An Unhelpful Introduction to Electricity & Magnetism

Part III b: Curvature, Charge and Two more! Maxwell Equations

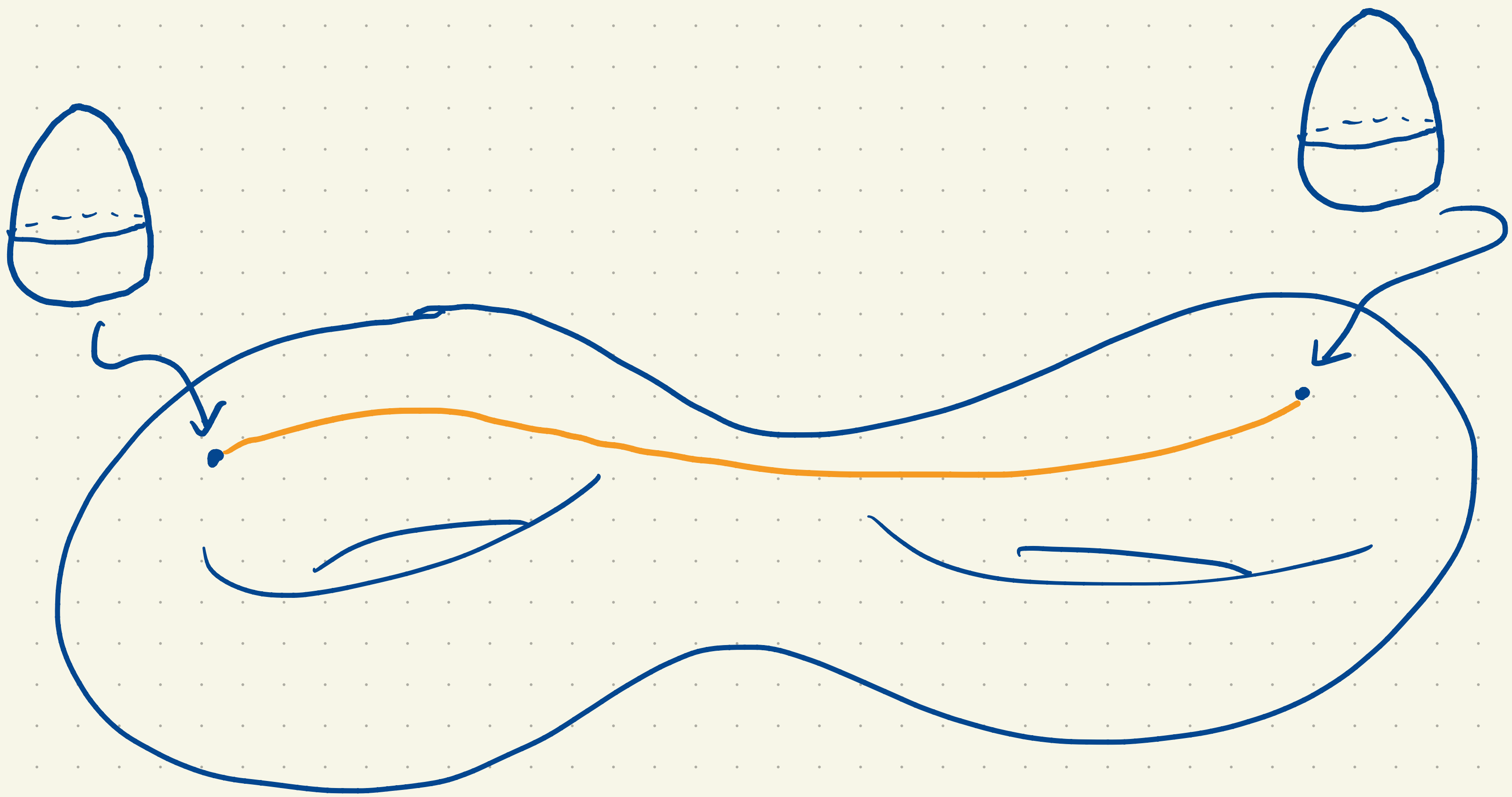
Dec 1, 2020

David Maxwell (damaxwell.github.io)

UAF Mathematics

Summary so Far

- Object of interest: U(1) connection



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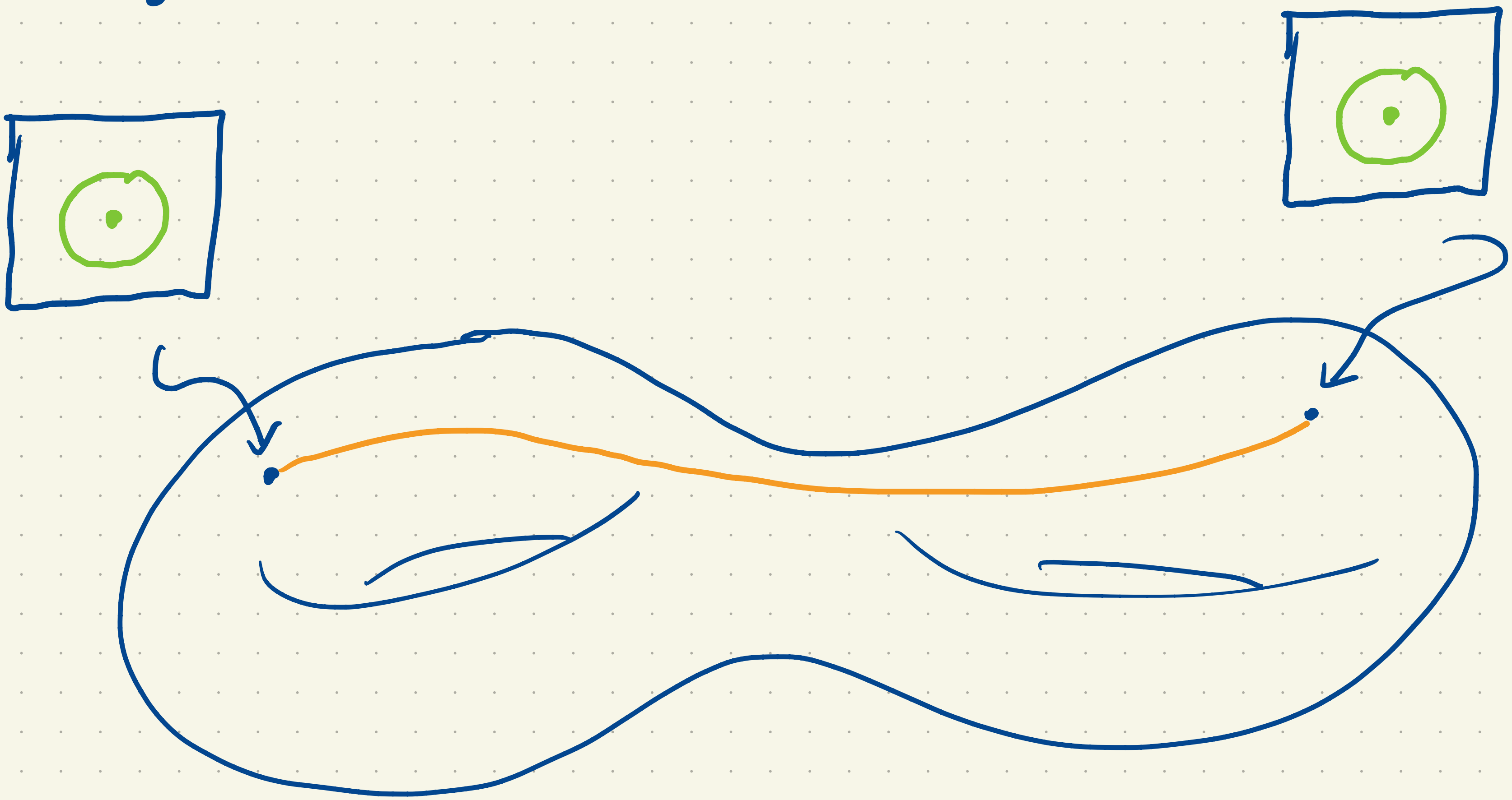

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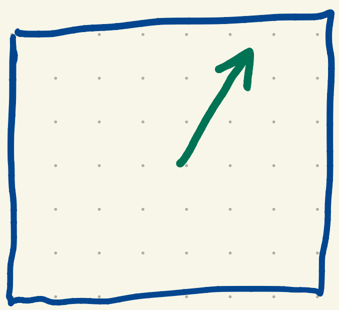
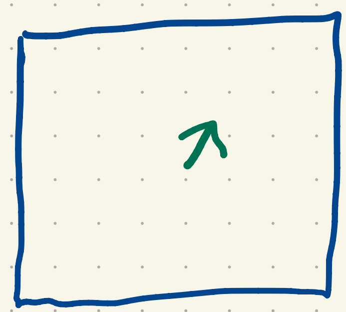
- It lets you compute derivatives of wave functions





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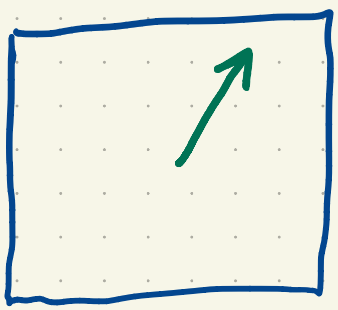

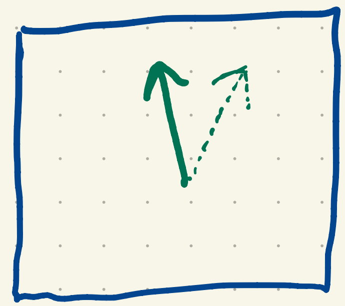

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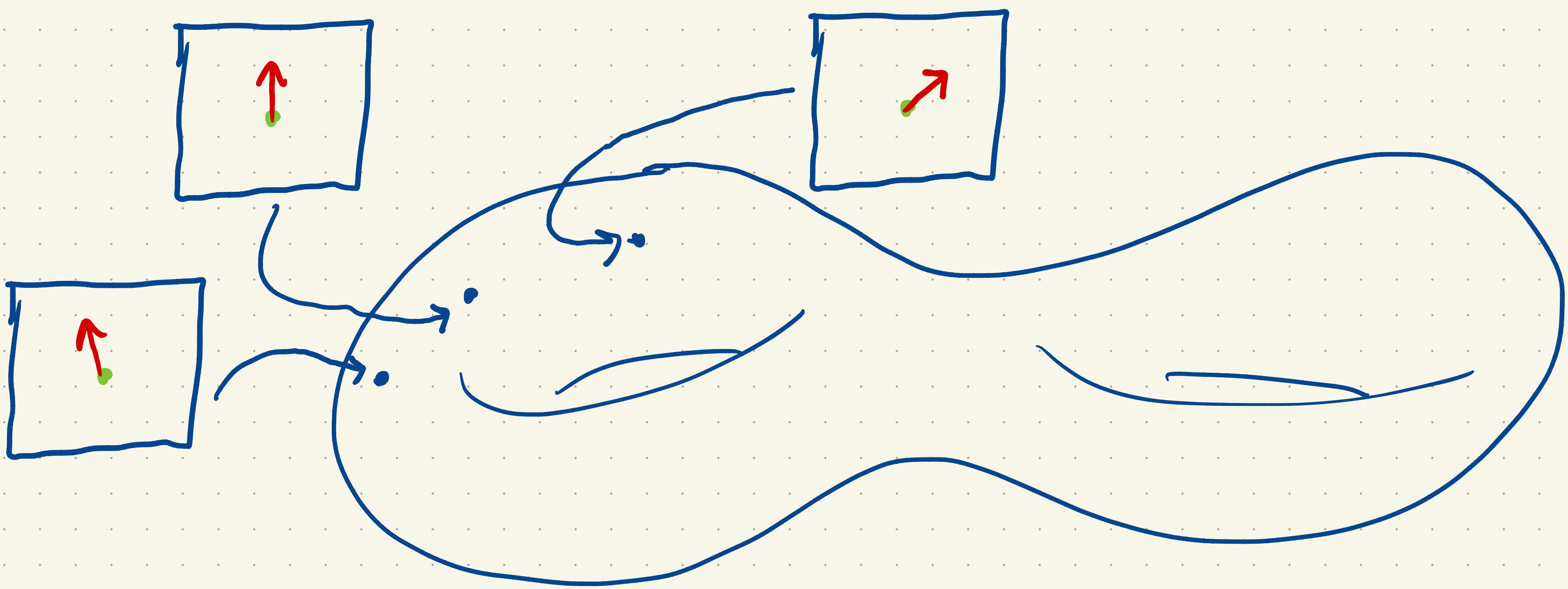

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- Object of interest: U(1) connection
- It lets you compute derivatives of wave functions
- An "egg structure" admits many connections
- A frame is an arbitrary local choice of mutually congruent "points"
- Once a choice of local frame is made, we can represent the connection by a 1-form A.

$$A[X] = \dot{\theta}$$ → how is frame rotating as I move with tangent vector X

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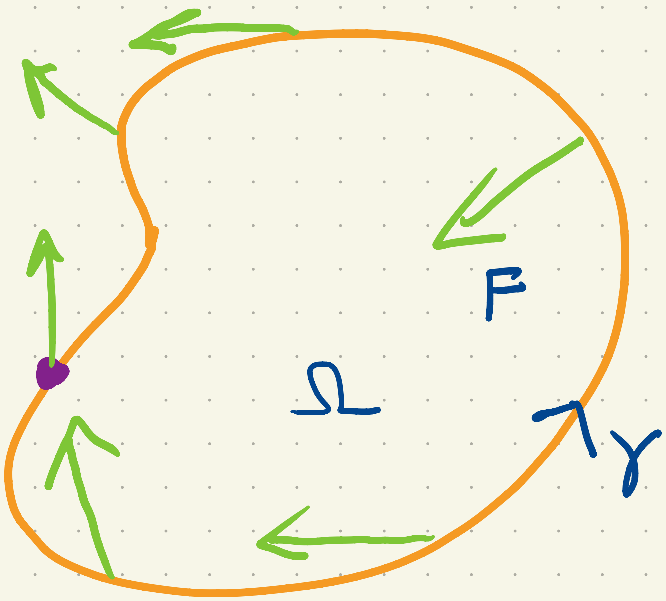
- $A + d\theta$ represents the same connection WRT a different frame.
- All frames agree on $\mathcal{F} = dA = d(A + d\theta)$.
- Two of Maxwell's equations: $d\mathcal{F} = 0$ $(d^2A = 0)$.

What is dA?



$$\int_\gamma A = \int_\Omega dA$$

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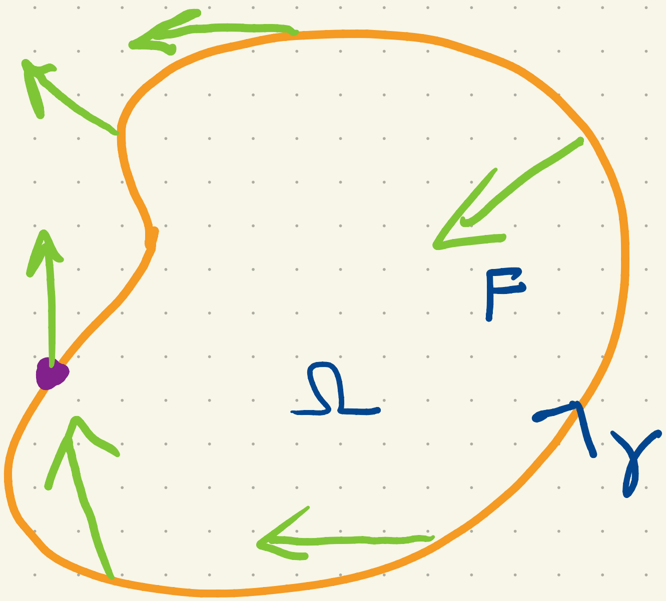

$$\int_\gamma A = \int_\Omega dA$$

$$\int_a^b A[\dot\gamma]\,dt = \int_a^b \dot\theta\,dt$$

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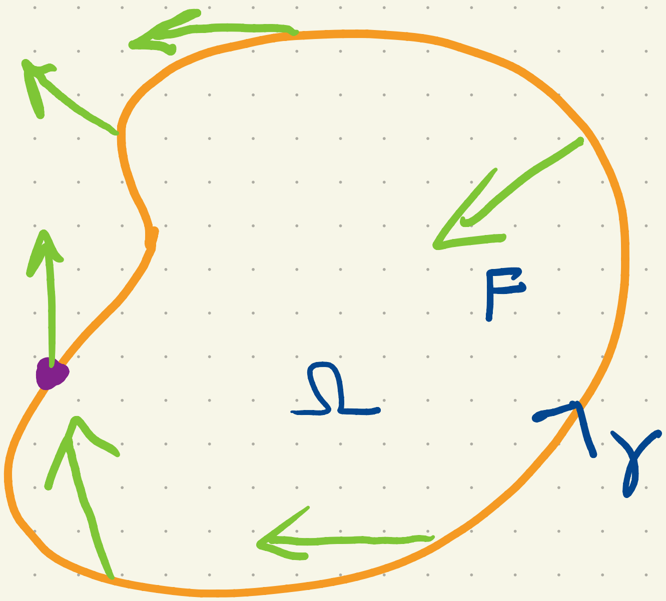
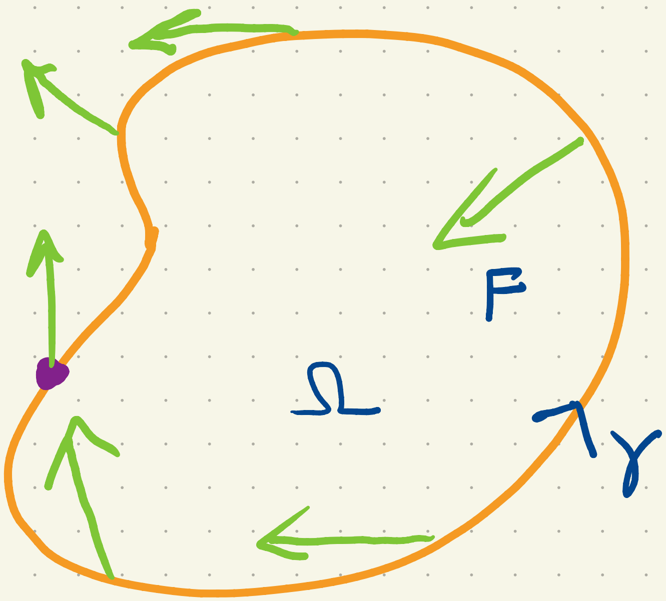


$$\int_\gamma A = \int_\Omega dA$$

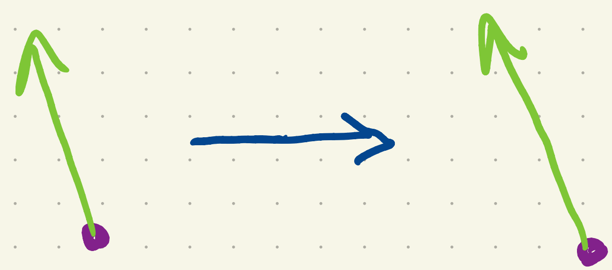
$$\int_a^b A[\dot\gamma]\,dt = \underbrace{\int_a^b \dot\theta\,dt}_{\Delta\theta}$$

Net rotation of frame.

What is $d\mathcal{A}$?



$\int_\Omega d\mathcal{A}$: $\Delta\theta$, net rotation of F around γ.



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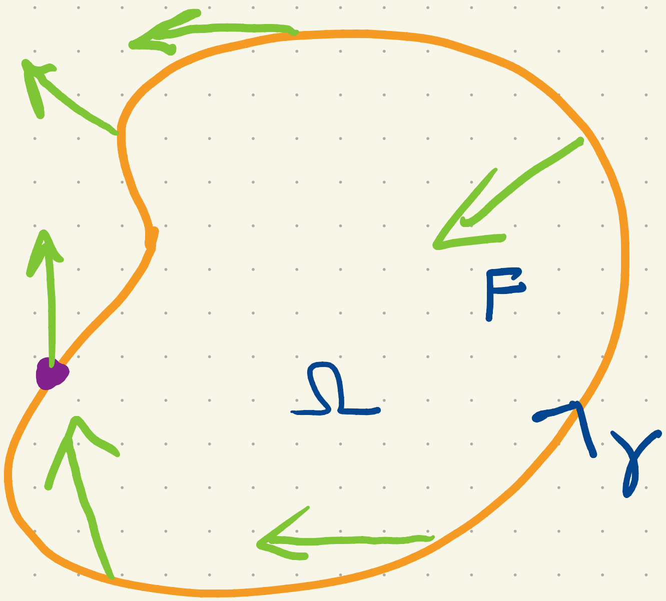

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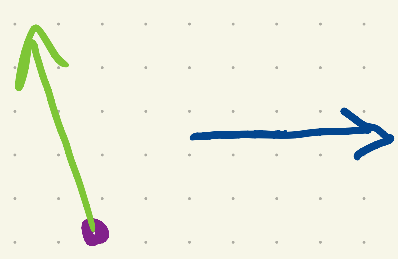
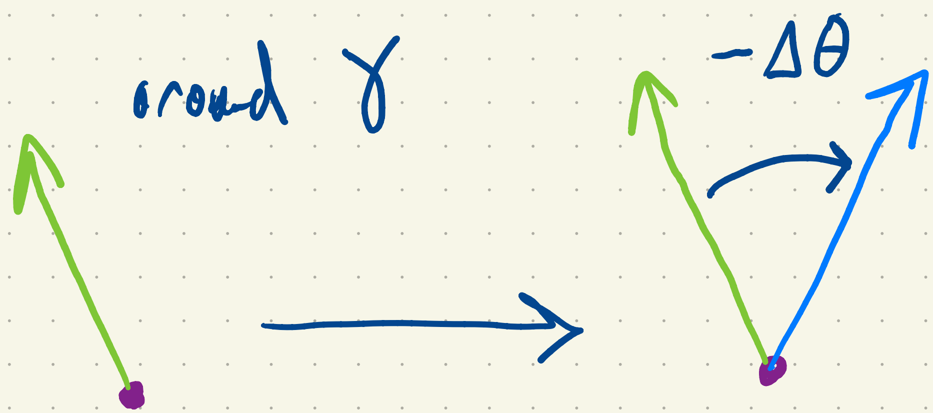


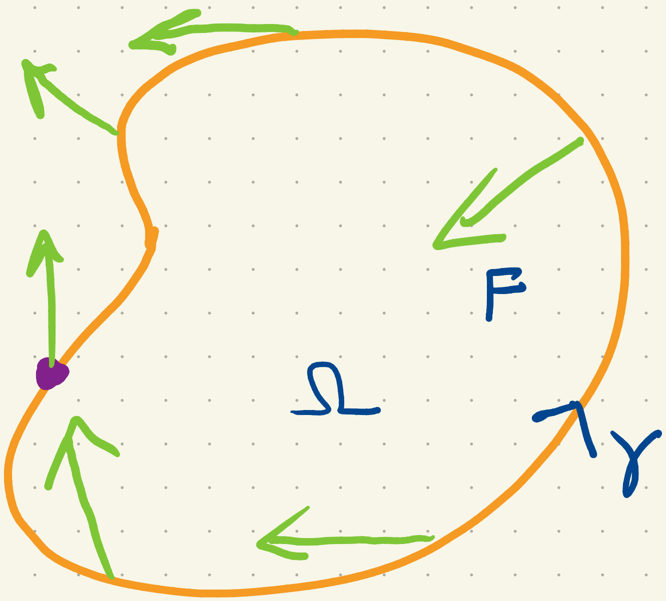
- Parallel transport of F around γ



What is $d\mathcal{A}$?

$-\int_\Omega d\mathcal{A}$ is the rotation anomaly from traversing the curve $\gamma = \partial\Omega$

What is dA?

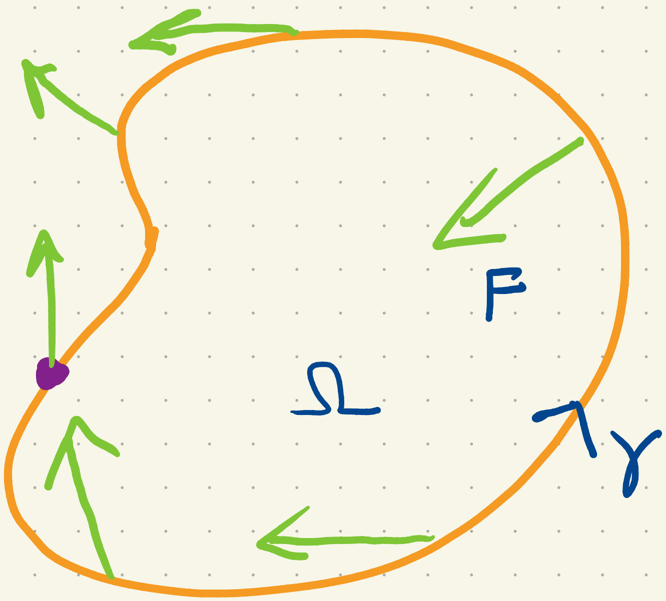


- Parallel transport of <u>anything</u> around γ rotates it by the "rotation anomaly"

$$-\int_\Omega dA = -\Delta\theta.$$

What is dA?

$\cancel{F}$ flat!



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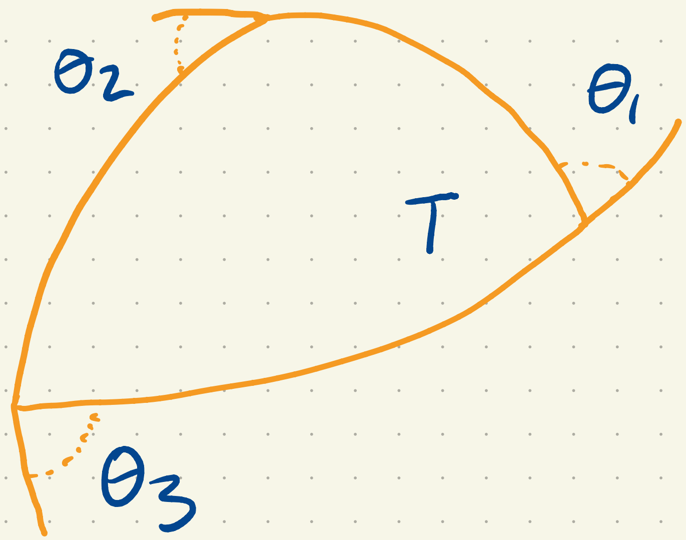
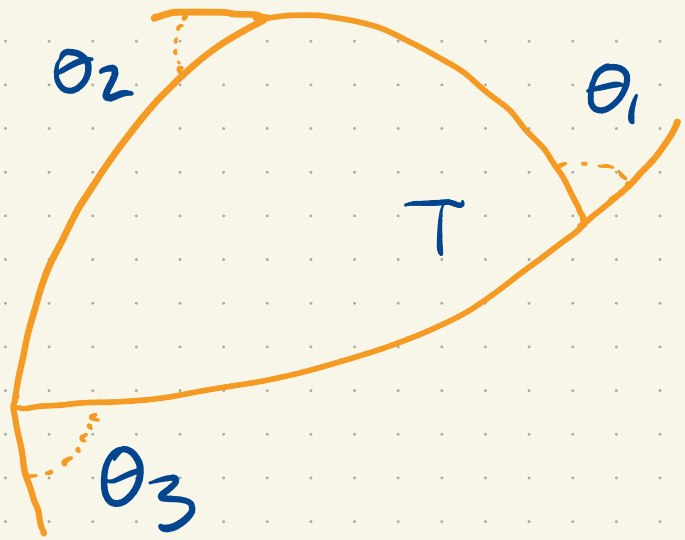
dA is Called Curvature



- On a sphere:

$$2\pi - (\theta_1 + \theta_2 + \theta_3) = 4\pi \frac{\text{Area}(T)}{\text{Area}(S)}$$

dA is Called Curvature



- On a sphere:

$$\underbrace{2\pi - (\theta_1 + \theta_2 + \theta_3)}_{\text{rotation anomoly}} = 4\pi \frac{\text{Area}(T)}{\text{Area}(S)}$$

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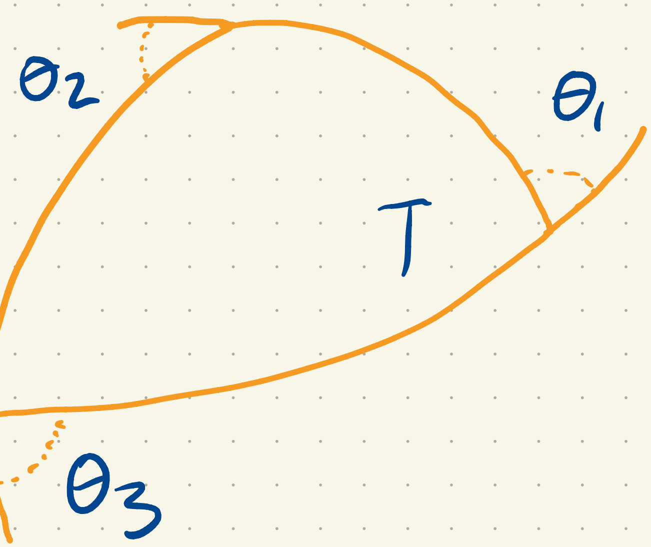
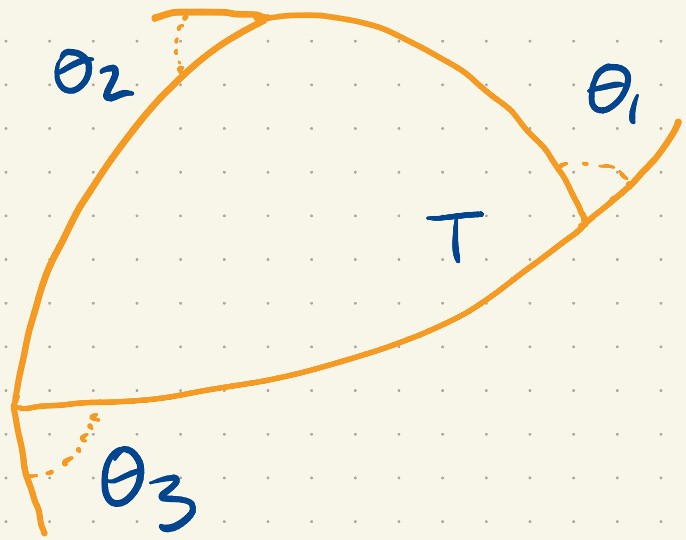


- On a sphere:

$$2\pi - \underbrace{(\theta_1 + \theta_2 + \theta_3)}_{\text{rotation anomaly}} = \frac{1}{R^2}\,\text{Area}(T)$$

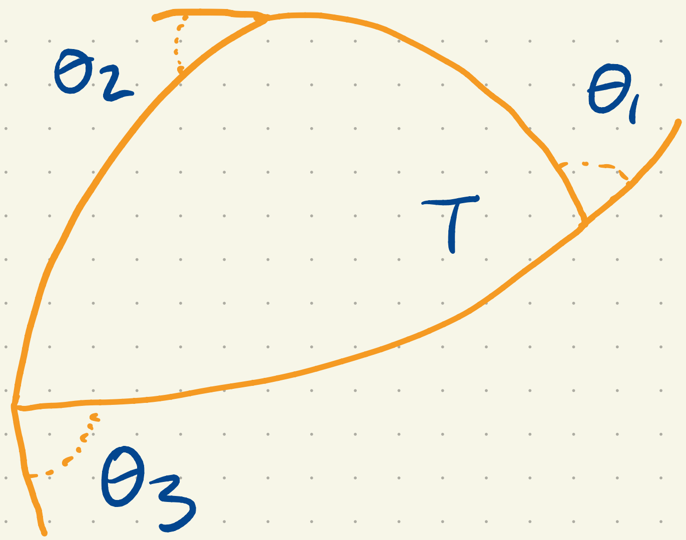
dA is Called Curvature



- On a sphere:

$$-\int_T dA = \frac{1}{R^2}\,\mathrm{Area}(T)$$

dA is Called Curvature

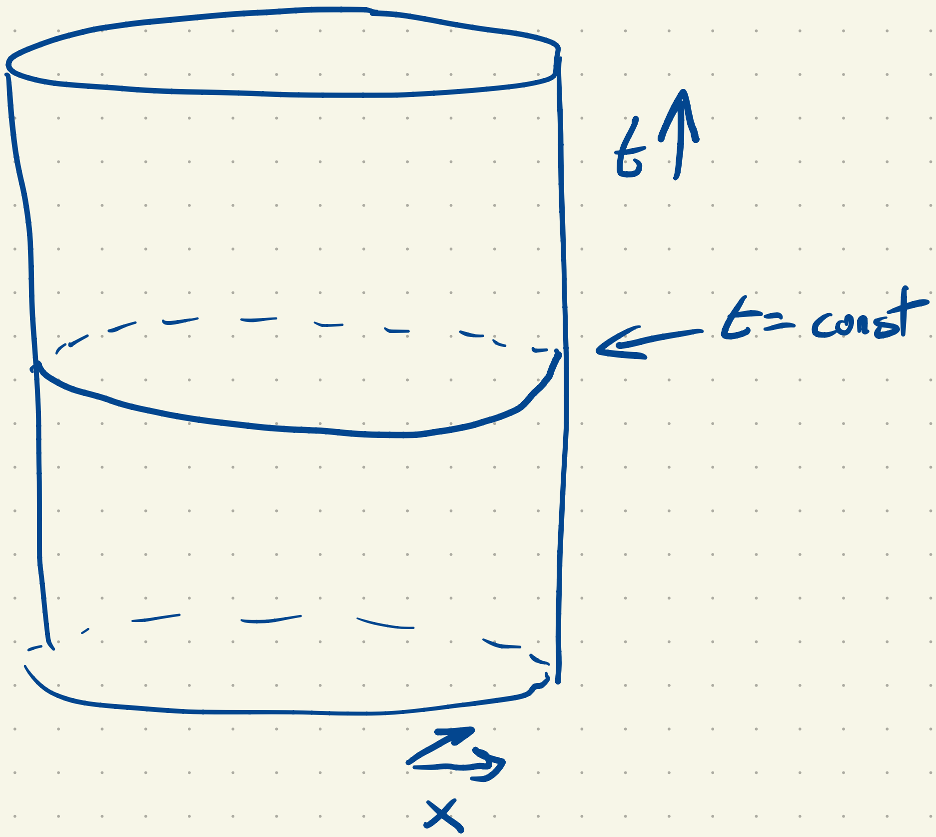


- On a sphere:

Gaussian curvature

$$-\int_T dA = \frac{1}{R^2}\,\text{Area}(T)$$

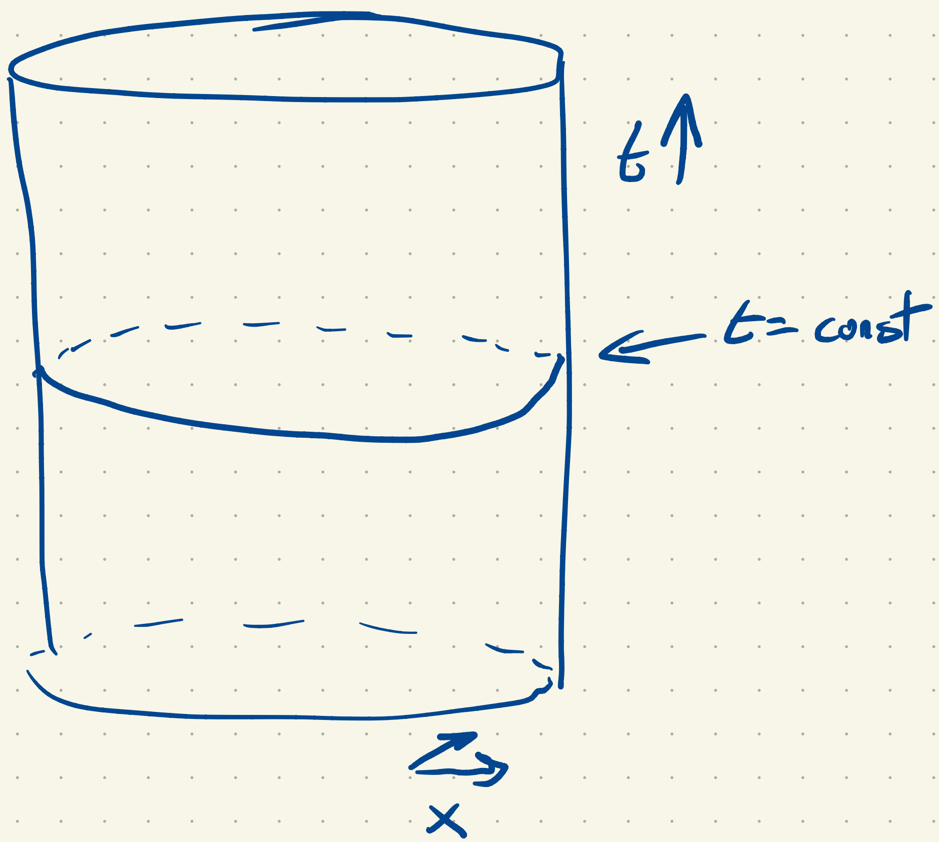
Magnetic Field



$$\mathcal{F} = dA = dt \wedge \text{crap}$$

$$- B_{ij}\, dx^i \wedge dx^j$$

Magnetic Field

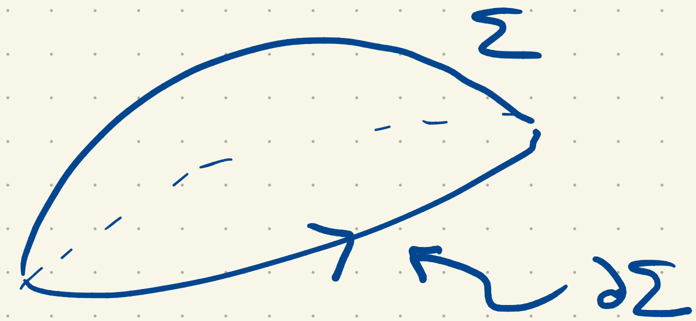


$$\mathcal{F} = dA = dt \wedge \text{crap} - \underbrace{B_{ij}\, dx^i \wedge dx^j}_{\downarrow}$$

B, magnetic 2-form

Magnetic Flux

$$\int_\Sigma B = -\int_\Sigma dA$$



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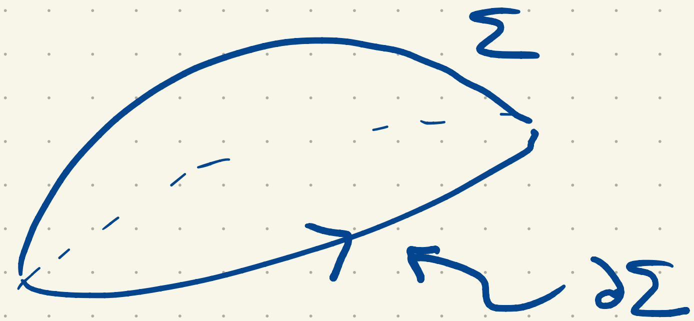
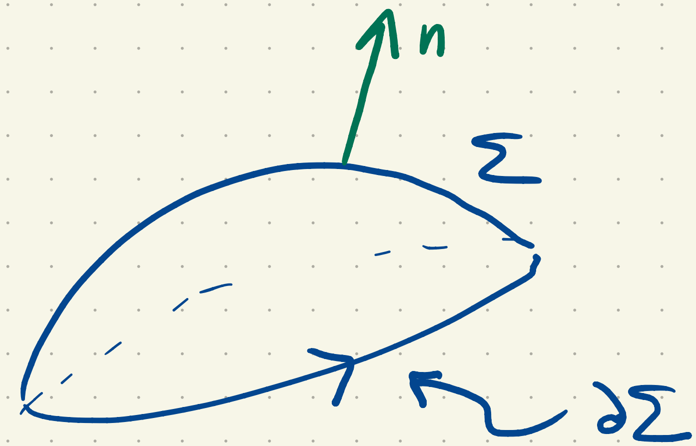


$= \text{rotation anomoly of } \partial\Sigma$

Magnetic Flux

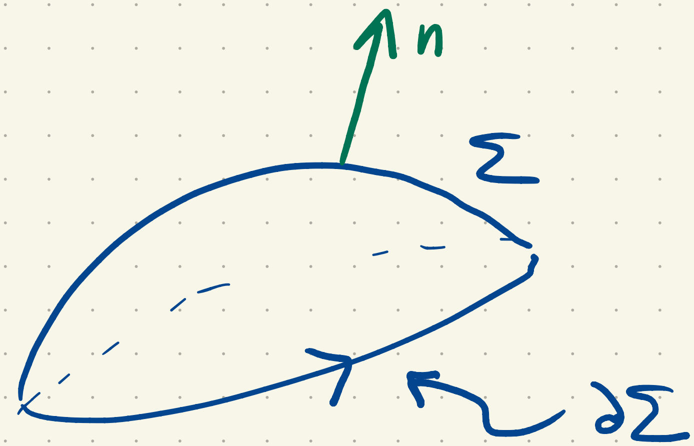


$$\int_\Sigma B = -\int_\Sigma dA$$

$=$ rotation anomoly of $\partial\Sigma$

"$= \int_\Sigma \vec{B}\cdot\vec{n}\, dA$"

Magnetic Flux



$$\int_\Sigma B = -\int_\Sigma dA$$

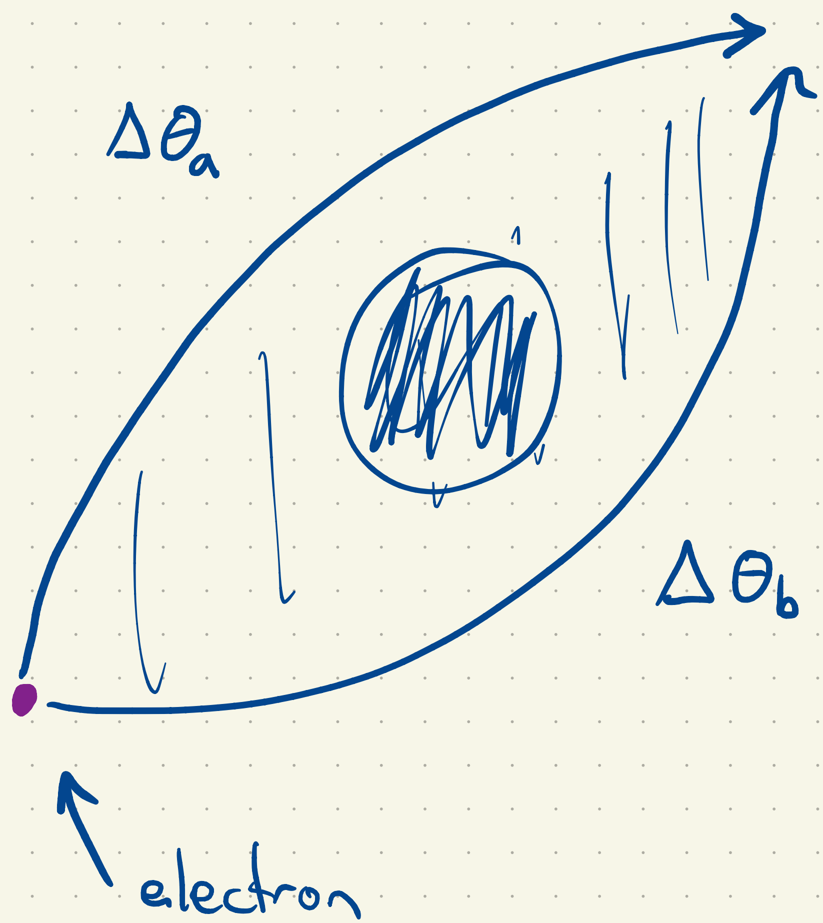
$$= \text{rotation anomoly of } \partial\Sigma$$

$$\text{"} = \int_\Sigma \vec{B}\cdot\vec{n}\, dA \text{"}$$

$$A[\gamma]$$

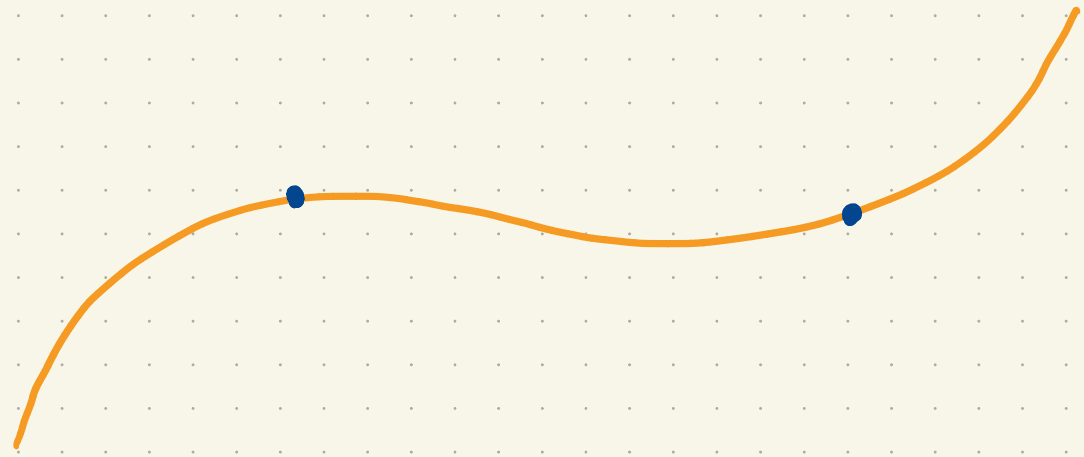
- B is the curvature of the connection "right now"

Magnetic Flux is Physical



$\int B$

Arc length



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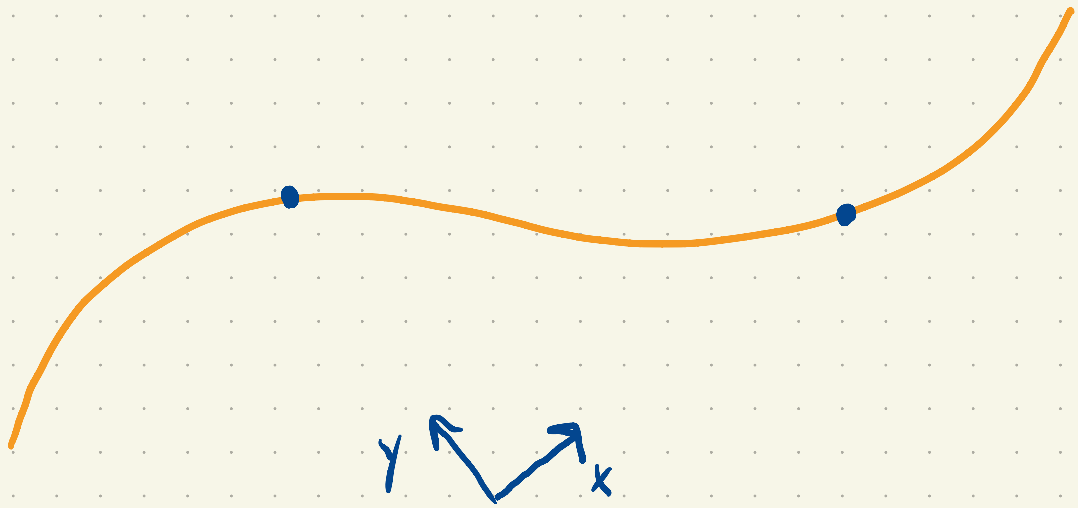

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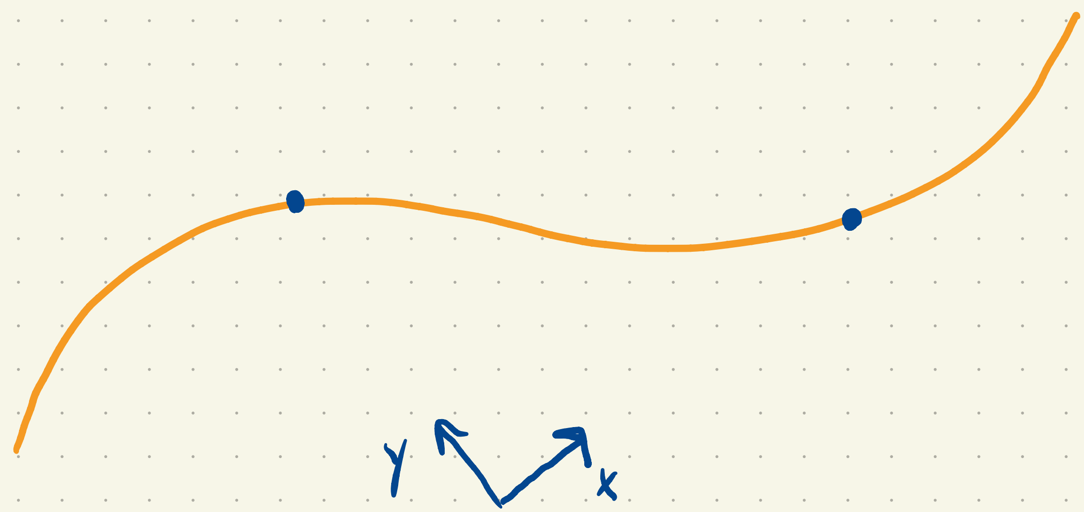
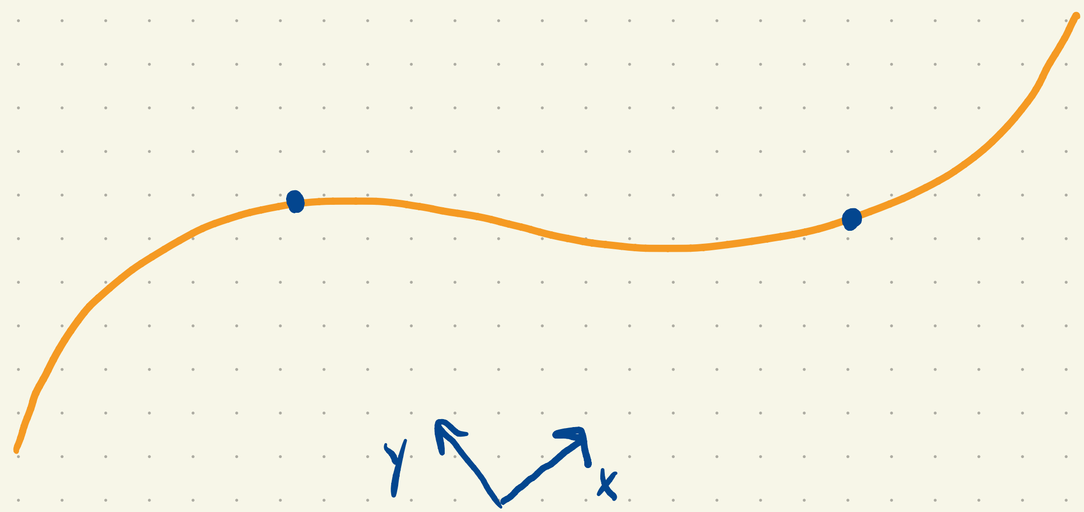


$$\gamma(x) = (x, y(x))$$

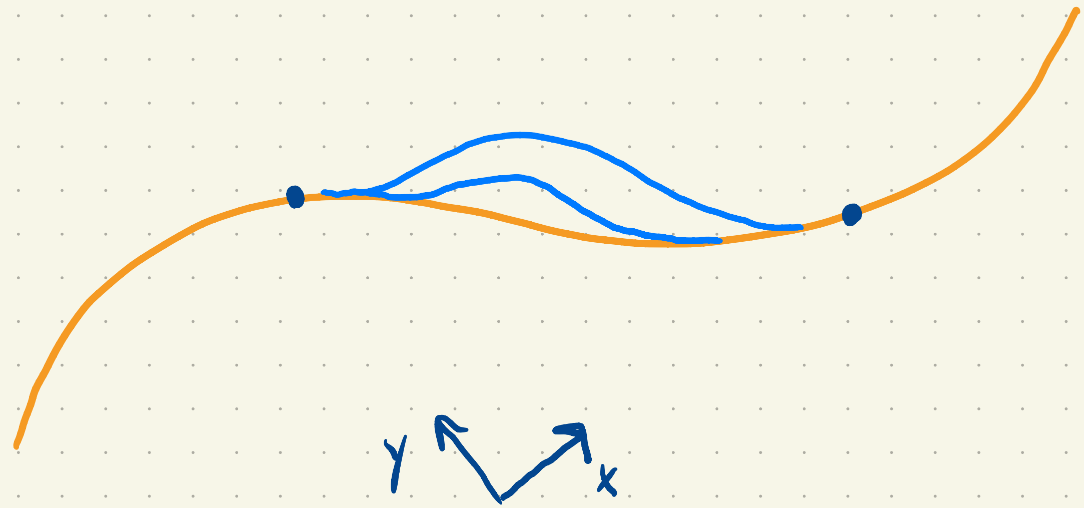
Arc length



$$\gamma(x) = (x, y(x))$$

$$S = \int_a^b \sqrt{1 + (y')^2}\, dx$$

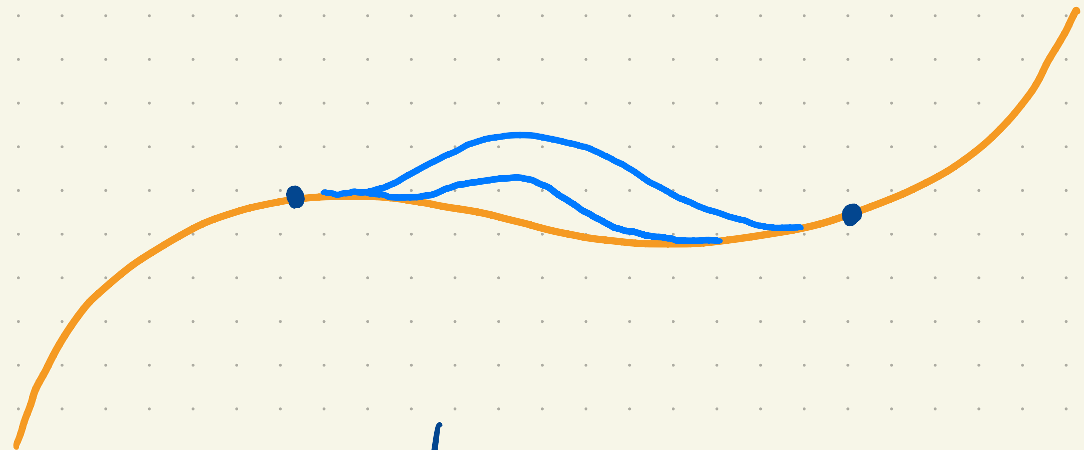
Arc length



$$\gamma_s(x) = (x, y_s(x))$$

$$S(s) = \int_a^b \sqrt{1 + (y_s')^2}\, dx$$

Arc length

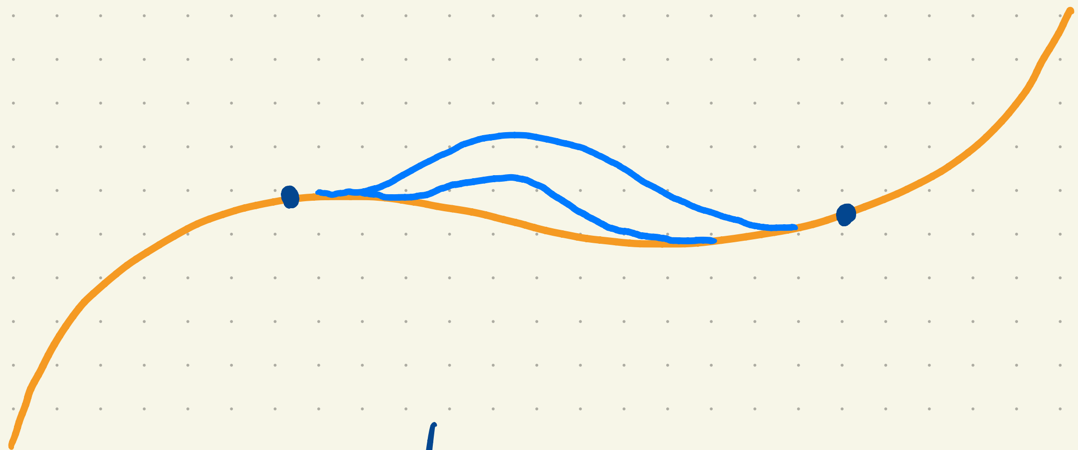


$$\delta y = \frac{d}{ds}\bigg|_{s=0} y_s$$

$$S = \int_a^b \sqrt{1 + (y_s')^2}\, dx$$

$$\frac{d}{ds}\bigg|_{s=0} S = \int_a^b \frac{y' \, (\delta y)'}{\sqrt{1 + (y')^2}}\, dx$$

Arc length

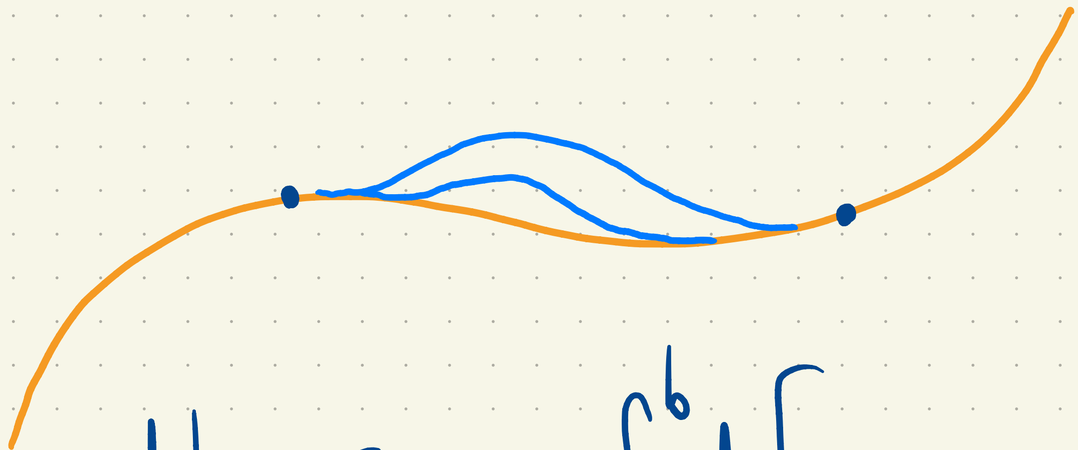


$$\delta y = \frac{d}{ds}\bigg|_{s=0} y_s$$

$$S = \int_a^b \sqrt{1 + (y_s')^2}\, dx$$

$$\frac{d}{ds}\bigg|_{s=0} S = -\int_a^b \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right]\delta y\, dx$$

Arc length



$$\left.\frac{d}{ds}\right|_{s=0} S = -\int_a^b \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right]\delta y \, dx$$

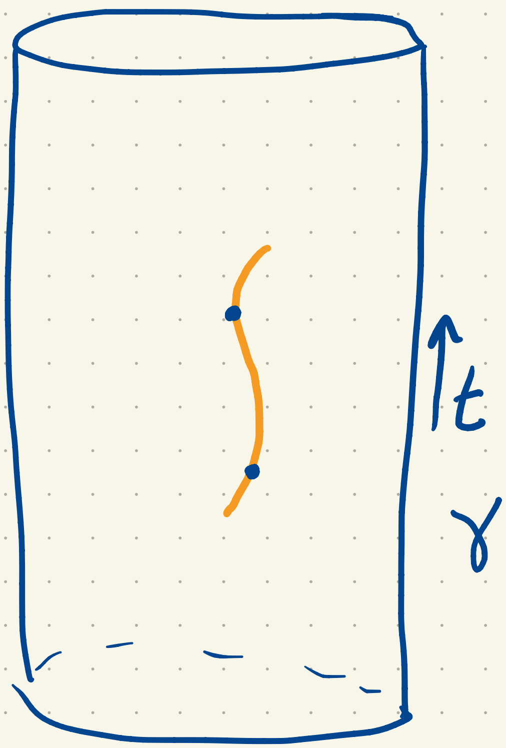
Stationary for all δy $\iff$ $\displaystyle \frac{d}{dx}\left[\frac{y'}{\sqrt{1+(y')^2}}\right] = 0$

$$\iff y'' = 0$$

Newtonian Lagrangian

- Particle traveling in a potential $V(x)$

$$S = \int_a^b \frac{m}{2} |\dot{x}|^2 - V(x) \, dt$$



$$\gamma(t) = (t, x(t))$$

Newtonian Lagrangian

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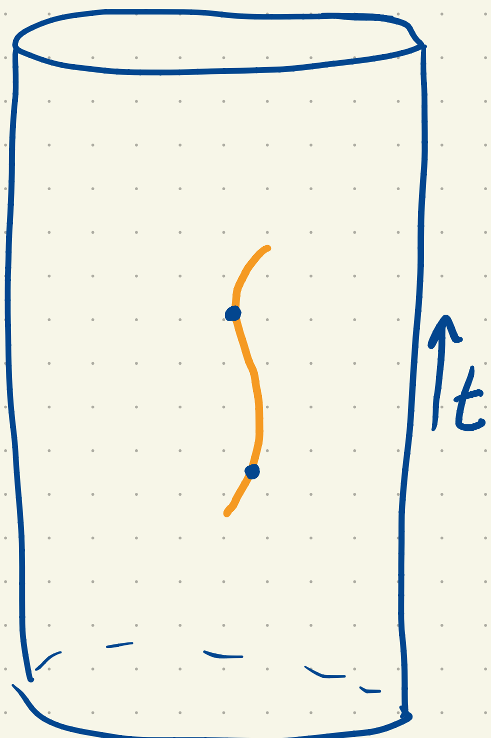


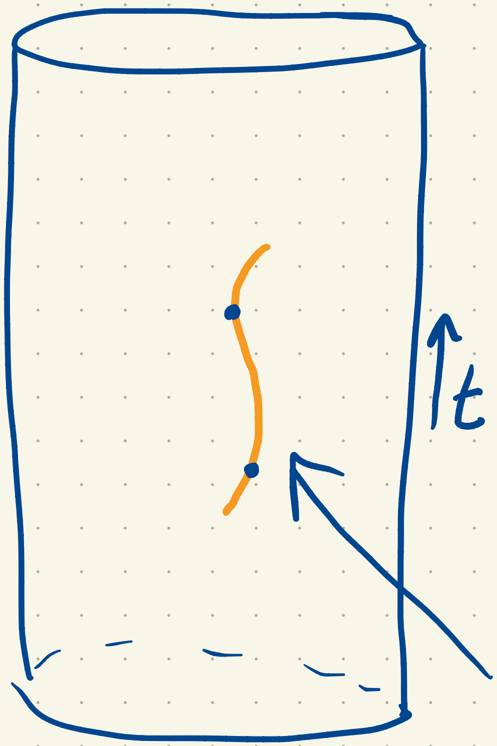
$$S = \int_a^b \frac{m}{2} |\dot{x}|^2 - V(x)\, dt$$

$$\delta S = \int_a^b m\,\dot{x}\,(\delta x)^{\cdot} - dV[\delta x]\, dt$$

force

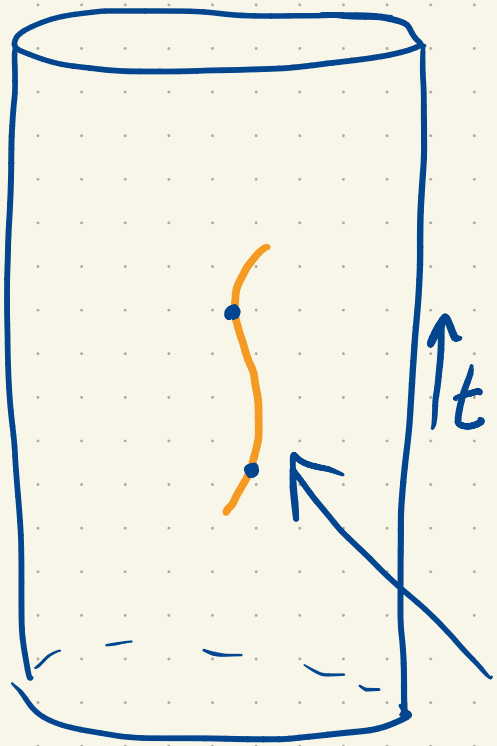
action stationary $\Leftrightarrow$ $\dfrac{d}{dt}\left[m\dot{x}\right] = -dV$

Special Relativistic Free Particle



$$\gamma(s) = \big(t(s),\ x(s)\big)$$

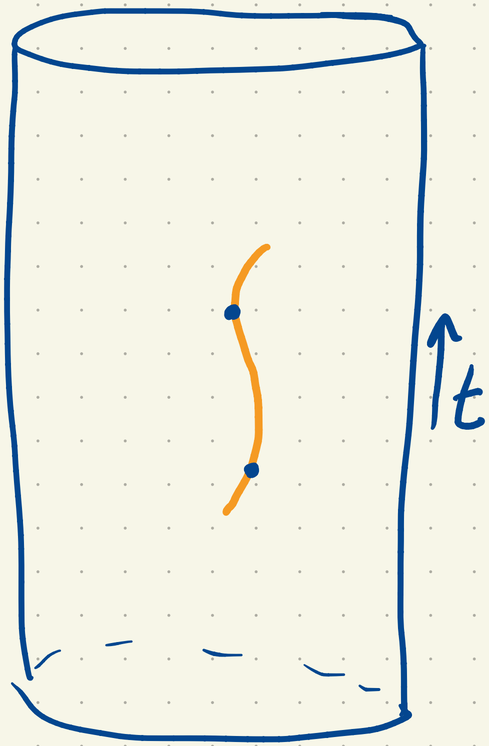
Special Relativistic Free Particle



$$S = -\int_a^b \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_g}\; ds$$

$\gamma(s) = \big(t(s),\ x(s) \big)$

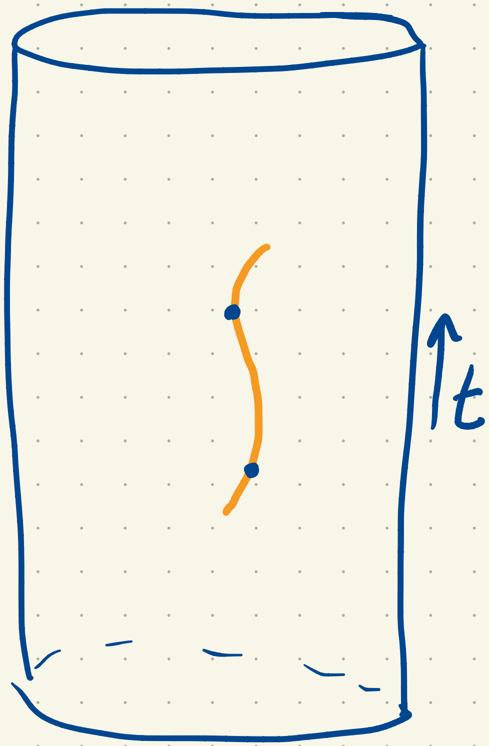
Special Relativistic Free Particle



$$S = -\int_a^b \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_g} \, ds$$

$$\langle \dot{\gamma}, \dot{\gamma} \rangle_g = (\dot{t})^2 - |\dot{x}|^2$$

Special Relativistic Free Particle



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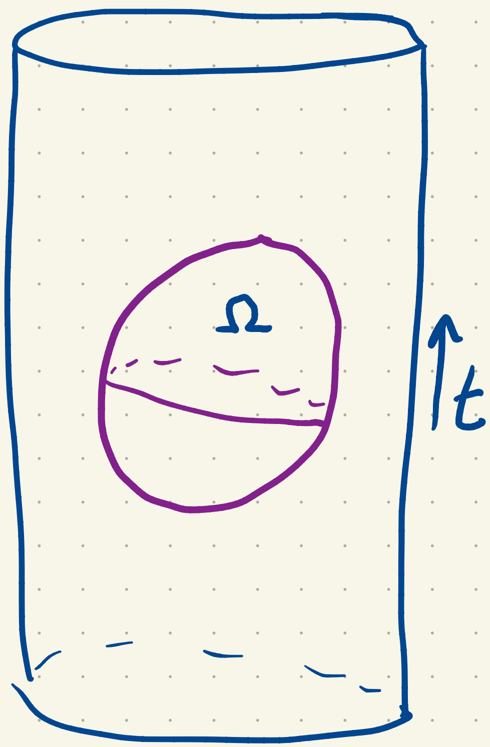
Exercise: $t(s) = s$

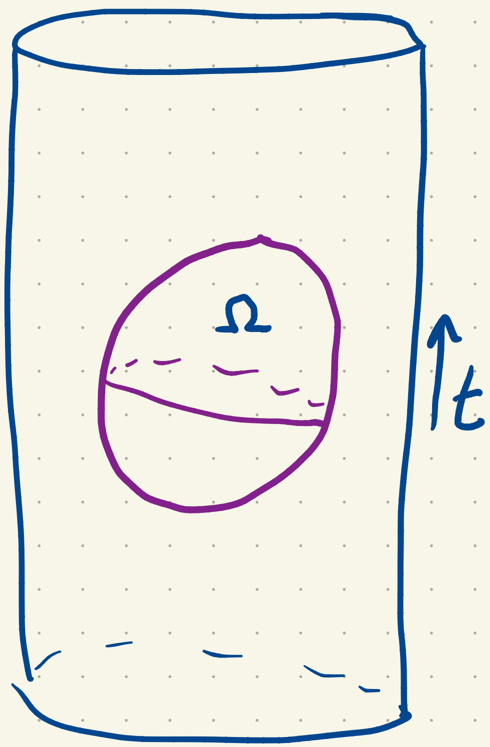
stationary $\Longleftrightarrow$ $\ddot{x} = 0$

Vacuum E+M Lagrangian



$$S = -\frac{1}{2}\int_{\Omega} \langle dA, dA \rangle_g \, dV_g$$

Vacuum E+M Lagrangian

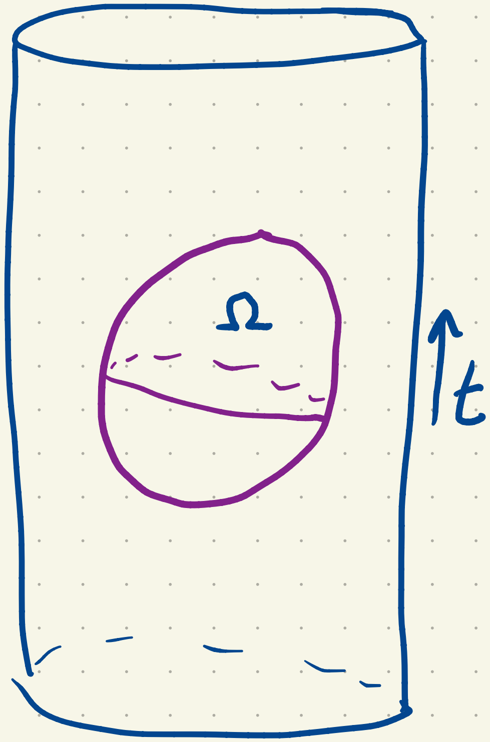


$$S = -\frac{1}{2}\int_{\Omega} \langle dA, dA \rangle_g \, dV_g$$

$\downarrow$

S.R. $dt \wedge dx^1 \wedge dx^2 \wedge dx^3$

Vacuum E+M Lagrangian



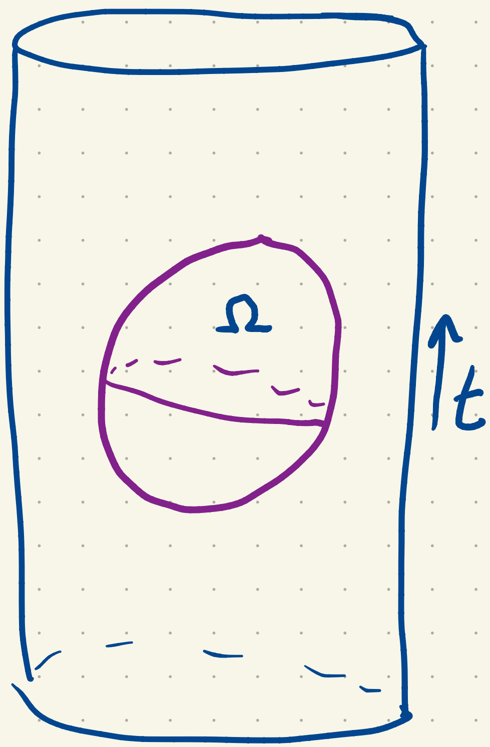
$$S = -\frac{1}{2}\int_{\Omega} \langle dA, dA\rangle_g \, dV_g$$

S.R., $c=1$:

$$dA = -E^i dx^i \wedge dt - B_{ij} dx^i \wedge dx^j$$

$$-\langle dA, dA\rangle_g = \sum_i (E^i)^2 - \sum_{i<j} (B_{ij})^2$$

Vacuum E+M Lagrangian



$$S = -\frac{1}{2}\int_{\Omega} \langle dA, dA \rangle_g \, dV_g$$

Maxwell's equations: $\delta S = 0$

Maxwell w/ Matter

$$S = -\frac{1}{2}\int_\Omega \langle dA, dA \rangle_g \, dV_g + \int_\Omega L(\Phi, \partial\Phi, A, g)$$

$$d(dA + d\theta)$$

Maxwell w/ Matter

$$S = \int_\Omega -\frac{1}{2}\langle dA, dA\rangle_g \, dV_g + \int_\Omega L(\Phi, \partial\Phi, A, g)$$

$$\delta S = -\int_\Omega \langle dA, d\delta A\rangle_g \, dV_g + \int_\Omega \underbrace{\frac{\partial L}{\partial A}}_{\text{force}}[\delta A]$$

Maxwell w/ Matter

$$S = \int_\Omega -\frac{1}{2}\langle dA, dA\rangle_g \, dV_g + \int_\Omega L(\Phi, \partial\Phi, A, g)$$

$$\delta S = -\int_\Omega \langle dA, d\delta A\rangle_g \, dV_g + \int_\Omega \underbrace{\frac{\partial L}{\partial A}}_{\substack{\text{force} \\ \text{3-form } J}} \wedge \delta A$$

Charge Conservation

$$\delta S = -\int_\Omega \langle dA, d\delta A \rangle_g \, dV_g - \int_\Omega J \wedge \delta A$$

Charge Conservation

$$\delta S = -\int_\Omega \langle dA, d\delta A \rangle_g \, dV_g - \int_\Omega \mathcal{J} \wedge \delta A$$

$$\delta A = d\theta$$

Charge Conservation

$$\delta S = -\int_{\Omega} \langle dA, d\,\delta A \rangle_g \, dV_g - \int_{\Omega} \mathcal{J} \wedge \delta A$$

$$\delta A = d\theta$$

$$\int_{\Omega} \mathcal{J} \wedge d\theta = 0$$

Charge Conservation

$$\delta S = -\int_\Omega \langle dA, d\delta A\rangle_g \, dV_g - \int_\Omega J \wedge \delta A$$

$$\delta A = d\theta$$

$$\int_\Omega J \wedge d\theta = 0$$

$$d(J\theta) = (dJ)\theta - J \wedge d\theta$$

Charge Conservation

$$\delta S = -\int_\Omega \langle dA, d\delta A\rangle_g \, dV_g - \int_\Omega \mathcal{J} \wedge \delta A$$

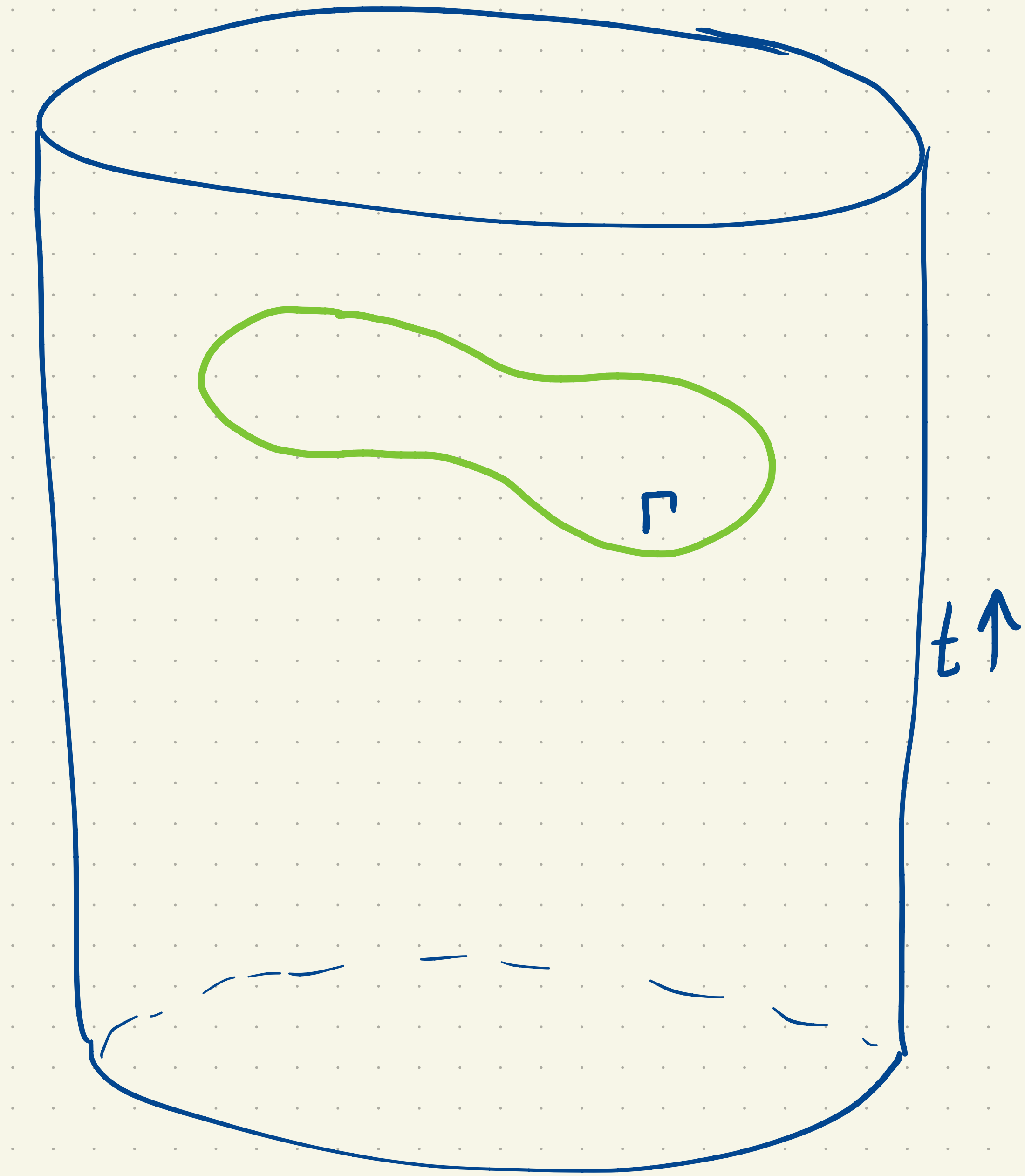
$\uparrow$ 3-form

$$\delta A = d\theta$$

$$\int_\Omega \mathcal{J} \wedge d\theta = 0 \Rightarrow d\mathcal{J} = 0$$

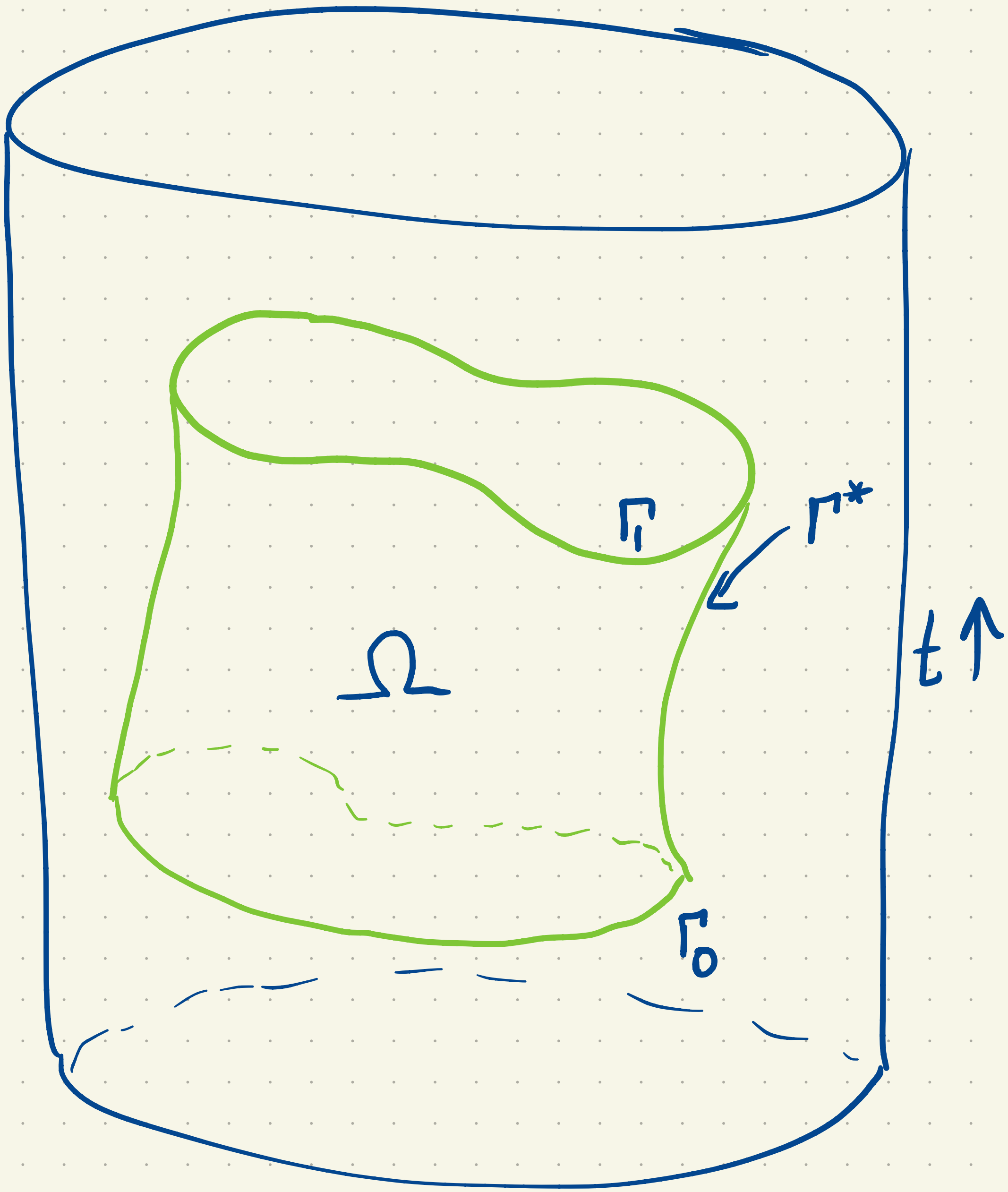
$$d(\mathcal{J}\theta) = (d\mathcal{J})\theta - \mathcal{J} \wedge d\theta$$

Charge Conservation



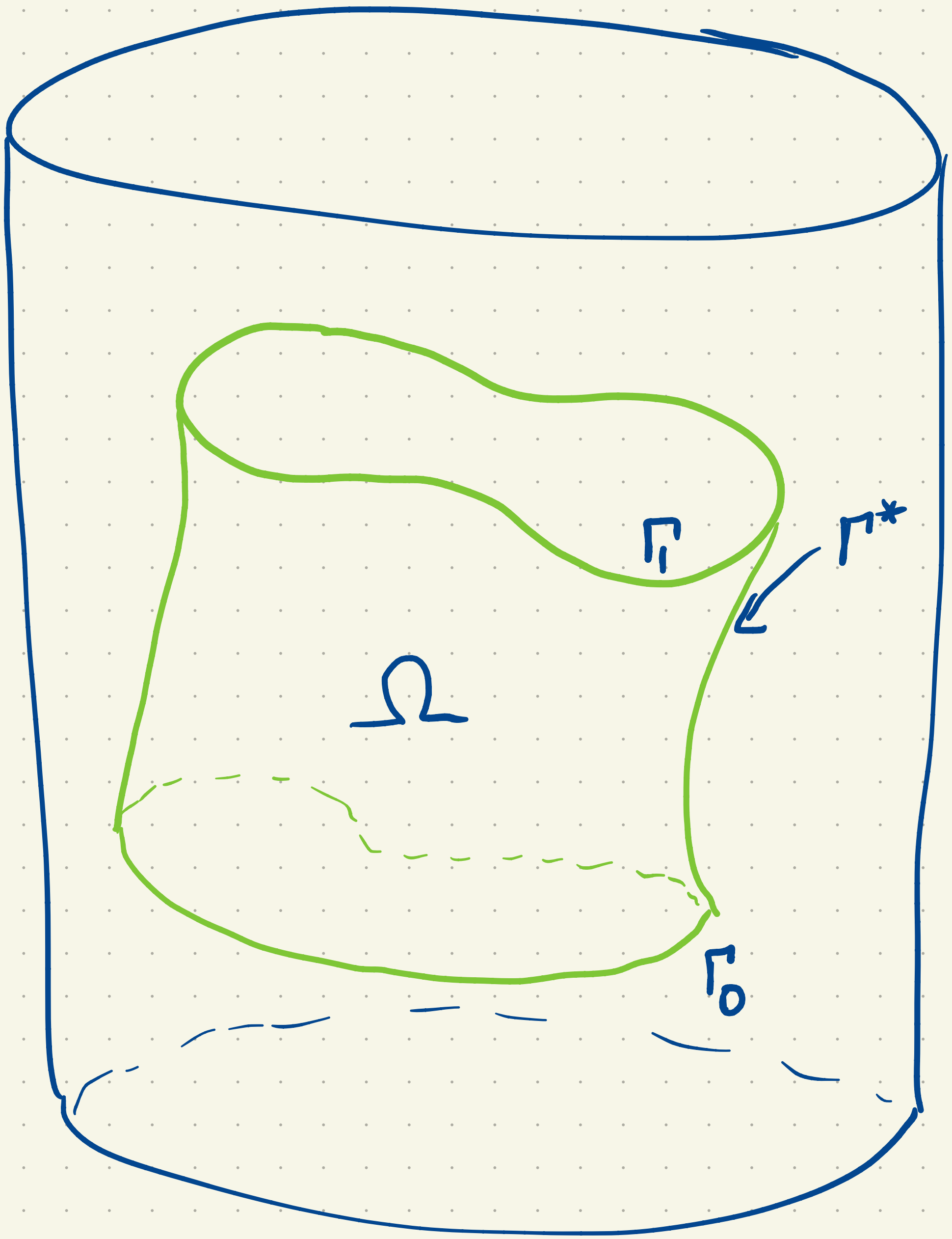
$\int_\Gamma \mathcal{J}$: flux of charge thru Γ

Charge Conservation



$\int_\Gamma \mathcal{J}$: flux of charge thru Γ

Charge Conservation



$\int_\Gamma \mathcal{J}$: flux of charge thru Γ

$$\int_\Omega d\mathcal{J} = 0$$

$$\int_{\Gamma_0} \mathcal{J} + \int_{\Gamma_1} \mathcal{J} + \int_{\Gamma^*} \mathcal{J} = 0$$

Charge Flux

SR, $c = 1$

$$\mathcal{J} = \rho \, dx^1 \wedge dx^2 \wedge dx^3$$
$$- j^1 \, dt \wedge dx^2 \wedge dx^3$$
$$- j^2 \, dt \wedge dx^3 \wedge dx^1$$
$$- j^3 \, dt \wedge dx^1 \wedge dx^2$$

Charge Flux

ρ : charge density

$\vec{j} = (j^1, j^2, j^3)$

: current density

SR, $c = 1$

$$\mathcal{J} = \rho \, dx^1 \wedge dx^2 \wedge dx^3$$

$$- j^1 \, dt \wedge dx^2 \wedge dx^3 \qquad \bar{\mathcal{J}}$$

$$- j^2 \, dt \wedge dx^3 \wedge dx^1$$

$$- j^3 \, dt \wedge dx^1 \wedge dx^2$$

$$d\mathcal{J} = (\dot{\rho} + \nabla \cdot \vec{j}) \, dt \wedge dx^1 \wedge dx^2 \wedge dx^3$$

On to Maxwell

$$-\int_\Omega \langle dA, d\delta A\rangle_g \, dV_g$$

$$\overbrace{\langle \underbrace{\mathcal{A}_1}_{\text{2-form}}, \underbrace{\mathcal{A}_2}_{\text{2-form}} \rangle_g \, dV_g}^{\text{4-form}} = \overbrace{\underbrace{*_g}_{\text{Hodge star}} \mathcal{A}_1}^{\text{2-form}} \wedge \overbrace{\mathcal{A}_2}^{\text{2-form}}$$

On to Maxwell

$$-\int_\Omega \langle dA, d\,\delta A\rangle_g \, dV_g$$

$$\overbrace{\left\langle \underset{\text{2-form}}{\underbrace{\mathcal{A}_1}}, \underset{\text{2-form}}{\underbrace{\mathcal{A}_2}} \right\rangle_g dV_g}^{\text{4-form}}$$

On to Maxwell

$$-\int_{\Omega} \langle dA, d\,\delta A \rangle_g \, dV_g$$

$$\overbrace{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle_g \, dV_g}^{\text{4-form}}$$

2-form

2-form

$$\mathcal{G} \wedge \mathcal{F}_2$$

On to Maxwell

$$-\int_{\Omega} \langle dA, d\delta A \rangle_g \, dV_g$$

$$\overbrace{\langle \underbrace{\mathcal{A}_1}_{\text{2-form}}, \underbrace{\mathcal{A}_2}_{\text{2-form}} \rangle_g \, dV_g}^{\text{4-form}} = \underbrace{*_g}_{\text{Hodge star}} \overbrace{\mathcal{A}_1}^{\text{2-form}} \wedge \overbrace{\mathcal{A}_2}^{\text{2-form}}$$

Hodge Star

SR, $c = 1$, $n = 3$

$$* \, dt \wedge dx^1 = -dx^2 \wedge dx^3$$

$$* \, dt \wedge dx^2 = -dx^3 \wedge dx^1$$

$$* \, dt \wedge dx^3 = -dx^1 \wedge dx^2$$

$$* \, dx^2 \wedge dx^3 = dt \wedge dx^1$$

$$* \, dx^3 \wedge dx^1 = dt \wedge dx^2$$

$$* \, dx^1 \wedge dx^2 = dt \wedge dx^3$$

On to Maxwell

$$-\int_{\Omega} \langle dA, d\,\delta A\rangle_g \, dV_g = -\int_{\Omega} *_g \, dA \wedge d(\delta A)$$

On to Maxwell

$$-\int_\Omega \langle dA, d\,\delta A\rangle_g \, dV_g = -\int_\Omega *_g\, dA \wedge d(\delta A)$$

$$= \int_\Omega d(*_g\, dA) \wedge \delta A$$

Maxwell's Equations

$$\delta S = 0 \qquad -\int_{\Omega} \star_g dA \wedge d(\delta A) + \int_{\Omega} J \wedge \delta A = 0$$

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$$\delta S = 0 \qquad -\int_{\Omega} \star_g dA \wedge d(\delta A) + \int_{\Omega} J \wedge \delta A = 0$$

$$d \star_g dA + J = 0$$

Maxwell's equations

Maxwell's Equations

$$\delta S = 0 \qquad -\int_{\Omega} *_g dA \wedge d(\delta A) + \int_{\Omega} J \wedge \delta A = 0$$

$$d *_g dA + J = 0$$

$$-d *_g dA = J$$

SR Maxwell's Equations

1) $dA = -E^i \, dx^i \wedge dt - B_{ij} \, dx^i \wedge dx^j$

2)

$* \, dt \wedge dx^1 = -dx^2 \wedge dx^3$	$* \, dx^2 \wedge dx^3 = dt \wedge dx^1$
$* \, dt \wedge dx^2 = -dx^3 \wedge dx^1$	$* \, dx^3 \wedge dx^1 = dt \wedge dx^2$
$* \, dt \wedge dx^3 = -dx^1 \wedge dx^2$	$* \, dx^1 \wedge dx^2 = dt \wedge dx^3$

3) Compute $-d * dA$

4) $J = \rho \, dx^1 \wedge dx^2 \wedge dx^3 - j^1 \, dt \wedge dx^2 \wedge dx^3 - \cdots$

$- j^3 \, dt \wedge dx^1 \wedge dx^2$

SR Maxwell's Equations

1) $dA = -E^i \, dx^i \wedge dt - B_{ij} \, dx^i \wedge dx^j$

2)

$* \, dt \wedge dx^1 = -dx^2 \wedge dx^3$	$* \, dx^2 \wedge dx^3 = dt \wedge dx^1$
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SR Maxwell's Equations

1) $dA = -E^i \, dx^i \wedge dt - B_{ij} \, dx^i \wedge dx^j$

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SR Maxwell's Equations

1) $dA = -E^i \, dx^i \wedge dt - B_{ij} \, dx^i \wedge dx^j$

2) $* \, dt \wedge dx^1 = -dx^2 \wedge dx^3$ | $* \, dx^2 \wedge dx^3 = dt \wedge dx^1$

$* \, dt \wedge dx^2 = -dx^3 \wedge dx^1$ | $* \, dx^3 \wedge dx^1 = dt \wedge dx^2$

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$- j^3 \, dt \wedge dx^1 \wedge dx^2$

SR Maxwell's Equations

1) $dA = -E^i dx^i \wedge dt - B_{ij}\, dx^i \wedge dx^j$

$$\vec{E} = (E^1, E^2, E^3)$$

$$\vec{B} = (B_{23}, B_{31}, B_{12})$$

$$\vec{\nabla} \cdot \vec{E} = \rho$$

$$\dot{\vec{E}} = -\vec{j} + \vec{\nabla} \times \vec{B}$$

Geometric Interpretation

$$-d\,\overbrace{\star_g\, dA}^{2} = \overbrace{J}^{3}$$

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$$-d \overbrace{*_g dA}^{2} = \overbrace{J}^{3}$$

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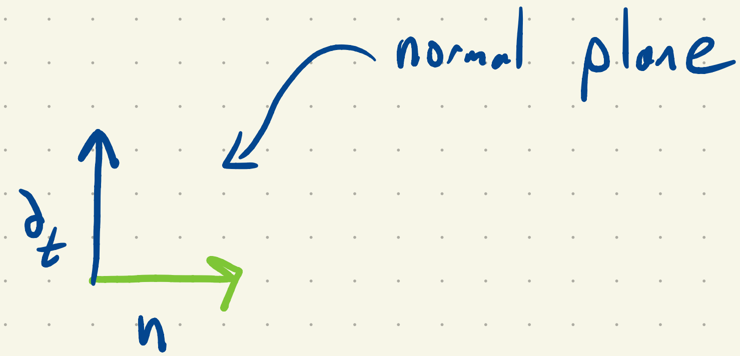
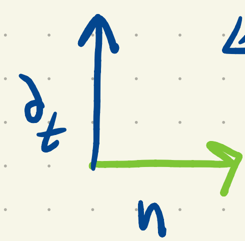
$$-d\,\overbrace{*_g\, dA}^{2} = \overbrace{J}^{3}$$



Geometric Interpretation

$$- d \overbrace{\star_g \, dA}^{2} = \overbrace{J}^{3}$$



normal plane

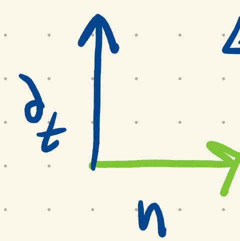
curvature density

$\vec{E} \cdot \vec{n}$

Geometric Interpretation

$$-\overbrace{d *_g dA}^{2} = \overbrace{J}^{3}$$

normal plane



curvature density

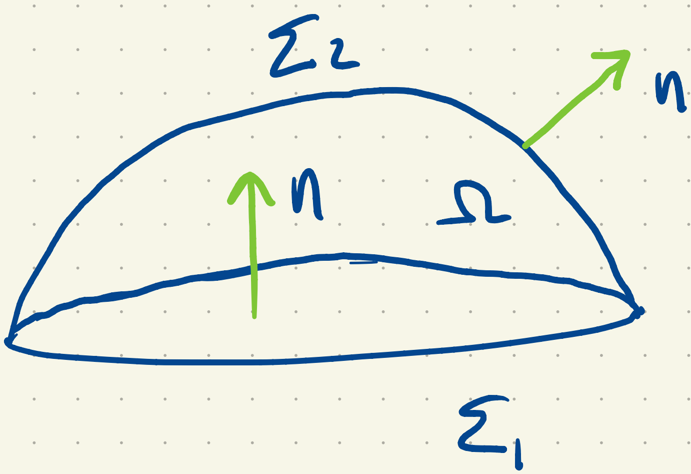
$\vec{E} \cdot \vec{n}$

$$-\int_{\Sigma} *_g dA = \text{total normal curvature}$$

$= \text{electric flux}$

Geometric Interpretation

$$-d \overbrace{*_g dA}^{2} = \overbrace{J}^{3}$$

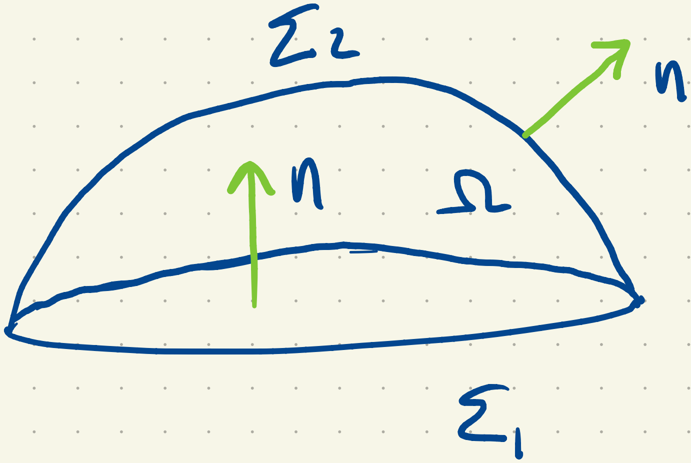


$$\int_{\Sigma_2} *_g dA - \int_{\Sigma_1} *_g dA = \int_{\Omega} J$$

choose passing thru Ω

Geometric Interpretation

$$-d\,\overbrace{\star_g dA}^{2} = \overbrace{J}^{3}$$



$$\int_{\Sigma_2} \star_g dA - \int_{\Sigma_1} \star_g dA = \int_{\Omega} J$$

charge passing thru Ω

"As charge passes through a surface it alters normal curvature"