps)!

Our approach to these difficulties will be to keep properties 1)-7), but to restrict the domain of our length function to "nice" sets for which these properties hold.

Outer measure

Suppose A is a set, and $\{I_k\}$ is a countable collection of bounded open intervals such that $A \subseteq \cup\{I_k\}$. We call such a collection of intervals a **measuring cover** for A. Every measuring cover for A gives an upper bound for its length: it should be no longer than

$$\sum_k \ell(I_k),$$

where we *define* $\ell((a,b)) = b - a$. With this intuition in mind, we define a function $m^* : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ that is a candidate length function.

Definition: Let $A \subseteq \mathbb{R}$. The **Lebesgue outer measure** of A, denoted by $m^*(A)$, is

$$\inf\{\sum_k \ell(I_k) : \{I_k\} \text{ is a measuring cover for } A\}.$$

We should consider $m^*(A)$ to be a kind of best estimate from above of the length of A (where covers by open intervals are used to form the estimate).

Exercise 4: Suppose A is countable. Prove that $m^*(A) = 0$. *Hint:* If $A = \{a_1, a_2, \ldots\}$, cover a_k with an interval of length $\epsilon/2^k$.

Sets with outer measure zero are important, and we say that such sets have **measure zero** or are **null sets**. It should be noted that not every set with zero outer measure is countable. Recall that the Cantor set is uncountable; we will soon be able to show that the Cantor set also has measure zero.

Since m^* is only an upper estimate for length, it would seem that it might not make a good candidate for a length function. However, it satisfies a remarkable number of the properties 1)-7).

Theorem 1: Let $[a, b]$ be a bounded closed interval. Then $m^*([a, b]) = b - a$.

Proof. Consider the measuring cover consisting of the single interval $\{(a - \epsilon, b + \epsilon)\}$. It follows that

$$m^*([a, b]) \leq b - a + 2\epsilon$$

for any $\epsilon > 0$ and hence $m^*([a, b]) \leq b - a$.

On the other hand, let $\{I_k\}$ be any measuring cover of $[a, b]$. We will show that $\sum \ell(I_k) \geq b - a$ to conclude that $m^*([a, b]) = b - a$. Since $[a, b]$ is compact, we can find a finite subcover $I_{k_1}, \ldots, I_{k_n}$, and it is enough to show that $\sum_{j=1}^{n} \ell(I_{k_j}) \geq b - a$.

Let $J_1 = (a_1, b_1)$ be an element of $\{I_{k_1}, \ldots, I_{k_n}\}$ that contains a. Now for each $k > 1$, if $b_{k-1} \in (a, b)$, let J_k be an element of $\{I_{k_1}, \ldots, I_{k_n}\}$ that contains b_{k-1}; such an element exists since $\{I_{k_1}, \ldots, I_{k_n}\}$ is a cover of $[a, b]$. Otherwise we stop this procedure with a set J_m such that $b_m > b$. Although we have been a little informal in the definition of the sets J_k, their definition can be made more precise using the Principle of Recursive Definition.

Notice that $a \in (a_1, b_1)$, so

$$\ell(J_1) \geq (b_1 - a).$$

Similarly, for each k with $1 < k < m$, $b_{k-1} \in (a_k, b_k)$, so

$$\ell(J_k) \geq (b_k - b_{k-1}).$$

Finally, J_m must satisfy $b_m > b$, so $(b_{k-1}, b] \subseteq J_m$ and

$$\ell(J_m) \geq (b - b_{m-1}).$$

It follows that

$$\sum_{k=1}^{m} \ell(J_k) \geq (b_1 - a) + (b_2 - b_1) + \cdots + (b - b_{m-1}) = b - a.$$

Since the sets J_k are clearly distinct (the sequence of their right endpoints is strictly increasing) we have

$$\sum_{j=1}^{n} \ell(I_{j_k}) \geq \sum_{k=1}^{m} J_k \geq b - a.$$

□

We have thus verified that m^* satisfies property 1) of length functions. That m^* satisfies 4) is obvious from its definition, and it is an easy verification to determine that m^* also satisfies 2) and 3).

Exercise 5: Show that m^* satisfies properties 2) and 3).

Exercise 6: If I is any bounded interval, prove that $m^*(I)$ is its length. If I is unbounded, prove that $m^*(I) = \infty$. *Hint:* Use monotonicity!

Now no function satisfying 1)-4) can also satisfy 5) and 6) simultaneously. Nor can it satisfy 7) as this is equivalent to 5) and 6). So m^* can satisfy at most one of 5) and 6), and it turns out that m^* is countably subadditive, but not finitely additive.

Proposition 2: Let $\{A_k\}$ be a countable collection of subsets of $\mathbb{R}$. Then

$$m^*(\cup A_k) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

Proof. Let $\epsilon > 0$ and for each A_k, let $\{I_{k,j}\}_{j=1}^{\infty}$ be a measuring cover for A_k such that

$$\sum_j \ell(I_{k,j}) \leq m^*(A_k) + \frac{\epsilon}{2^k}.$$

Then $\{I_{k,j}\}_{k,j=1}^{\infty}$ is a measuring cover for $\cup_k A_k$ and hence

$$m^*(\cup A_k) \leq \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \ell(I_{k,j} \leq \sum_{k=1}^{\infty} m^*(A_k) + \frac{\epsilon}{2^k} = \left[\sum_{k=1}^{\infty} m^*(A_k)\right] + \epsilon.$$

Since this is true for every $\epsilon > 0$,

$$m^*(\cup A_k) \leq \sum_{k=1}^{\infty} m^*(A_k).$$

□

Restoring finite additivity

We have established that m^* satisfies properties 1)-4) and 6), and we have claimed that it cannot satisfy 5) for no such function exists. Hence there exist disjoint sets A and B such that

$$m^*(A \cup B) \neq m^*(A) + m^*(B).$$

Since m^* is countably subadditive, it follows then that we have the strict inequality

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

The structure of the sets A and B must be so complicated that our estimation procedure defined by m^* must be assigning some of the length of $A \cup B$ to both A and B, and hence m^*

must be determining an overestimate for their length. We would like to restrict our attention to sets that don't exhibit this phenomenon.

Suppose E is a set contained in an interval (a,b). How can we determine if m^* is determining an overestimate for the length of E? We can decompose (a,b) into two pieces, $(a,b) \cap E = E$ and $(a,b) \cap E^c$. If $m^*(E)$ and $m^*(E^c \cap (a,b))$ are not "overestimates", then we should have finite additivity for E and $E^c \cap (a,b)$ and hence

$$m^*(E) + m^*((a,b) \cap E^c) = m^*((a,b)) = b - a. \tag{1}$$

We can perform a similar construction even if E is not contained in (a,b) by replacing $m^*(E)$ with $m^*(E \cap (a,b))$ in (1). This leads us to:

Condition CC' We say E satisfies condition CC' if for every bounded open interval (a,b),

$$b - a = m^*((a,b) \cap E) + m^*((a,b) \cap E^c).$$

Condition CC' is a minimal requirement for a set to have a length well estimated by m^*. A seemingly stronger condition is:

Condition CC We say E satisfies condition CC (the Caratheodory Condition) if for every set $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Since $m^*([a,b]) = b - a$, it is clear that condition CC implies condition CC'. It is perhaps surprising that condition CC' also implies condition CC.

Proposition 3: A set $E \subseteq \mathbb{R}$ satisfies condition CC if and only if it satisfies condition CC'.

Proof. We need only show that condition CC' implies condition CC. Let E be a set satisfying condition CC', and let $A \subseteq \mathbb{R}$. By countable subadditivity, we know that

$$m^*(A) \le m^*(A \cap E) + m^*(A \cap E^c).$$

So it is enough to show that $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^c)$.

Let $\epsilon > 0$ and let $\{I_k\}$ be a measuring covering for A such that

$$\sum \ell(I_k) \le m^*(A) + \epsilon.$$

Since E satisfies condition CC', for every k, $m^*(I_k) = m^*(I_k \cap E) + m^*(I_k \cap E^c)$. Now $A \cap E \subseteq \cup(I_k \cap E)$ and $A \cap E^c \subseteq \cup(I_k \cap E^c)$. So by countable subadditivity,

$$m^*(A \cap E) \le \sum_k m^*(I_k \cap E), \text{ and}$$
$$m^*(A \cap E^c) \le \sum_k m^*(I_k \cap E^c).$$

But then

$$\begin{aligned} m^*(A\cap E)+m^*(A\cap E^c) &\le \sum_k \left[m^*(I_k\cap E)+m^*(I_k\cap E^c)\right] \\ &= \sum_k m^*(I_k) = \sum_k \ell(I_k) \le m^*(A)+\epsilon. \end{aligned}$$

Since this inequality holds for every $\epsilon > 0$,

$$m^*(A\cap E)+m^*(A\cap E^c)\le m^*(A)$$

as required. □

Exercise 7: We chose to use open intervals rather than closed intervals in the definition of condition CC' because it simplified the previous proof. Historically, however, closed intervals were used. Let CC'' be the analogue of condition CC' using closed bounded intervals. Show that conditions CC' and CC'' are equivalent.

Condition CC (or equivalently condition CC') is exactly the condition we need to have to restore finite additivity to m^*.

Proposition 4: Let E_1 and E_2 be disjoint sets that satisfy condition CC. Then

$$m^*(E_1\cup E_2) = m^*(E_1)+m^*(E_2).$$

Proof. Notice that $(E_1\cup E_2)\cap E_1 = E_1$ and $(E_1\cup E_2)\cap(E_1)^c = E_2$ since E_1 and E_2 are disjoint. Then, since E_1 satisfies condition CC,

$$m^*(E_1\cup E_2) = m^*((E_1\cup E_2)\cap E_1)+m^*((E_1\cup E_2)\cap(E_1)^c) = m^*(E_1)+m^*(E_2).$$

□

Although condition CC is unintuitive at first, the underlying idea is easy – a set E is measurable if it can "carve" an arbitrary set A into two pieces $A\cap E$ and $A\cap E^c$ such that the sum of the lengths of the pieces equals the length of A.

Definition: A set $E\subseteq\mathbb{R}$ is **measurable** if it satsifies condition CC. The collection of all measurable sets is denoted by $\mathcal{M}$. If E is measurable, the value of $m^*(E)$ is called the **Lebesgue measure** of E. The restriction of m^* to $\mathcal{M}$ is denoted by m.

Although verifying condition CC can be difficult in practice, it is easy to show that null sets satisfy this condition.

Exercise 8: Suppose $m^*(N)=0$. Prove that N is measurable.

We are are well on our way to showing that m satisfies properties of 1)-7) of a length function. One question arises, however. Are the sets appearing in properties 1)-7) measurable? That is, if E and F are measurable, are $E+c$, rE, and $E\cup F$ measurable? And if $\{E_k\}$ is a collection of measurable sets, is $\cup E_k$ measurable? The answers to these questions will all be affirmative, and we turn next to showing that the collection of measurable sets is closed under countable set operations.

Measurable sets and set operations

A collection $\mathcal{A}$ of subsets of a set X is called (set) **algebra** in X if it is closed under finite set operations (unions, intersections, and complements). That is, if $A, B \in \mathcal{A}$, then so are $A \cup B$, $A \cap B$, and A^c. An algebra is the right domain for performing finite set operations. To verify that a collection of sets $\mathcal{A}$ is an algebra, it is only necessary to verify that it is closed under unions and complements – deMorgan's Laws imply $A \cap B = (A^c \cup B^c)^c$.

Exercise 9: Let $\mathcal{A}$ be the collection of subsets of $\mathbb{R}$ that are either finite, or have finite complement. Show that $\mathcal{A}$ is an algebra. This is known as the algebra of finite and co-finite sets.

An algebra is the right domain for performing finite set operations, but we also want to consider unions of the form $\cup_{k=1}^{\infty} E_k$. That is, we wish to consider countably many set operations. A σ-algebra is an algebra that is also closed under countable unions and countable intersections. Just as for algebras, in practice one only needs to verify that a σ-algebra is closed under countable unions since deMorgan's Laws imply the formula $\cap_k E_k = (\cup_k E_k^c)^c$.

Exercise 10: Let $\{\mathcal{A}_\alpha\}_{\alpha \in \mathcal{I}}$ be a family of σ-algebras in X. Show that $\cap_\alpha A_\alpha$ is also a σ-algebra in X.

Exercise 11: Let $\mathcal{C}$ be a collection of subsets of X. Show that there exists a unique smallest σ-algebra containing $\mathcal{C}$ (i.e. a σ-algebra that contains $\mathcal{C}$ and is contained in any other σ-algebra containing $\mathcal{C}$.) We call this the σ-algebra generated by $\mathcal{C}$.

We will now show that, $\mathcal{M}$, the collection of measurable sets is a σ-algebra. To do so, we need to verify that $\mathcal{M}$ is closed under taking complements, finite unions, and countable unions. Showing that $\mathcal{M}$ is closed under taking complements is nearly immediate from the definition.

Lemma 5: If E is measurable, so is E^c.

Proof. Suppose E is measurable. Then for any set A

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

But $E = (E^c)^c$, so for any set A,

$$m^*(A) = m^*(A \cap (E^c)^c) + m^*(A \cap E^c).$$

Hence E^c is measurable. □

Showing that $\mathcal{M}$ is closed under finite unions is a little harder since it is easy to get bogged down in the set notation. But the underlying idea is easy. Suppose E and F are measurable and A is an arbitrary set. We want to show that $E \cup F$ can "carve" A, so

$$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap ((E \cup F)^c)).$$

We proceed as follows. First carve A using F, so

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^c)). \tag{2}$$

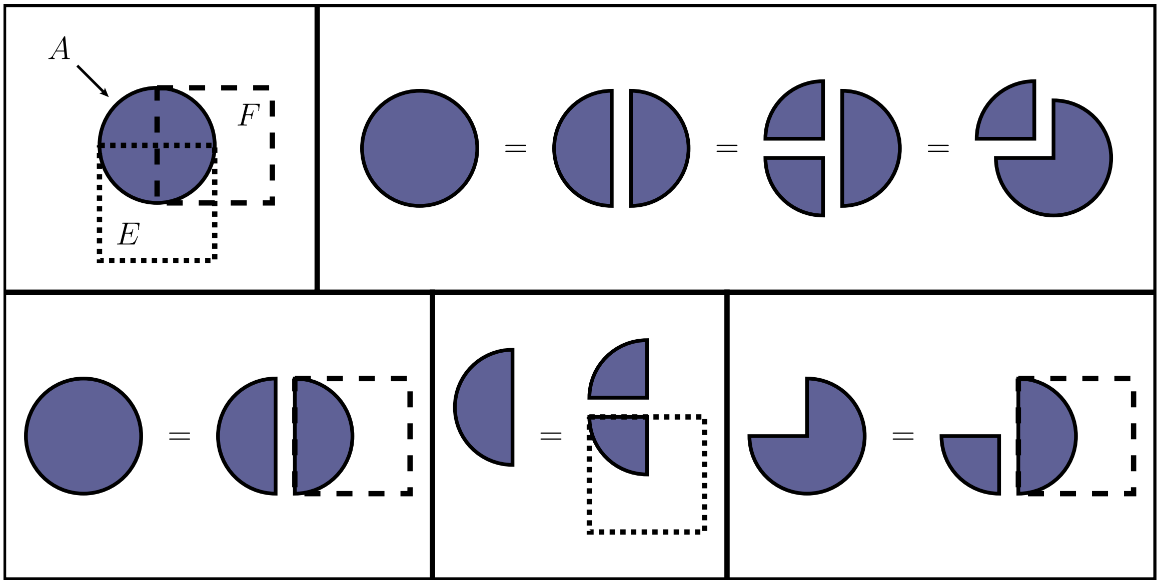

Figure 1: $E \cup F$ is measurable.

Now carve $A \cap F^c$ using E, so

$$\begin{aligned} m^*(A \cap F^c) &= m^*(A \cap F^c \cap E) + m^*(A \cap F^c \cap E^c) \\ &= m^*(A \cap F^c \cap E) + m^*(A \cap (F \cup E)^c). \end{aligned} \tag{3}$$

Finally, carve $A \cap (E \cup F)$ using F to get

$$\begin{aligned} m^*(A \cap (E \cup F)) &= m^*(A \cap (E \cup F) \cap F) + m^*(A \cap (E \cup F) \cap F^c) \\ &= m^*(A \cap F) + m^*(A \cap E \cap F^c). \end{aligned} \tag{4}$$

From (2) and (3) we have

$$m^*(A) = m^*(A \cap F) + m^*(A \cap F^c \cap E) + m^*(A \cap (F \cup E)^c),$$

and from (4) we conclude that

$$m^*(A) = m^*(A \cap (E \cup F)) + m^*(A \cap (F \cup E)^c).$$

We have therefore proved the following (and hence that $\mathcal{M}$ is an algebra).

Proposition 6: Suppose E and F are measurable. Then $E \cup F$ is measurable.

To work with countable unions, we need the following technical lemma.

Lemma 7: Suppose $E_1, \ldots, E_n$ are disjoint and measurable. Then for any set $A \subseteq \mathbb{R}$,

$$m^*(A \cap (\cup_{k=1}^n E_k)) = \sum_{k=1}^n m^*(A \cap E_k).$$

Proof. We proceed by induction on n. The case where $n = 1$ is obvious. Suppose that for some $n \in \mathbb{N}$ that if $E_1, \ldots, E_n$ are measurable and disjoint, then for any set A

$$m^*(A \cap (\cup_{k=1}^n E_k)) = \sum_{k=1}^n m^*(A \cap E_k).$$

Now consider a collection of $n+1$ disjoint measurable sets $E_1, \ldots, E_{n+1}$. Then since E_{n+1} is measurable and is disjoint from the other sets E_k,

$$\begin{aligned} m^*(A \cap (\cup_{k=1}^{n+1} E_k)) &= m^*(A \cap (\cup_{k=1}^{n+1} E_k) \cap E_{n+1}) + m^*(A \cap (\cup_{k=1}^{n+1} E_k) \cap E_{n+1}^c) \\ &= m^*(A \cap E_{n+1}) + m^*(A \cap (\cup_{k=1}^n E_k)). \end{aligned}$$

From the inductive hypothesis we have

$$m^*(A \cap (\cup_{k=1}^n E_k)) = \sum_{k=1}^n m^*(A \cap E_k).$$

So we conclude that

$$m^*(A \cap (\cup_{k=1}^{n+1} E_k)) = \sum_{k=1}^{n+1} m^*(A \cap E_k).$$

□

Proposition 8: Suppose $\{E_k\}$ is a countable collection of disjoint measurable sets. Then $\cup E_k$ is measurable.

Proof. Let $E = \cup_{k=1}^\infty E_k$. Let A be an arbitrary subset of $\mathbb{R}$; to show that E is measurable it is enough to show that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

Now for each n we have

$$m^*(A) = m^*(A \cap \cup_{k=1}^n E_k) + m^*(A \cap (\cup_{k=1}^n E_k)^c).$$

From monotonicity, it follows that

$$m^*(A \cap (\cup_{k=1}^n E_k)^c) \geq m^*(A \cap E^c)$$

and from Lemma 7 it follows that

$$m^*(A \cap \cup_{k=1}^n E_k) = \sum_{k=1}^n m^*(A \cap E_k).$$

Hence

$$m^*(A) \geq \sum_{k=1}^n m^*(A \cap E_k) + m^*(A \cap E^c)$$

for each n. Taking the limit as $n \to \infty$ we conclude

$$m^*(A) \geq \sum_{k=1}^\infty m^*(A \cap E_k) + m^*(A \cap E^c).$$

Now $A \cap E = \cup_{k=1}^{\infty}(A \cap E_k)$ so countable subadditivity implies

$$\sum_{k=1}^{\infty} m^*(A \cap E_k) \geq m^*(A \cap \cup_{k=1}^{\infty} E_k) = m^*(A \cap E).$$

We conclude that

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

□

We have now shown that a countable union of disjoint measurable sets is measurable. The case where the sets are not necessarily disjoint reduces to the disjoint case, however, as the following exercise shows.

Exercise 12: If $\{E_k\}$ is a countable collection of measurable sets, show that there is a countable collection of disjoint measurable sets $\{F_k\}$ such that $\cap E_k = \cap F_k$. Conclude that $\cap E_k$ is measurable.

In summary, in this section we have proved the following.

Theorem 9: $\mathcal{M}$ is a σ-algebra.

Measurable sets and topology

Determining if a given set is measurable from the definition can be difficult in practice, and it will be helpful to have other characterizations of measurablility. Until now, we have only determined that null sets (and in particular finite and countable sets) are all measurable. What about intervals?

Proposition 10: For any $a \in \mathbb{R}$, (a, ∞) is measurable.

Proof. Let $E = (a, \infty)$. To show E is measurable, it is enough to show that E satisfies condition CC'. Let $I = (c, d)$ be an interval. We want to show that

$$m^*(I) = m^*(I \cap E) + m^*(I \cap E^c). \tag{5}$$

There are three cases. If $a \geq d$, then $I \cap E = \varnothing$ and $I \cap E^c = I$. So

$$m^*(I \cap E) + m^*(I \cap E^c) = 0 + m^*(I) = m^*(I).$$

Similarly, if $a \leq c$, then $I \cap E^c = \varnothing$ and $I \cap E = I$, so

$$m^*(I \cap E) + m^*(I \cap E^c) = m^*(I) + 0 = m^*(I).$$

Finally, if $c < a < d$, then $I \cap E = (a, d)$ and $I \cap E^c = (c, a]$, so

$$m^*(I \cap E) + m^*(I \cap E^c) = m^*((a, d)) + m^*((c, a]) = d - a + a - c = d - c = m^*(I).$$

So in all cases, (5) holds, so (a, ∞) is measurable. □

To show that arbitrary intervals are measurable is now easy using the fact that $\mathcal{M}$ is a σ-algebra and that null sets (and hence finite sets) are measurable.

Corollary 11: Every interval I is measurable.

Exercise 13: Prove Corollary 11.

Every open subset of $\mathbb{R}$ can be written as a countable disjoint union of open intervals. Since $\mathcal{M}$ is a σ-algebra containing the open intervals, we conclude that $\mathcal{M}$ contains every open set.

Theorem 12: Every open set is measurable.

Recall from Exercise 11 that every collection of subsets of $\mathbb{R}$ is contained in a smallest σ-algebra. The smallest σ-algebra containing the open sets is called the collection of **Borel sets**, and is denoted by $\mathcal{B}$. Examples of Borel sets include open sets, closed sets, F_σ sets, G_δ sets, $F_{\sigma\delta}$ sets and so forth. Since $\mathcal{M}$ is a σ-algebra containing the open sets, it must contain every Borel set

Theorem 13: $\mathcal{B} \subseteq \mathcal{M}$.

Exercise 14: Let $E \subseteq \mathbb{R}$. Prove that for any $\epsilon > 0$ there exist an open set $U \supseteq E$ such that $m^*(U) < m^*(E) + \epsilon$ and a G_δ set $G \supseteq E$ such that $m^*(G) = m^*(E)$.

The previous exercise shows that every set can be approximated from above, in terms of outer measure, by an open set or a G_δ set. However, it is not true in general that one can find a G_δ set containing E such that

$$m^*(G \setminus E) = 0.$$

Indeed, suppose this were possible, and let $N = G \setminus E$, so N is a null set. We can write

$$E = G \setminus (G \setminus E) = G \cap N^c.$$

Since G and N are measurable, then so is E! That is, although we can estimate any set E from above by a G_δ set G, we cannot ensure that the left over bits $G \setminus E$ have measure zero unless E is measurable. This leads us to a characterization of measurable sets in terms of open sets.

Theorem 14: The following are equivalent.

1. E is measurable.
2. For every $\epsilon > 0$ there is an open set $U \supseteq E$ such that $m^*(U \setminus E) < \epsilon$.
3. There exists a G_δ set $G \supseteq E$ such that $m^*(G \setminus E) = 0$.

Proof. We have already argued that 3) implies 1). That 2) implies 3) is also easy to see. For each $n \in \mathbb{N}$, find an open set U_n containing E such that $m^*(U_n \setminus E) < 1/n$. Let $G = \cap_n U_n$.

Then $G \setminus E$ is contained in $U_n \setminus E$ for every n and hence $m^*(G \setminus E) < 1/n$ for every n. So $m^*(G \setminus E) = 0$.

It remains to show that 1) implies 2). First suppose that E is measurable and has finite measure. Let $\epsilon > 0$ and let U be an open set containing E such that $m^*(U \setminus E) < \epsilon$ – such an open set exists by Exercise 14. Notice that $U = E \cup (U \setminus E)$. Since U and E are measurable,

$$m^*(U) = m^*(E) + m^*(U \setminus E).$$

Since $m^*(U) < m^*(E) + \epsilon$ we conclude that

$$m^*(U \setminus E) < \epsilon.$$

Now suppose E is measurable with possibly infinite measure. For each $n \in \mathbb{N}$, let $E_n = E \cap [-n, n]$. Each E_n is measurable. Let $\epsilon > 0$. For each n, find an open set U_n containing E_n such that $m^*(U_n \setminus E_n) < \epsilon/2^n$. Let $U = \cup U_n$, so $E \subseteq U$. Then from countable subadditivity and monotonicity we conclude

$$m^*(U \setminus E) = m^*(\cup_n(U_n \setminus E)) \leq \sum m^*(U_n \setminus E) \leq \sum m^*(U_n \setminus E_n) < \sum \frac{\epsilon}{2^n} = \epsilon.$$

$\square$

The previous result is a rigorous statement of the sentiment that every measurable set is "nearly" an open set. Our characterization of measurability in terms of open sets can also be used to show that every measurable set is "nearly" a closed set by using the fact that E is measurable if and only if E^c is measurable.

Corollary 15: The following are equivalent.

1. E is measurable.
2. For every $\epsilon > 0$ there is an closed set $V \subseteq E$ such that $m^*(E \setminus V) < \epsilon$.
3. There exists a F_σ set $F \subseteq E$ such that $m^*(E \setminus F) = 0$.

Exercise 15: Prove Corollary 15.

Exercise 16: Suppose $m^*(E)$ is finite. Show that E is measurable if and only if for any $\epsilon > 0$ there is a set A that is a finite union of bounded intervals such that $m^*(A \Delta E) < \epsilon$. Show that the intervals can be taken as either open or closed.

Exercise 17: Show that a set $E \subseteq \mathbb{R}$ is measurable if and only if for every $\epsilon > 0$ there exists an open set U and a closed set V such that $V \subseteq E \subseteq U$ and such that

$$m^*(U \setminus V) < \epsilon.$$

Exercise 18: If E is measurable, show that $E + h$ is measurable and that rE is measurable.

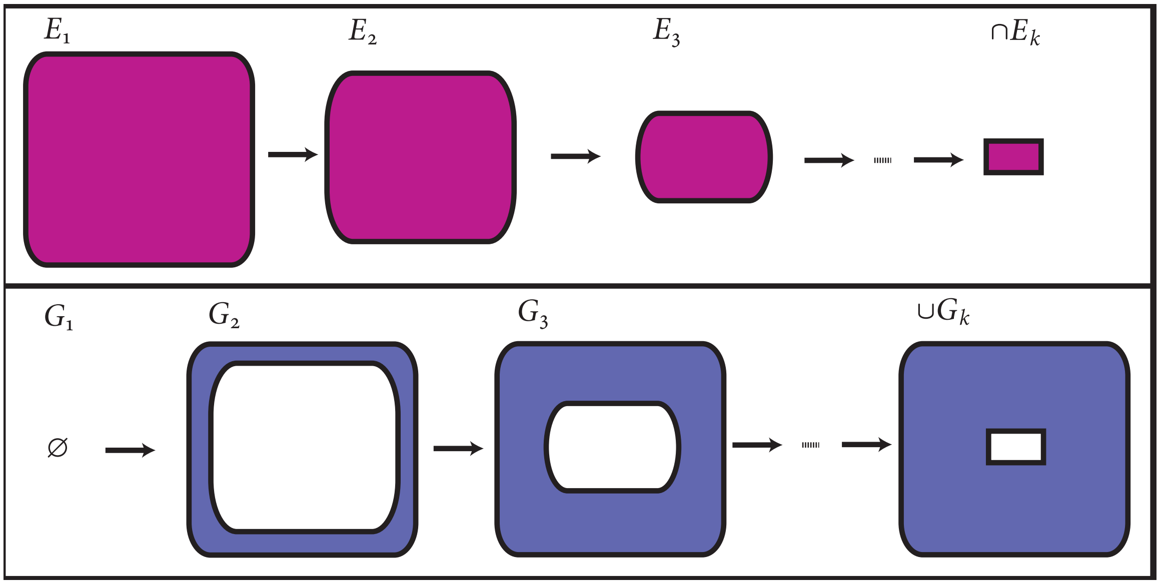

Figure 2: Corollary 17. $G_k = E_1 \setminus E_k$.

Continuity properties of m

Consider a nested increasing sequence of measurable sets $E_1 \subseteq E_2 \subseteq \cdots$. In some sense the limit of this sequence should be $\cup E_k$. Lebesgue measure satisfies a kind of continuity (**continuity from below**) inasmuch as the measure of this limit is equal to the limit of the measures of the sets E_k.

Proposition 16: Let $E_1 \subseteq E_2 \subseteq \cdots$ be an increasing sequence of measurable sets and let $E = \cup_{k=1}^{\infty} E_k$. Then

$$\lim_n m(E_n) = m(E).$$

Proof. Define $G_1 = E_1$ and $G_k = E_k \setminus E_{k-1}$ for $k > 1$. Then the sequence $\{G_k\}$ consists of disjoint measurable sets satisfying $\cup_{k=1}^{n} G_k = E_n$ for every n and $\cup_{k=1}^{\infty} G_k = E$. But then by countable additivity,

$$m(E) = \sum_{k=1}^{\infty} m(G_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(G_k).$$

By finite additivity we have, since $\cup_{k=1}^{n} G_k = E_n$,

$$\sum_{k=1}^{n} m(G_k) = m(\cup_{k=1}^{n} G_k) = m(E_n).$$

Hence

$$\lim_{n\to\infty} m(E_n) = m(E).$$

□

There is a corresponding **continuity from above**. Given nested measurable sets $E_1 \supseteq E_2 \supseteq \cdots$, we would like to claim that $m(\cap E_k) = \lim_n m(E_n)$. But this is false, in general. For example, if $E_k = (k, \infty)$, then $m(E_k) = \infty$ for every k but $m(\cap E_k) = 0$. The lurking infinite measure prevents us from seeing what is occurring at the finite scales. Fortunately, this problem can be avoided by requiring that E_1 (or in fact any single E_k) has finite measure.

Corollary 17: Let $E_1 \supseteq E_2 \supseteq \cdots$ be a decreasing sequence of measurable sets such that $m(E_1) < \infty$ and let $E = \cap_{k=1}^{\infty} E_k$. Then

$$\lim_{n\to\infty} m(E_n) = m(E).$$

Exercise 19: Prove Corollary 17. *Hint:* Let $G_k = E_1 \setminus E_k$ and apply Proposition 16. Use Figure 2 to help visualize the setup.

Summary

In these notes we have attempted to construct a length function on $\mathcal{P}(\mathbb{R})$ satisfying properties 1)-7). Although we were not able to do so, we were able to find a candidate function m^* that satisfied 1)-4) and 5) (i.e. everything except finite additivity). We restricted the domain of m^* to the σ-algebra of sets $\mathcal{M}$ for which finite additivity holds (a large σ-algebra containing the Borel sets) and obtained Lebesgue measure.

Theorem 18: Lebesgue measure $m : \mathcal{M} \to \mathbb{R}$ satisfies the following properties.

1. $m([a,b]) = b - a$ if $a \le b$.
2. If E is measurable, so is $E + c$ for any $c \in \mathbb{R}$ and $m(E + c) = m(E)$.
3. If E is measurable, so is rE for any $r \in \mathbb{R}$ and $m(rE) = |r| m(E)$.
4. If E and F are measurable and $E \subseteq F$, then $m(E) \le m(F)$.
5. If E and F are measurable and disjoint then $m(E \cup F) = m(E) + m(F)$.
6. If $\{E_k\}$ is a countable collection of measurable sets, then
$$m(\cup E_k) \le \sum_k m(E_k).$$
7. If $\{E_k\}$ is a countable collection of disjoint measurable sets, then
$$m(\cup E_k) = \sum_k m(E_k).$$