

An Unhelpful Introduction

to Electricity & Magnetism

Part I: Really Unhelpful

Nov 10, 2020

David Maxwell

UAF Mathematics

Background

- Gauss' Law:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

Background

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$$\vec{\nabla} \cdot \vec{E} = \rho$$

- "part of shape of now" = "stuff density"

Background

- Gauss' Law:

$$\vec{\nabla} \cdot \vec{E} = \rho$$

- "part of shape of now" = "stuff density"
- How do you describe the stuff in the universe before it has a shape?

No:

- angles
- lengths
- volumes
- dot products

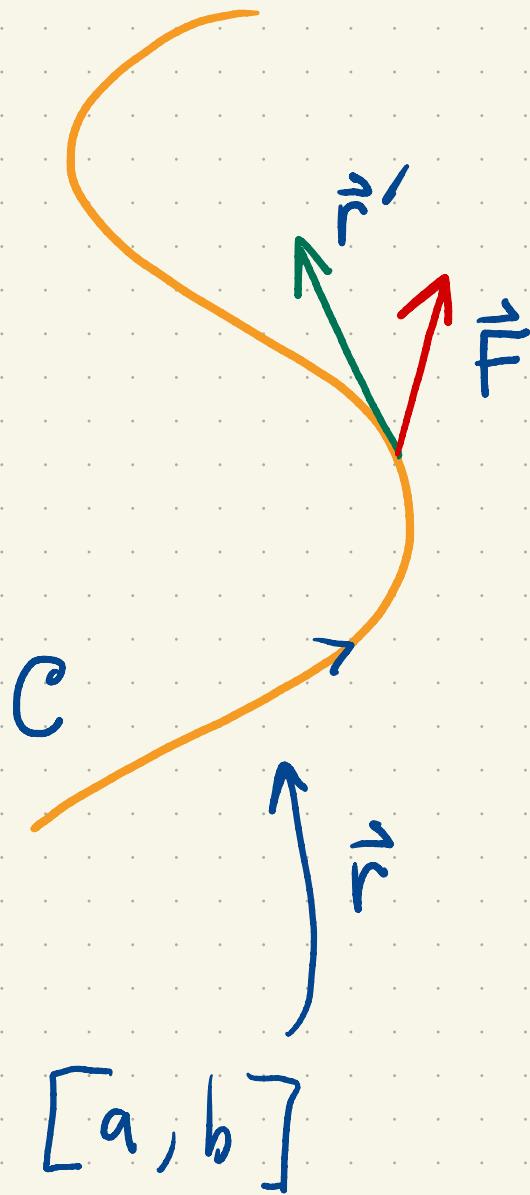
Outline

- Today: exterior derivative
covectors and their kin

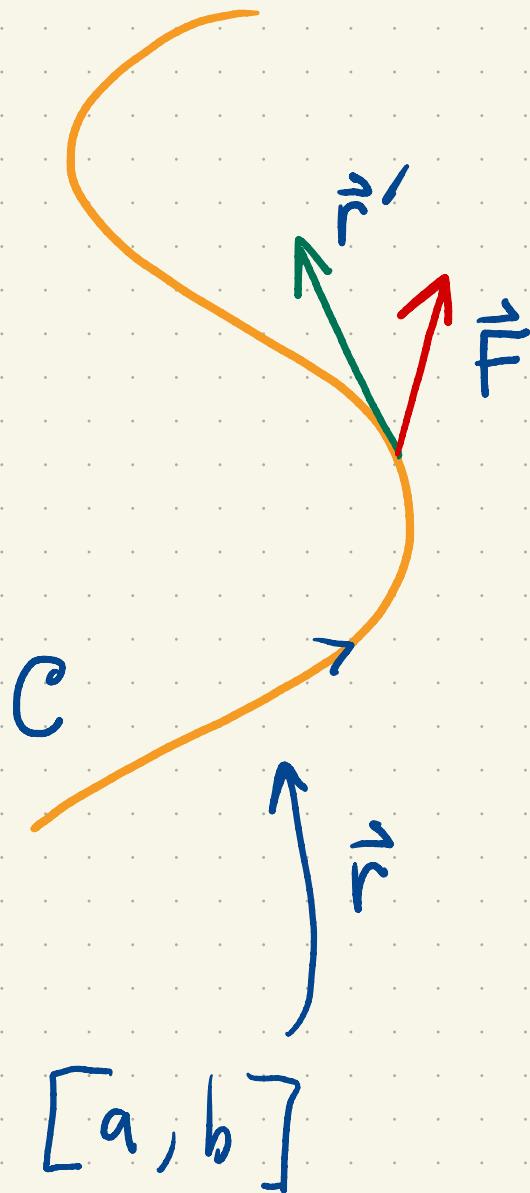
11/17 • What is the object described by
electromagnetism?

11/24 • Where do Maxwell's equations come from?

Line Integrals

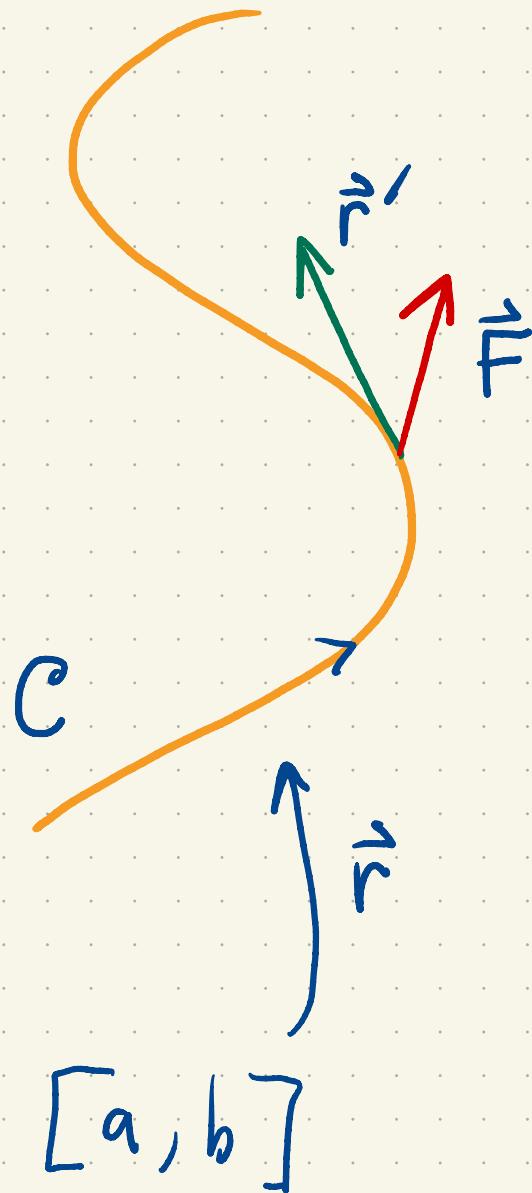


Line Integrals



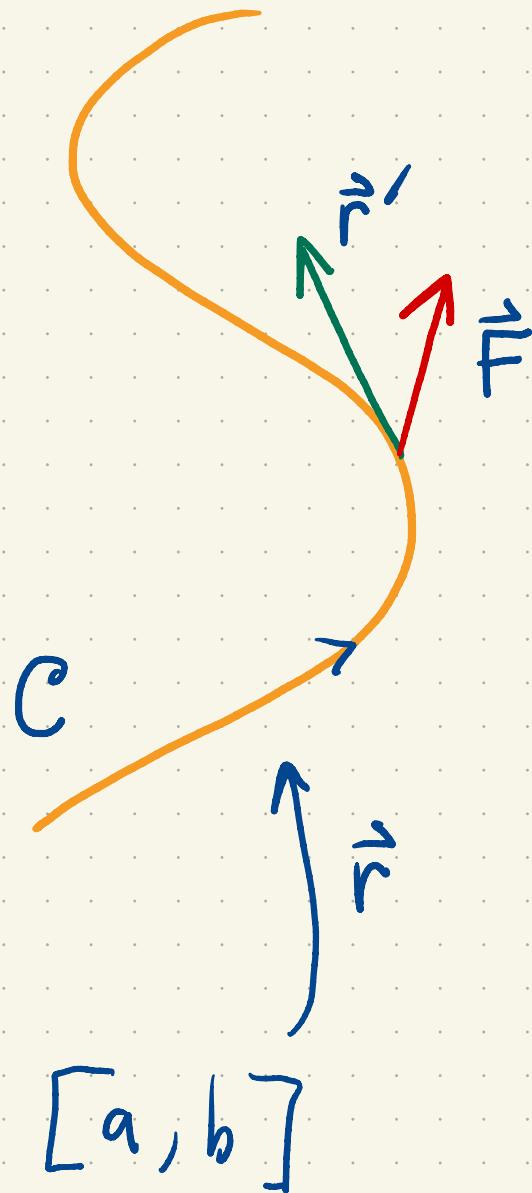
$$\int_C \vec{F} \cdot d\vec{r}$$

Line Integrals



$$\int_C \vec{F} \cdot d\vec{r} \quad \downarrow$$
$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Line Integrals

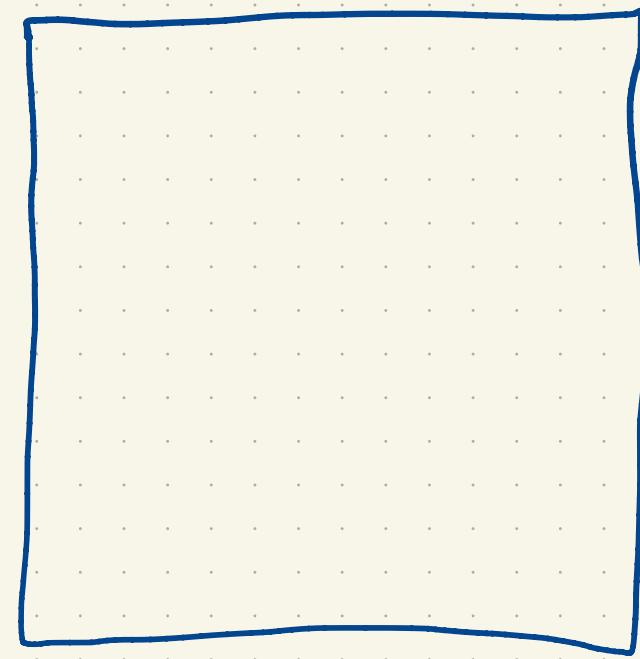


$$\int_C \vec{F} \cdot d\vec{r}$$

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

uh oh

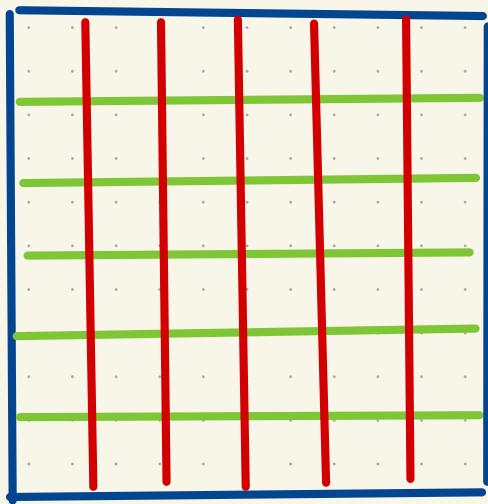
Coordinates



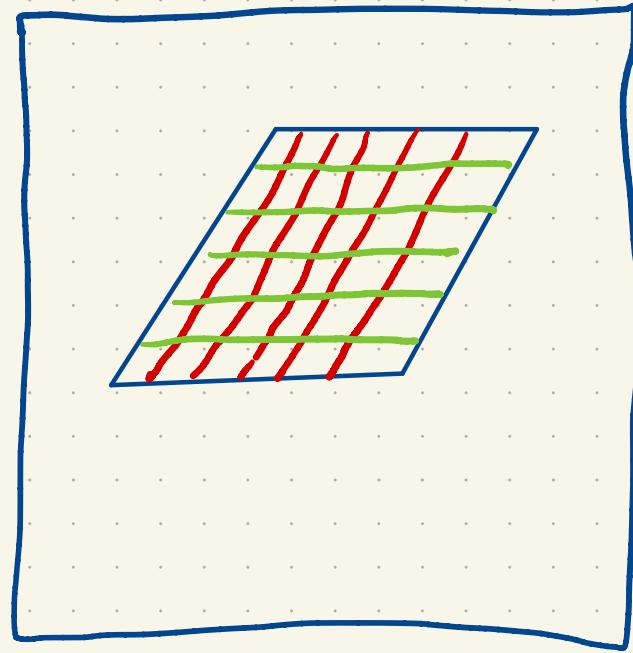
The true thing

Coordinates

(u, v)



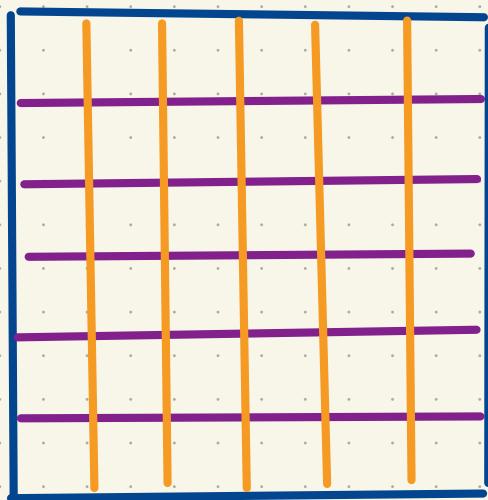
Your coordinates



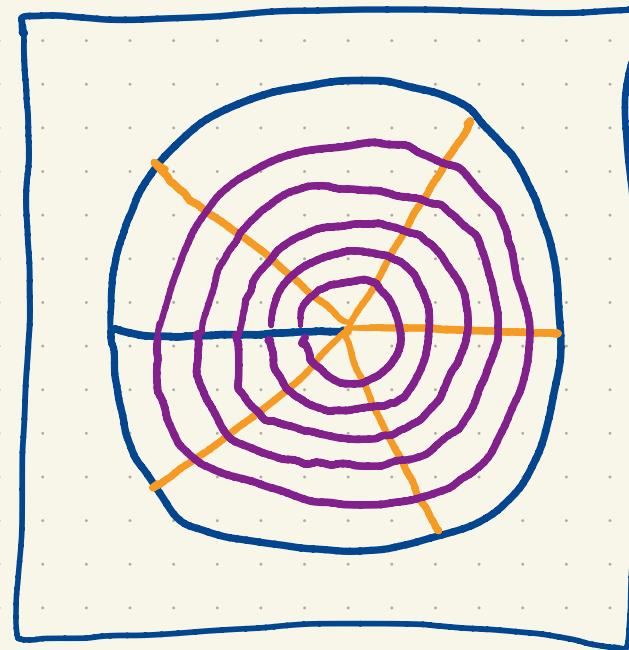
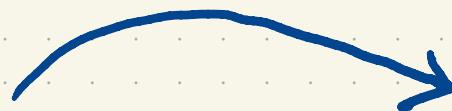
The true thing

Coordinates

(r, θ)



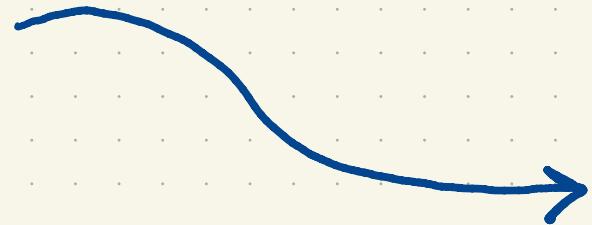
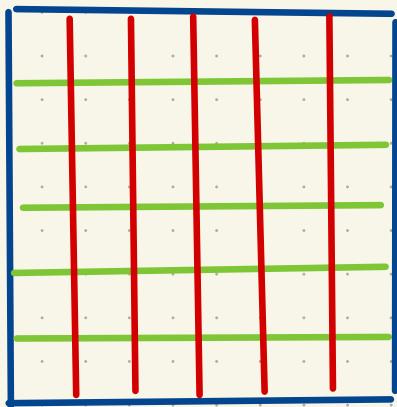
my coordinates



The true thing

Change of Coordinates

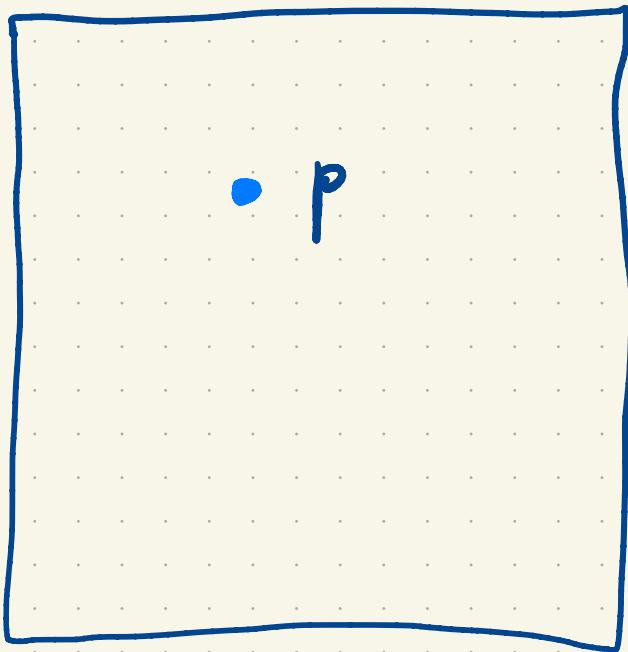
(u, v)



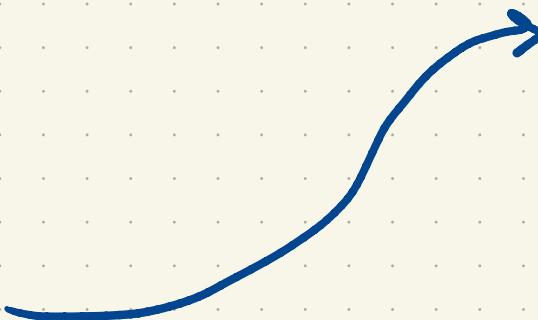
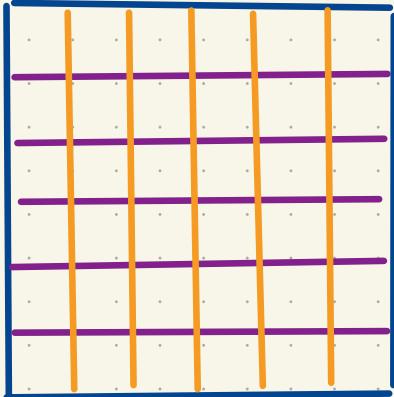
The true thing



$\bullet P$

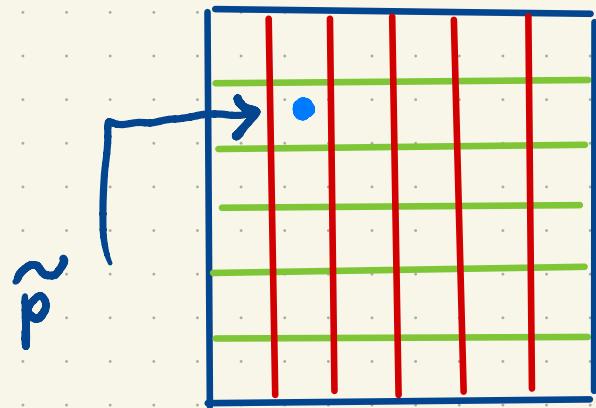


(r, θ)

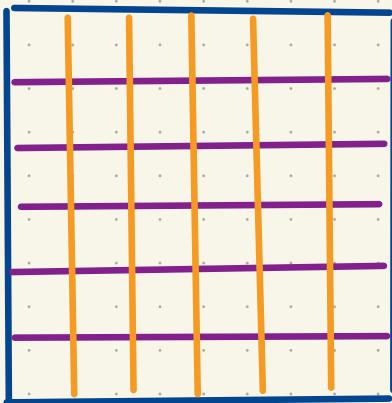


Change of Coordinates

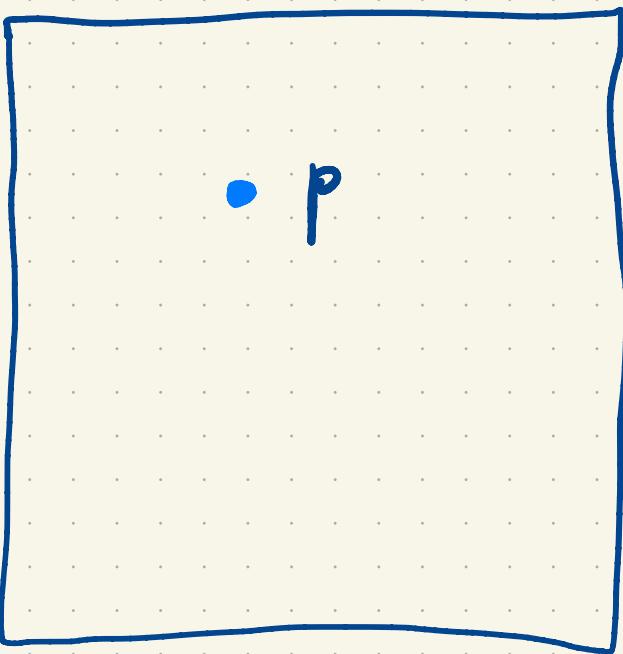
(u, v)



(r, θ)

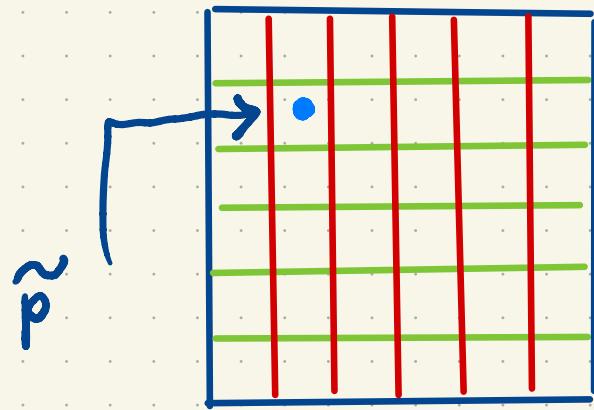


The true thing

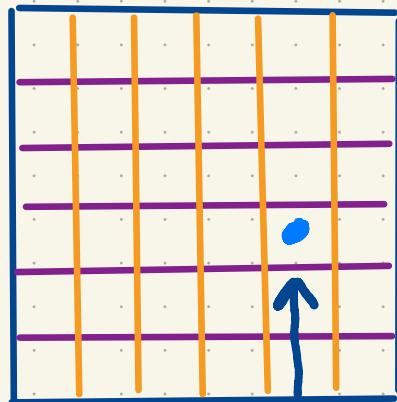


Change of Coordinates

(u, v)

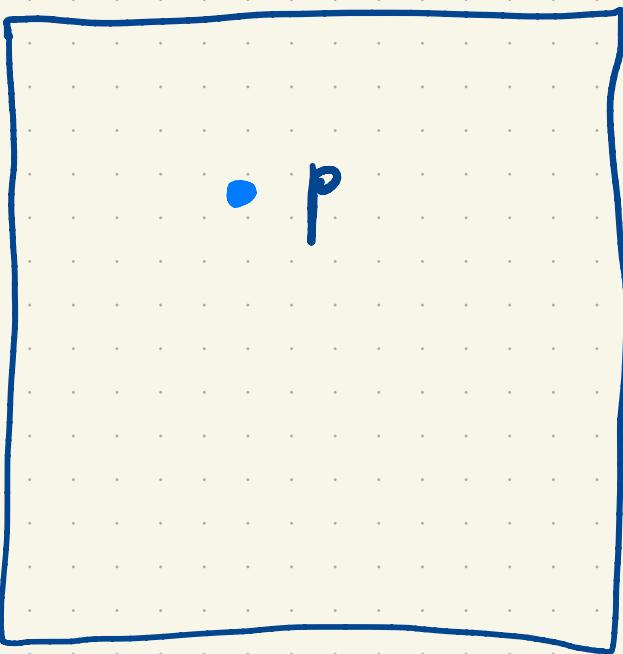


(r, θ)



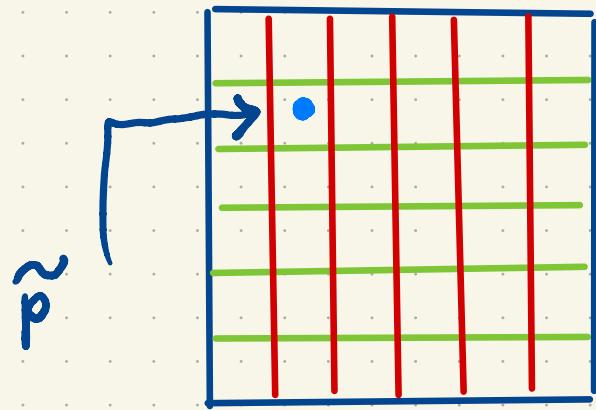
\hat{p}

The true thing

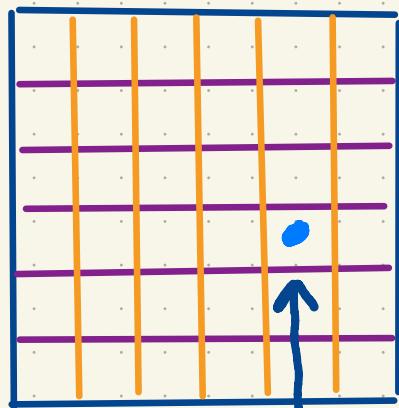


Change of Coordinates

(u, v)

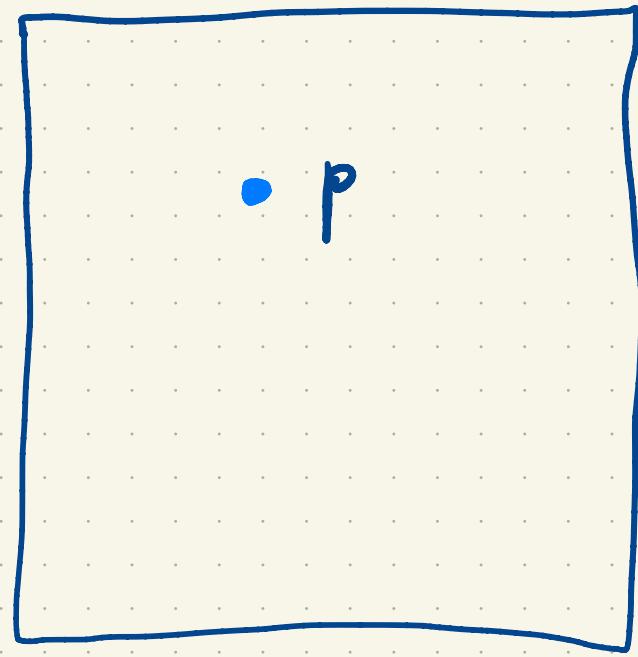


(r, θ)



The true thing

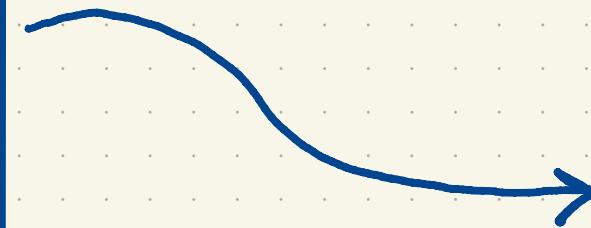
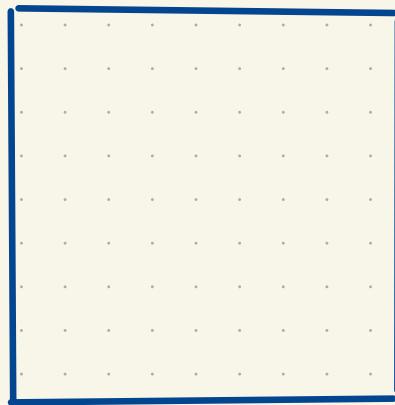
p



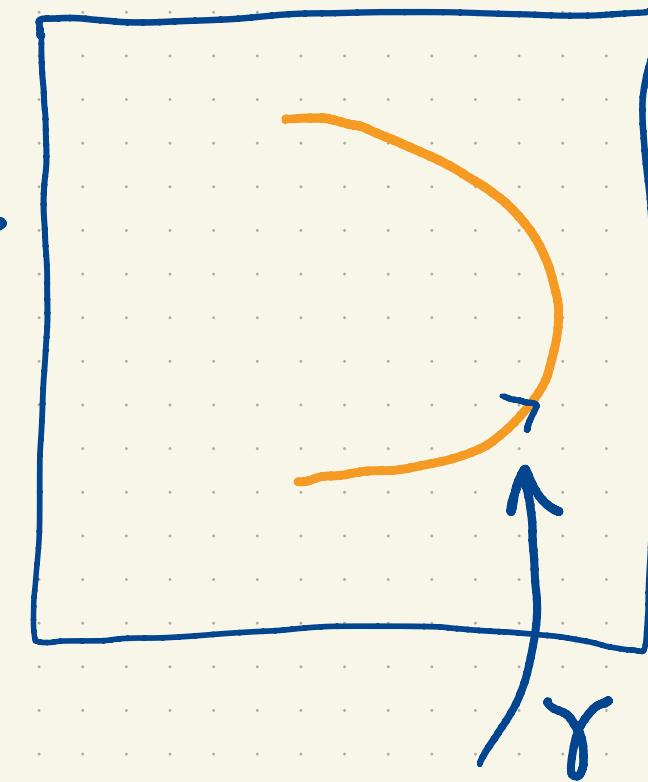
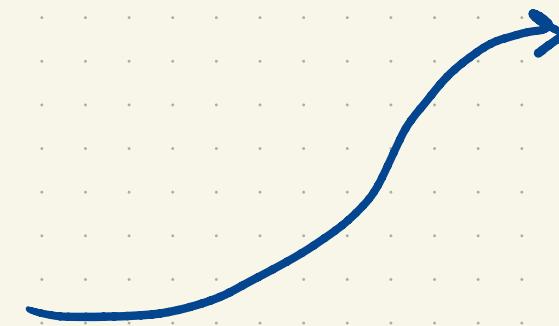
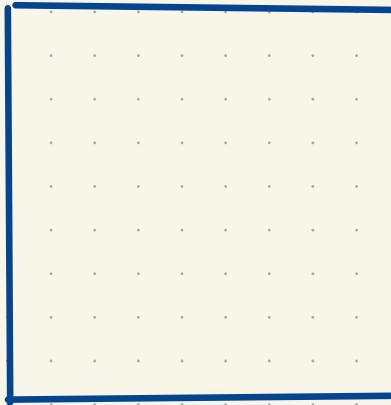
$$(u, v) = (U(r, \theta), V(r, \theta))$$

Tangent Vector = Infinitesimal Curve

(u, v)



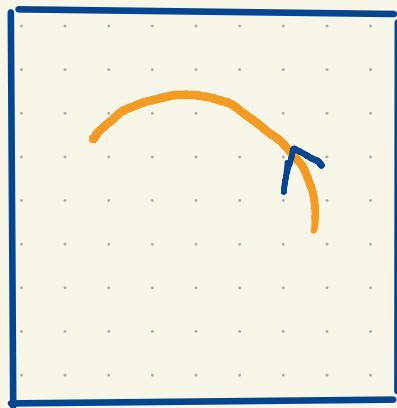
(r, θ)



$[a, b]$

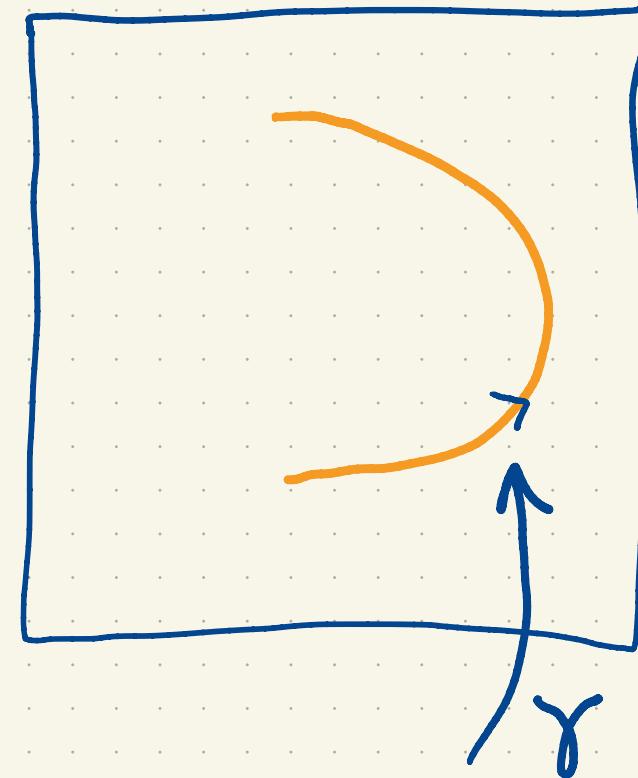
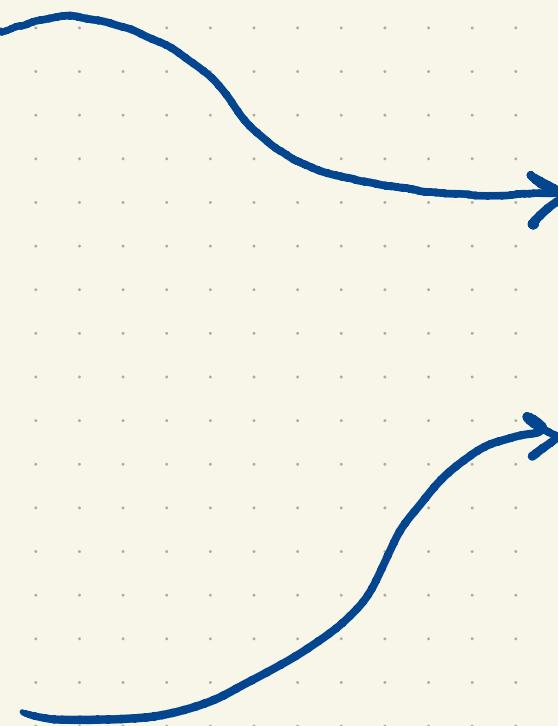
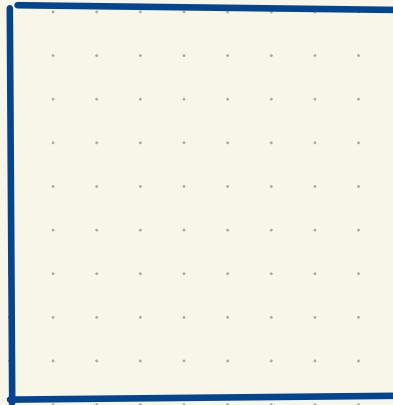
Tangent Vector = Infinitesimal Curve

(u, v)



$$\tilde{\gamma}(t) = (u(t), v(t))$$

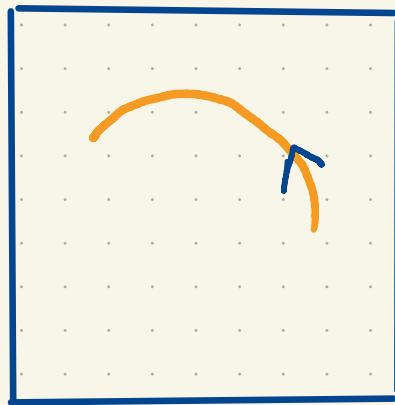
(r, θ)



$[a, b]$

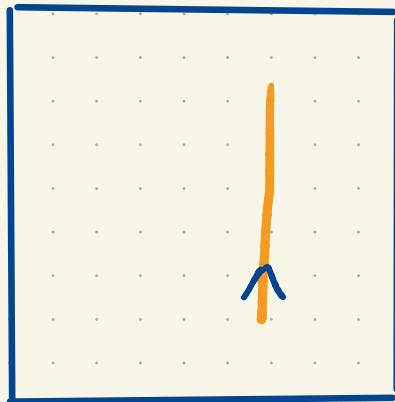
Tangent Vector = Infinitesimal Curve

(u, v)

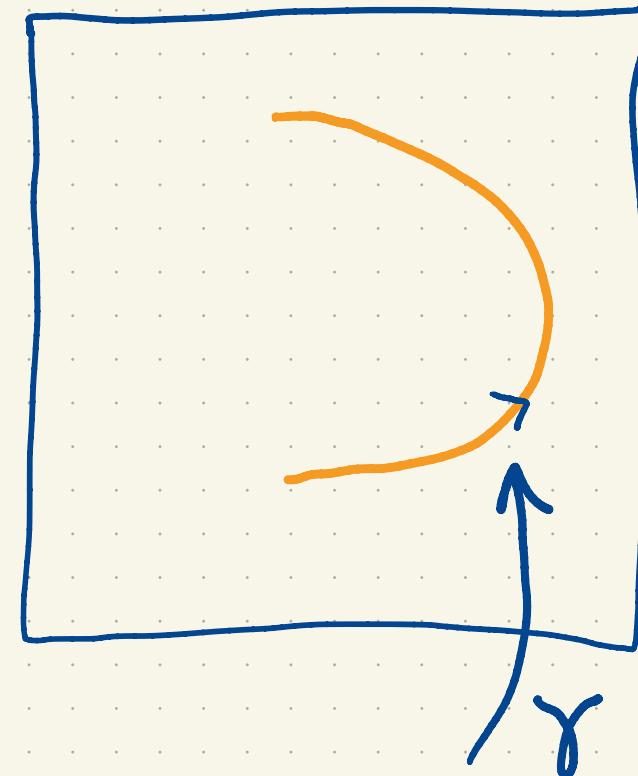


$$\tilde{\gamma}(t) = (u(t), v(t))$$

(r, θ)

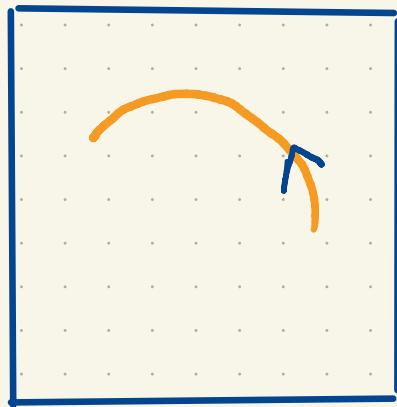


$$\hat{\gamma}(t) = (r(t), \theta(t))$$



Tangent Vector = Infinitesimal Curve

(u, v)

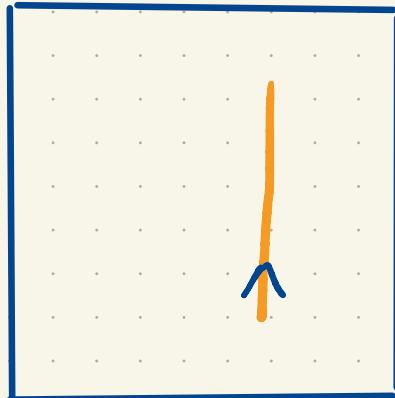


$$\tilde{\gamma}(t) = (u(t), v(t)) = (U(r(\varepsilon), \theta(\varepsilon))), V(r(\varepsilon), \theta(\varepsilon)))$$

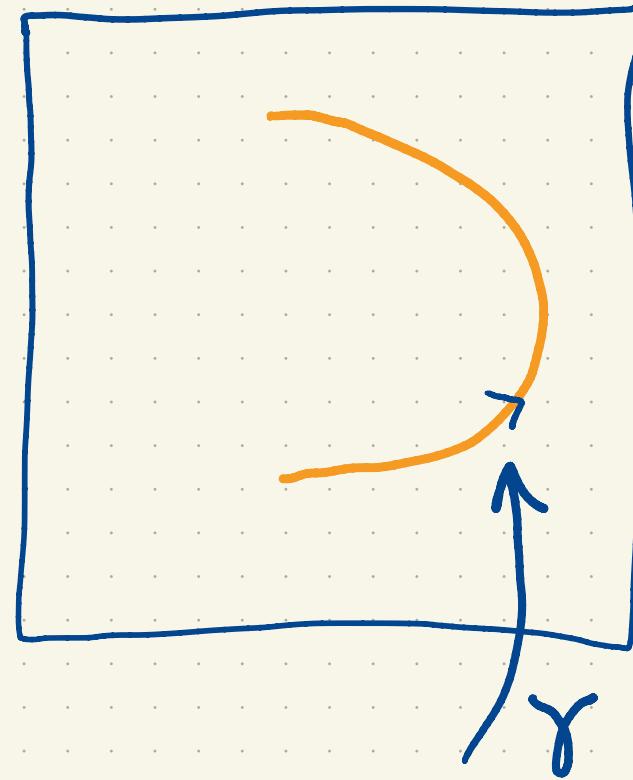
$\tilde{\gamma}(t)$

$\tilde{\gamma}(\varepsilon)$

(r, θ)



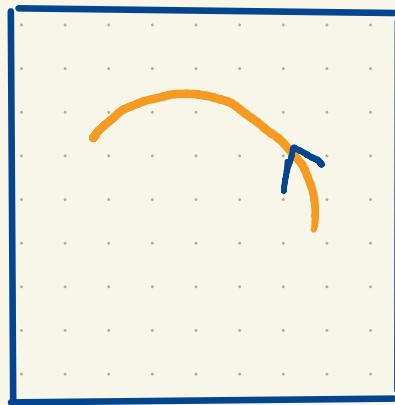
$$\hat{\gamma}(t) = (r(t), \theta(t))$$



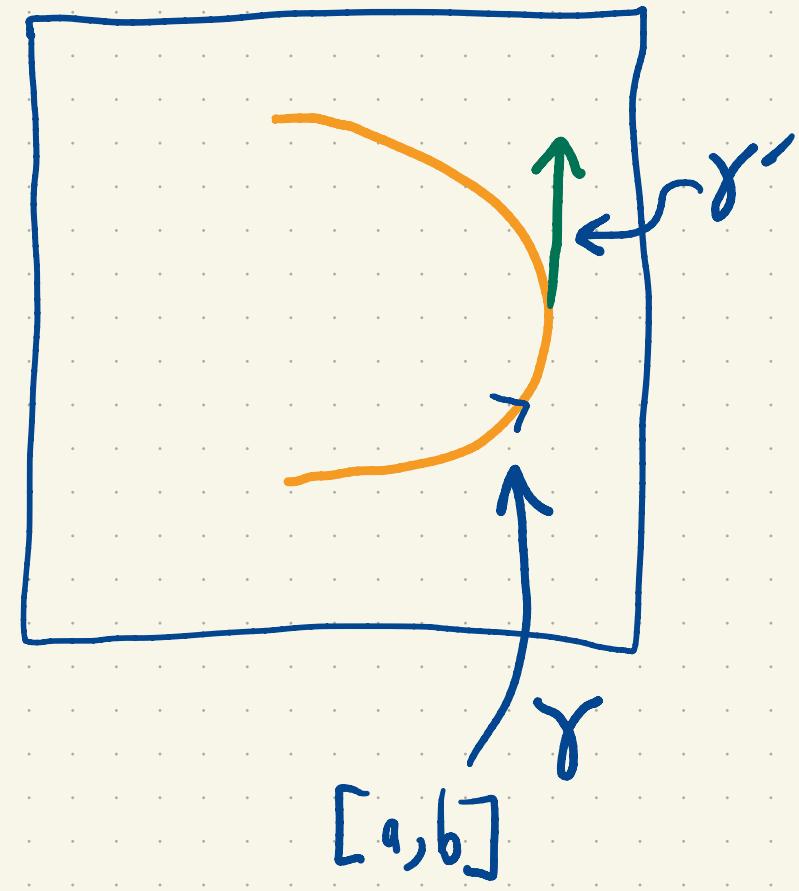
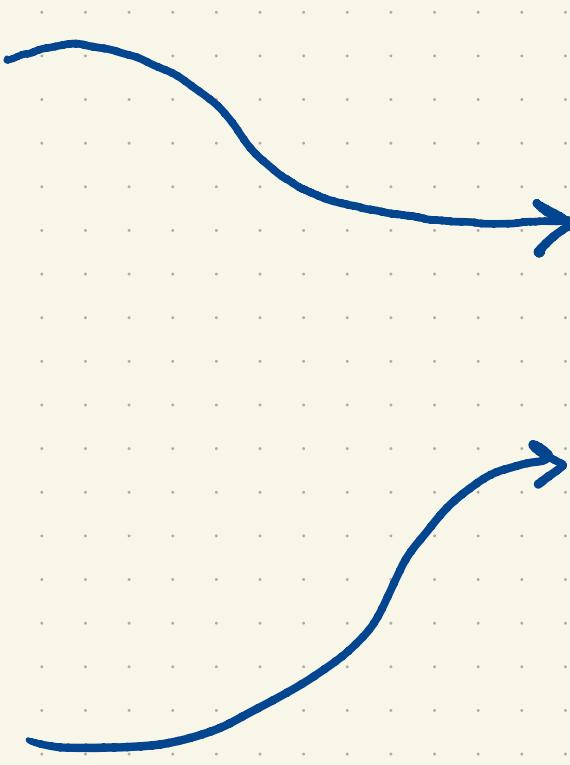
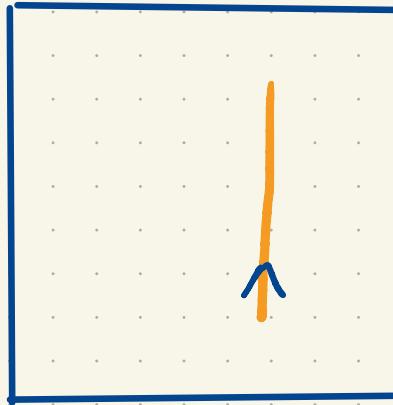
$[a, b]$

Tangent Vector = Infinitesimal Curve

(u, v)

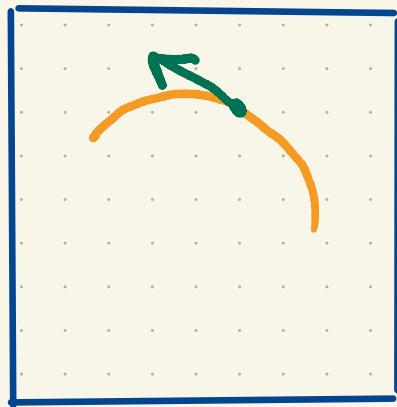


(r, θ)



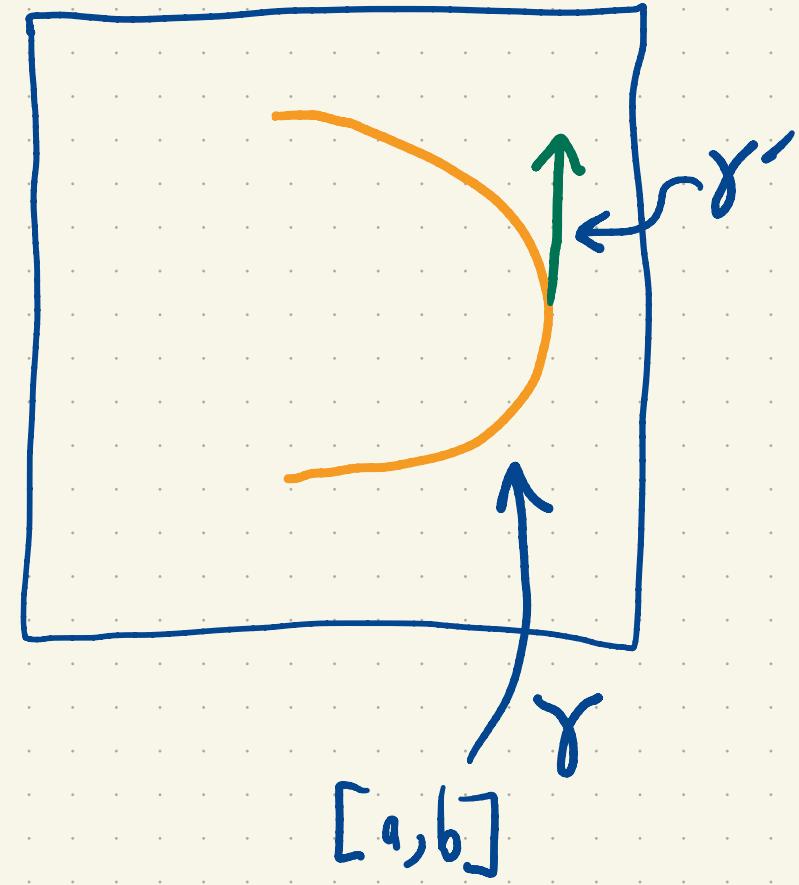
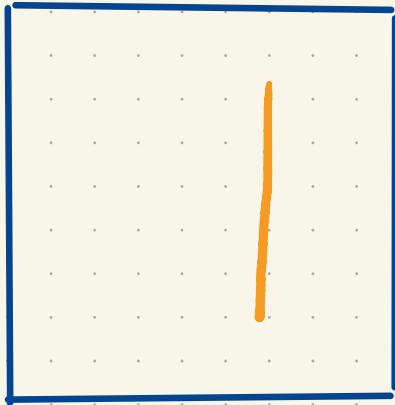
Tangent Vector = Infinitesimal Curve

(u, v)



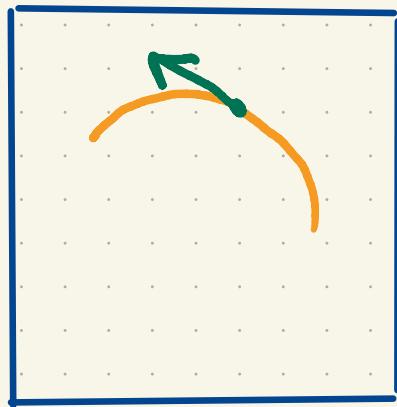
$$\tilde{\gamma}' = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

(r, θ)



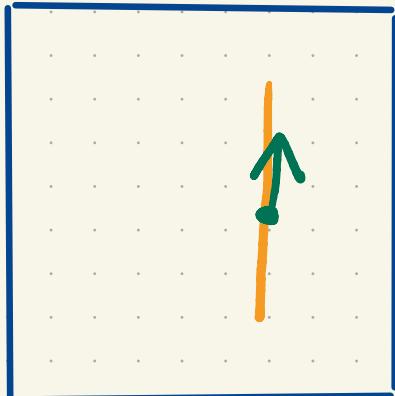
Tangent Vector = Infinitesimal Curve

(u, v)

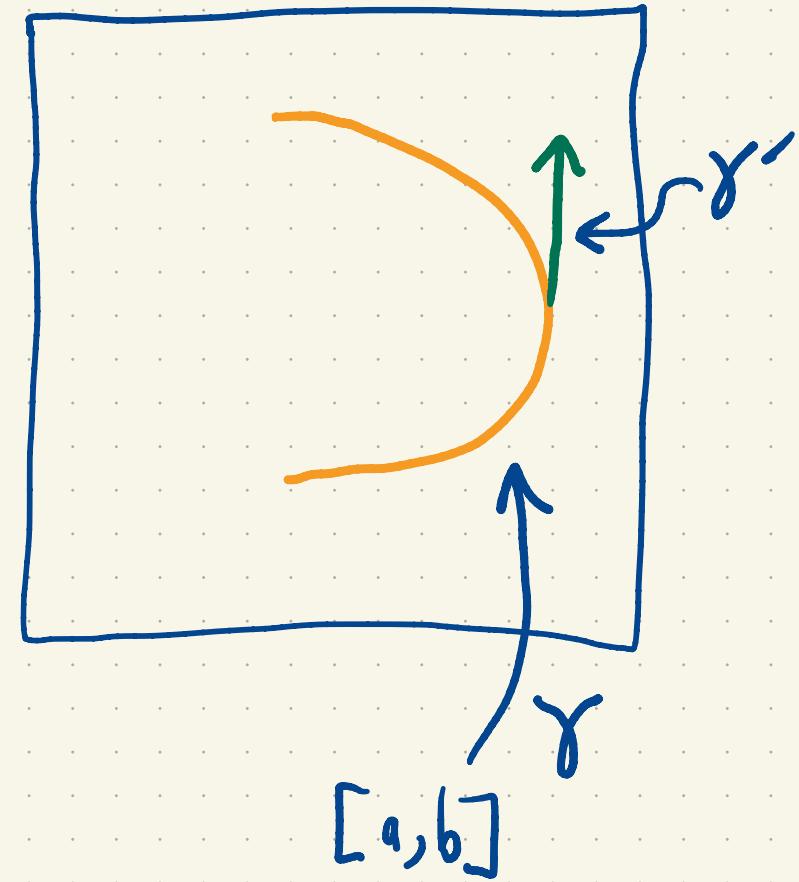


$$\hat{\gamma}' = \begin{bmatrix} u' \\ v' \end{bmatrix}$$

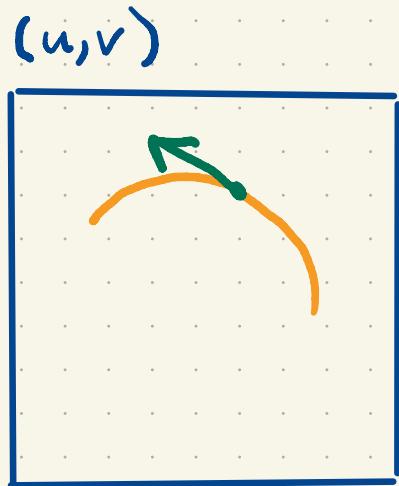
(r, θ)



$$\hat{\gamma}' = \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

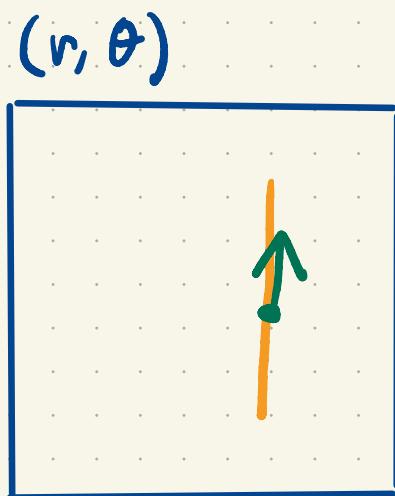


Tangent Vector = Infinitesimal Curve



$$u(t) = U(r(t), \theta(t))$$

$$v(t) = V(r(t), \theta(t))$$

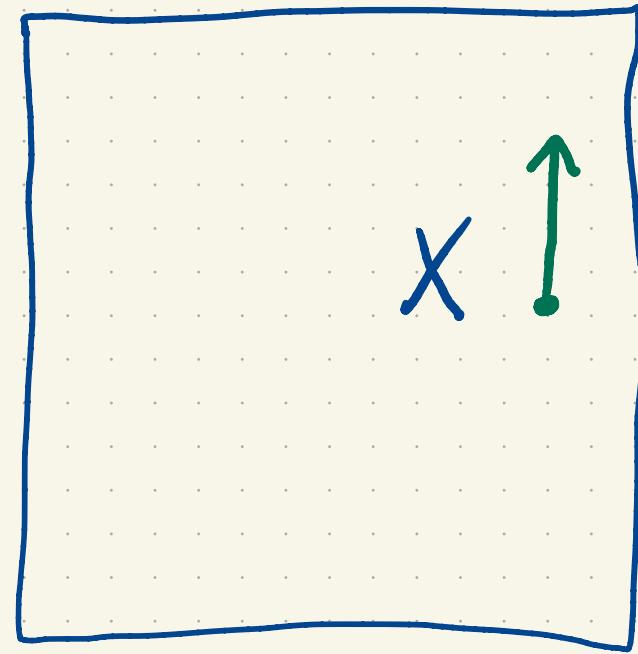
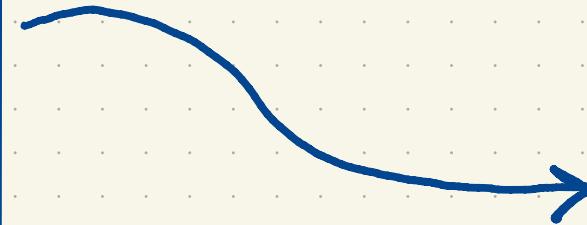
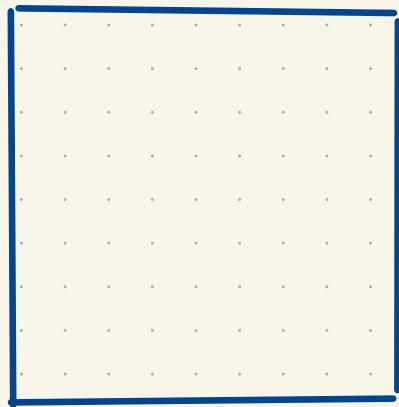


$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \partial U / \partial r & \partial U / \partial \theta \\ \partial V / \partial r & \partial V / \partial \theta \end{bmatrix} \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

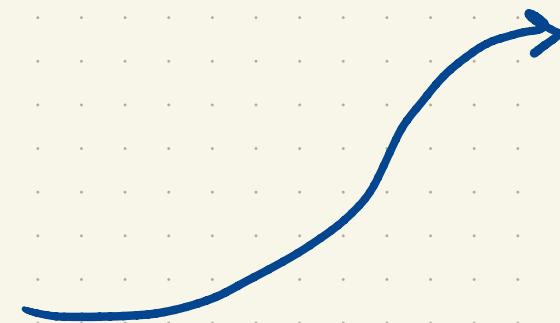
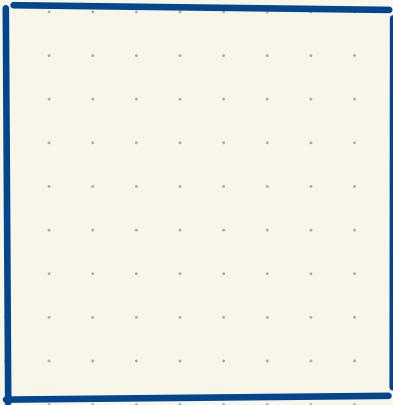
\uparrow J

Tangent Vector = Infinitesimal Curve

(u, v)

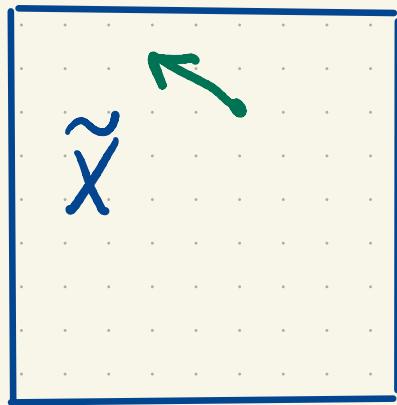


(r, θ)

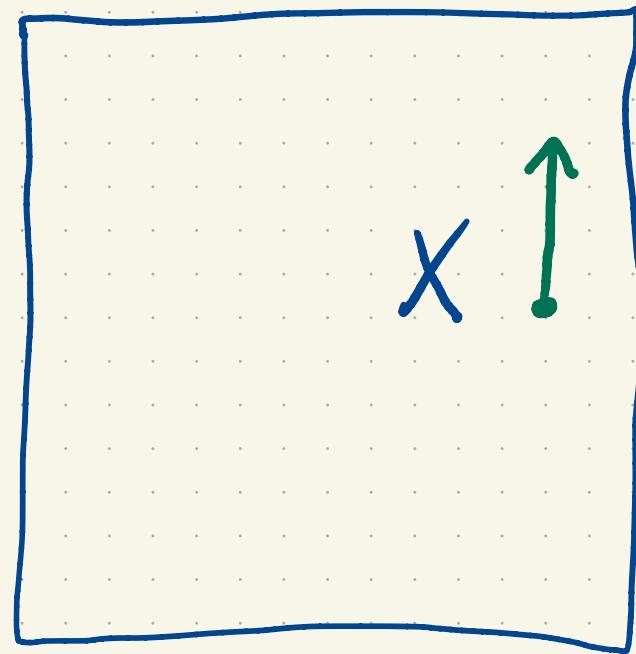
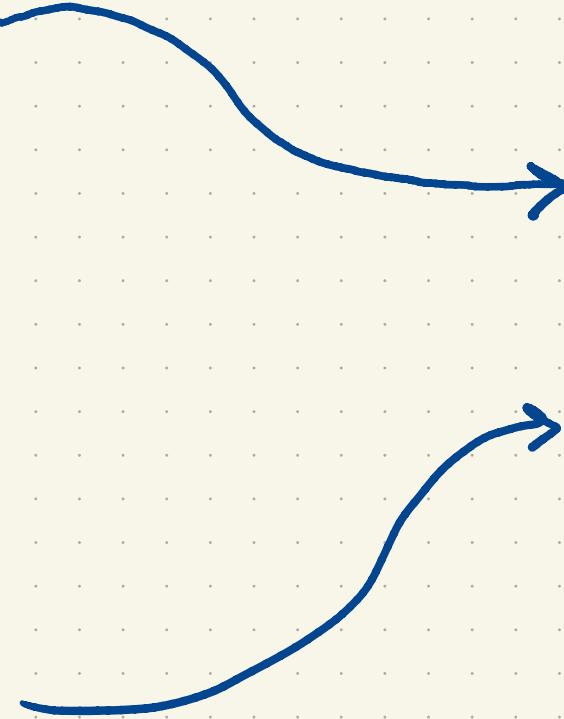
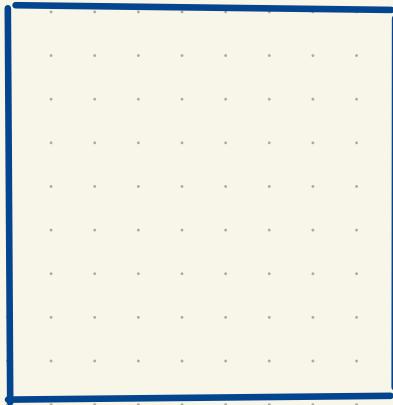


Tangent Vector = Infinitesimal Curve

(u, v)

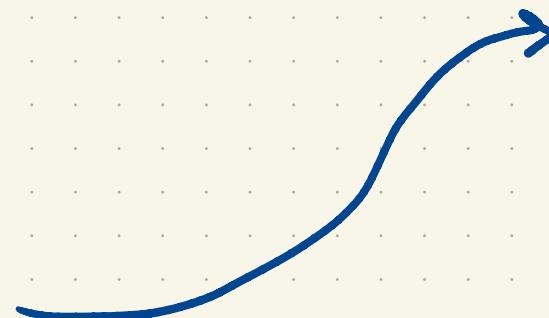
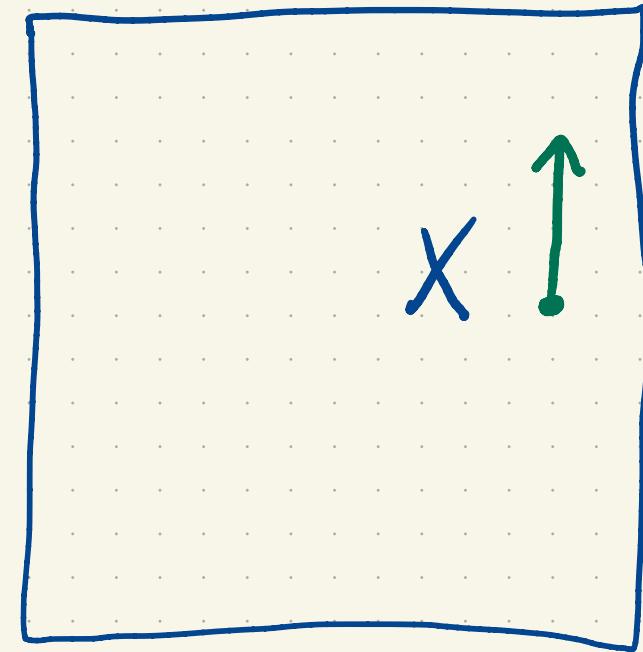
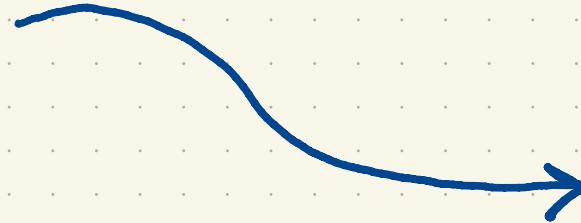
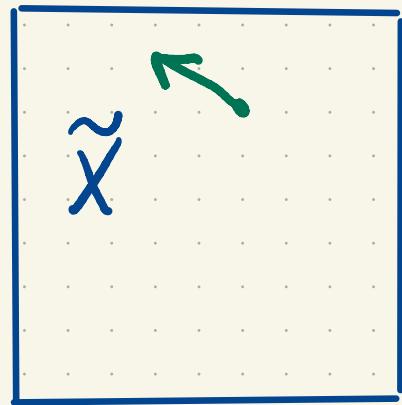


(r, θ)



Tangent Vector = Infinitesimal Curve

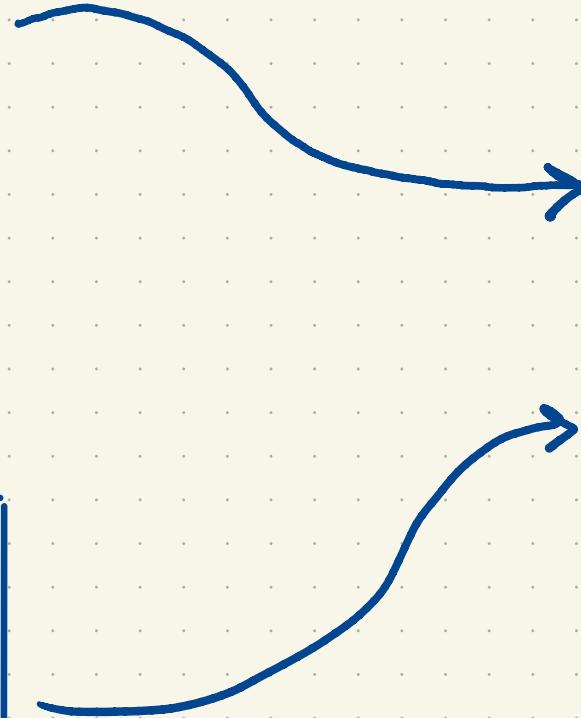
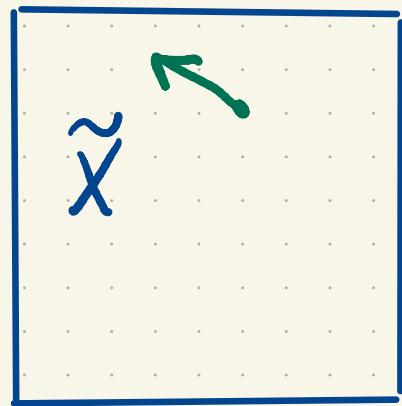
(u, v)



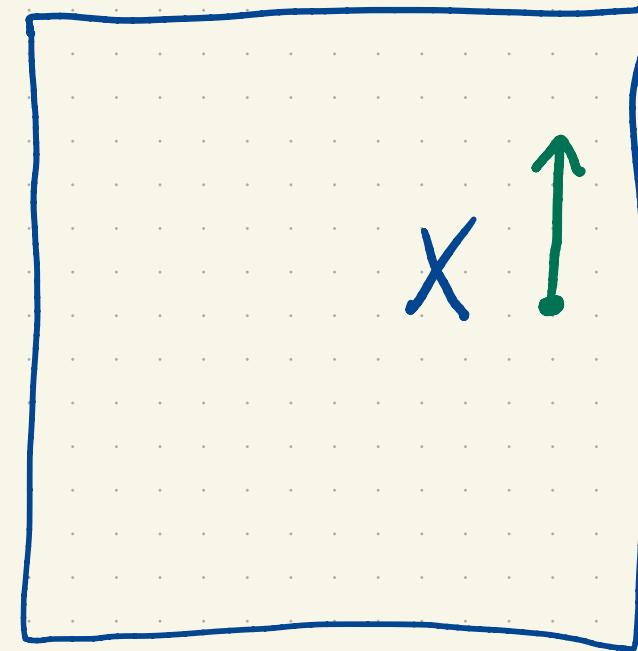
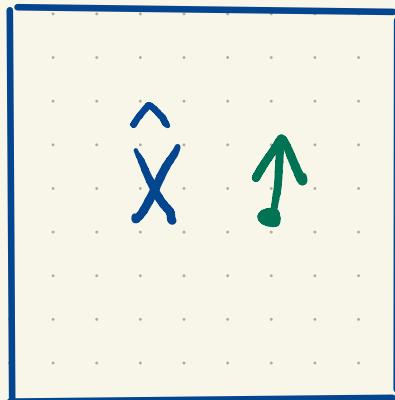
(r, θ)

Tangent Vector = Infinitesimal Curve

(u, v)



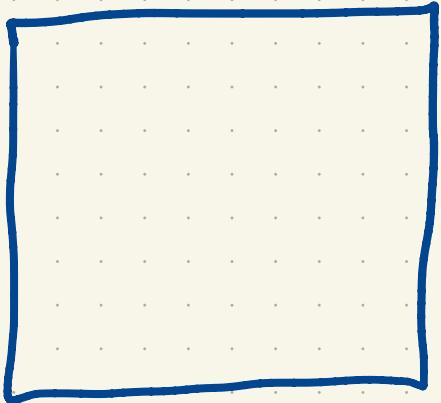
(r, θ)



$$\tilde{X} = J \hat{X}$$

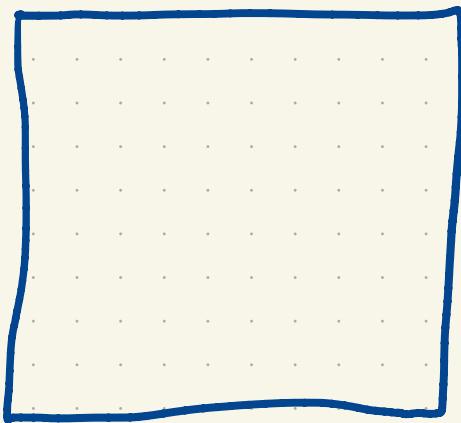
Covector = Infinitesimal Function

(u, v)



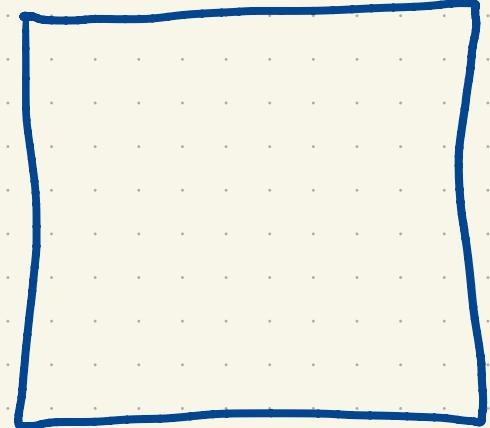
\tilde{T}

(r, θ)



\hat{T}

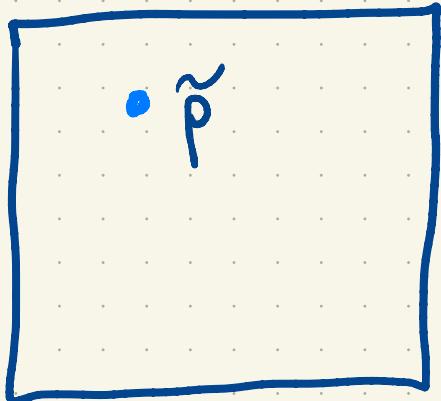
True thing



T (e.g., temp)

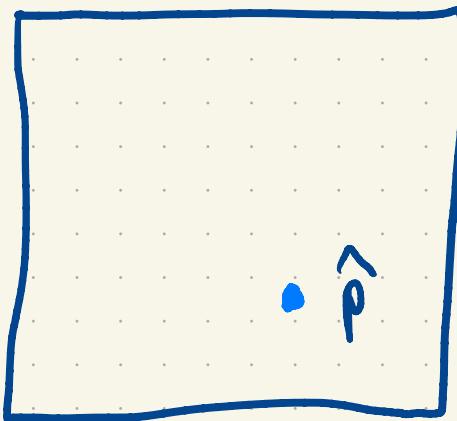
Covector = Infinitesimal Function

(u, v)



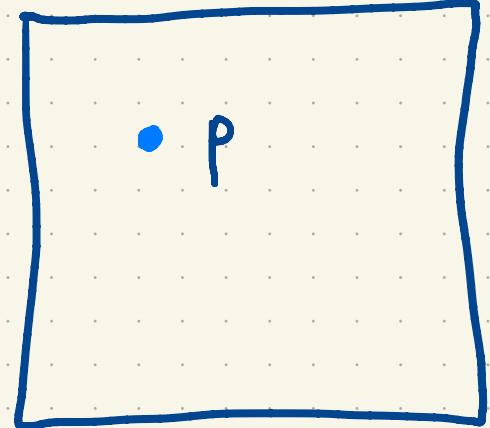
\tilde{T}

(r, θ)



\hat{T}

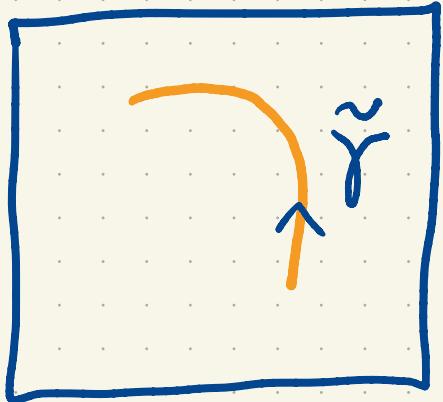
True thing



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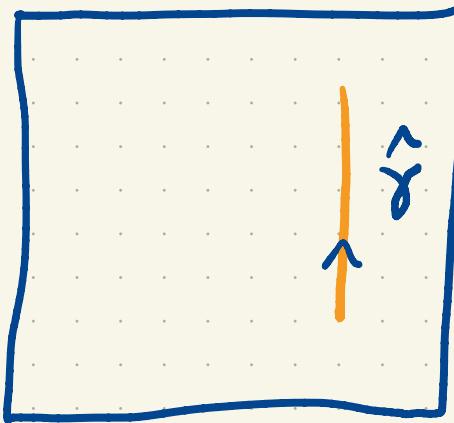
Covector = Infinitesimal Function

(u, v)



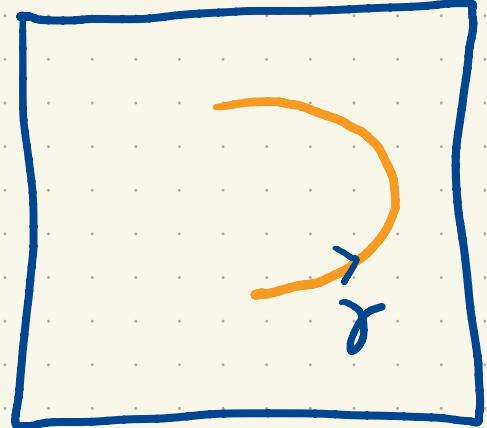
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

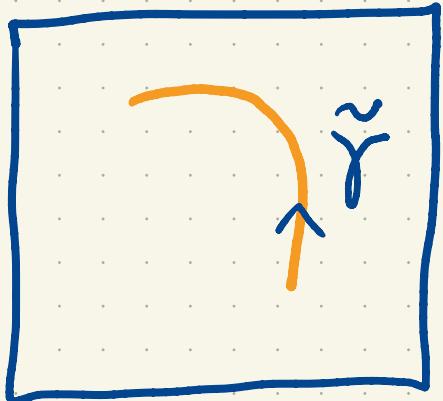
True thing



T (e.g., temp)

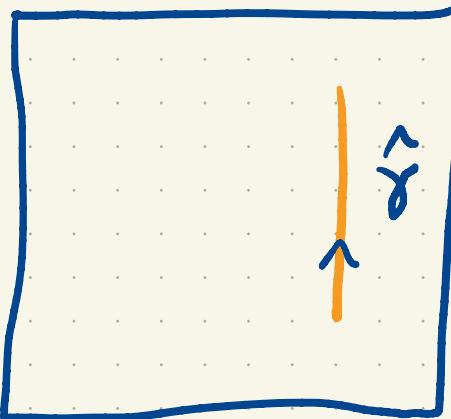
Covector = Infinitesimal Function

(u, v)



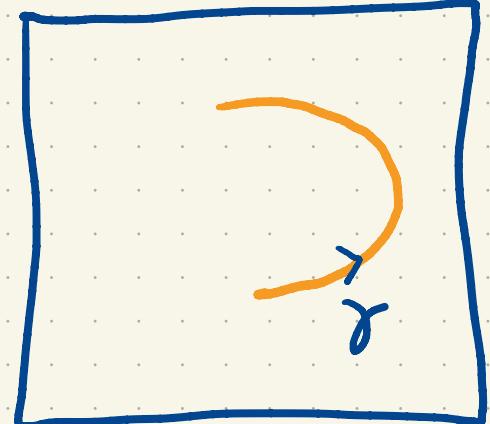
\tilde{T}

(r, θ)



\hat{T}

True thing



T (e.g., temp)

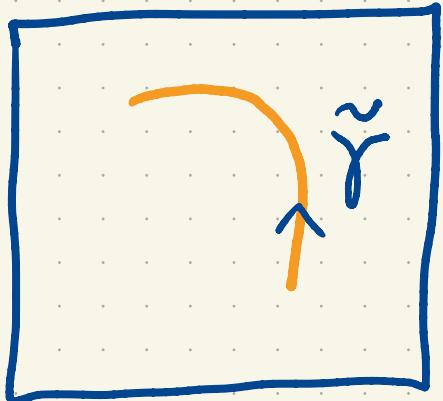
$T(\gamma(t))$



$\tilde{T}(\tilde{\gamma}(t))$

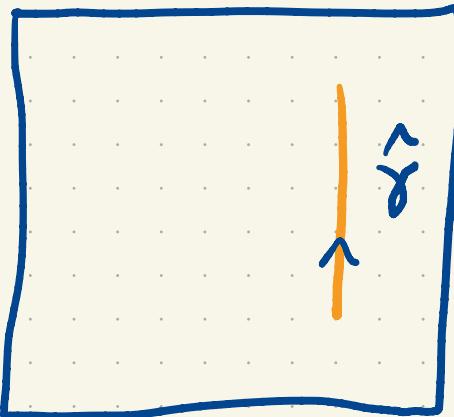
Covector = Infinitesimal Function

(u, v)



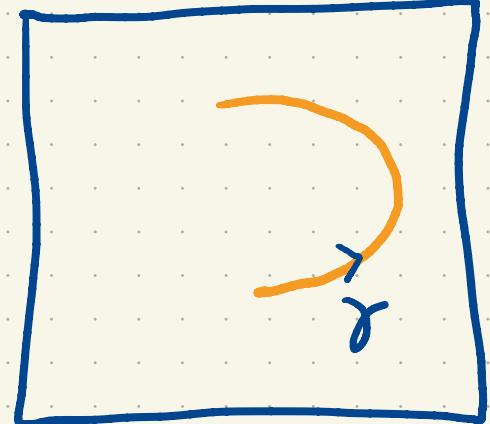
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

True thing



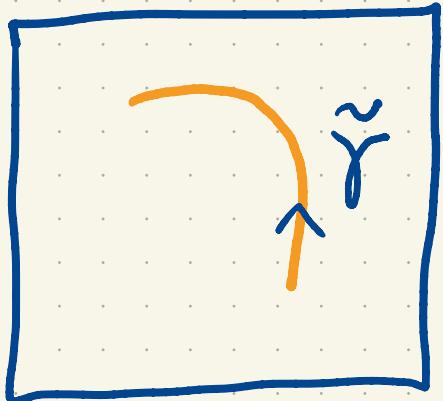
T (e.g., temp)

$T(\gamma(t))$

$$\begin{array}{ccc} & T(\gamma(t)) & \\ \swarrow & & \searrow \\ \tilde{T}(\tilde{\gamma}(t)) & & \hat{T}(\hat{\gamma}(t)) \end{array}$$

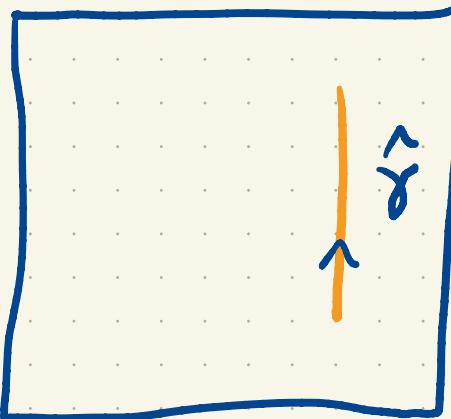
Covector = Infinitesimal Function

(u, v)



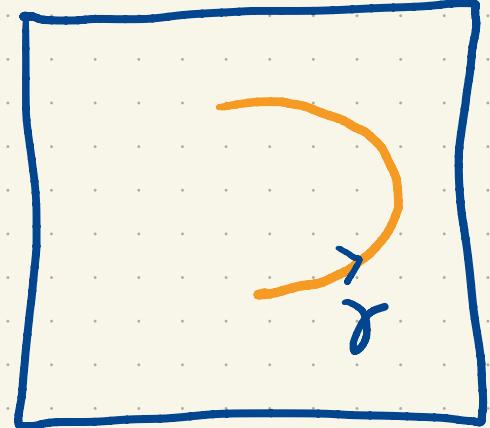
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

True thing



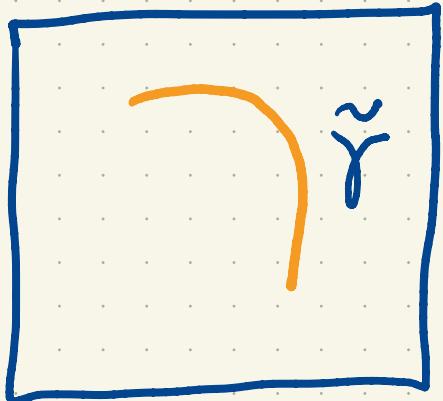
T (e.g., temp)

$T(\gamma(t))$

$$\tilde{\tau}(\tilde{\gamma}(t)) = \hat{\tau}(\hat{\gamma}(t))$$

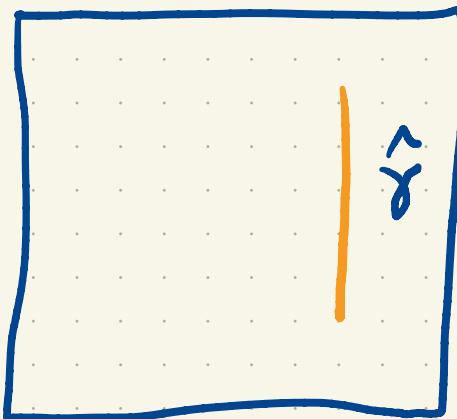
Covector = Infinitesimal Function

(u, v)



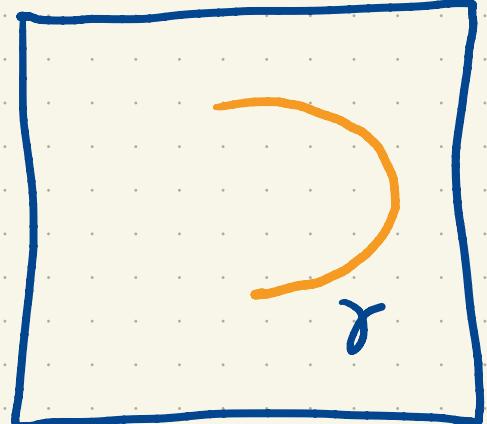
\tilde{T}

(r, θ)



$\hat{\gamma}$

True thing

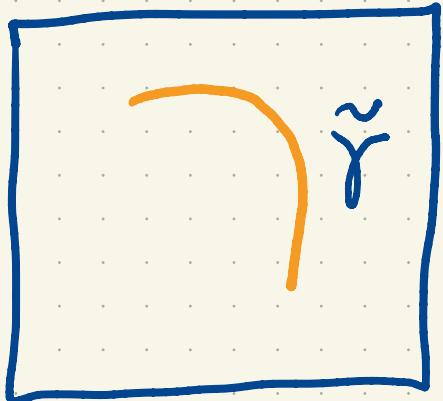


T (e.g., temp)

$$\frac{d}{dt} T(\gamma(t))$$

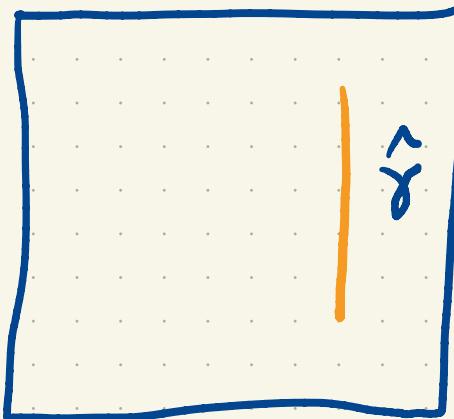
Covector = Infinitesimal Function

(u, v)



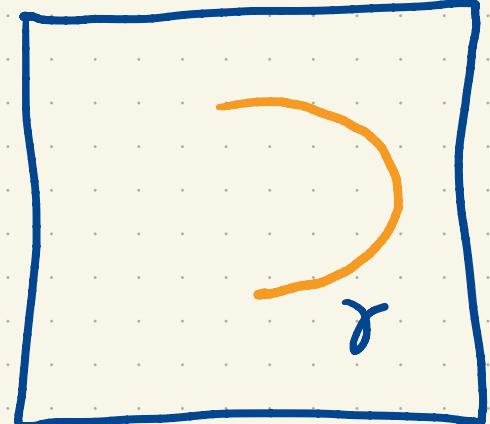
\tilde{T}

(r, θ)



\hat{T}

True thing



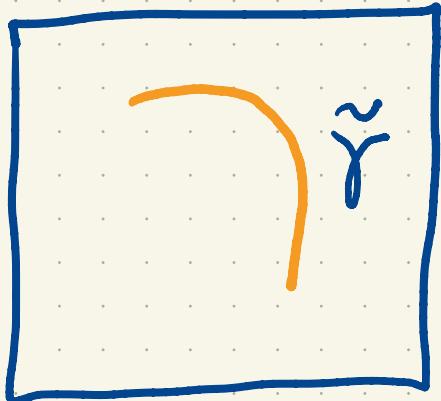
T (e.g., temp)

$$\frac{d}{dt} T(\gamma(t))$$

$$\frac{d}{dt} \tilde{T}(\tilde{\gamma}(t)) = \frac{d}{dt} \hat{T}(\hat{\gamma}(t))$$

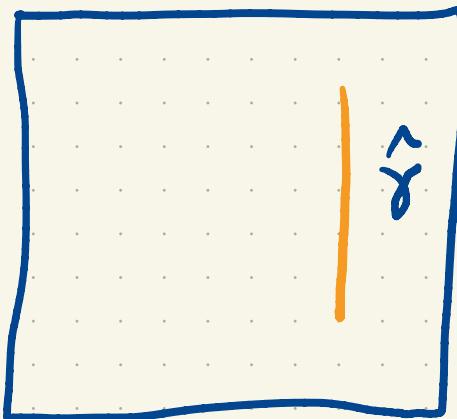
Covector = Infinitesimal Function

(u, v)



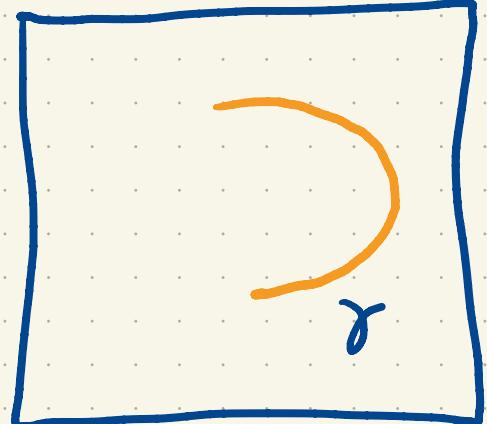
\tilde{T}

(r, θ)



\hat{T}

True thing



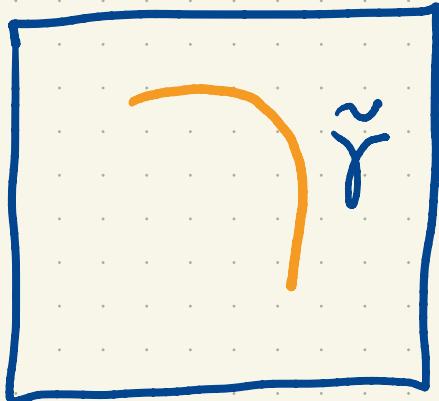
T (e.g., temp)

$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \begin{bmatrix} u' \\ v' \end{bmatrix}$$

$$\frac{d}{dt} \tilde{T}(\tilde{\gamma}(t))$$

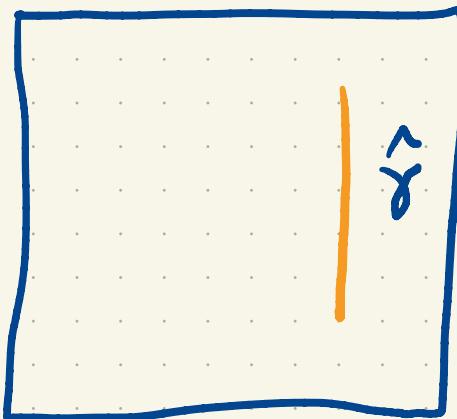
Covector = Infinitesimal Function

(u, v)



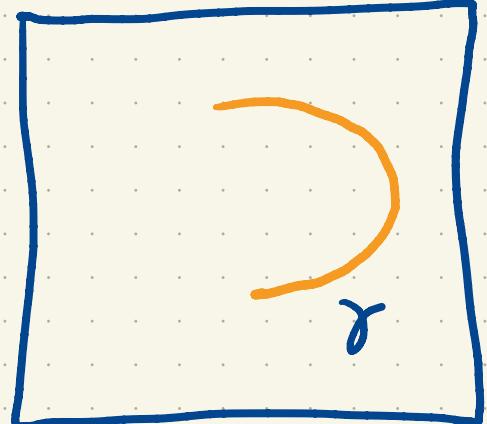
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

True thing



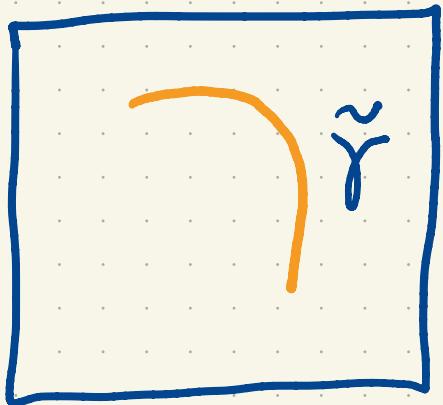
T (e.g., temp)

$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \begin{bmatrix} u' \\ v' \end{bmatrix} = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right] \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

\uparrow \uparrow
 $\frac{d}{dt} \tilde{T}(\tilde{\gamma}(t))$ $\frac{d}{dt} \hat{T}(\hat{\gamma}(t))$

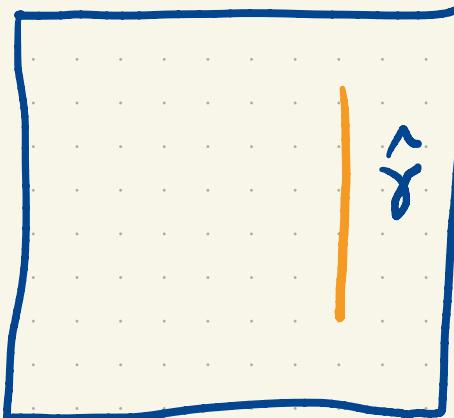
Covector = Infinitesimal Function

(u, v)



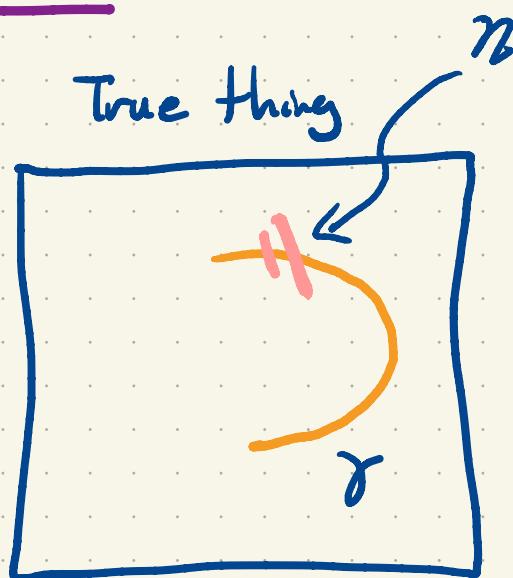
\tilde{T}

(r, θ)



\hat{T}

True thing

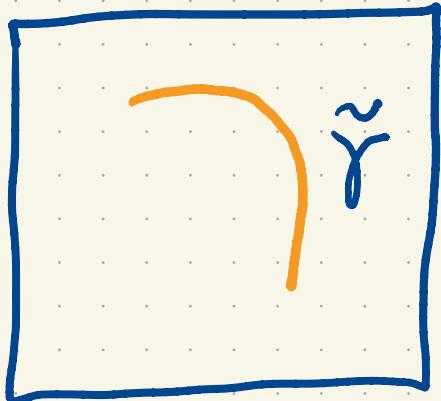


T (e.g., temp)

$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \begin{bmatrix} u' \\ v' \end{bmatrix} = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right] \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

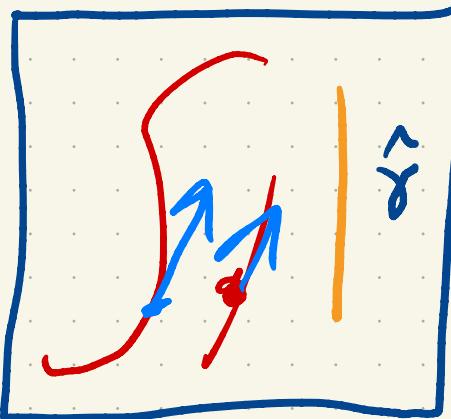
Covector = Infinitesimal Function

(u, v)



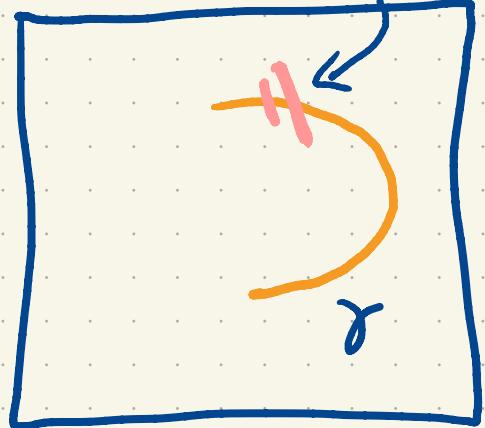
\tilde{n}

(r, θ)



\hat{n}

True thing



T (e.g., temp)

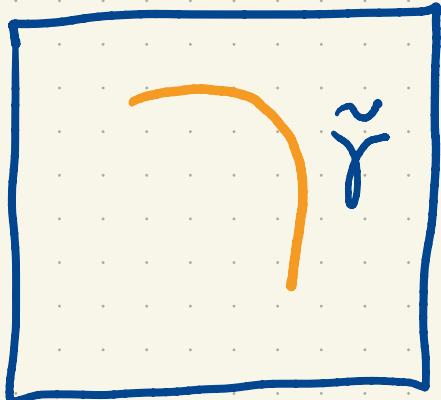
$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \begin{bmatrix} u' \\ v' \end{bmatrix} = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right] \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

\tilde{n} \hat{n}



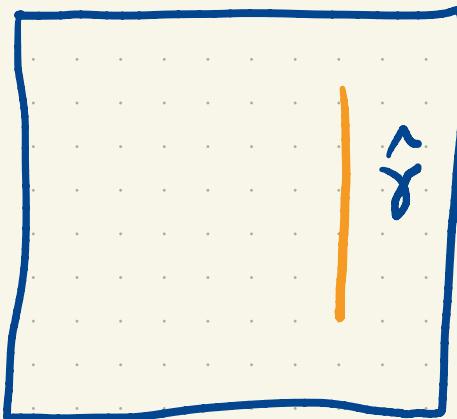
Covector = Infinitesimal Function

(u, v)



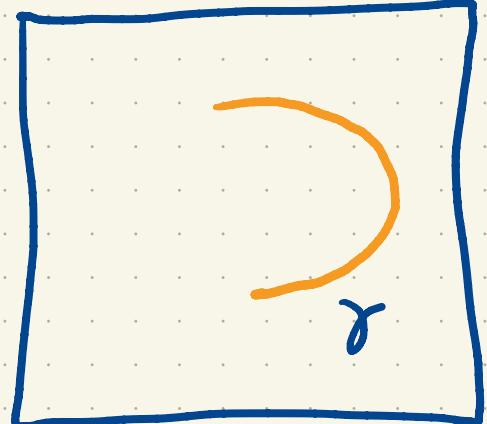
\tilde{T}

(r, θ)



\hat{T}

True thing

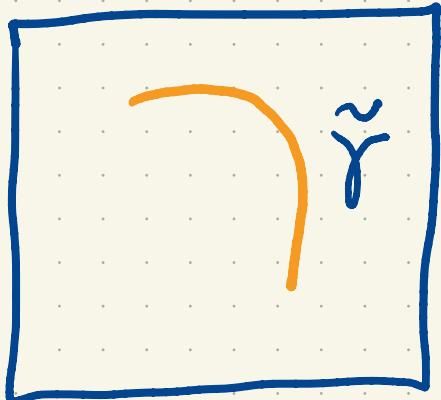


T (e.g., temp)

$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] \circ J \begin{bmatrix} r' \\ \theta' \end{bmatrix} = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right] \begin{bmatrix} r' \\ \theta' \end{bmatrix}$$

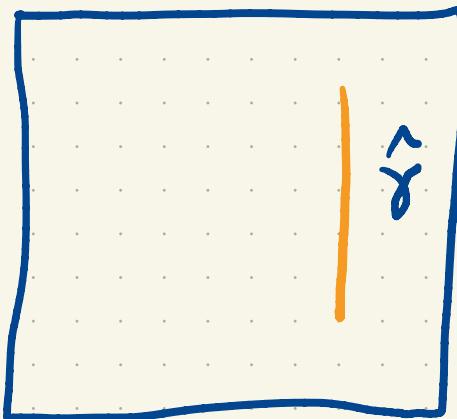
Covector = Infinitesimal Function

(u, v)



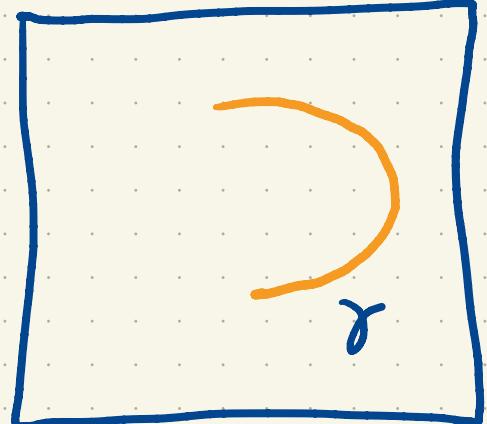
\tilde{T}

(r, θ)



\hat{T}

True thing

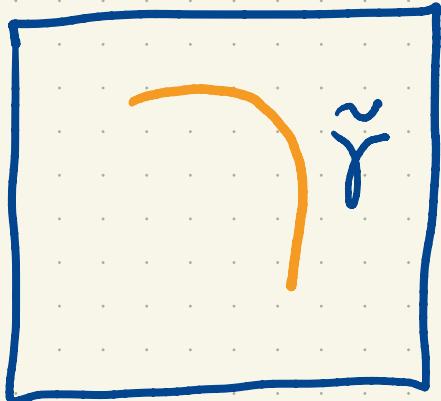


T (e.g., temp)

$$\left[\frac{\partial \tilde{T}}{\partial u}, \frac{\partial \tilde{T}}{\partial v} \right] J = \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right]$$

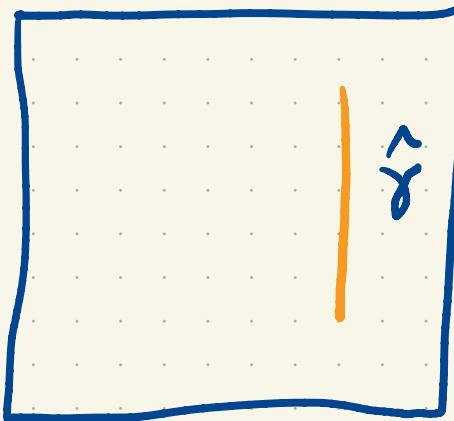
Covector = Infinitesimal Function

(u, v)



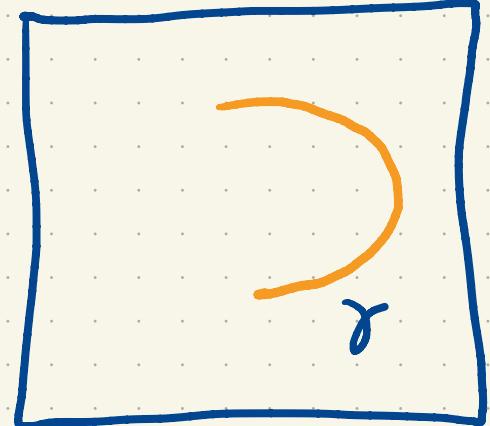
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

True thing

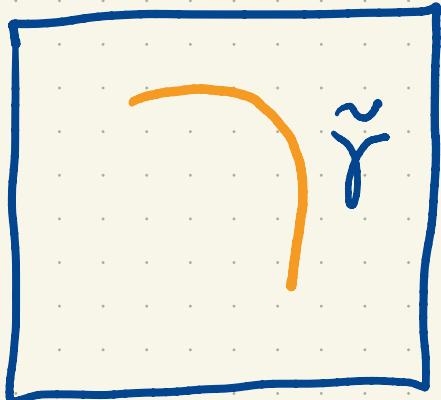


T (e.g., temp)

$$\tilde{n} J = \hat{n}$$

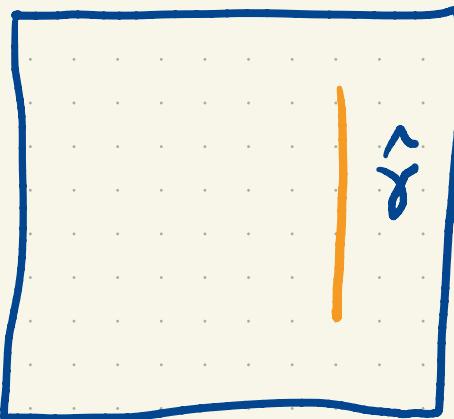
Covector = Infinitesimal Function

(u, v)



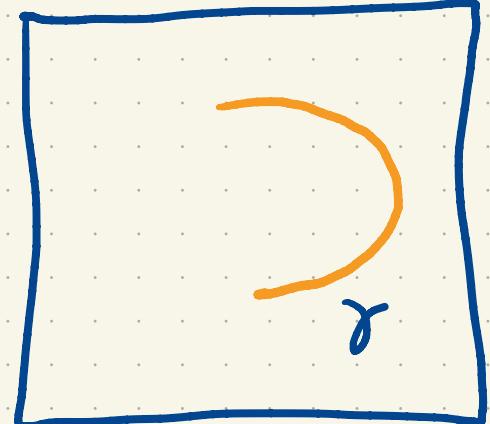
$\tilde{\tau}$

(r, θ)



$\hat{\tau}$

True thing



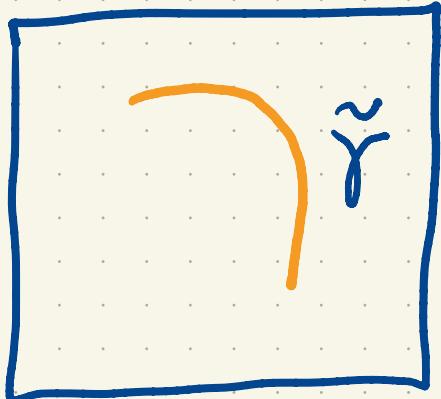
T (e.g., temp)

$$\tilde{n} J = \hat{n}$$

$$\tilde{X} = J \hat{X}$$

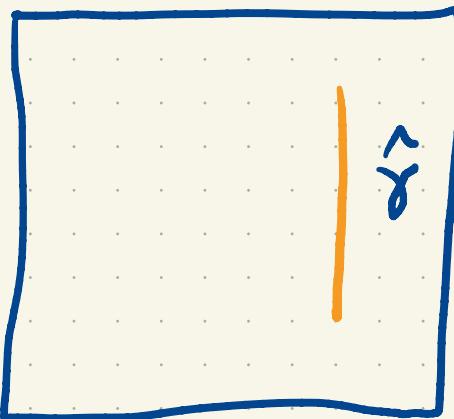
Covector = Infinitesimal Function

(u, v)



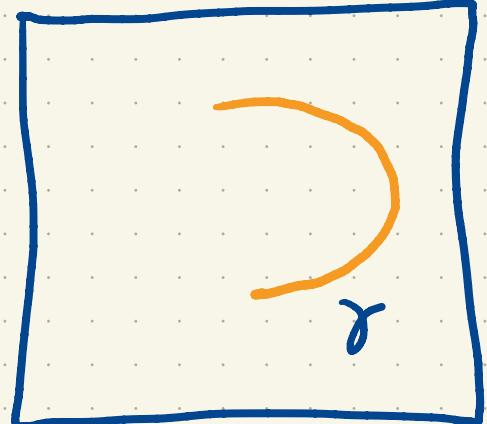
$\tilde{\tau}$

(r, θ)



$\hat{\gamma}$

True thing



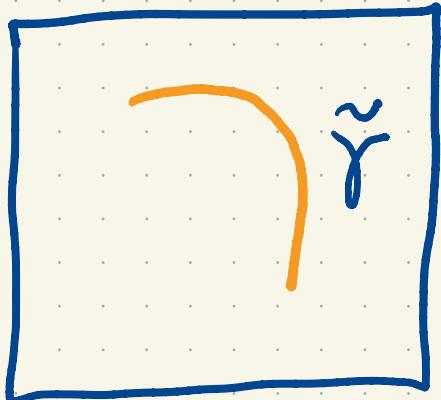
T (e.g., temp)

$$\tilde{n} = \hat{n} J^{-1}$$

$$\tilde{x} = J \hat{x}$$

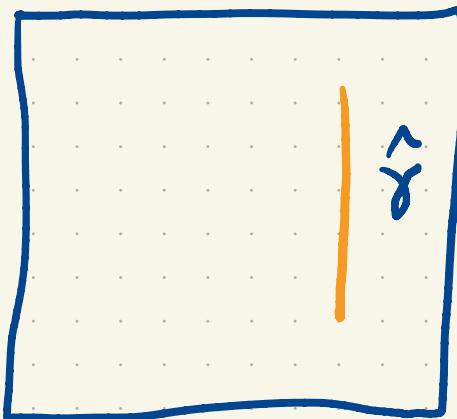
Covector = Infinitesimal Function

(u, v)



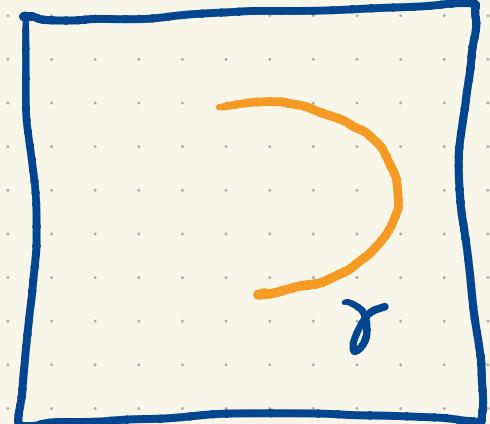
$\tilde{\tau}$

(r, θ)



$\hat{\gamma}$

True thing



T (e.g., temp)

$$\tilde{n} \tilde{X} = \hat{n} J^{-1} J \hat{X} = \hat{n} \hat{X}$$

Job of a Covector

Covectors eat vectors and give back numbers.

$$n[x] = \hat{n} \hat{x} = \tilde{n} \tilde{x}$$

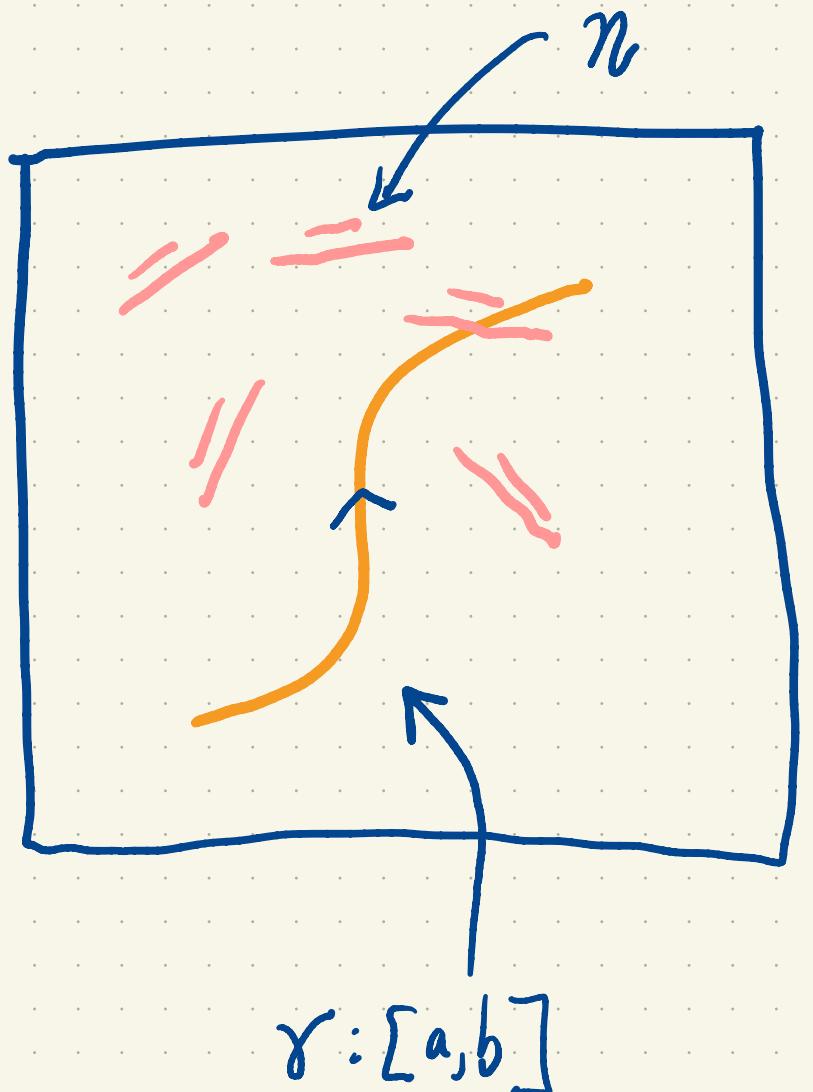
Job of a Covector

Covectors eat vectors and give back numbers.

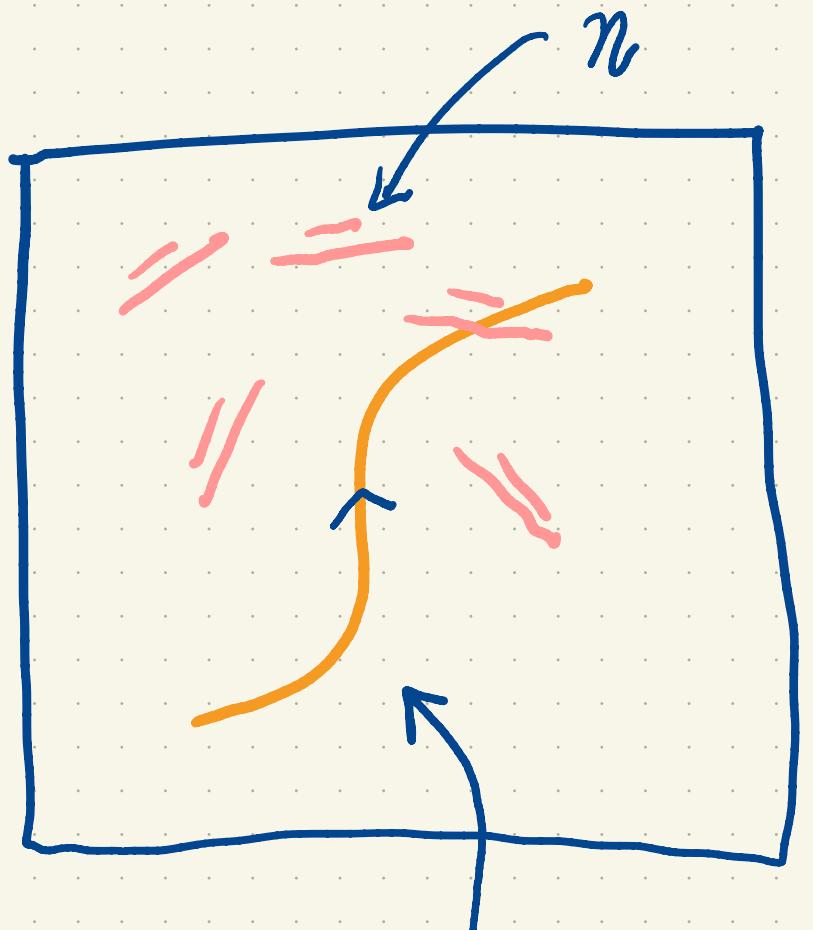
$$n[x] = \hat{n} \hat{x} = \tilde{n} \tilde{x}$$

"How fast is the infinitesimal function changing as I move with tangent vector X ?"

Fields of Covectors Eat Curves

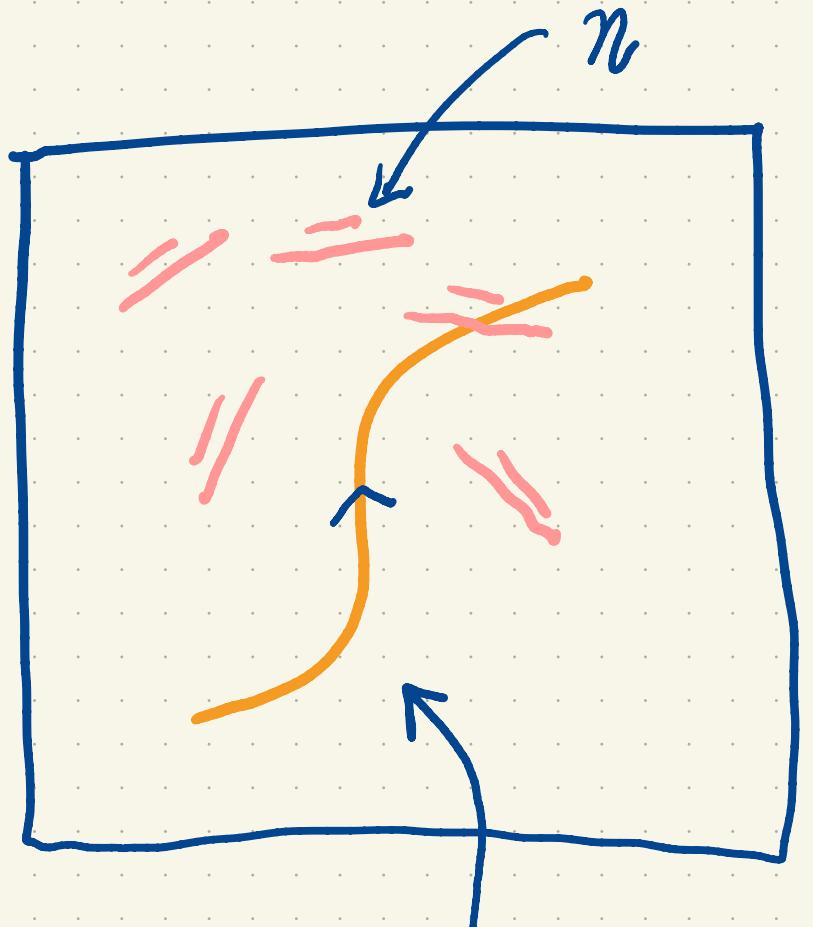


Fields of Covectors Eat Curves



$$\int_{\gamma} n = \int_a^b n[\gamma'(t)] dt$$

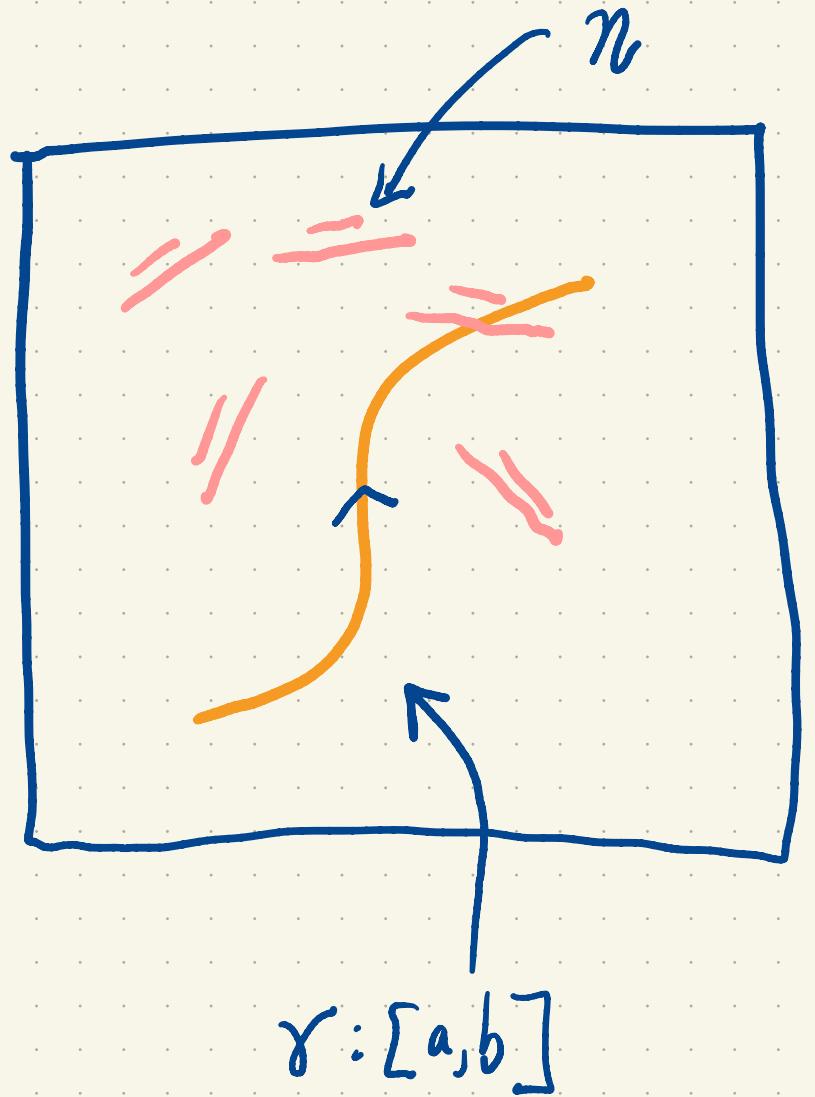
Fields of Covectors Eat Curves



$$\gamma: [a, b]$$

$$\int_{\gamma} n = \int_a^b n[\gamma'(t)] dt$$
$$= \int_a^b \hat{n} \hat{\gamma}' dt$$

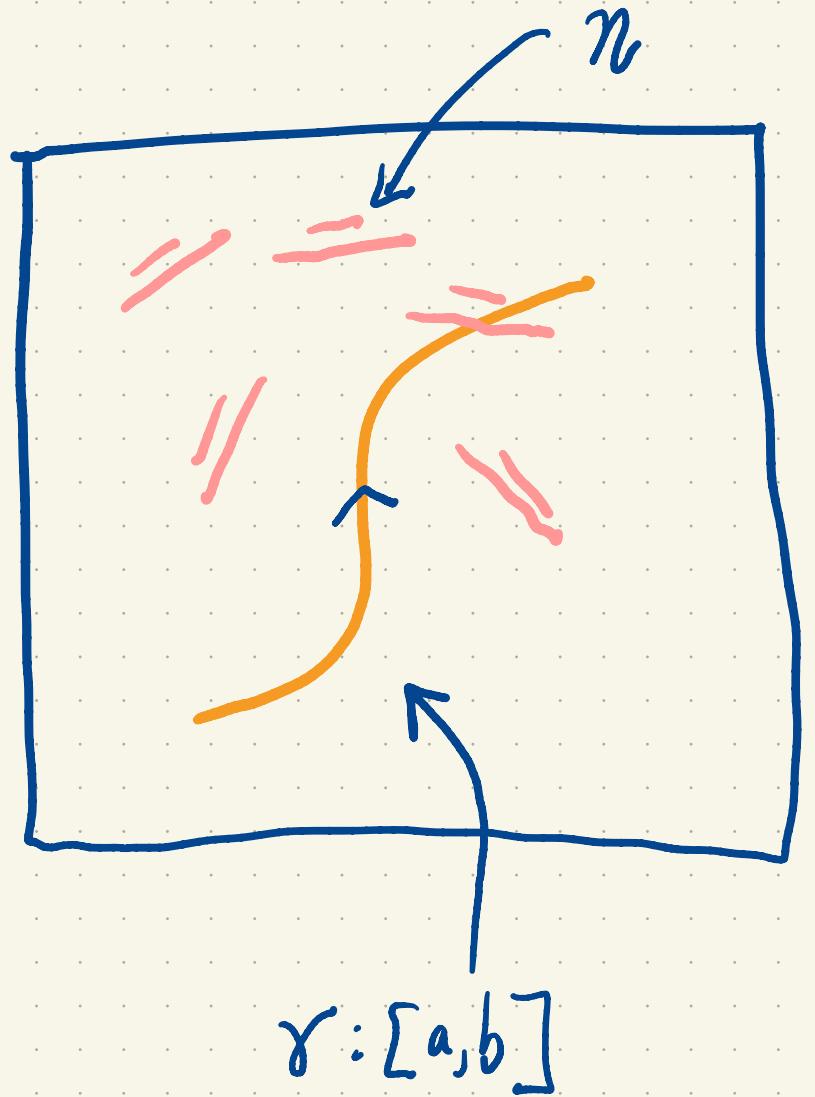
Fields of Covectors Eat Curves



$$\int_{\gamma} n = \int_a^b n[\gamma'(t)] dt$$
$$= \int_a^b \hat{n} \hat{\gamma}' dt$$
$$= \int_a^b \tilde{n} \tilde{\gamma}' dt$$

Fields of Covectors Eat Curves

X



$$\int_{\gamma} n = \int_a^b n[\gamma'(t)] dt$$
$$= \int_a^b \hat{n} \hat{\gamma}' dt$$
$$= \int_a^b \tilde{n} \tilde{\gamma}' dt$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function

covector

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function

covector

$$dT[X] = ?$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function

covector

$$dT[\gamma'] = \frac{d}{dt} T \circ \gamma$$

Exterior Derivative (part 0)

$$T \xrightarrow{d} dT$$

function

covector

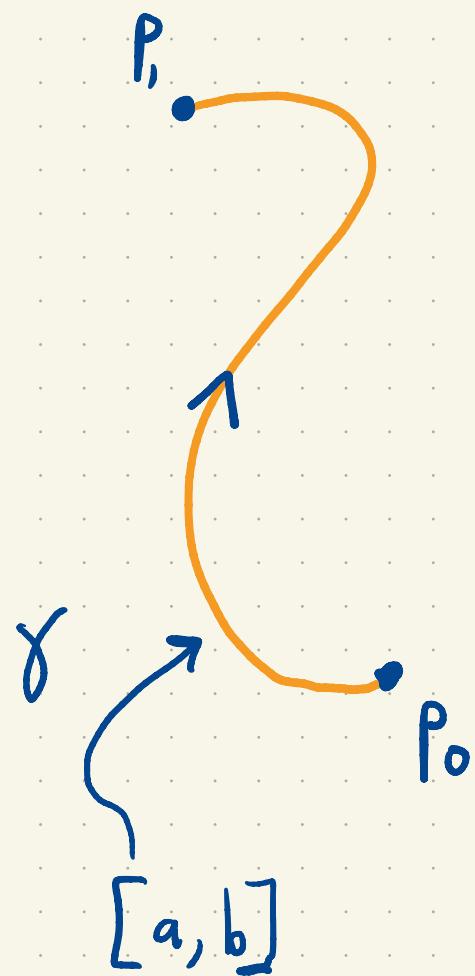
$$dT[\gamma'] = \frac{d}{dt} T \circ \gamma$$

$$\left[\widetilde{\frac{\partial T}{\partial u}}, \widetilde{\frac{\partial T}{\partial v}} \right] \longleftrightarrow \left[\frac{\partial \hat{T}}{\partial r}, \frac{\partial \hat{T}}{\partial \theta} \right]$$

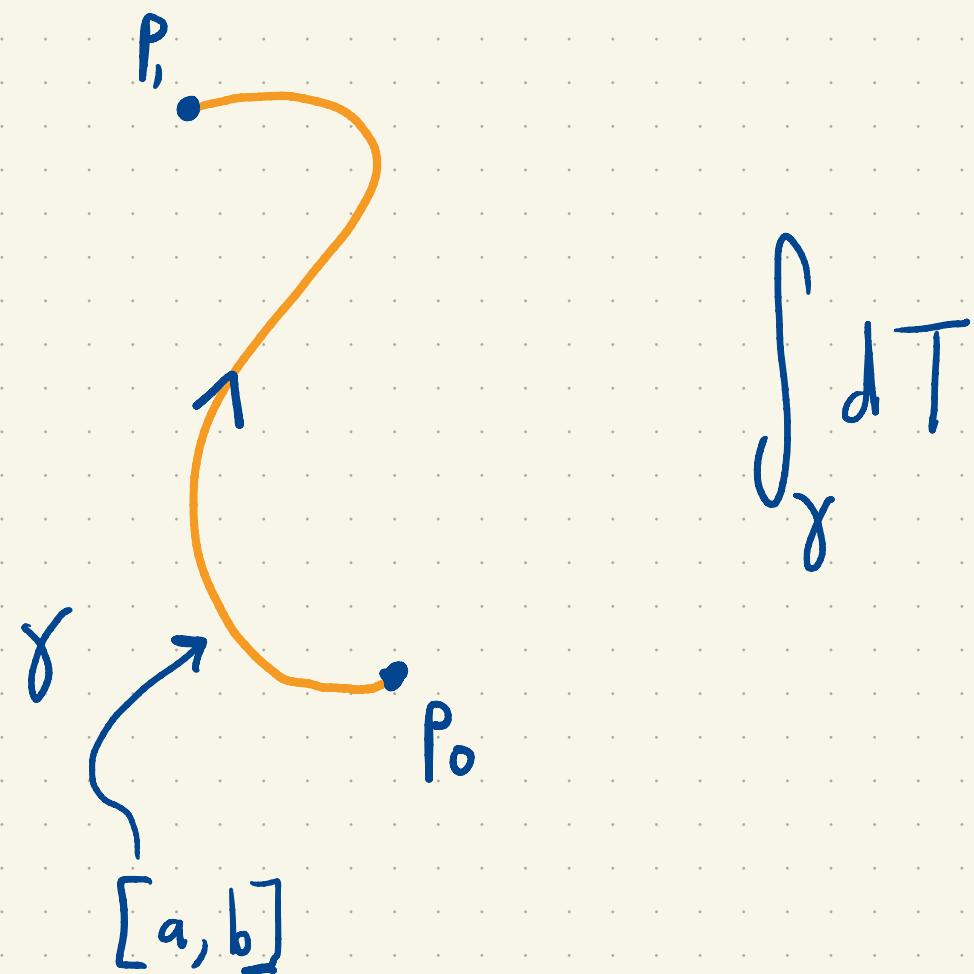
$$\widetilde{dT}$$

$$\widehat{dT}$$

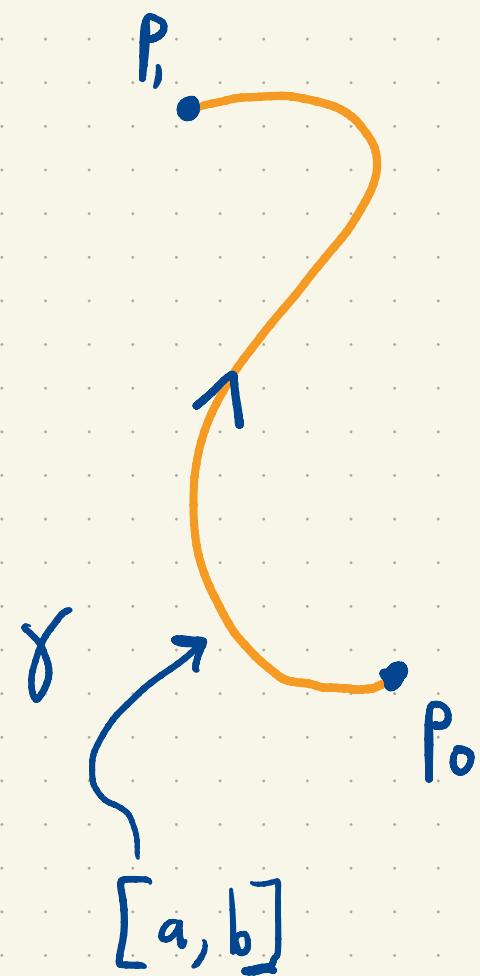
Exterior Derivative (part 0)



Exterior Derivative (part 0)

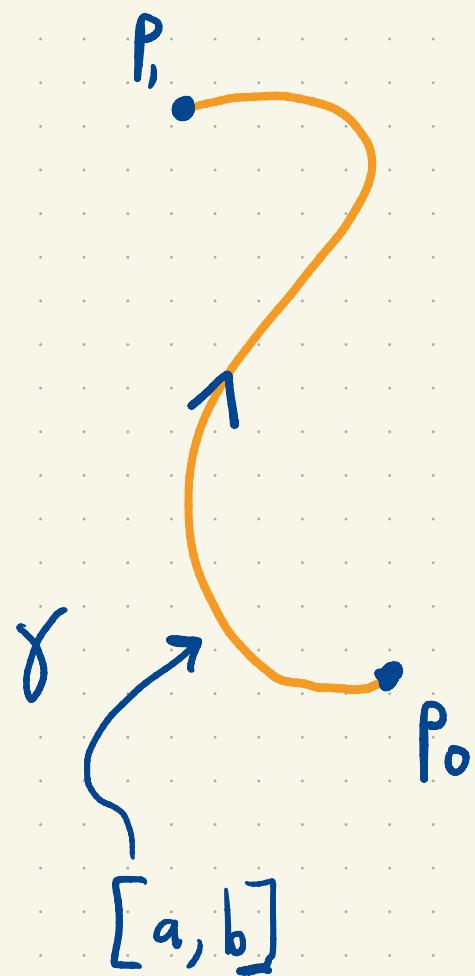


Exterior Derivative (part 0)



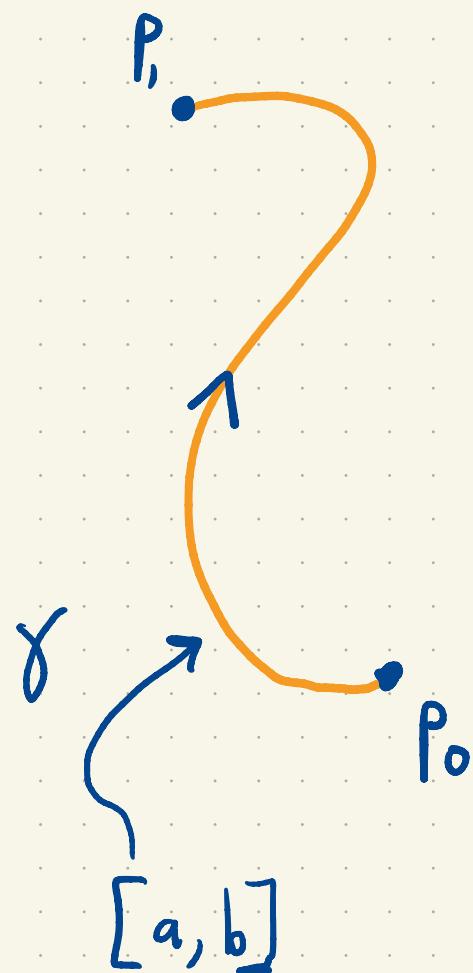
$$\int_{\gamma} dT = \int_a^b dT(\gamma'(t)) dt$$

Exterior Derivative (part 0)



$$\int_{\gamma} dT = \int_a^b \frac{d}{dt} T(\gamma(t)) dt$$

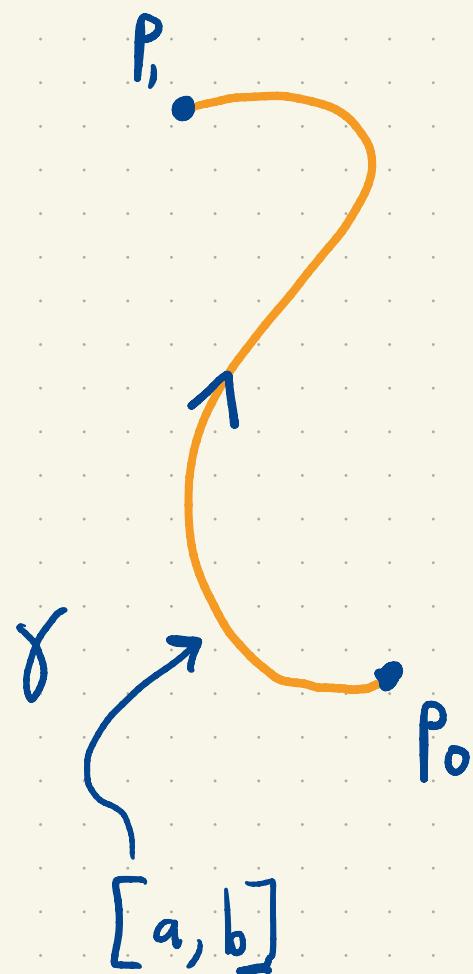
Exterior Derivative (part 0)



$$\int_{\gamma} dT = \int_a^b \frac{d}{dt} T(\gamma(t)) dt$$
$$= T(\gamma(b)) - T(\gamma(a))$$

FTC: $\int_a^b f'(s) ds = f(b) - f(a)$

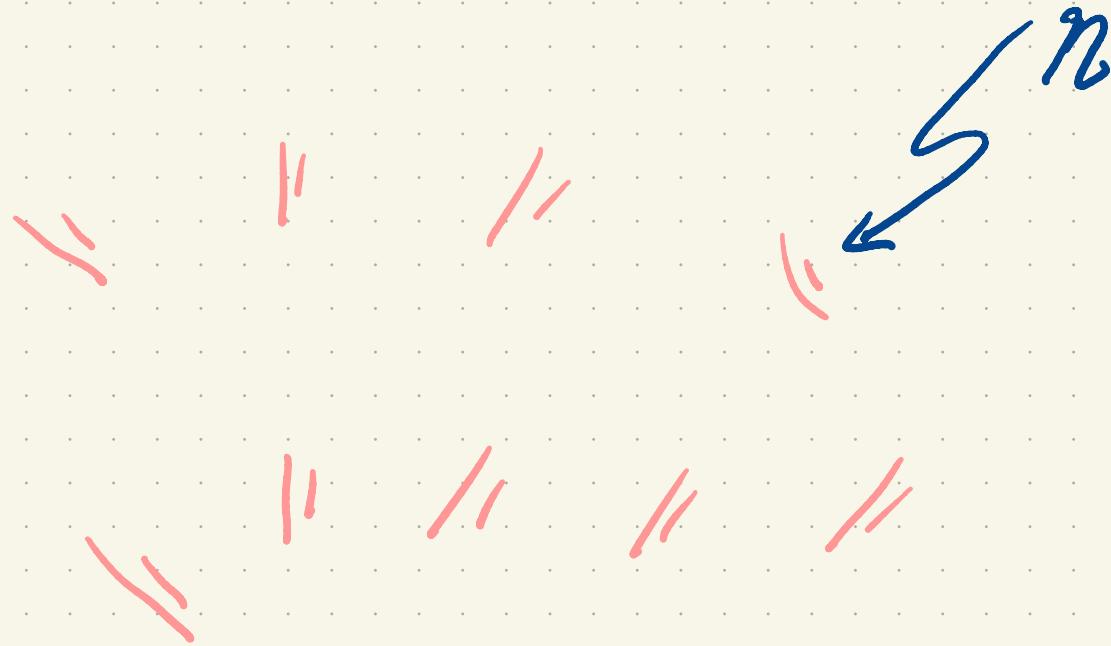
Exterior Derivative (part 0)



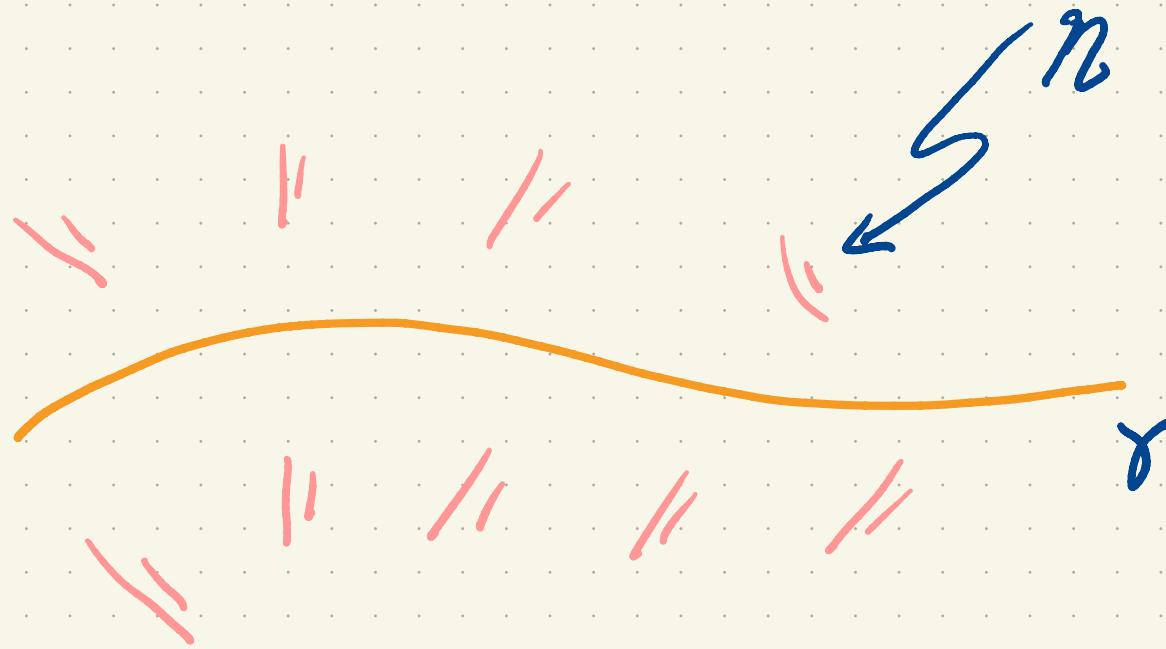
$$\begin{aligned}\int_{\gamma} dT &= \int_a^b \frac{d}{dt} T(\gamma(t)) dt \\ &= T(\gamma(b)) - T(\gamma(a)) \\ &= T(P_1) - T(P_0)\end{aligned}$$

FTC: $\int_a^b f'(s) ds = f(b) - f(a)$

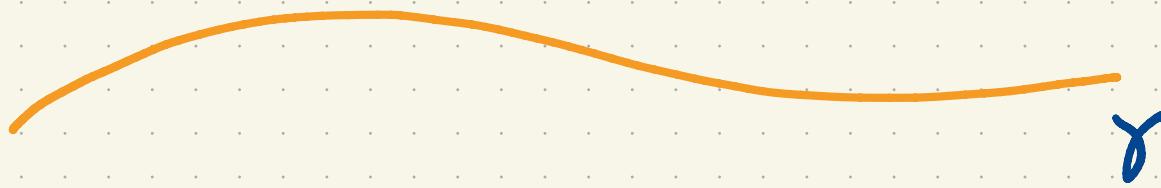
Derivative of a Covector Field



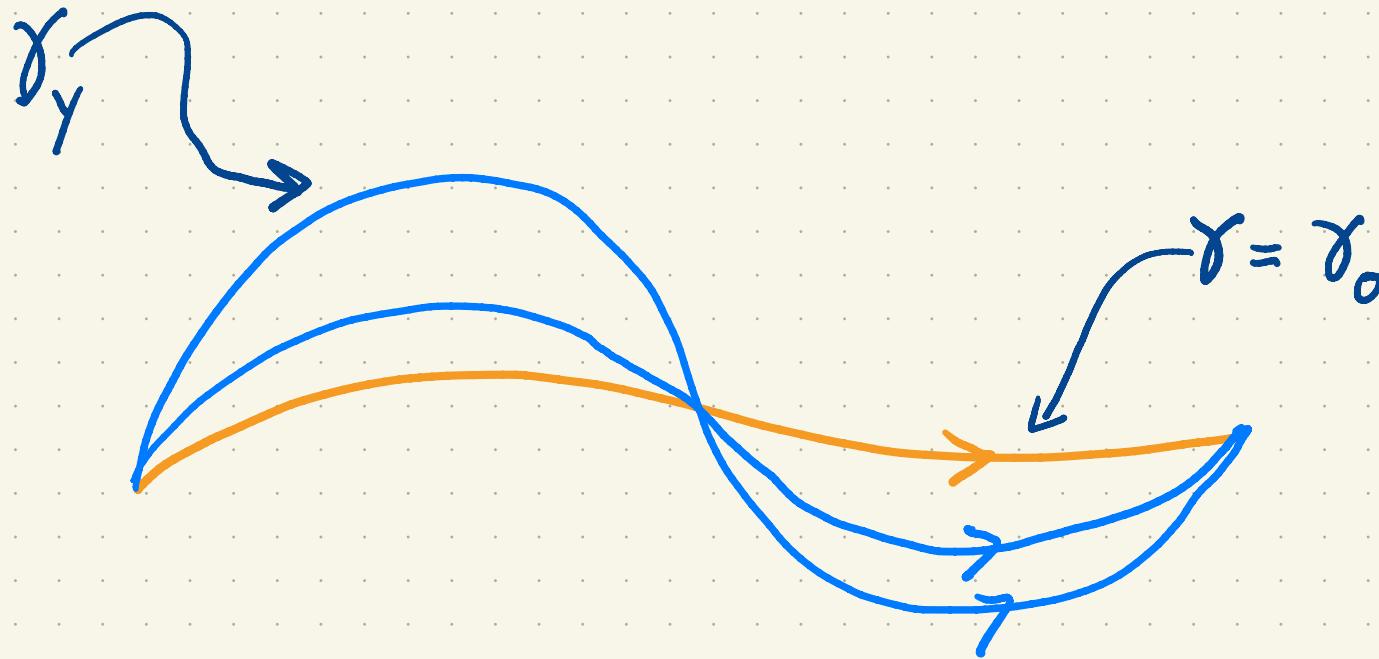
Derivative of a Covector Field



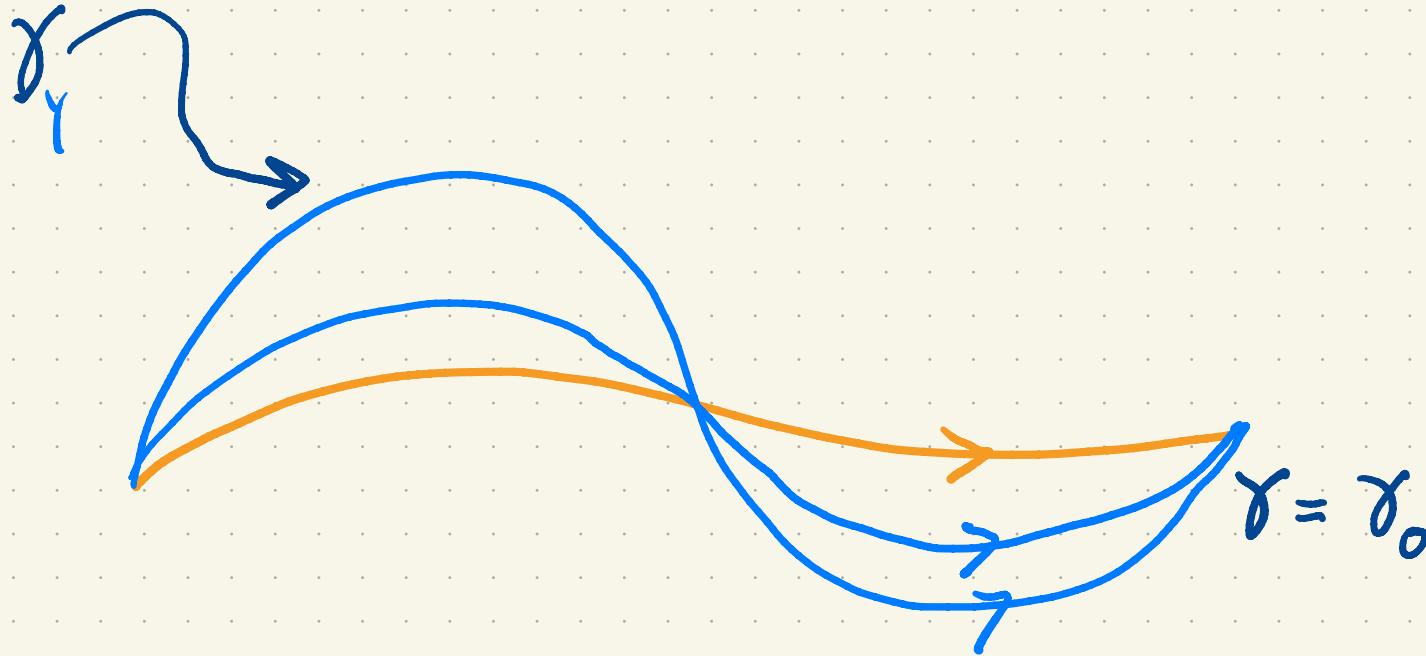
Derivative of a Covector Field



Derivative of a Covector Field

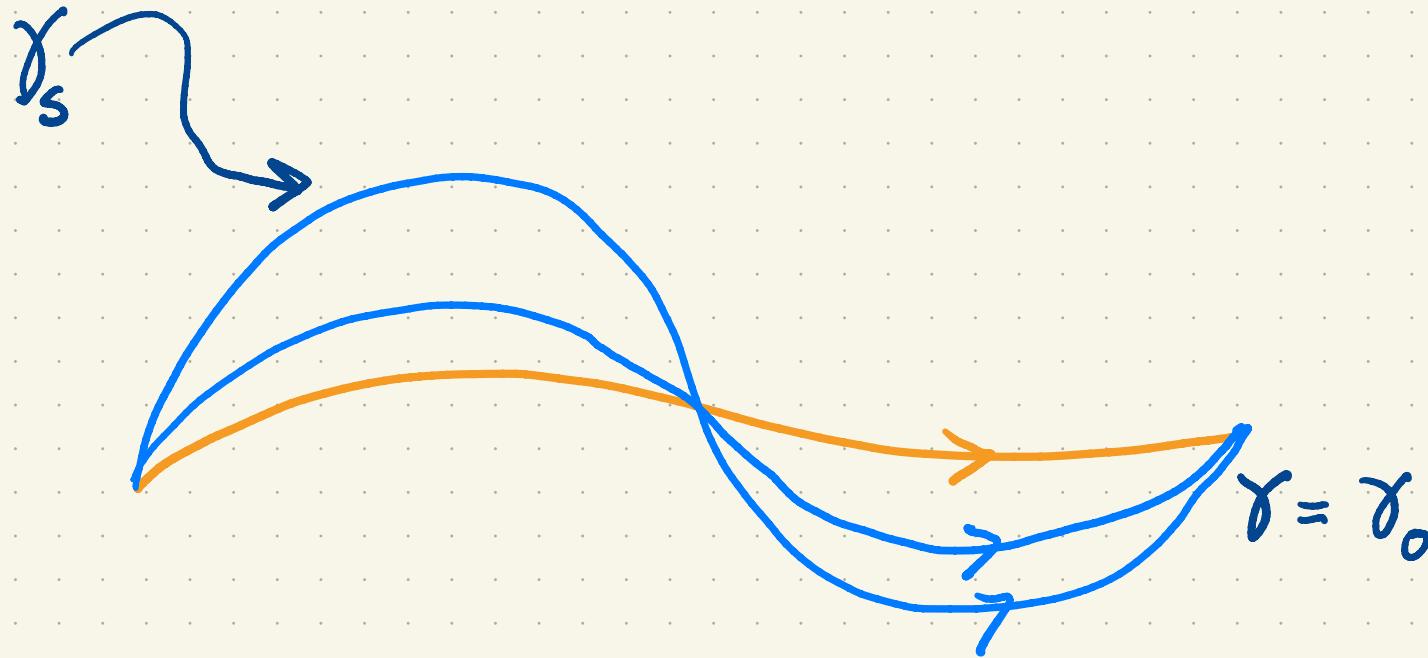


Derivative of a Covector Field



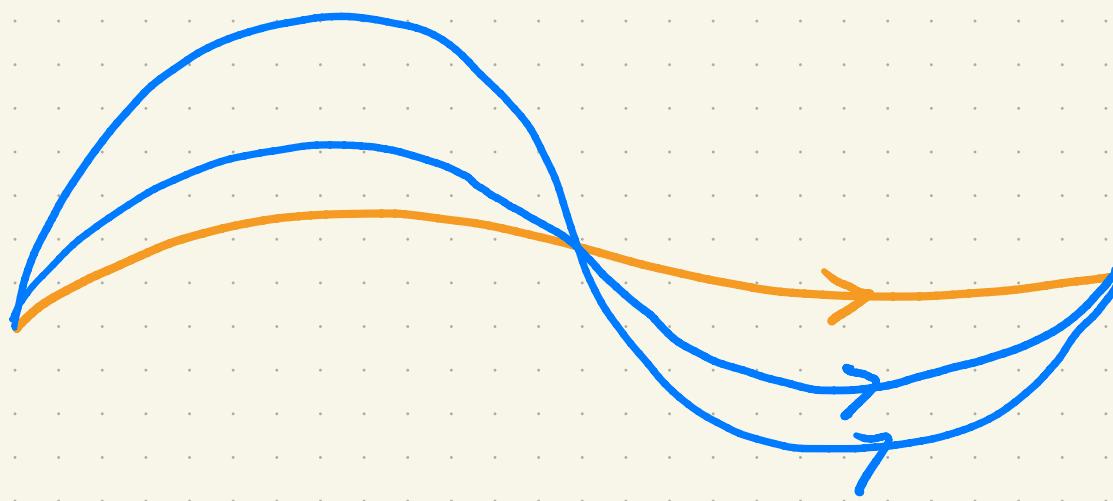
$$h(\gamma) = \int_{\gamma_y}^{\gamma} n$$

Derivative of a Covector Field

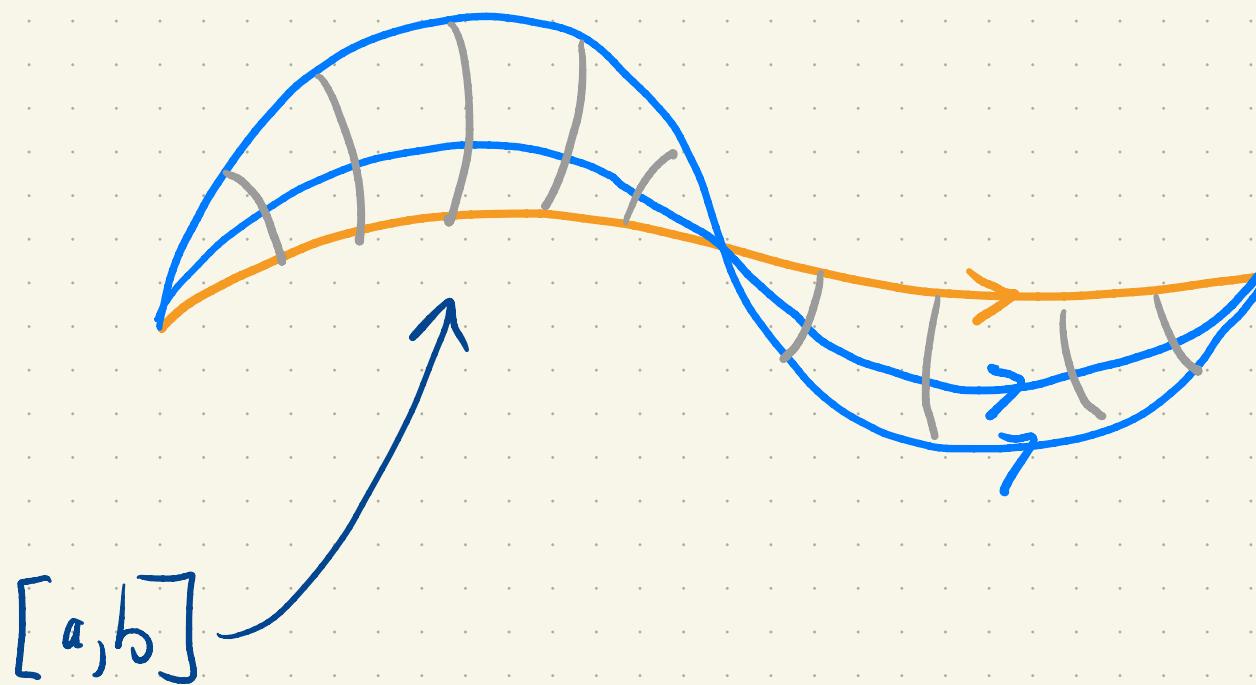


$$h(s) = \int_{\gamma_s}^{\gamma} n \quad | \quad h'(0) = ?$$

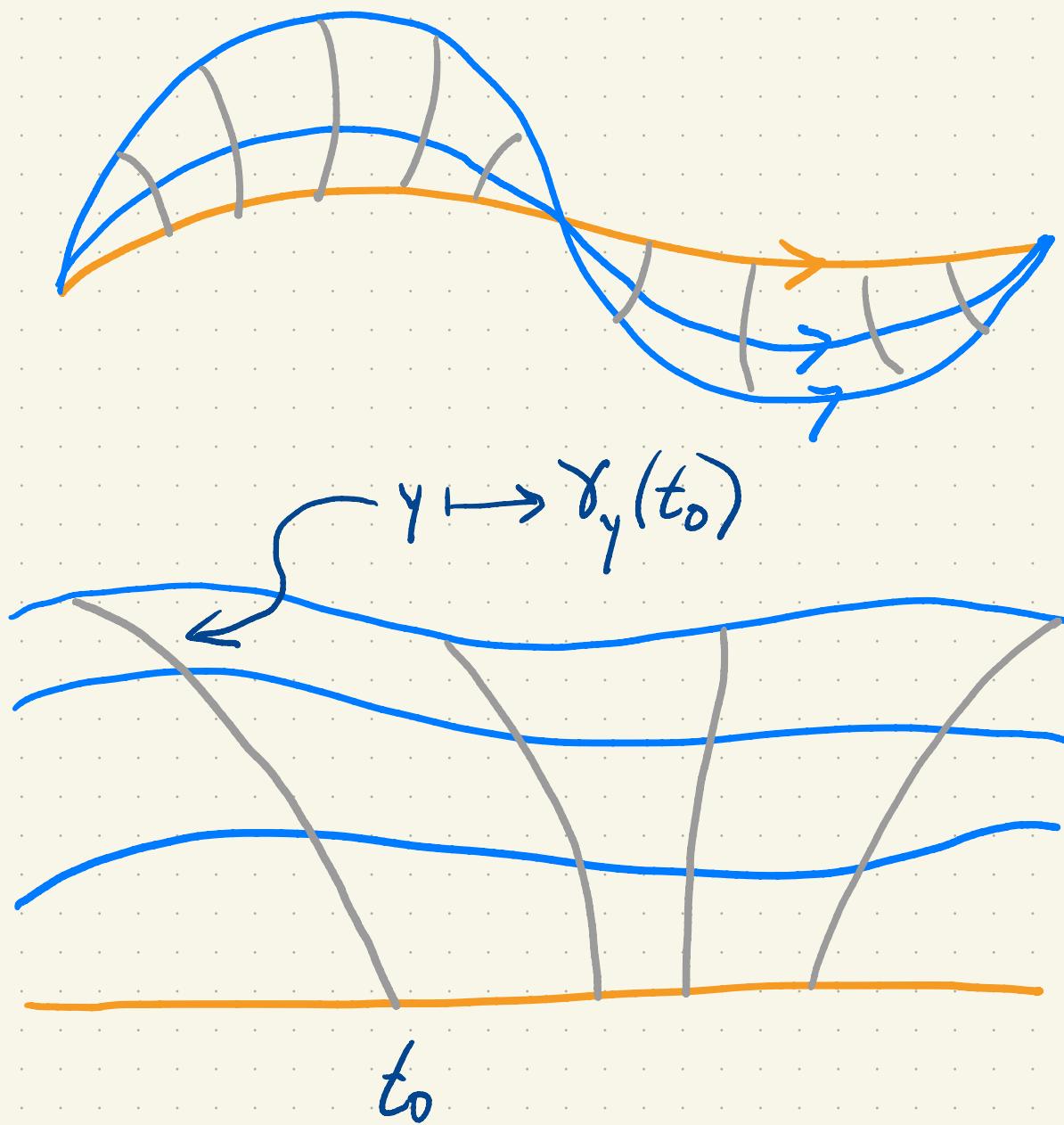
Derivative of a Covector Field



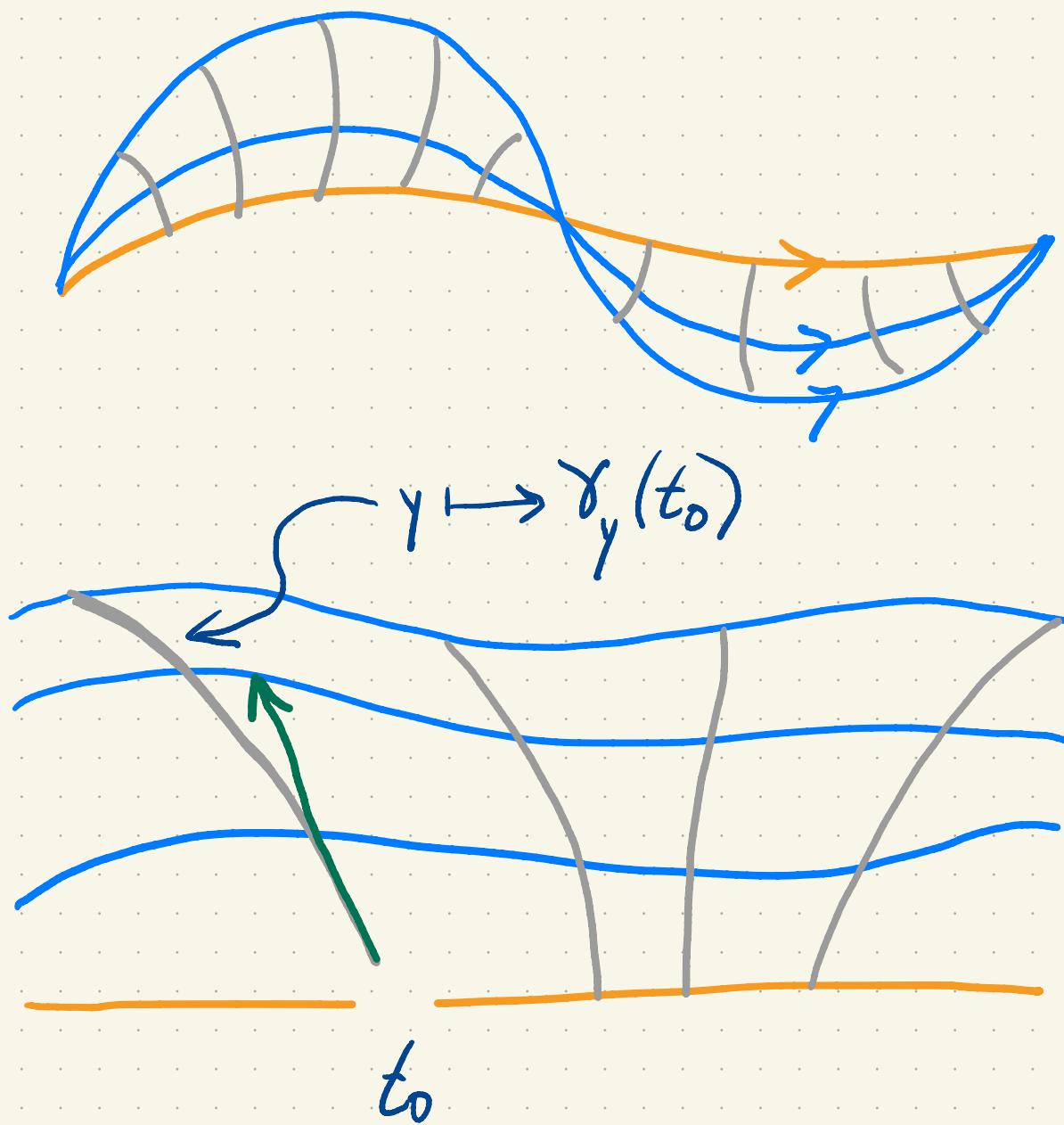
Derivative of a Covector Field



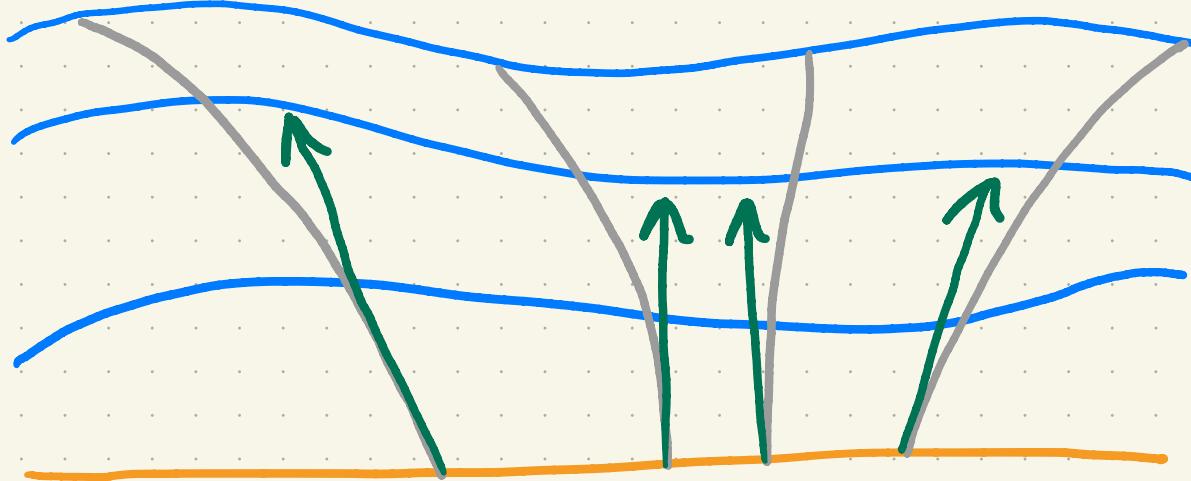
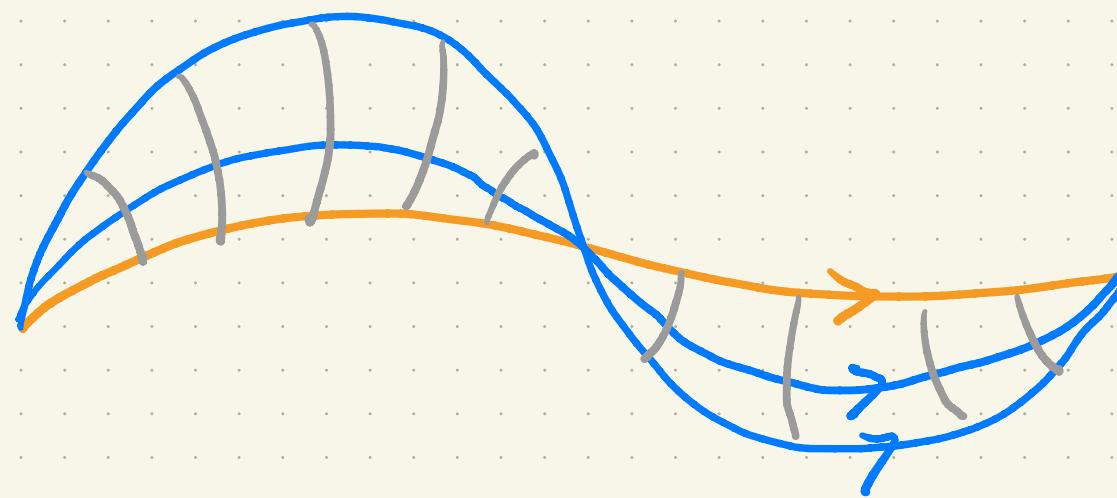
Derivative of a Covector Field



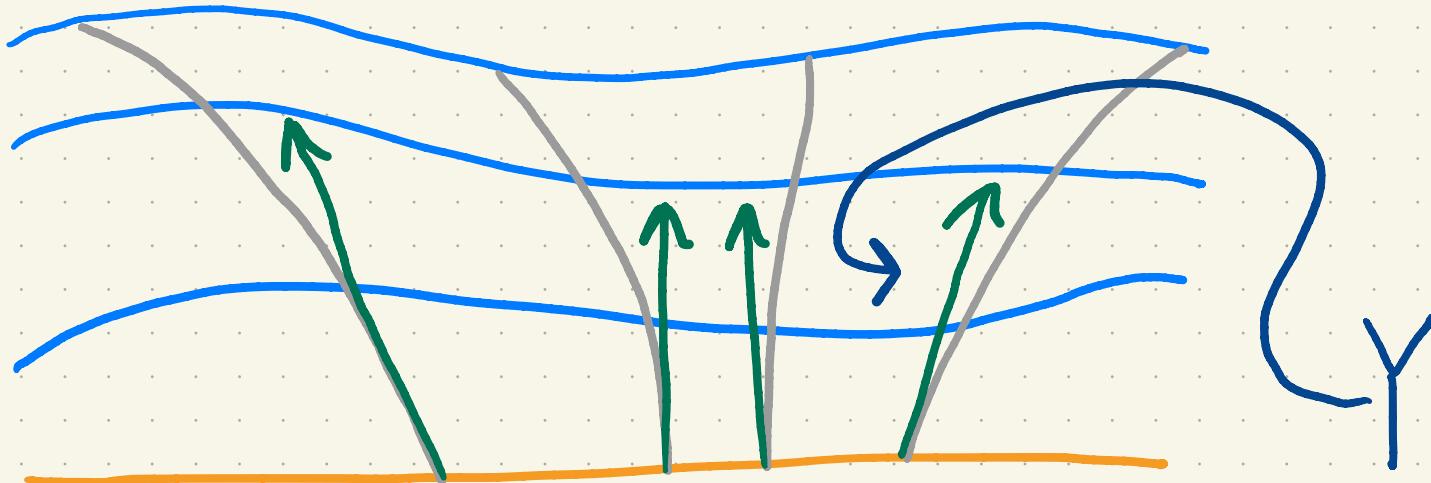
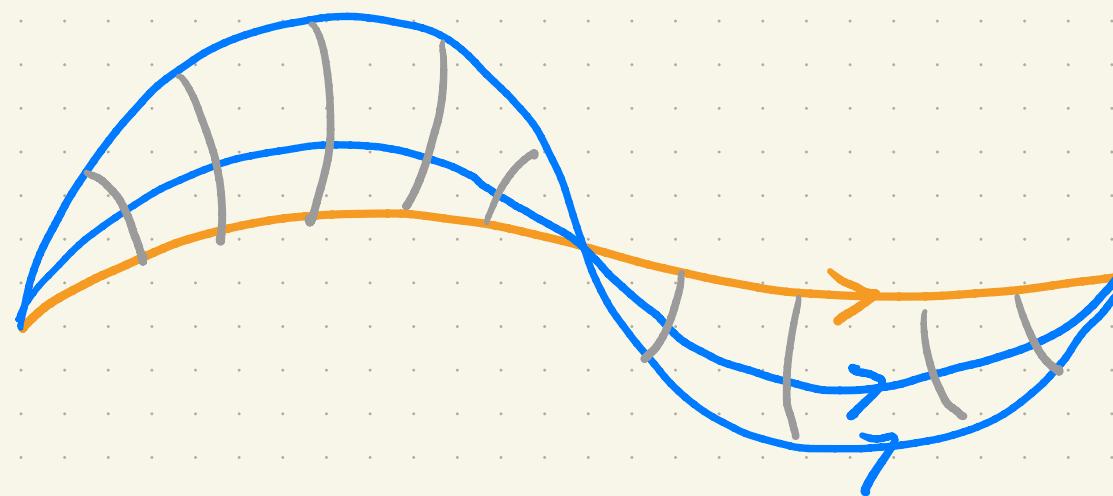
Derivative of a Covector Field



Derivative of a Covector Field



Derivative of a Covector Field



Exterior Derivative (Part 1)

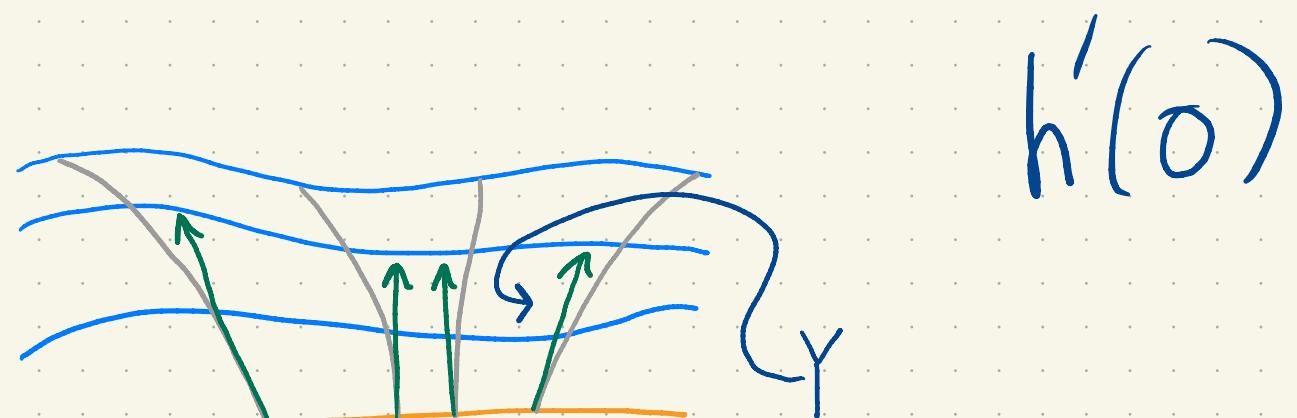
- $n \xrightarrow{d} dn$

Exterior Derivative (Part 1)

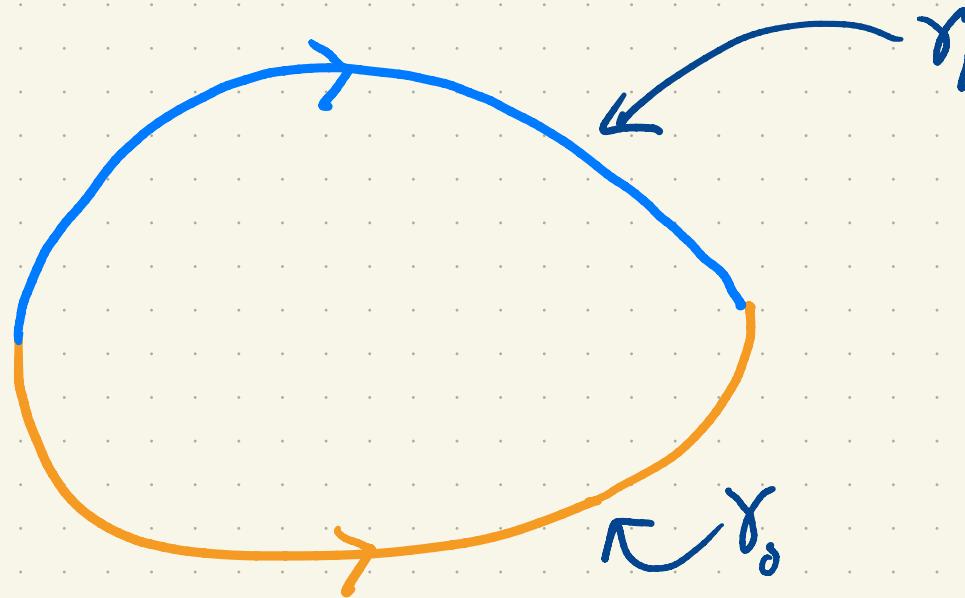
- $n \xrightarrow{d} dn$
- $dn[y, \cdot]$ is a covector

Exterior Derivative (Part 1)

- $n \xrightarrow{d} dn$
- $\int_Y dn [Y, \cdot]$ is a covector
- $\int_Y dn [Y, \cdot] = \left. \frac{d}{dy} \right|_{y=0} \int_Y n$

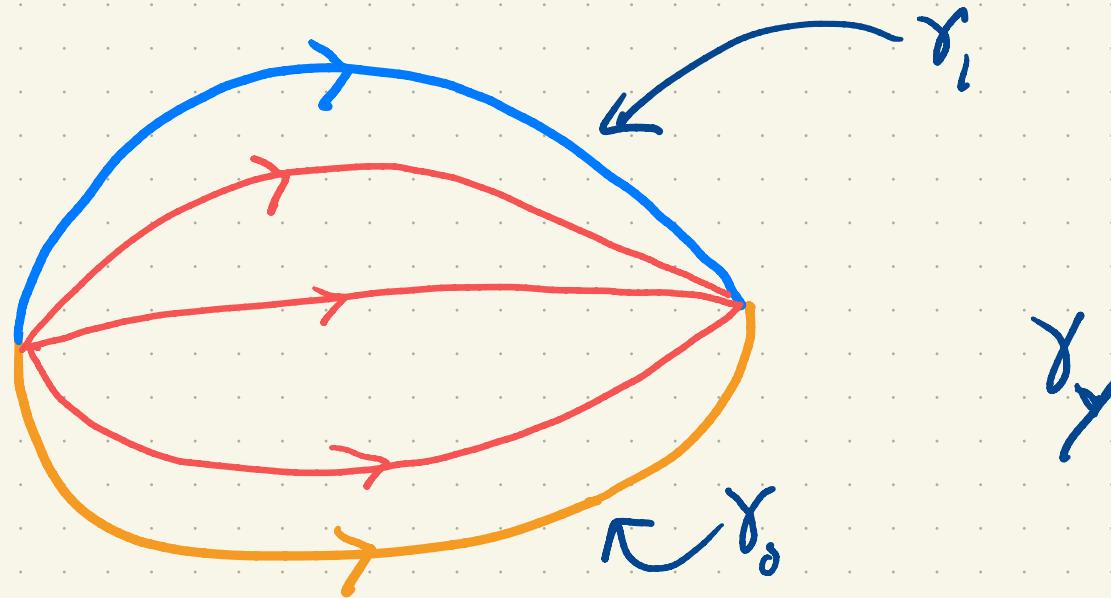


Stokes' Theorem (Preview)



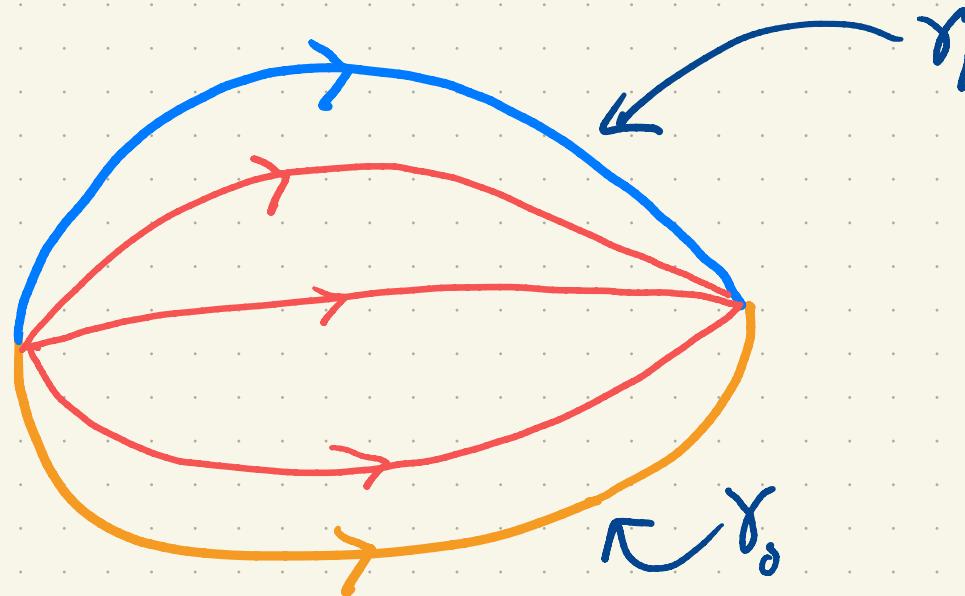
$$\int_{\gamma_1} \mathbf{r} - \int_{\gamma_0} \mathbf{r} = ?$$

Stokes' Theorem (Preview)



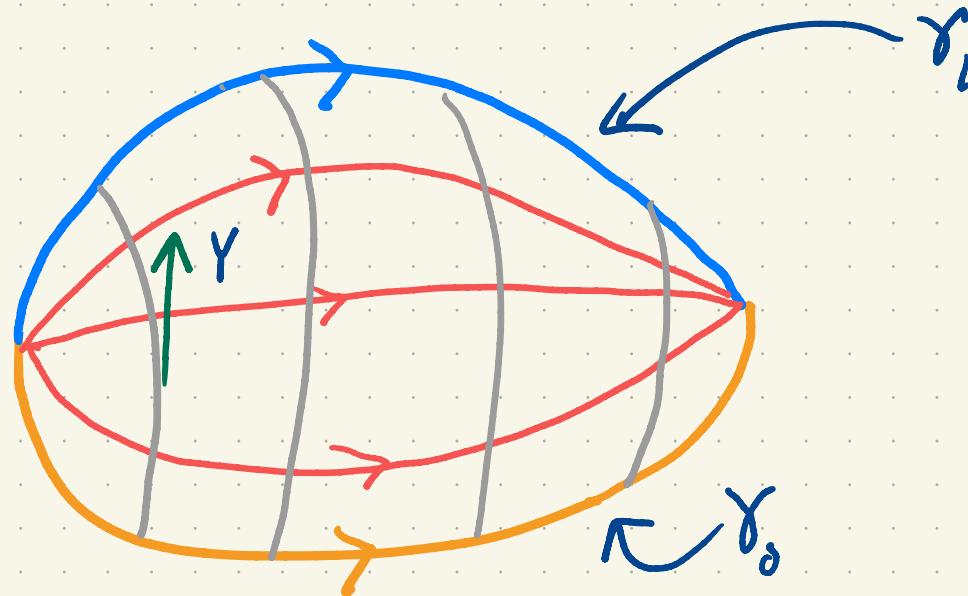
$$\int_{\gamma_1} n - \int_{\gamma_3} n = ?$$

Stokes' Theorem (Preview)



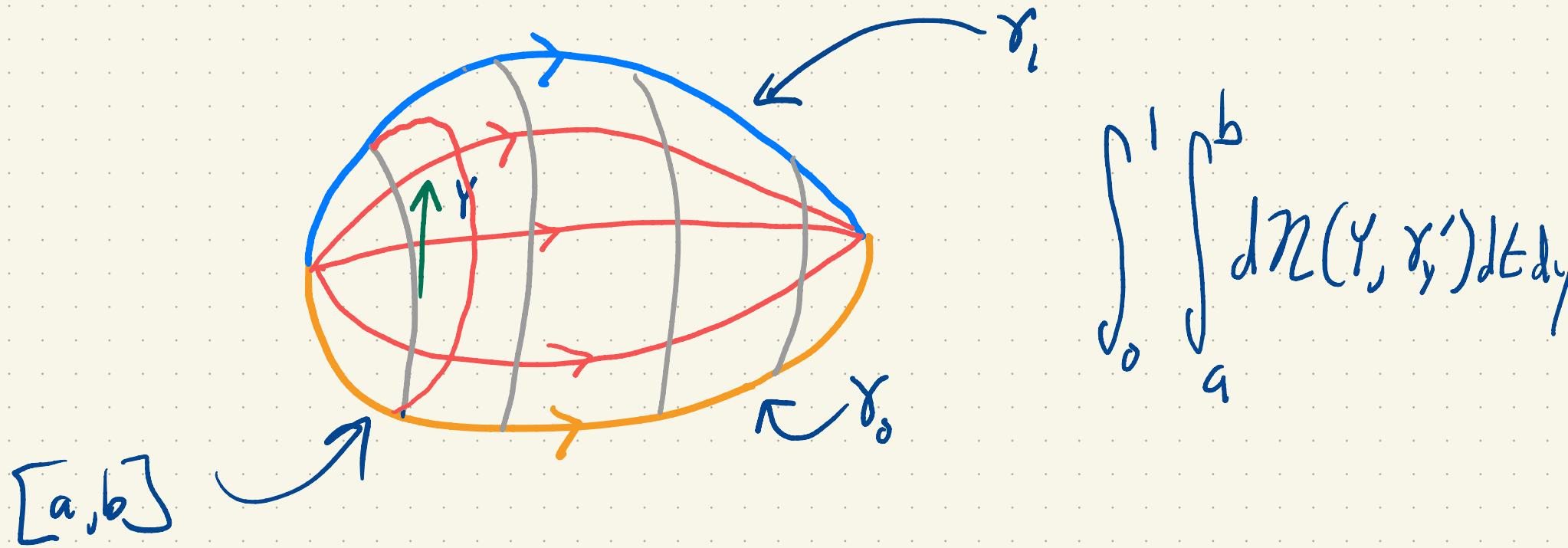
$$\int_{\gamma_1} n - \int_{\gamma_0} n = \int_0^1 \left[\frac{d}{dy} \int_{\gamma_y} n \right] dy$$

Stokes' Theorem (Preview)



$$\int_{\gamma_1} n - \int_{\gamma_0} n = \int_0^1 \left[\frac{d}{dy} \int_{\gamma_y} n \right] dy = \int_0^1 \int_{\gamma_y} d n(\gamma_y, y) dy$$

Stokes' Theorem (Preview)



$$\int_0^1 \int_a^b d\mathcal{N}(y, y') dt dy$$

$$\int_{\gamma_1} n - \int_{\gamma_0} n = \int_0^1 \left[\frac{d}{dy} \int_{\gamma_y} n \right] dy = \int_0^1 \int_a^b d\mathcal{N}(y, \gamma_y') dt dy$$