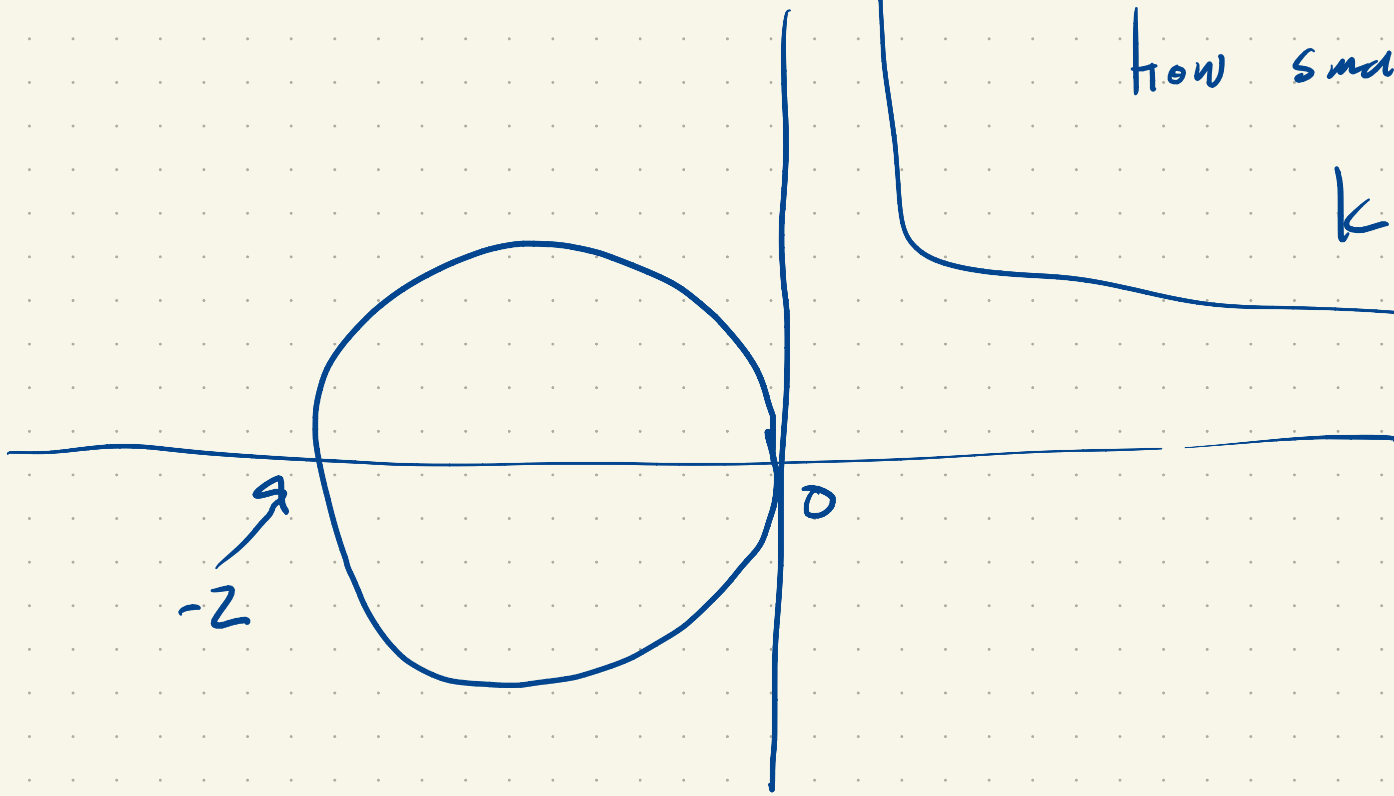
Region of abs stability:

$-\pi^2, -4\pi^2, \ldots, -N^2\pi^2$

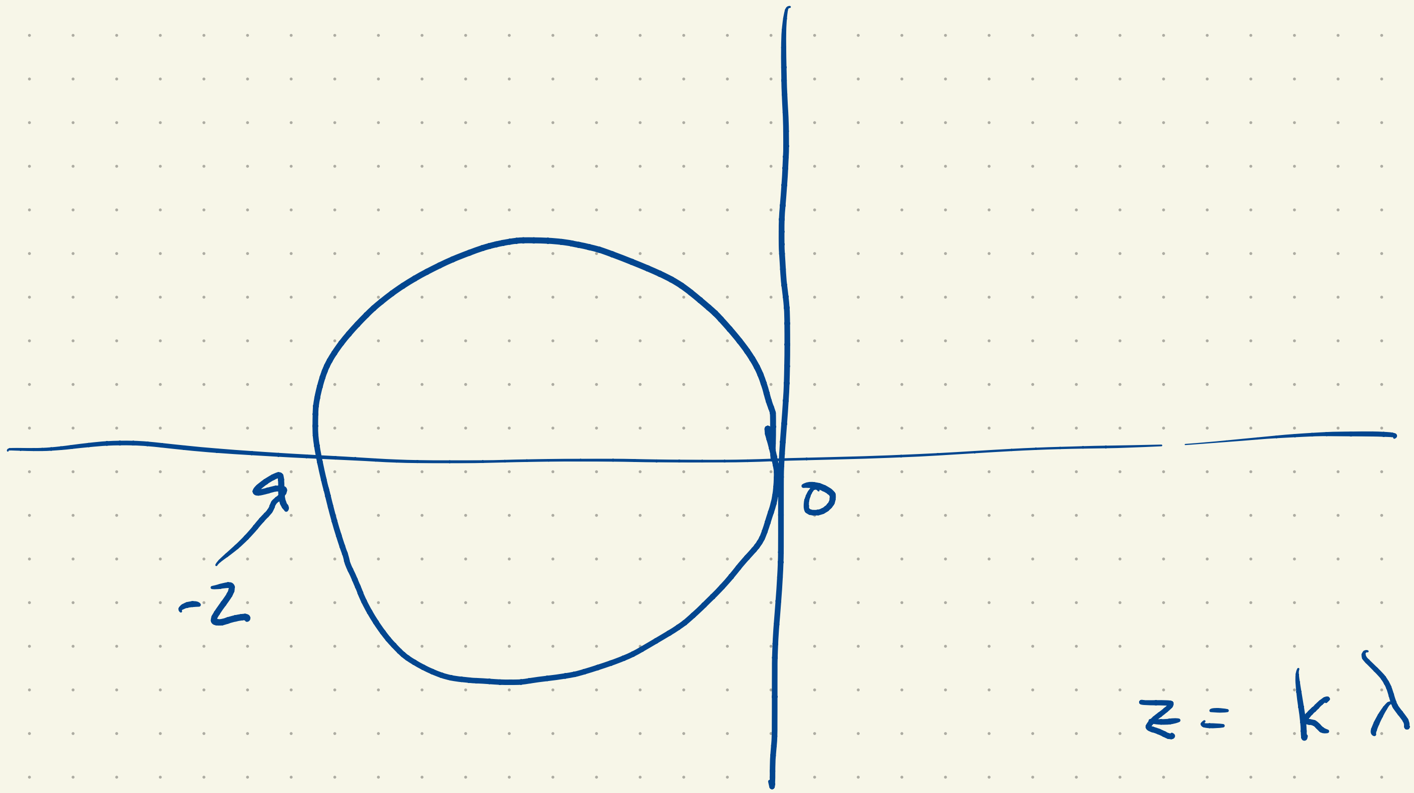
How small should k be?



$z = h\lambda$

$k\eta$

Region of abs stability:



$z = k\lambda > -2$

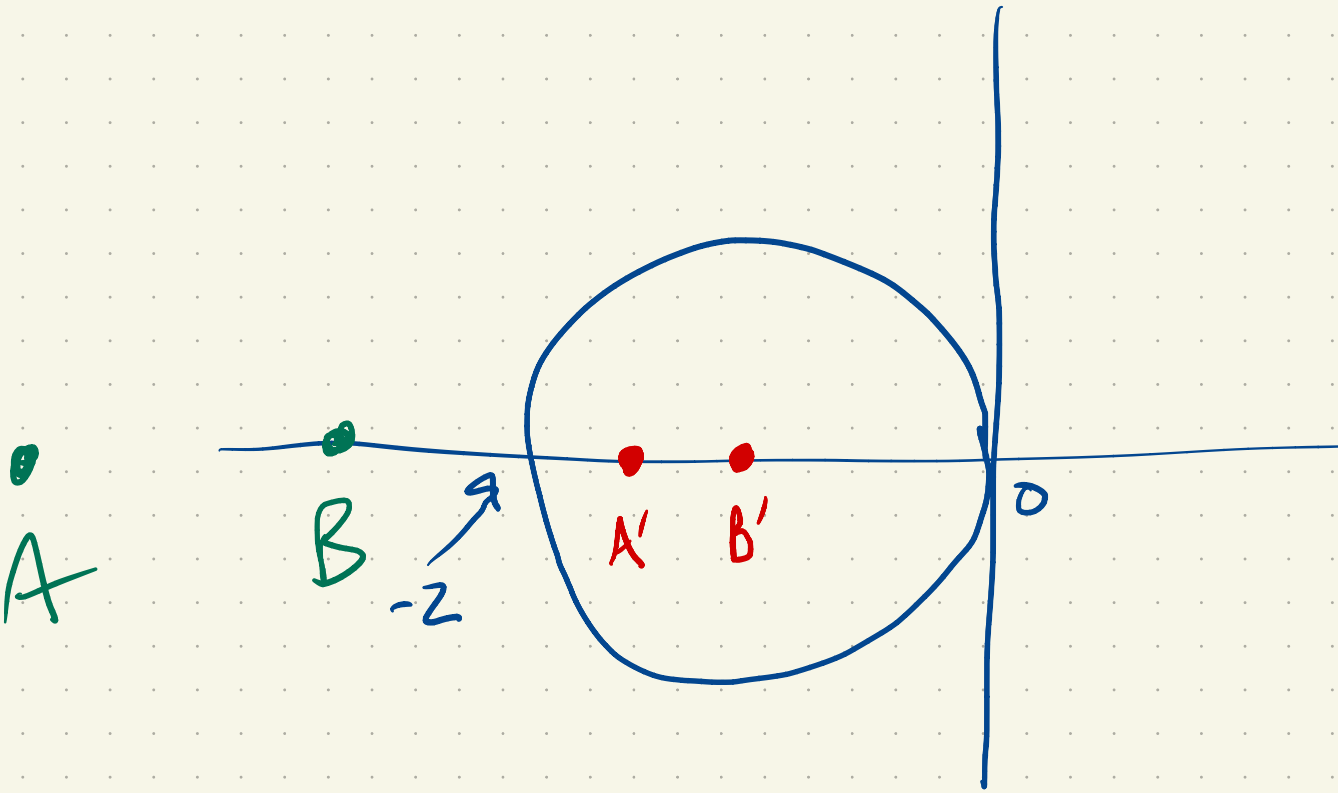
Region of abs stability:

$-\pi^2, -4\pi^2, -9\pi^2, \ldots, -N^2\pi^2$

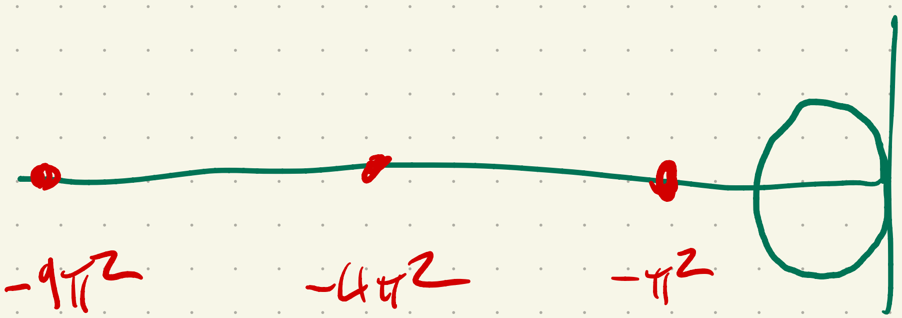
$-\pi^2 j^2$

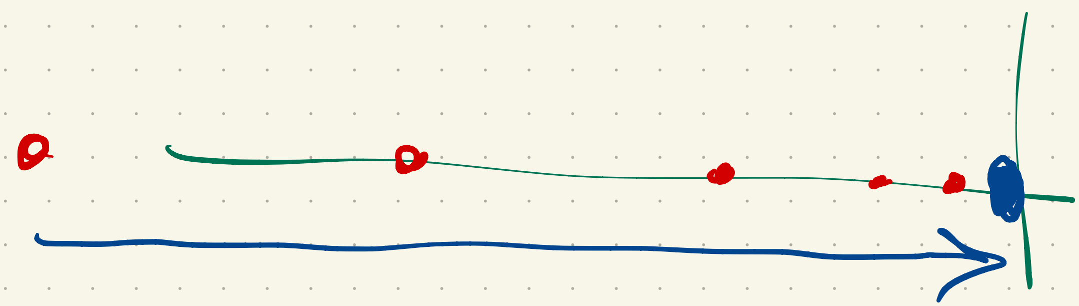
$\downarrow$

$j = 1, 2, \ldots, N$

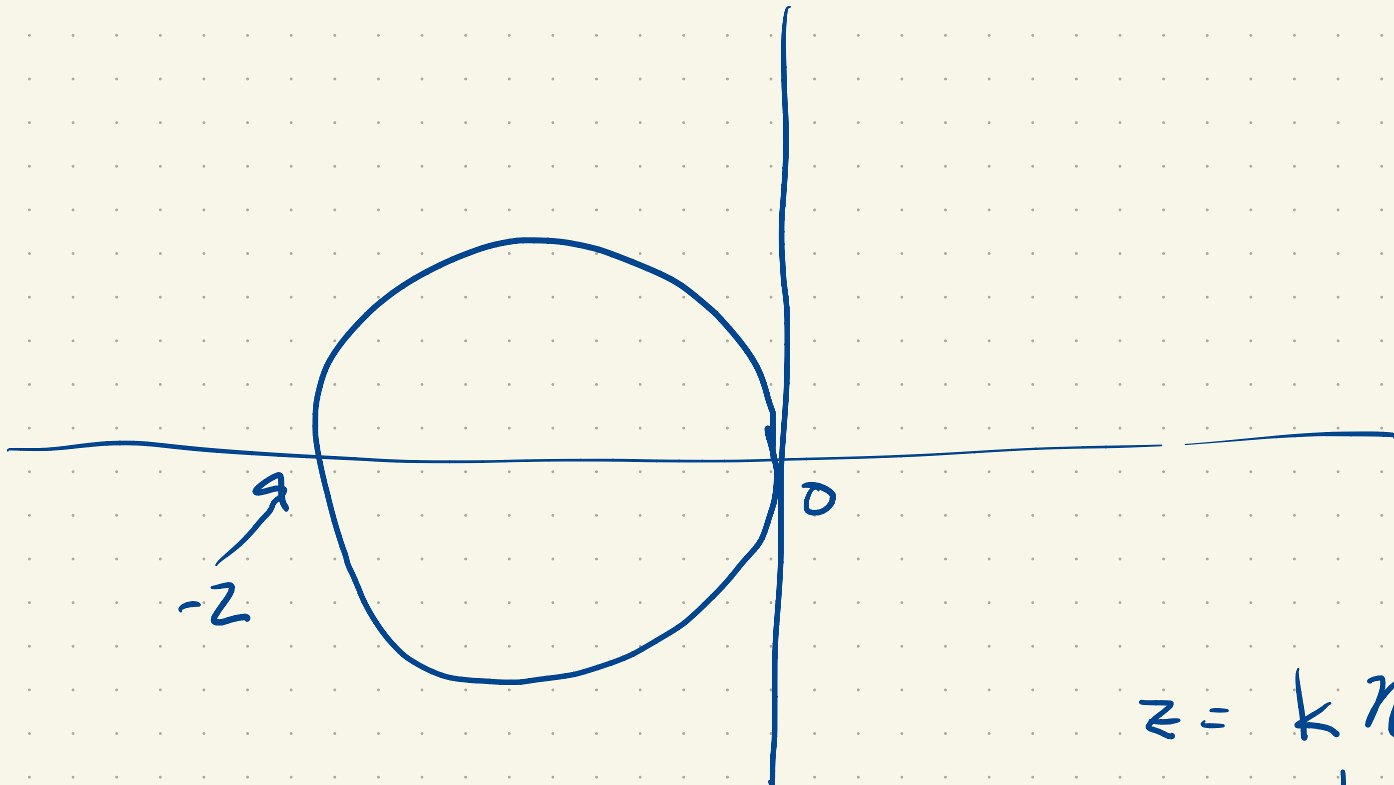


$z = k\,\overset{n}{\cancel{\lambda}} > -2$





Region of abs stability:



$$z = k\lambda > -2$$

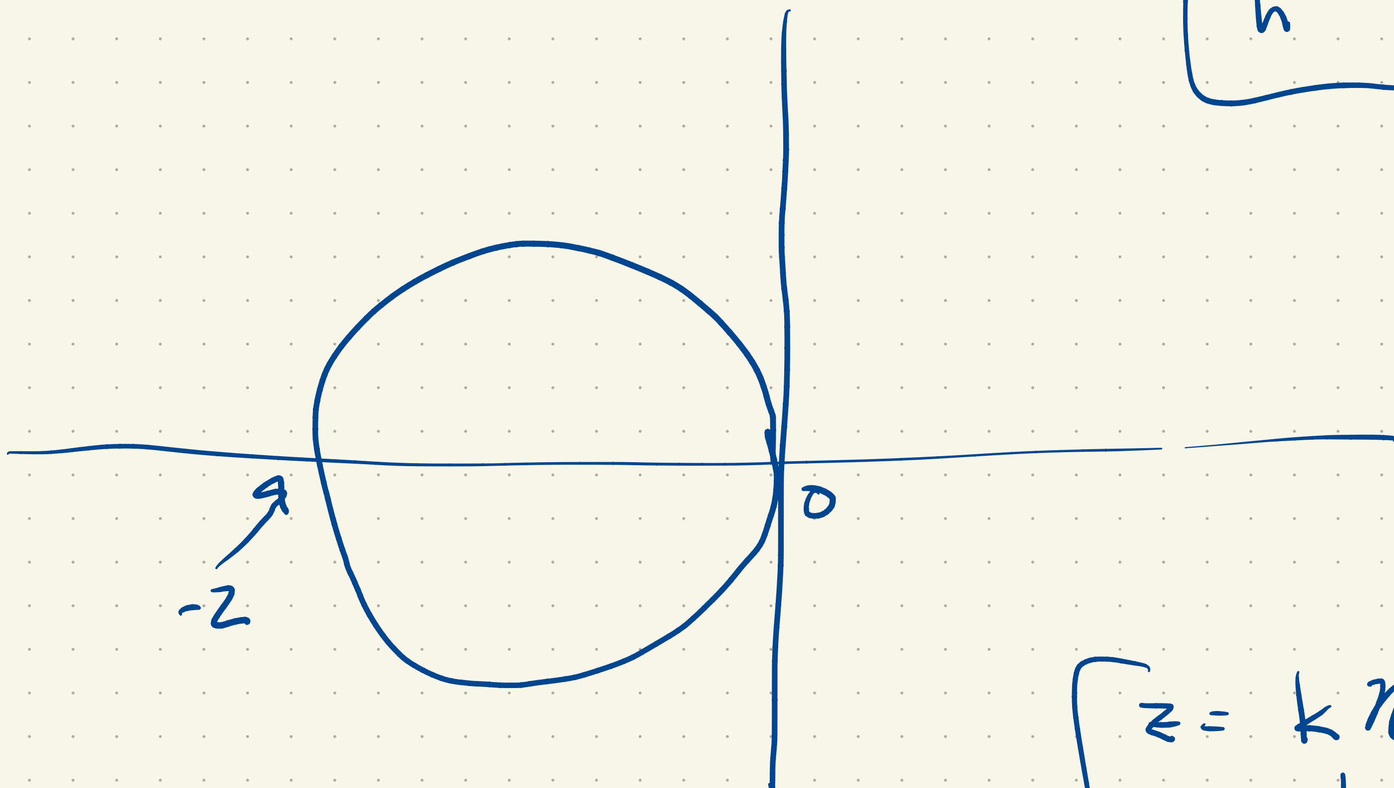
$$\downarrow$$

$$k(-\pi^2 N^2) > -2$$

$$\frac{k}{h^2} < \frac{2}{\pi^2}$$

Region of abs stability:

$$\boxed{\frac{k}{h^2} = \frac{1}{2}}$$



heat wave

$$z = k\lambda > -2$$

$$\downarrow$$

$$k(-\pi N^2) > -2$$

$$\frac{k}{h^2} < \frac{2}{\pi^2}$$

Eigenvalues of $\frac{1}{h^2}D$.

$\cancel{e^{Jrx_j}}$

We're gonna guess them!

$\vec{w} \rightarrow w_j = e^{Jrx_j}$ $\quad J^2 = -1 \quad$ r TBA

Eigenvalues of $\frac{1}{h^2}D$. $\quad 0 \quad 1 \quad -2 \quad 1 \quad 0$

We're gonna guess them!

$$\vec{w}_j = e^{J r x_j} \qquad J^2 = -1 \qquad r \text{ TBA}$$

$$(D\vec{w})_j = e^{J r x_{j-1}} - 2e^{J r x_j} + e^{J r x_{j+1}}$$

$$x_{j-1} = x_j - h$$

$$x_{j+1} = x_j + h$$

$$e^{J r (x_j - h)} = e^{J r x_j} \cdot e^{-J r h}$$

Eigenvalues of $\frac{1}{h^2}D$.

We're gonna guess them!

$$\vec{w}_j = e^{J r x_j} \qquad J^2 = -1 \qquad r \text{ TBA}$$

$$(D\vec{w})_j = e^{J r x_{j-1}} - 2e^{J r x_j} + e^{J r x_{j+1}} \qquad [j \neq 1, N]$$

Eigenvalues of $\frac{1}{h^2} D$.

We're gonna guess them!

$$\vec{w}_j = e^{J r x_j} \qquad J^2 = -1 \qquad r \text{ TBA}$$

$$x_{N+1} = 1$$

$$(D\vec{w})_j = e^{J r x_{j-1}} - 2 e^{J r x_j} + e^{J r x_{j+1}}$$

$$= e^{J r x_j}\left[e^{-J r h} - 2 + e^{J r \cdot h}\right]$$

$$e^{J\theta} + e^{-J\theta} = 2\cos\theta$$

$$e^{J\theta} = \cos\theta + J\sin\theta$$

Eigenvalues of $\frac{1}{h^2}D$.

We're gonna guess them!

$$\vec{w}_j = e^{\mathcal{J} r x_j} \qquad \mathcal{J}^2 = -1 \qquad r \text{ TBA}$$

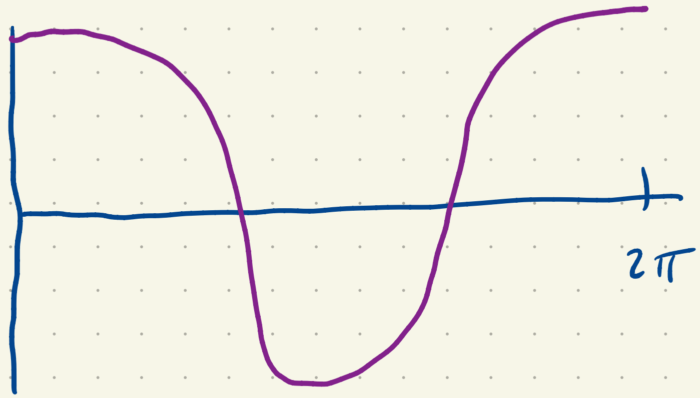
$$\left(D\vec{w}\right)_j = e^{\mathcal{J} r x_{j-1}} - 2e^{\mathcal{J} r x_j} + e^{\mathcal{J} r x_{j+1}}$$

$$= e^{\mathcal{J} r x_j}\left[e^{-\mathcal{J} r h} - 2 + e^{\mathcal{J} r h}\right]$$

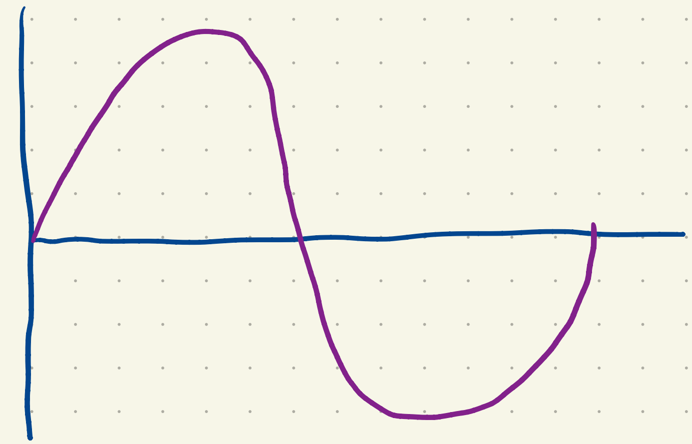
$$= 2e^{\mathcal{J} r x_j}\left[\cos(rh) - 1\right]$$

$$= -4\underbrace{e^{\mathcal{J} r x_j}}_{w_j}\sin^2\left(\frac{rh}{2}\right)$$

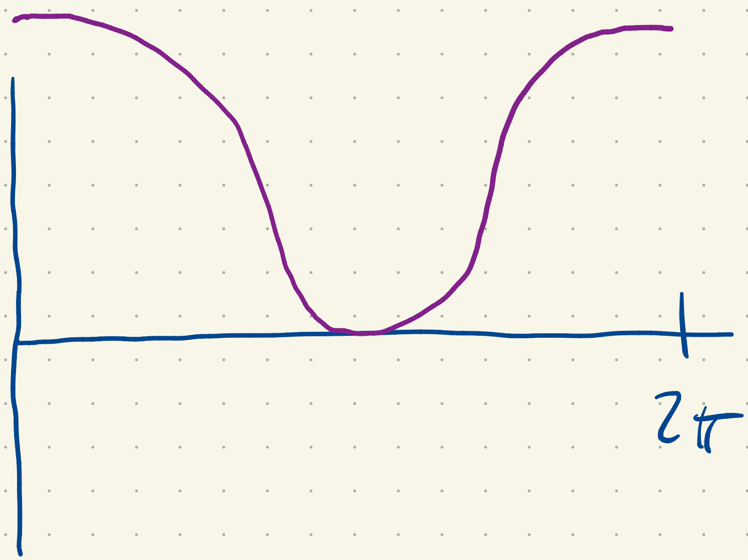
cos(x)

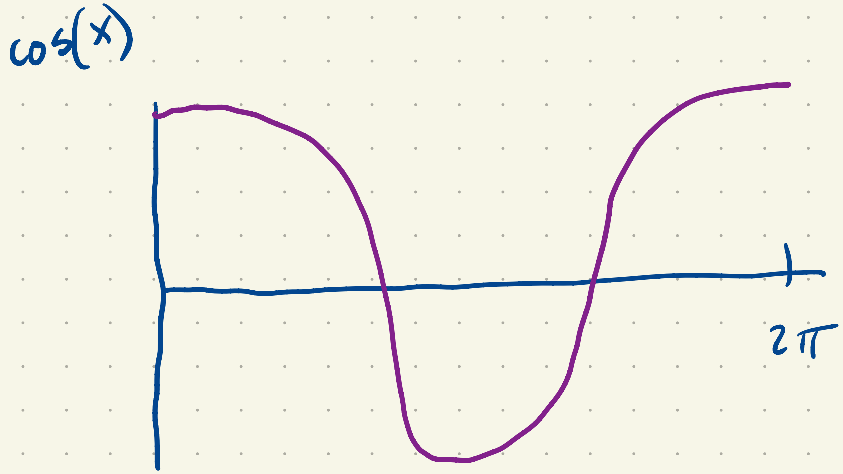


sin(x)

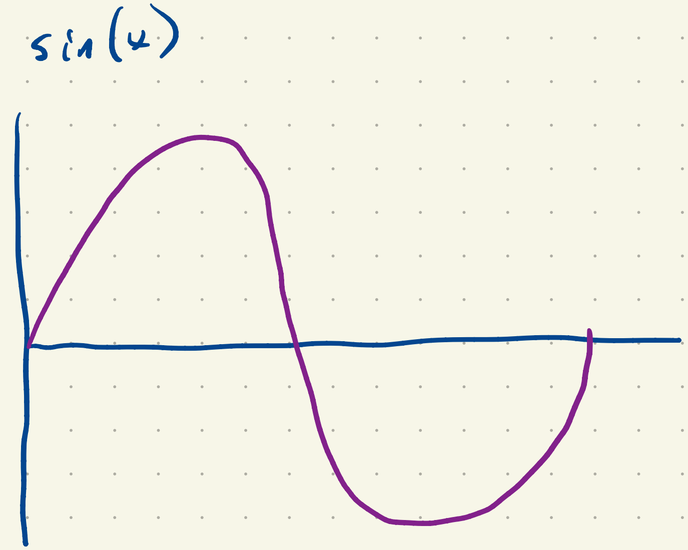


1+cos(x)

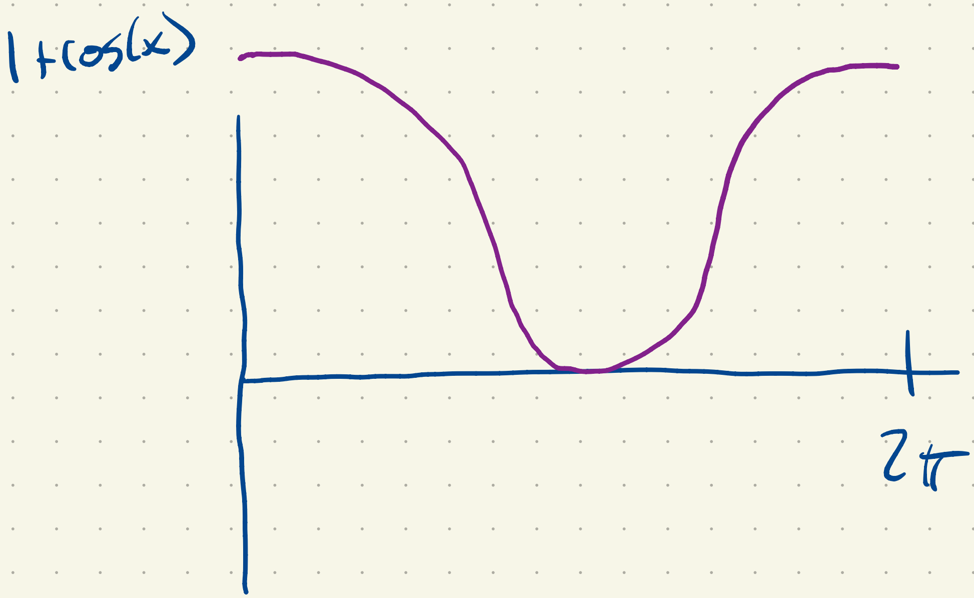




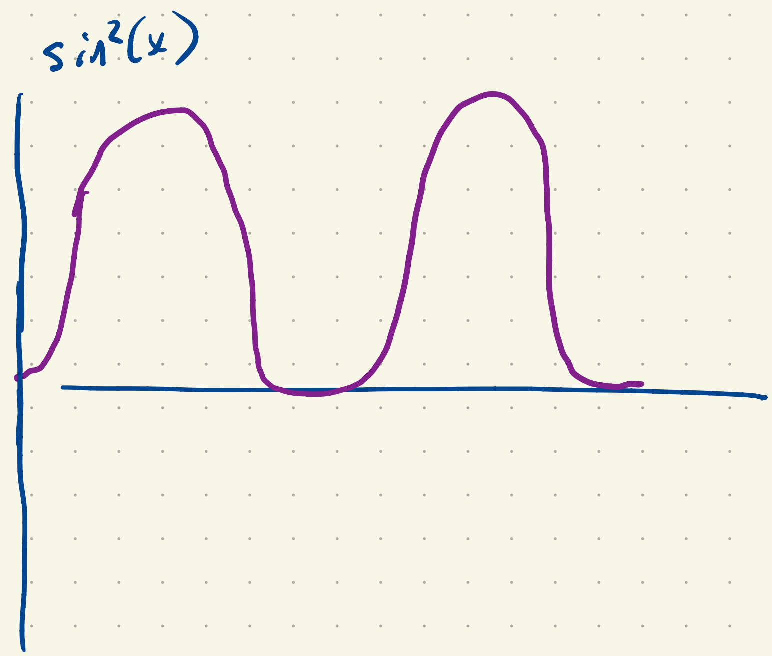
cos(x)
2π



sin(x)

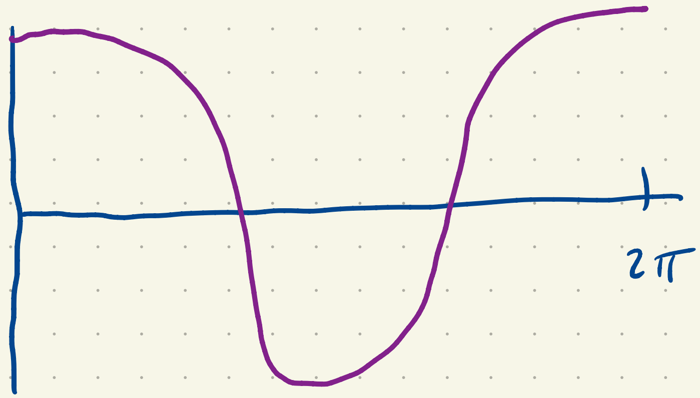


1+cos(x)
2π

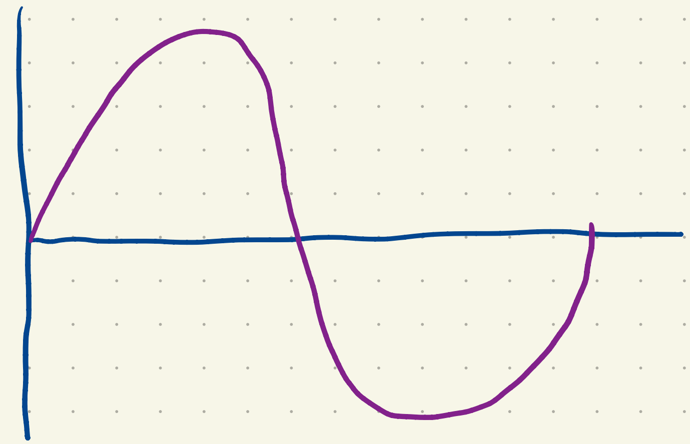


sin²(x)

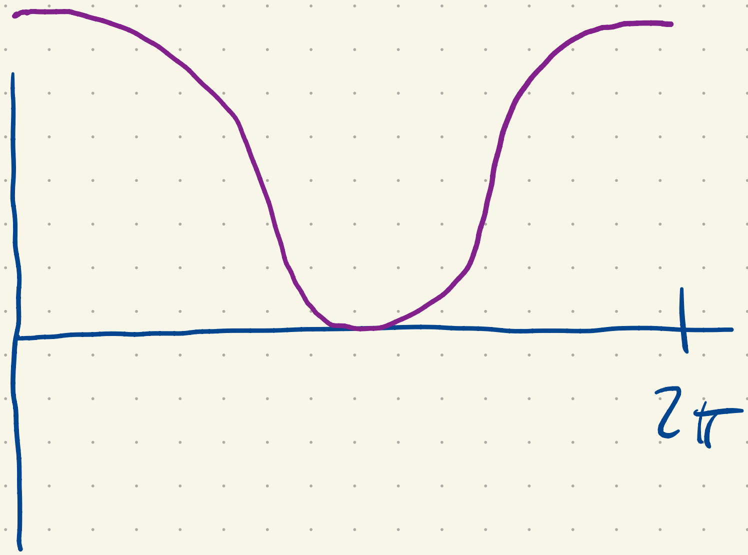
$\cos(x)$



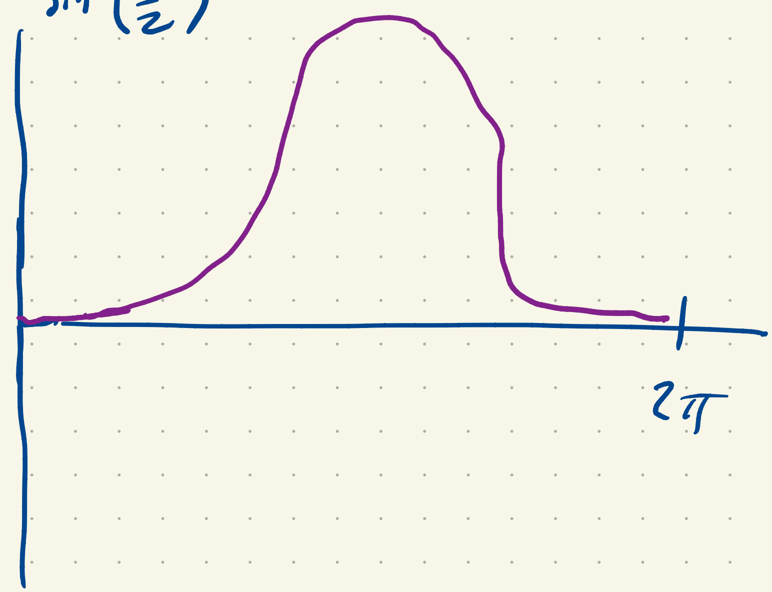
$\sin(x)$

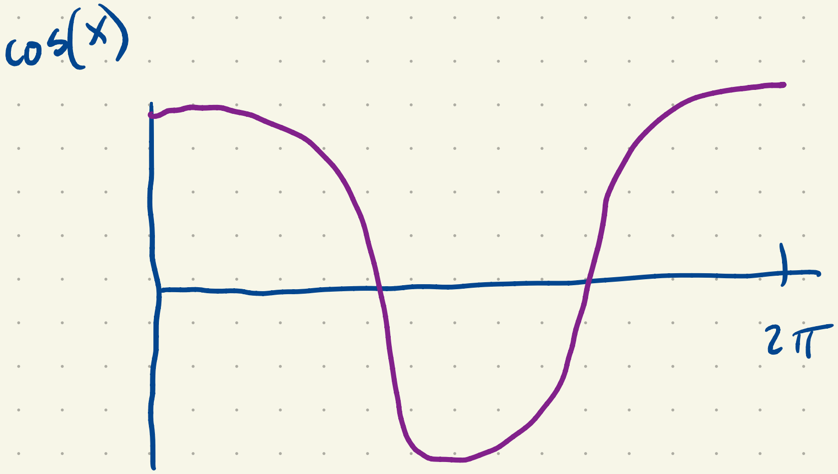


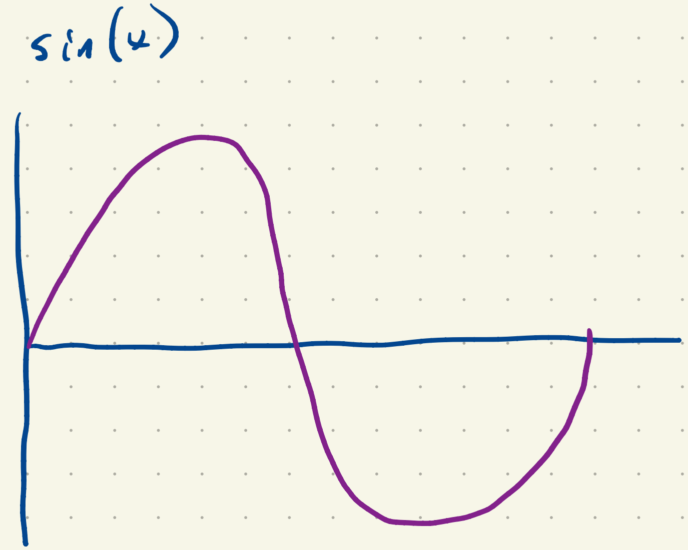
$1+\cos(x)$

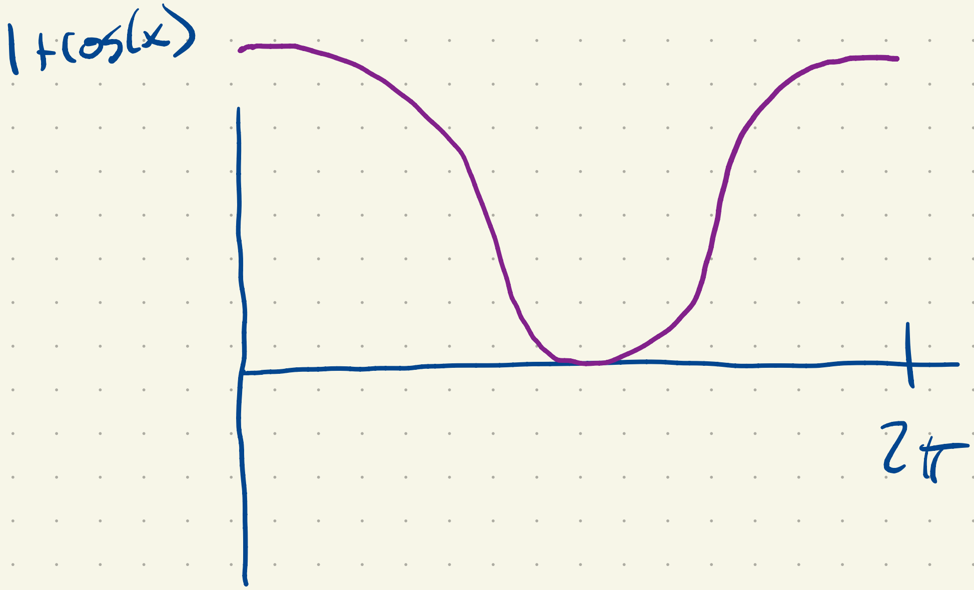


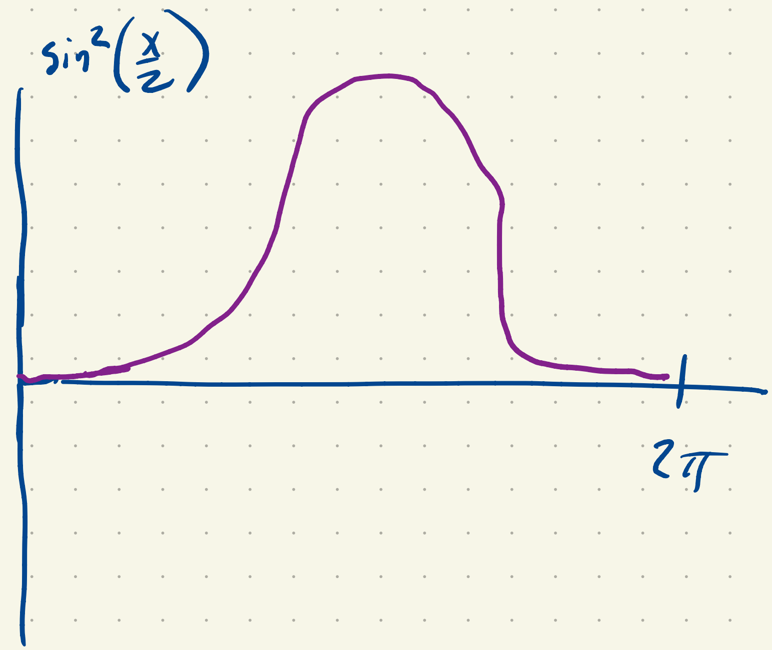
$\sin^2\left(\frac{x}{2}\right)$



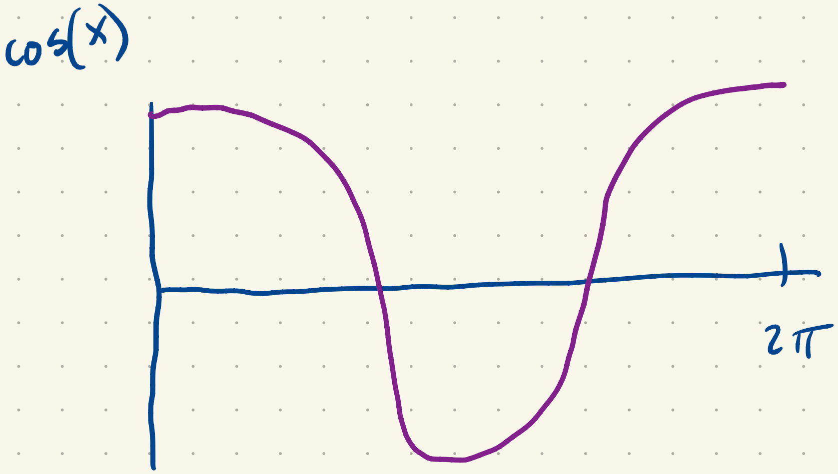


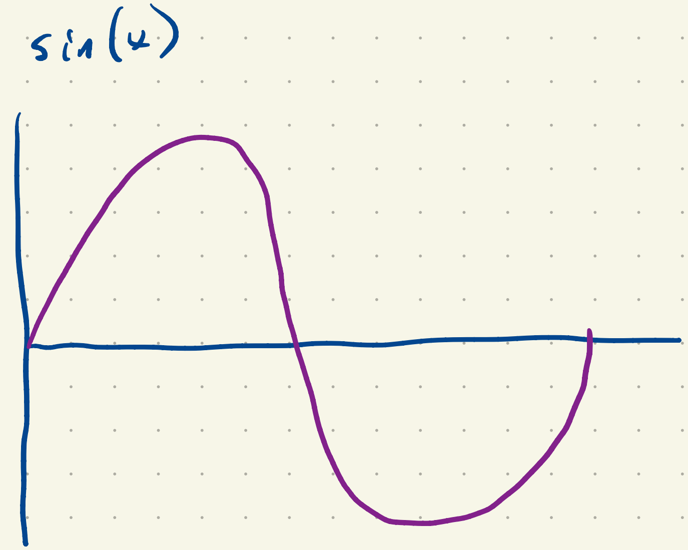


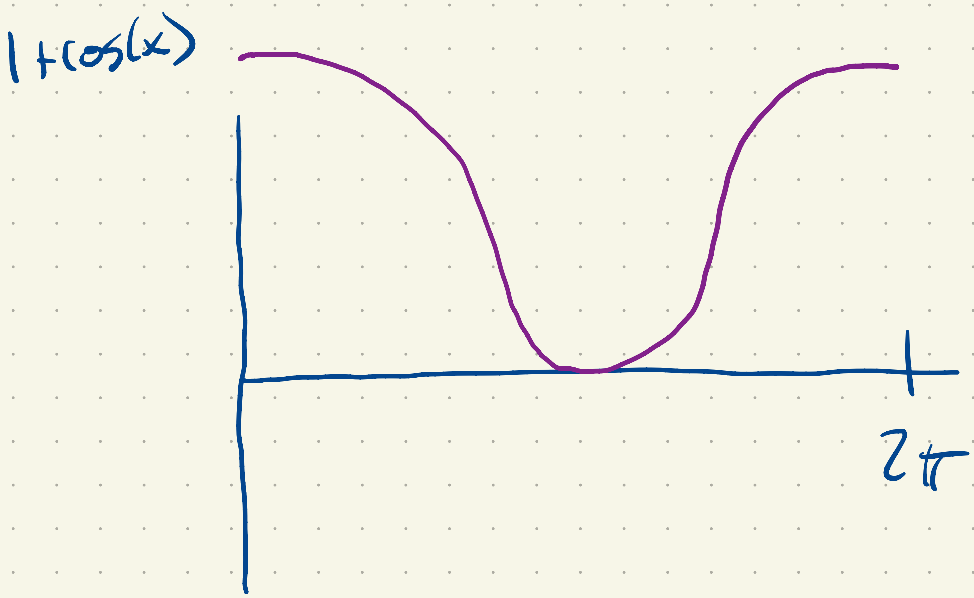


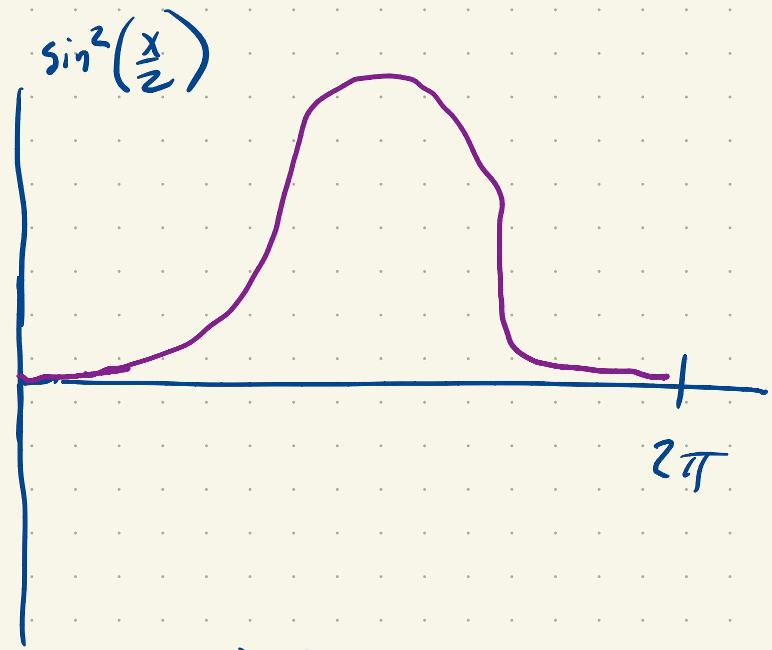


$$\frac{1}{2}\left[1+\cos(x)\right] = 1 - \sin^2\left(\frac{x}{2}\right)$$









$$\sin^3\left(\frac{x}{2}\right) = \frac{1}{2}\left[1 - \cos(x)\right]$$

$$-2\sin^2\left(\frac{x}{2}\right) = \cos(x) - 1$$

Eigenvalues of $\frac{1}{h^2}D$.

We're gonna guess them!

$$\vec{w}_j = e^{\mathcal{J} r x_j} \qquad \mathcal{J}^2 = -1 \qquad r \text{ TBA}$$

$$\begin{aligned}
(D\vec{w})_j &= e^{\mathcal{J} r x_{j-1}} - 2e^{\mathcal{J} r x_j} + e^{\mathcal{J} r x_{j+1}} \\
&= e^{\mathcal{J} r x_j}\left[e^{-\mathcal{J} r h} - 2 + e^{\mathcal{J} r \cdot h}\right] \\
&= 2e^{\mathcal{J} r x_j}\left[\cos(rh) - 1\right] \\
&= -4e^{\mathcal{J} r x_j}\sin^2\left(\frac{rh}{2}\right)
\end{aligned}$$

except for $j=1$, $j=N$

$$v_j = \mathrm{Im}(w_j)$$

$$D\vec{v} = D\,\mathrm{Im}(\vec{w}) = \mathrm{Im}(D\vec{w})$$

$$= \mathrm{Im}\left(-4\sin^2\left(\frac{rh}{2}\right)\vec{w}\right)$$

$$= -4\sin^2\left(\frac{rh}{2}\right)\mathrm{Im}(\vec{w})$$

$$= -4\sin^2\left(\frac{rh}{2}\right)\vec{v}$$

$$\vec{v}_j = \mathrm{Im}(\vec{w}_j) = \sin(r x_j) \qquad e^{J r x_j}$$

End points:

$1 \quad -2 \quad 1$

$$\vec{V}_j = \sin(r x_j)$$

$x_0 = 0$

$x_{N+1} = 1$

Above is correct if

$$``\vec{V}_0" = 0 \rightarrow \text{free}$$

$$``\vec{V}_{N+1}" = 0 \qquad \sin(r x_{N+1}) = \sin(r) = 0$$

$$\Rightarrow r = n\pi$$

$-\pi^2, -4\pi^2, -9\pi^2, \ldots, -N^2\pi^2$

Upshot:

$\vec{v}$

D

- eigenvectors $\quad v_j = \sin(n\pi x_j) \quad n = 1, \ldots, N$

- eigenvalues $\quad -4\sin^2\left(\frac{n\pi h}{2}\right) \approx -4\frac{(n\pi h)^2}{4} \quad h$ small

$$\begin{bmatrix} -2 & 1 & 0 & - & & \\ 1 & -2 & 1 & 0 & - & \\ 0 & 1 & -2 & 1 & 0 & - \\ & & & \ddots & & \end{bmatrix}$$

$$\frac{1}{h^2} D \sim \partial_x^2$$

$$-\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right) \approx -\frac{4}{h^2}\left(\frac{n\pi h}{2}\right)^2$$

$$- n^2\pi^2$$

$\sin(\theta) \approx \theta$

θ small

Upshot:

eigenvectors $\vec{V}_j = \sin(n\pi x_j)$ $\quad n = 1, \dots, N$

eigenvalues $-4\sin^2\left(\frac{n\pi h}{2}\right) \approx -4\frac{(n\pi h)^2}{4}$ $\quad h$ small

---

For fixed n, the "n^{th}" eigenvalue of $\frac{1}{h^2} D$

$\longrightarrow -n^2\pi^2$ as $h \to 0$.

Analysis from Euler's Method

$\frac{1}{h^2} D$

$$-4\frac{k}{h^2}\sin^2\left(\frac{n\pi h}{2}\right) > -2$$

no control ($h \not\to 0$)

$$\boxed{\frac{k}{h^2} < \frac{1}{2}}$$

$$k\lambda > -2$$

$\cancel{-\pi^2 n^2}$

$-\frac{4}{h^2}\sin^2\left(\frac{n\pi h}{2}\right)$

$$-4\frac{k}{h^2} > -2$$

Summary:

- The direct method of solving $u_t = u_{xx}$ can be interpreted as Euler's method applied to $\vec{u}' = \frac{1}{h^2} D \vec{u}$

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  $-k^2\pi^2, \quad k = 1, \ldots, N.$

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- Actual eigenvalues are $-\frac{4}{h^2} \sin^2\left(\frac{k\pi h}{2}\right)$

- To keep all $\uparrow$ in region of abs stab, $\frac{k}{h^2} \le \frac{1}{2}$