$u_t + a u_x = 0$

Lax Wendroff

$$u_{i,j+1} = A u_{i+1,j} + B u_{i,j} + C u_{i-1,j}$$

A, B, C to be chosen

$$\left(A = 0,\ B = (1-\lambda),\ C = \lambda \right. \Rightarrow \text{upwinding}$$

$$\lambda = \frac{ka}{h} \qquad u_t + a u_x = 0$$

$$(x_i, t_j)$$

$$u(x_i, t_j + k) = u + u_t k + u_{tt} \frac{k^2}{2} + u_{ttt} \frac{k^3}{6} + O(k^4)$$

$$u(x_i, t_j+k) = u + u_t k + u_{tt}\frac{k^2}{2} + u_{ttt}\frac{k^3}{6} + O(k^4)$$

$$u_t + a u_x = 0$$

$$u_t = -a u_x$$

$$u_{tt} = \partial_t(-a u_x) = -a\partial_x(\partial_t u) = a^2 \partial_x^2 u$$

$$u_{ttt} = -a^3 \partial_x^3 u$$

$$u(x_i, t_j+k) = u - ak\,u_x + \frac{k^2}{2} a^2 u_{xx} - \frac{a^3 k^3}{6} u_{xxx} + O(k^4)$$

$$u(x_i, t_j+k) = u + u_t k + u_{tt}\frac{k^2}{2} + u_{ttt}\frac{k^3}{6} + O(k^4)$$

$$u_t = -a u_x$$

$$u(x_i, t_j+k) = u + u_t k + u_{tt} \frac{k^2}{2} + u_{ttt} \frac{k^3}{6} + O(k^4)$$

$$u_t = -a u_x$$

$$u_{tt} = \partial_t(-a u_x) = -a\partial_x(\partial_t u) = a^2 \partial_x^2 u$$

$$u(x_i, t_j+k) = u + u_t k + u_{tt} \frac{k^2}{2} + u_{ttt} \frac{k^3}{6} + O(k^4)$$

$$u_t = -a u_x$$

$$u_{tt} = \partial_t(-a u_x) = -a \partial_x(\partial_t u) = a^2 \partial_x^2 u$$

$$u_{ttt} = -a^3 \partial_x^3 u$$

$$u(x_i, t_j+k) = u + u_t k + u_{tt}\frac{k^2}{2} + u_{ttt}\frac{k^3}{6} + O(k^4)$$

$$u_t = -a u_x$$

$$u_{tt} = \partial_t(-a u_x) = -a\partial_x(\partial_t u) = a^2\partial_x^2 u$$

$$u_{ttt} = -a^3\partial_x^3 u$$

$$u(x_i, t_j+k) = u - a k u_x + \frac{k^2}{2}a^2 u_{xx} - \frac{a^3k^3}{6}u_{xxx} + O(k^4)$$

$u_{i+1,j} \qquad u_{i,j} \qquad u_{i-1,j}$

$$u(x_i, t_j + k) = u - aku_x + \frac{k^2}{2} a^2 u_{xx} - \frac{a^3 k^3}{6} u_{xxx} + O(k^4)$$

$$u(x_i \pm h, t_j) = u \pm u_x h + \frac{u_{xx} h^2}{2} \pm u_{xxx} \frac{h^3}{6} + O(h^4)$$

$$\tau = \frac{1}{k} \left[u(x_i, t_{j+1}) - A u(x_{i+1}, t_j) - B u(x_i, t_j) - C u(x_{i-1}, t_j) \right]$$

$$u(x_i, t_j + k) = u - a k u_x + \frac{k^2}{2} a^2 u_{xx} - \frac{a^3 k^3}{6} u_{xxx} + O(k^4)$$

$$u(x_i \pm h, t_j) = u \pm u_x h + u_{xx} \frac{h^2}{2} \pm u_{xxx} \frac{h^3}{6} + O(h^4)$$

$$\tau = \frac{1}{k} \left[u(x_i, t_{j+1}) - A u(x_{i+1}, t_j) - B u(x_i, t_j) - C u(x_{i-1}, t_j) \right]$$

$$\frac{1}{k} \left[\underset{\substack{\downarrow \\ 1-(A+B+C)}}{c_0} u + \underset{\downarrow}{c_1} u_x + c_2 u_{xx} + c_3 u_{xxx} + O(k^4) + O(h^4) \right]$$

$c_1: -ak - Ah + Ch$

$\frac{c_0}{k}: \frac{1-(A+B+C)}{k}$

$\frac{c_1}{k} = \frac{h}{k}\left[-\lambda - A + C\right]$

LTE:

$$\frac{1}{k}\left[\, c_0 u + c_1 u_x + c_2 u_{xx} + c_3 u_{xxx} + O(k^4) + O(h^4) \right.$$

$$\frac{c_0}{k} : \frac{1}{k}\left[1 - (A + B + C)\right]$$

$$\frac{c_1}{k} : -\frac{h}{k}\left[-\frac{ak}{h} - A + C\right] = -\frac{h}{k}\left[-\lambda - A + C\right]$$

$$\frac{c_2}{k} : \frac{h^2}{2k}\left[\lambda^2 - A - C\right]$$

$$\frac{c_3}{k} : -\frac{h^3}{6k}\left[\lambda^3 + A - C\right]$$

Consistency:

$$A + B + C = 1$$

$$-A + C = \lambda$$

Consistency:

$$A + B + C = 1$$
$$-A + C = \lambda$$

$$\left[\begin{array}{l} A = 0 \Rightarrow \\ C = \lambda, \quad B = 1 - \lambda \end{array}\right]$$

Consistency:

$$A + B + C = 1$$
$$-A + C = \lambda$$

$$\left[\begin{array}{l} A = 0 \Rightarrow \\ C = \lambda, \quad B = 1 - \lambda \end{array} \right]$$

Extra decay

$$A + C = \lambda^2$$

$$C = \frac{\lambda(1+\lambda)}{2} \qquad A = C - \lambda = \frac{\lambda}{2}(-1+\lambda)$$

$$B = 1 - \lambda^2$$

$$\tau = \frac{-h^3}{6k}\left[\lambda^3 + A - C\right] u_{xxx} + \cdots$$

$$\tau = -\frac{h^3}{6k}\left[\lambda^3 + A - C\right] u_{xxx} + \cdots$$

$$= -\frac{h^3}{6k}\left[\lambda^3 - \lambda\right] u_{xxx}$$

$$= -\frac{h^3}{6k}\left[\left(\frac{ak}{h}\right)^3 - \frac{ak}{h}\right] u_{xxx} + \cdots$$

$$\tau = -\frac{h^3}{6k}\left[\lambda^3 + A - C\right] u_{xxx} + \cdots$$

$$= -\frac{h^3}{6k}\left[\lambda^3 - \lambda\right] u_{xxx}$$

$$= -\frac{h^3}{6k}\left[\left(\frac{ak}{h}\right)^3 - \frac{ak}{h}\right] u_{xxx} + \cdots$$

$$= -\frac{1}{6}\left[a^3 k^2 - a h^2\right] u_{xxx} + \cdots = O(k^2) + O(h^2)$$

$$\tau = -\frac{h^3}{6k}\left[\lambda^3 + A - C\right] u_{xxx} + \cdots$$

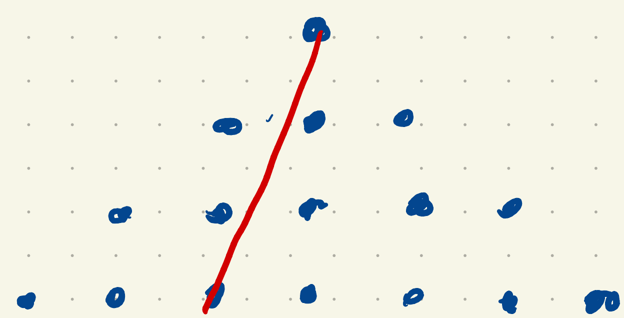
$$= -\frac{h^3}{6k}\left[\lambda^3 - \lambda\right] u_{xxx}$$

$$= -\frac{h^3}{6k}\left[\left(\frac{ak}{h}\right)^3 - \frac{ak}{h}\right] u_{xxx} + \cdots$$

$$= -\frac{1}{6}\left[a^3k^2 - ah^2\right] u_{xxx} + \cdots = O(k^2) + O(h^2)$$

[Compare: $O(k) + O(h)$ for upwinding]

Domain of Dependence (CFL)

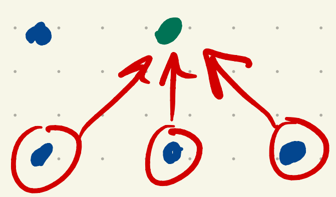


$$|a| \le \frac{h}{k}$$

$$|\lambda| \le 1 \qquad \lambda = \frac{ak}{h}$$

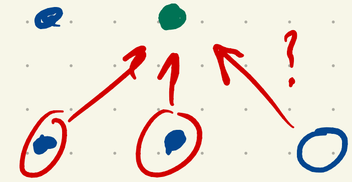
But: waves can travel in wrong direction!

Boundary conditions $(a > 0)$



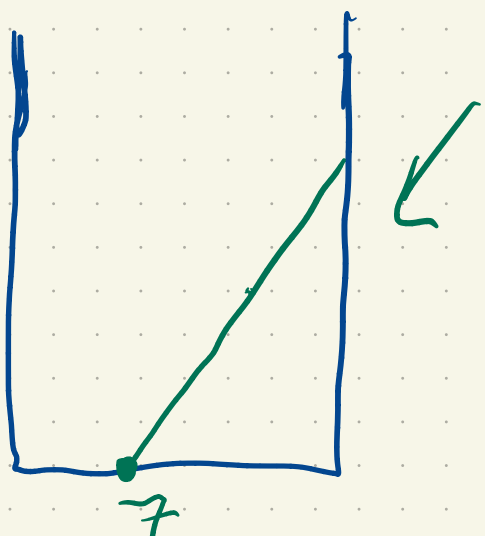
$x_0 \quad x_1 \quad x_2$

$$u_t + a u_x = 0$$



x_N

Use upwinding here



$$u_t + a u_x = 0$$

Stability?

(Without stability, high order LTE is useless)

Stability?

(Without stability, high order LTE is useless)

Von Neumann

$$K^M$$

$$u_{i,j} = K^j e^{rihI}$$

$$u_{i,j+1} = A u_{i+1,j} + B u_{i,j} + C u_{i-1,j}$$

$$K^{j+1} e^{rihI} = K^j e^{rihI} \left[A e^{irhI} + B + C e^{-irhI} \right]$$

$$K^{j+1} e^{r i h I} = K^{j} e^{r i h I} \left[A e^{i r h I} + B + C e^{-i r h I} \right]$$

$$K^{j+1} e^{rihI} = K^{j} e^{rihI} \left[A e^{irhI} + B + C e^{-irhI} \right]$$

$$K = (1-\lambda^2) + \left(\frac{\lambda^2 - \lambda}{2}\right) e^{rIh} + \left(\frac{\lambda^2 + \lambda}{2}\right) e^{-rIh}$$

$$= 1 - \lambda^2 + \lambda^2 \cos(rh) - \lambda \sin(rh) I$$

$$= 1 + \lambda^2 \left[\cos(rh) - 1\right] - \lambda I \sin(rh)$$

$$K^{j+1} e^{rihI} = K^{j} e^{rihI}\left[A e^{irhI} + B + C e^{-irhI}\right]$$

$$K = (1-\lambda^2) + \left(\frac{\lambda^2-\lambda}{2}\right) e^{rIh} + \left(\frac{\lambda^2+\lambda}{2}\right) e^{-rIh}$$

$$= 1-\lambda^2 + \lambda^2 \cos(rh) - \lambda \sin(rh) I$$

$$= 1 + \lambda^2 \left[\cos(rh) - 1\right] - \lambda I \sin(rh)$$

Stability: $|K| \leq 1$ [if $K > 1$ then growth K^{n} by n'th timestep]

$$K = 1 + \lambda^2 [\cos(rh) - 1] - \lambda I \sin(rh)$$

$$|K|^2 = 1 - 4\lambda^2(1-\lambda^2) \sin^4\left(\frac{rh}{2}\right)$$

$a < 0$

$a > 0$

$$-1 \leq \lambda \leq 1$$

$$K = 1 + \lambda^2 [\cos(rh) - 1] - \lambda I \sin(rh)$$

$$|K|^2 = 1 - 4\lambda^2(1-\lambda^2)\sin^4\left(\frac{rh}{2}\right)$$

$\Rightarrow$ stable if $-1 \leq \lambda \leq 1$

(well resolved modes: h small compared to $\frac{1}{r}$)

Last class:

Upwinding: $$|K| \approx 1 - 2\lambda(1-\lambda^2)\sin^2\left(\frac{\theta}{2}\right)$$

$$\theta = \frac{2\pi}{L} n h$$

Last class:

Upwinding: $$|K| \approx 1 - 2\lambda(1-\lambda^2)\sin^2\left(\frac{\theta}{2}\right)$$

$$\theta = \frac{2\pi}{L} n h$$

L-W

$$|K|^2 = 1 - 4\lambda^2(1-\lambda^2)\sin^4\left(\frac{\nu h}{2}\right)$$

$$|K| \simeq 1 - 2\lambda^2(1-\lambda^2)\sin^4\left(\frac{\nu h}{2}\right) \qquad \left(\frac{\nu h}{2} \text{ small}\right)$$

Last class:

Upwinding: $$|K| \approx 1 - 2\lambda(1-\lambda^2)\overbrace{\boxed{\sin^2\left(\frac{\theta}{2}\right)}}^{O(h^2)}$$

$$\theta = \frac{2\pi}{L} n h$$

L-W

$$|K|^2 = 1 - 4\lambda^2(1-\lambda^2)\sin^4\left(\frac{\eta h}{2}\right)$$

$$|K| \approx 1 - 2\lambda^2(1-\lambda^2)\underbrace{\boxed{\sin^4\left(\frac{\eta h}{2}\right)}}_{O(h^4)} \qquad \left(\frac{\eta h}{2} \text{ small}\right)$$

DEMO