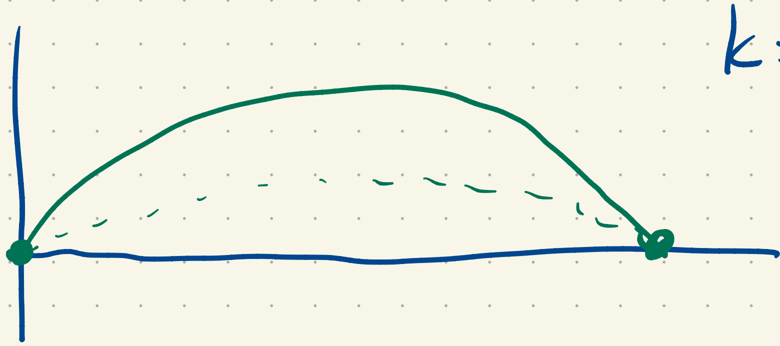
HW 4

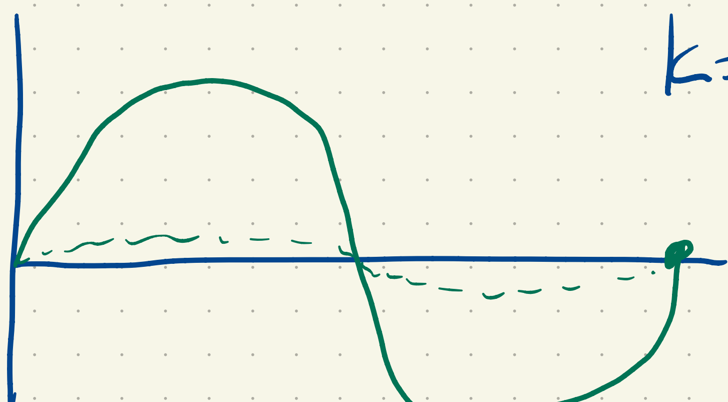
$v_k(x) = \sin(k\pi x)$, $\lambda = -k^2\pi^2$

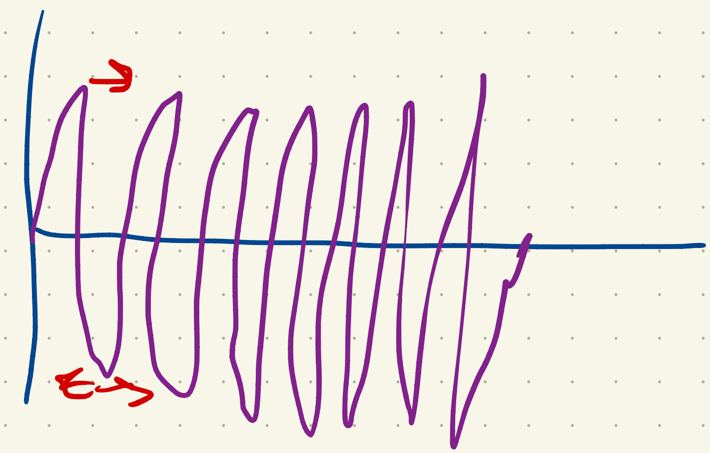
solution: $u(x,t) = e^{-k^2\pi^2 t} \sin(k\pi x)$

$k=1$



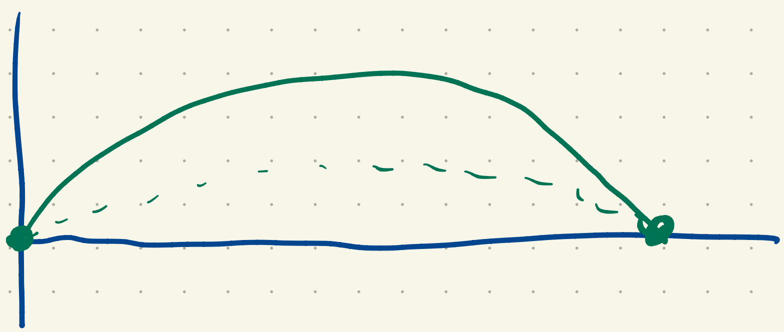
$k=2$

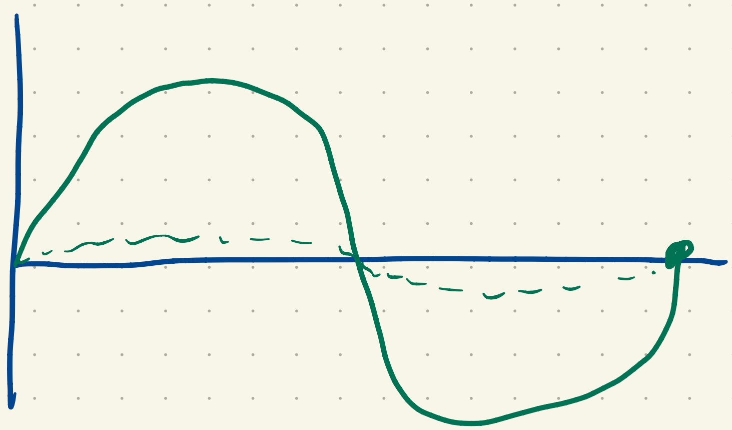




$$v_k(x) = \sin(k\pi x) \quad , \quad \lambda = -k^2\pi^2$$

solution: $\quad u(x,t) = e^{-k^2\pi^2 t} \sin(k\pi x)$





Large k:

a) faster oscillation

b) faster decay.

$u_t = u_{xx}$

If $u(x,t) = \sum_{k=1}^{n} c_k e^{-k^2\pi^2 t} \sin(k\pi x)$

then 1) u satisfies $u_t = u_{xx}$

2) $u(0,t) = 0 \quad u(1,t) = 0$

3) $u(x,0) = \sum_{k=1}^{n} c_k \sin(k\pi x)$

If $\quad u(x,t) = \sum_{k=1}^{n} c_k \, e^{-k^2\pi^2 t} \sin(k\pi x)$

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One is tempted to take $n \to \infty$.

This is domain of Fourier Analysis

Is every $u_0(x) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$

what sense of $=$ here?

(what sense of convergence?)

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$$\left(\text{Is } \partial_t \sum_{k=1}^{\infty} c_k e^{-k^2\pi^2 t} \sin(k\pi x) \overset{?}{=} \sum_{k=1}^{\infty} c_k (-k^2\pi^2) e^{-k^2\pi^2 t} \sin(k\pi x) \right)$$

Maximum Principle

key physical and numerical property

"under diffusion we don't generate isolated peaks"

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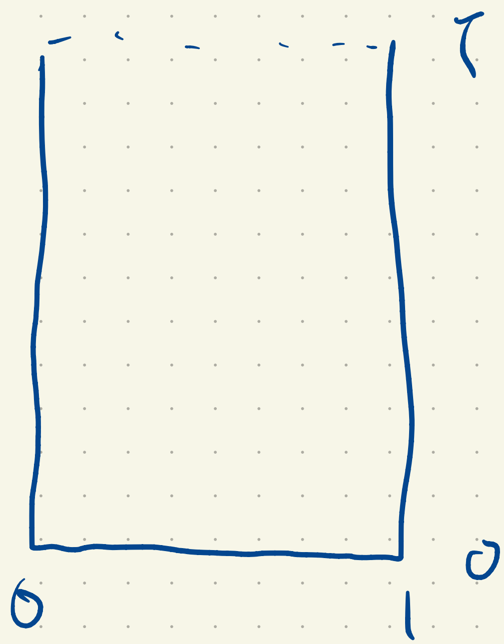

Claim: If $u_t = u_{xx}$

then

max u occurs either

at $t=0$ or

$x=0$ or $x=1$.

$\Omega = [0,1] \times [0,T]$

$\partial\Omega$: boundary

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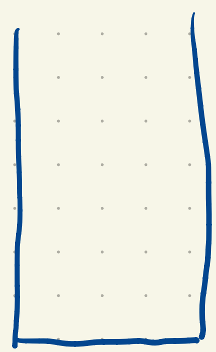
$\partial\Omega^* =$ boundary except $(0,1) \times \{T\}$



$\leftarrow \partial\Omega^*$

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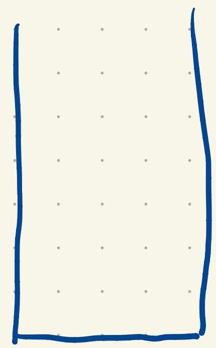


$\leftarrow \partial\Omega^*$

Weak Maximum Principle:

If $u_t - u_{xx} \leq 0$ then

$$\max_{\Omega} u = \max_{\partial\Omega^*} u$$

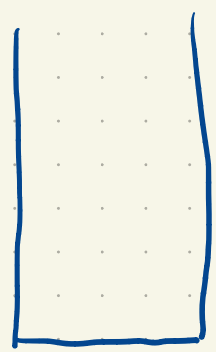
$\Omega = [0,1] \times [0,T]$

$f(x) = x^2$

$f''(x) = 2$

$\partial\Omega$: boundary

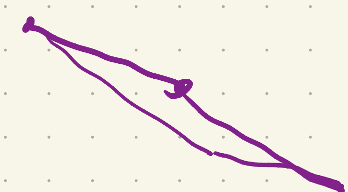
$\partial\Omega^* =$ boundary except $(0,1) \times \{T\}$



$\leftarrow \partial\Omega^*$

$-u_{xx} < 0$

$u_t = 0$



$u_{xx} = 0$

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Weak Maximum Principle:

If $u_t - u_{xx} \leq 0$ then

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Cor: If $u_t - u_{xx} \geq 0$

$$\min_{\Omega} u \geq \min_{\partial\Omega^*} u$$

(Apply above to $-u$)

u

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(Apply above to $-u$)

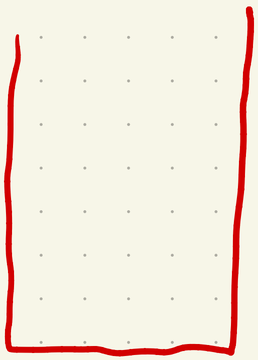
Cor: If $u_t - u_{xx} = 0$

$$\min_{\partial\Omega^*} u \leq u(x,t) \leq \max_{\partial\Omega^*} u \qquad \forall (x,t) \in \Omega$$

C^2, e.g.

Cor: If u_1, u_2 are two solutions of

$$u_t = u_{xx} \quad \text{on } \Omega = [0,1] \times [0,T]$$



with $u_1(x,0) = u_2(x,0) \quad x \in [0,1]$ and

$$u_1(0,t) = u_2(0,t) \quad t \in [0,T] \text{ and}$$

$$u_1(1,t) = u_2(1,t) \quad t \in [0,T]$$

then $u_1 = u_2$ on Ω.

$$w = u_1 - u_2 \qquad (\partial_t - \partial_x^2)\, w = 0$$

$$w\big|_{\partial\Omega^*} = 0$$

$$0 \leq w(x,t) \leq 0$$

There is at most one solution of the heat eq. with those boundary conditions

Pf of ~~maximum~~ weak maximum principle:

Suppose first $u_t - u_{xx} < 0$ on Ω

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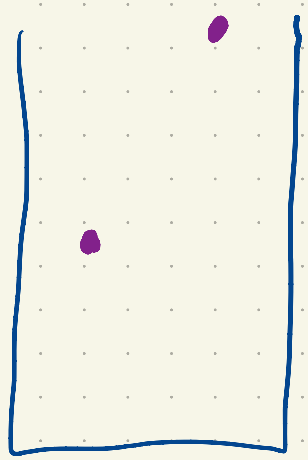
If a max in $\Omega \setminus \partial\Omega^*$:



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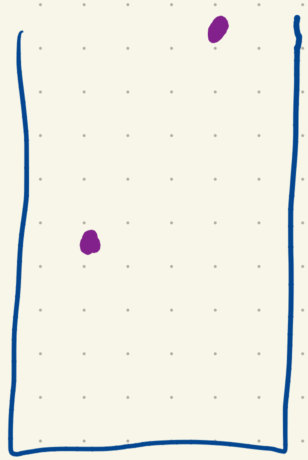
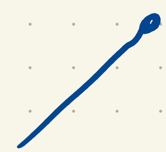


1) $u_t = 0$, $u_t \geq 0$  (not at $t=0$!)

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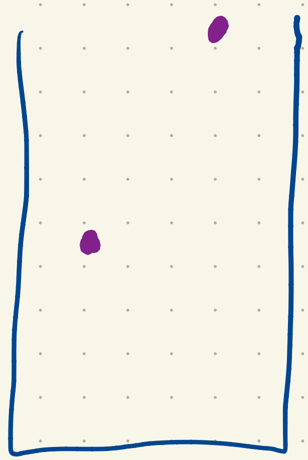

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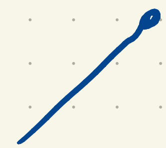

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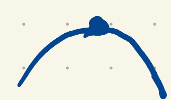
2) $u_x = 0$ (not at $x=0,1$)

3) $u_{xx} \leq 0$



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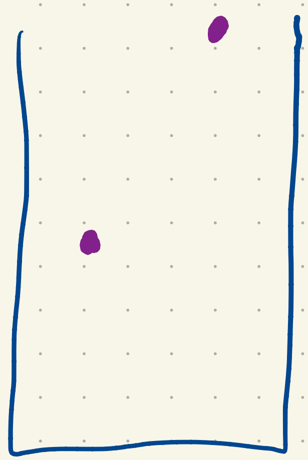
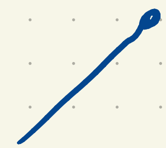


1) $u_t = 0$, $u_t \geq 0$  (not at $t=0$!)

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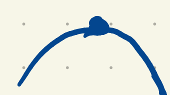
3) $u_{xx} \leq 0$ 
(uses $u_x = 0$)

$$\underbrace{u_t}_{\geq 0} \underbrace{- u_{xx}}_{\geq 0} \geq 0$$

$u_t \geq 0$
$u_{xx} \leq 0$ and $u_t = u_{xx}$ (oops!)

Now what if $u_t - u_{xx} \leq 0$ only?

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Let $v_\varepsilon = u - \varepsilon t$ $\quad \varepsilon > 0$

$$\partial_t v_\varepsilon = u_t - \varepsilon$$

$$\partial_x^2 v_\varepsilon = u_{xx}$$

$$\left(\partial_t - \partial_x^2\right) v_\varepsilon = \left(u_t - u_{xx}\right) - \varepsilon < 0.$$

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max

So $\quad V_\varepsilon(x,t) \leq \sup_{\partial\Omega^*} V_\varepsilon \leq \sup_{\partial\Omega^*} u$

Now what if $u_t - u_{xx} \leq 0$ only?

Let $v_\varepsilon = u - \varepsilon t$

$z \leq 5 + \varepsilon$

$\forall \varepsilon > 0$

$\partial_t v_\varepsilon = u_t - \varepsilon$

$\partial_x^2 v_\varepsilon = u_{xx}$

$(\partial_t - \partial_x^2) v_\varepsilon = (u_t - u_{xx}) - \varepsilon < 0.$

So $v_\varepsilon(x,t) \leq \sup_{\partial\Omega^*} v_\varepsilon \leq \sup_{\partial\Omega^*} u$

$u(x,t) = v_\varepsilon(x,t) + \varepsilon t \leq \sup_{\partial\Omega^*} u + \varepsilon T$

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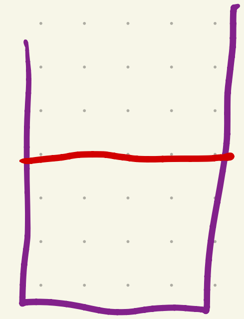
$$u(x,t) = v_\varepsilon(x,t) + \varepsilon t \leq \sup_{\partial\Omega^*} u + \varepsilon T$$

Now send $\varepsilon \to 0$.

Energy $E(t) = \frac{1}{2} \int_0^1 u_x(s,t)^2 \, ds$



$\partial_t E \leq 0$ under homogeneous

D or N conditions

E is decaying in t

"getting smoother"

You prove this on HW

(and show uniqueness from it)

Our Model Heat Equation

$$u_t = u_{xx} + f(x,t)$$

$$u(x,0) = u_0(x)$$

$$u(0,t) = 0$$

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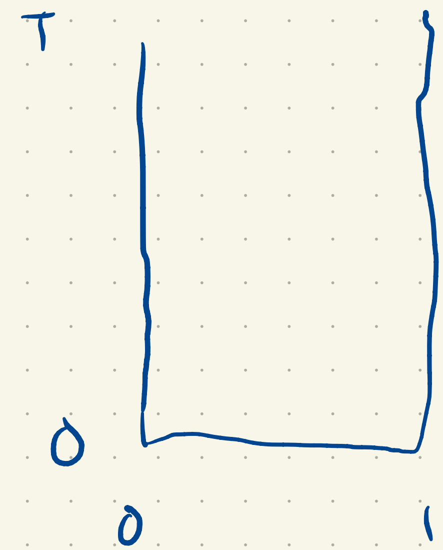

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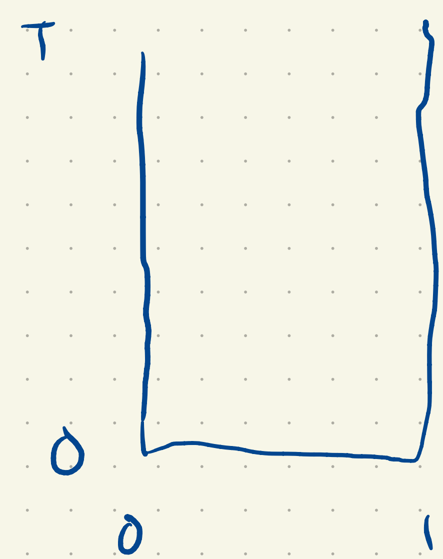
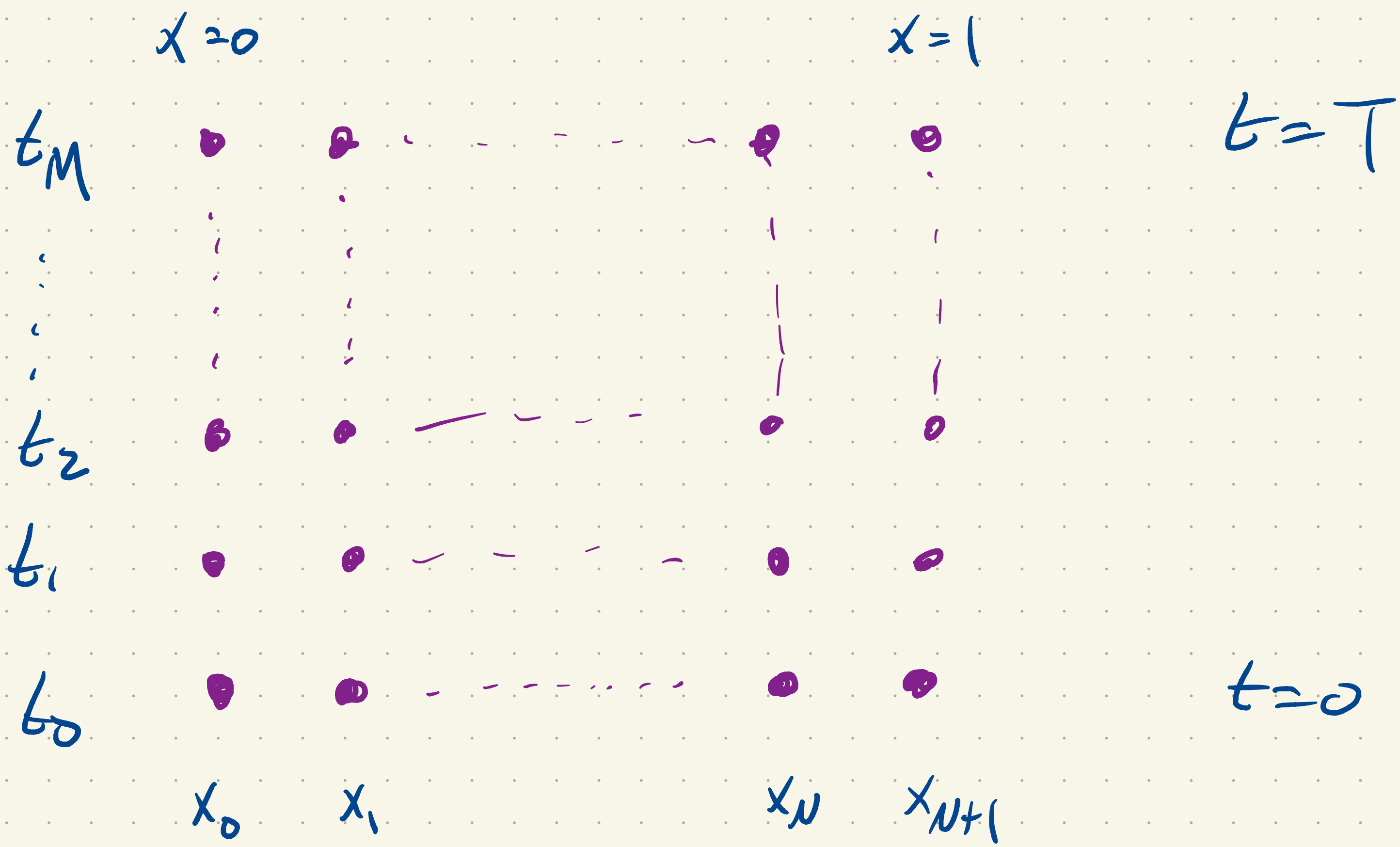


The $f(x,t)$ is less motivated but easy to code and practical (see method of manufactured solutions)

Grid



$$k = T/M \qquad h = \frac{1}{N+1}$$

Direct Method

Approximate Derivatives

$$u_t(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} + O(k)$$

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$$v(x+h) = v(x) + v'(x)h + \frac{1}{2}v''(x)h^2 + \frac{v'''(x)h^3}{6} + O(h^4)$$

$$v(x-h) = v(x) - v'(x)h + \frac{1}{2}v''(x)h^2 - \frac{v'''(x)}{6}h^3 + O(h^4)$$

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$$\frac{v(x+h) - 2v(x) + v(x-h)}{h^2} = v''(x) + O(h^2)$$

$$u_t(x_i, t_j) = \frac{u(x_i, t_j + k) - u(x_i, t_j)}{k} + O(k)$$

$$u_{xx}(x_i, t_j) = \frac{u(x_i + h, t_j) - 2u(x_i, t_j) + u(x - h, t_j)}{h^2} + O(h^2)$$

Replace: $u(x_i, t_j)$ with unknowns u_{ij}

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$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + f(x_i, t_j)$$

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$$u_{i,j+1} = u_{i,j} + \left(\frac{k}{h^2}\right)\left[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right] + kf_{i,j}$$

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