Elliptic PDEs (AKA steady state)

$$\partial_t u = \partial_x (a u_x) + F$$ ← forced heat equation

If f is independent of t steady state ($u_t = 0$) is possible

$$-\partial_x (a u_x) = F$$

$$-a u_{xx} - a_x u_x = F$$

Prototype:

$$-a\, u_{xx} - a_x\, u_x = F$$

Prototype:

$$-a\, u_{xx} - a_x u_x = F$$

$$\downarrow$$

$$u_{xx} + p u_x = f$$

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$$-a\, u_{xx} - a_x\, u_x = F$$

$$\downarrow$$

$$u_{xx} + p u_x = f$$

$$\downarrow$$

$$u_{xx} + p u_x + q u = f$$

who's this?

$$u_t = \partial_x (a\, u_x) + bu + F$$

$a \geq 0$

sign on b?

$$u_t = \partial_x (a\, u_x) + bu + F$$

$a \geq 0$

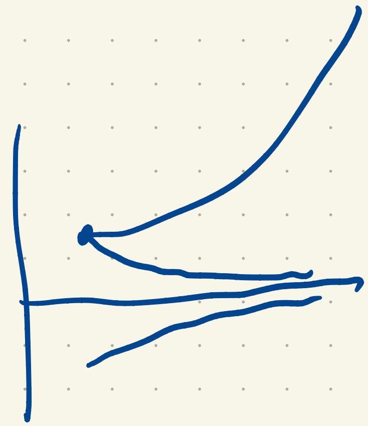
$b \leq 0$ for stability

$(u_t = bu)$

$$u_t = \partial_x (a\, u_x) + bu + F \qquad a \geq 0$$

$b \leq 0$ for stability

$$u_t = bu$$



$(u_t = bu)$

$$0 = a\, u_{xx} + a_x u_x + bu + F$$

$$u_{xx} + p\, u_x + q u = f$$

$q = \frac{b}{a} \leq 0$ for stability (in cts case)

Similar for wave equation:

$$u_{tt} = u_{xx} + f$$

$$u_{tt} \equiv 0 \qquad u_{xx} = -f$$

1-d Elliptic Problem

$$u_{xx} + p u_x + q u = f \qquad 0 \le x \le \ell$$

$$q \le 0$$

1-d Elliptic Problem

$$u_{xx} + p u_x + q u = f \qquad 0 \le x \le \ell$$

$$u(0) = \alpha$$

$$u(\ell) = \beta$$

$$u_{xx} + 3u_x - 7u = x$$

$u(0)$

$u_x(0)$

1-d Elliptic Problem

$$u_{xx} + p u_x + q u = f \qquad 0 \le x \le \ell$$

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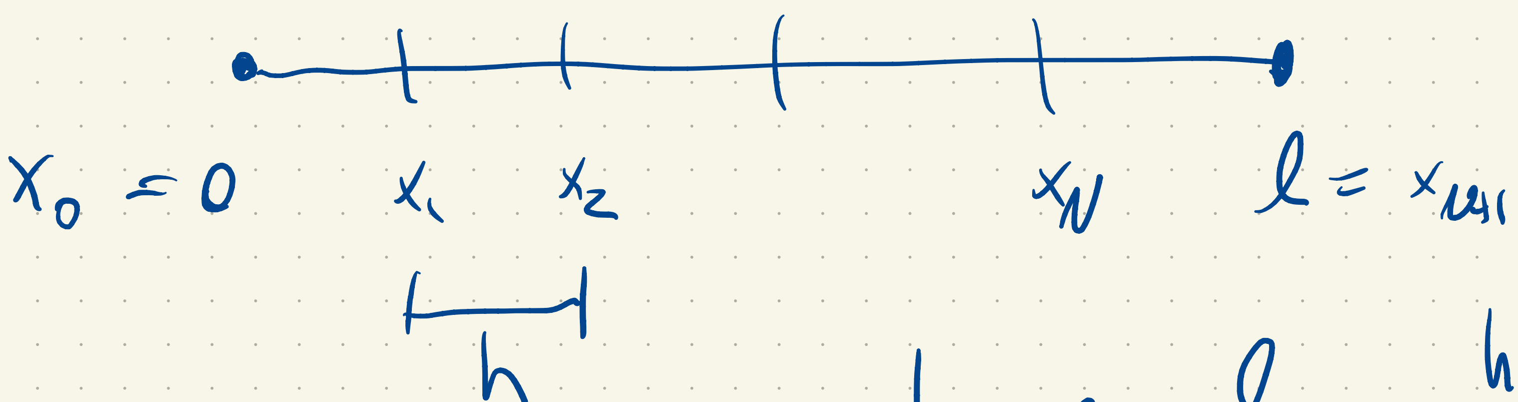

h vs ℓ $\qquad h = \dfrac{\ell}{N+1}$

Centered Differences

$$u_{xx} + p u_x + q u = f$$

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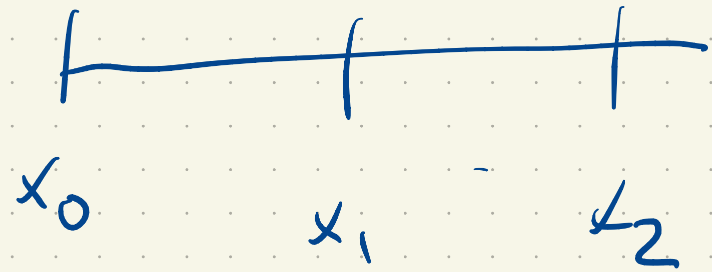
$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \underbrace{p(x_i)}_{p_i} \frac{u_{i+1} - u_{i-1}}{2h} + \underbrace{q(x_i)}_{q_i} u_i = \underbrace{f(x_i)}_{f_i}$$

Centered Differences



$$u_{xx} + p u_x + q u = f$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i = f_i$$

$$\tau = \frac{h^2}{12} u_{xxxx}(\eta_i) + p(x_i) u_{xxx}(\xi_i) \frac{h^2}{6} = O(h^2)$$

$$(2 \le i \le N-1)$$

$$u_{xx} + p u_x + q u = f \qquad \partial_x (a a_x)$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i = f_i$$

$$b_i u_{i-1} + a_i u_i + c_i u_{i+1} = f_i h^2$$

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Except:

$$a_1 u_1 + c_1 u_2 = f_1 h^2 - b_1 \overset{\to u_0}{\alpha}$$

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Except:

$$a_1 u_1 + c_1 u_2 = f_1 h^2 - b_1 \overset{u_0}{\alpha}$$

$$b_N u_{N-1} + a_N u_N = f_N h^2 - c_N \underset{u_{N+1}}{\beta}$$

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Except:

$$a_1 u_1 + c_1 u_2 = f_1 h^2 - b_1 \underbrace{\alpha}_{u_0}$$

$$b_N u_{N-1} + a_N u_N = f_N h^2 - c_N \underbrace{\beta}_{u_{N+1}}$$

$$\begin{bmatrix} a_1 & c_1 & 0 & \cdots & & 0 \\ b_2 & a_2 & c_2 & 0 & \cdots & 0 \\ 0 & b_3 & a_3 & c_3 & 0 \cdots & 0 \\ & & & \ddots & & \\ 0 & \cdots & & & b_N & a_N \end{bmatrix} \vec{u} = h^2 \vec{f} - \begin{bmatrix} b_1 \alpha \\ 0 \\ \vdots \\ 0 \\ c_N \beta \end{bmatrix}$$

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- Can we solve this?
- How efficiently?
- Does the solution converge?

Linear Solvers

$$A\vec{x} = \vec{b}$$

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$$Ax = b$$

lower tri

upper tri

- LU factorization $A = LU$

Linear Solvers

$$Ax = b$$

- LU factorization $\quad A = \underbrace{L\,U}_{\hookrightarrow O(N^3)}$

lower tri $\quad$ upper tri

1) $Lc = b$ $\quad O(N^2)$

solve these individually

2) $Ux = c$

$O(N^2)$

Upper Triangular solves are easy:

$$N \times N \quad \begin{bmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_N \end{bmatrix} \vec{x} = \vec{b}$$

$$O(N^2)$$

$$a_N x_N = b_N$$

$$a_{N-1} x_{N-1} + * x_N = b_{N-1}$$

Upper Triangular solves are easy:

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(back substitution!)

Cost: $O(N^2)$

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(back substitution!)

Cost: $O(N^2)$

LU factorization $O(N^3)$

Even easier:

$$u_i + d_i u_{i+1} = r_i$$

$$\begin{bmatrix} 1 & d_1 & & & 0 \\ & \ddots & d_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & d_{N-1} \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ \vdots \\ r_N \end{bmatrix}$$

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$$u_N = r_N$$

$$\left.\begin{aligned} u_{N-1} &= r_{N-1} - d_{N-1} u_N \\ &\vdots \\ u_1 &= r_1 - d_1 u_2 \end{aligned}\right\} 2(N-1) \qquad O(N)$$

Goal:

Convert

$$\begin{bmatrix} a_1 & c_1 & & 0 \\ b_2 & a_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ 0 & & b_N & a_N \end{bmatrix} \vec{u} = \vec{f}$$

into

$$\begin{bmatrix} 1 & d_1 & & \\ & \ddots & \ddots & \\ & & \ddots & d_{N-1} \\ & & & 1 \end{bmatrix} \vec{u} = \vec{r}$$

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into

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(This is really Gaussian elimination again, just specialized!)

Step 1

$$a_1 u_1 + c_1 u_2 = f_1 \longrightarrow u_1 + d_1 u_2 = r_2$$

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$$a_1 u_1 + d_1 u_2 = r_1$$

$$a_1 u_1 + c_1 u_2 = f_1$$

$$w := a_1$$

$$u_1 + \underbrace{\frac{c_1}{w}}_{d_1} u_2 = \underbrace{\frac{f_1}{w}}_{r_1}$$

2 ops

Step 2:

$$u_1 + d_1 u_2 = r_1 \qquad u_1 = r_1 - d_1 u_2$$

$$b_2 u_1 + a_2 u_2 + c_2 u_3 = f_2 .$$

$$u_2 + d_2 u_3 = r_3$$

Step 2:

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$$\uparrow$$

$$b_2 (r_1 - d_1 u_2)$$

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$$\underbrace{[a_2 - b_2 d_1]}_{w} u_2 + c_2 u_3 = f_2 - b_2 r_1$$

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$$b_2 u_1 + a_2 u_2 + c_2 u_3 = f_2$$

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$$b_2 \left(r_1 - d_1 u_2 \right)$$

$$\underbrace{\left[a_2 - b_2 d_1 \right]}_{w} u_2 + c_2 u_3 = f_2 - b_2 r_1$$

$$u_2 + \overbrace{\frac{c_2}{w}}^{d_2} u_3 = \overbrace{\frac{f_2 - b_2 r_1}{w}}^{r_2}$$

Step k:

Using: $u_{k-1} + d_{k-1} u_k = r_{k-1}$

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$$u_k + d_k u_{k+1} = r_k$$

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$$\left.\begin{aligned} w &= a_k - b_k d_{k-1} \\ r_k &= \frac{f_k - b_k r_{k-1}}{w} \; ; \quad d_k = \frac{c_k}{w} \end{aligned}\right\} \text{6 ops}$$

$$u_k + d_k u_{k+1} = r_k$$

Last step:

$$b_{N-1} u_{N-1} + a_N u_N = f_N$$

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Same as before but "$c_N = 0$"

so 5 operations. $\left(\dfrac{c_N}{w} \text{ not needed} \right)$

Total Operation count

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Step 1: 2

$$d_1 = \frac{c_1}{a_1} \qquad r_1 = \frac{f_1}{a_1}$$

Total Operation count

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Steps $2, \ldots, N-1$: 6

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Step N: 5

Total Operation count

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Step N: 5

rewrite:

$$2 + 6(N-2) + 5 = 6(N-2) + 7$$

Total Operation count

Step 1: 2

Steps 2, ..., N-1: 6

Step N: 5

rewrite:

$$2 + 6(N-2) + 5 = 6(N-2) + 7$$

solve: $2(N-1)$

Total Operation count

Step 1: 2

Steps $2, \ldots, N-1$: 6

Step N: 5

rewrite:

$$2 + 6(N-2) + 5 = 6(N-2) + 7$$

solve: $2(N-1)$

total: $8N - 7 = O(N)$ operations.

What can go wrong?

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$w = 0$ $\quad\quad\quad$ $w = a_k - b_k d_{k-1}$

(or w close to 0, also bad!)