absolute sability if $z = \lambda h$

and the method is a. s. for step size h.

Last class: $\underbrace{\text{absolute}}$ stability

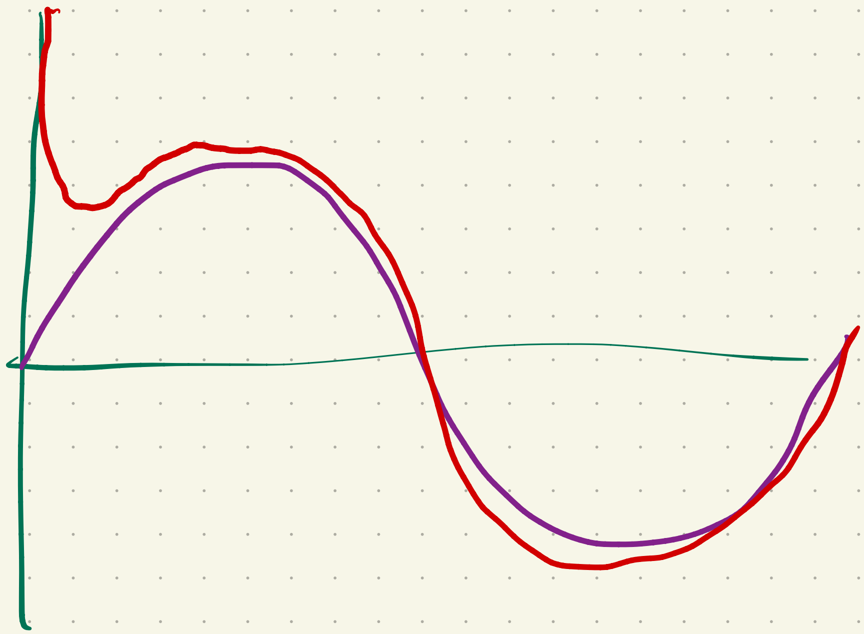
Idea: $u' = \lambda u$

$\lambda < 0 \to$ decay.

$$u_{k+1} = (1 + \lambda h)^k u_k$$

Even though the method is convergent

in practice one worries about step size.



Apply LMM to $u' = \lambda u$ $\lambda \in \mathbb{C}$

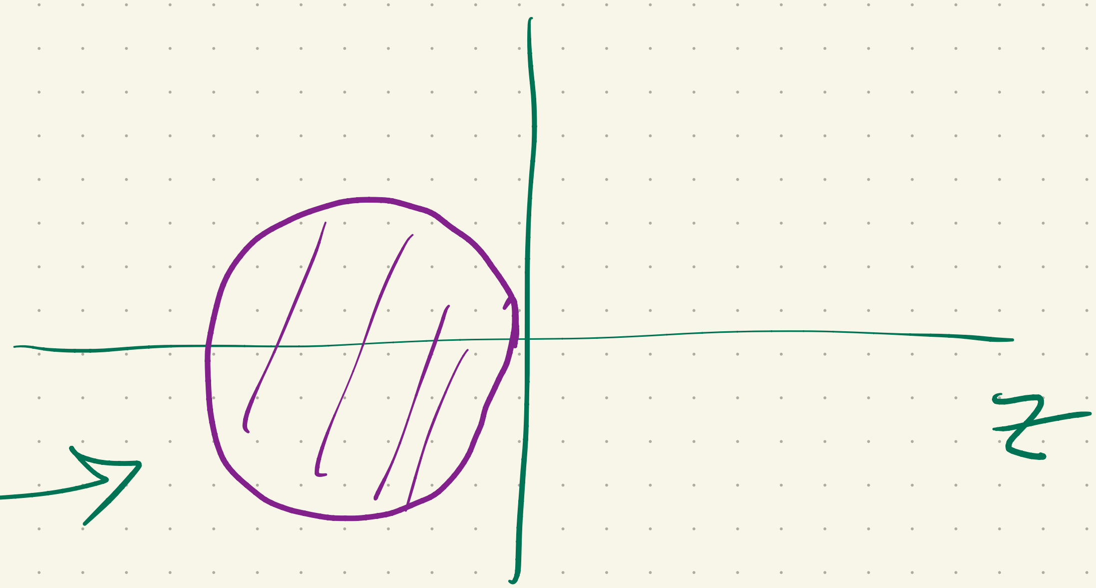
Method is abs. stable for step size

h, if the solutions remain bounded
↳ decay
for all time

$$u_k = (1 + \underbrace{\lambda h}_{z})^k u_0$$

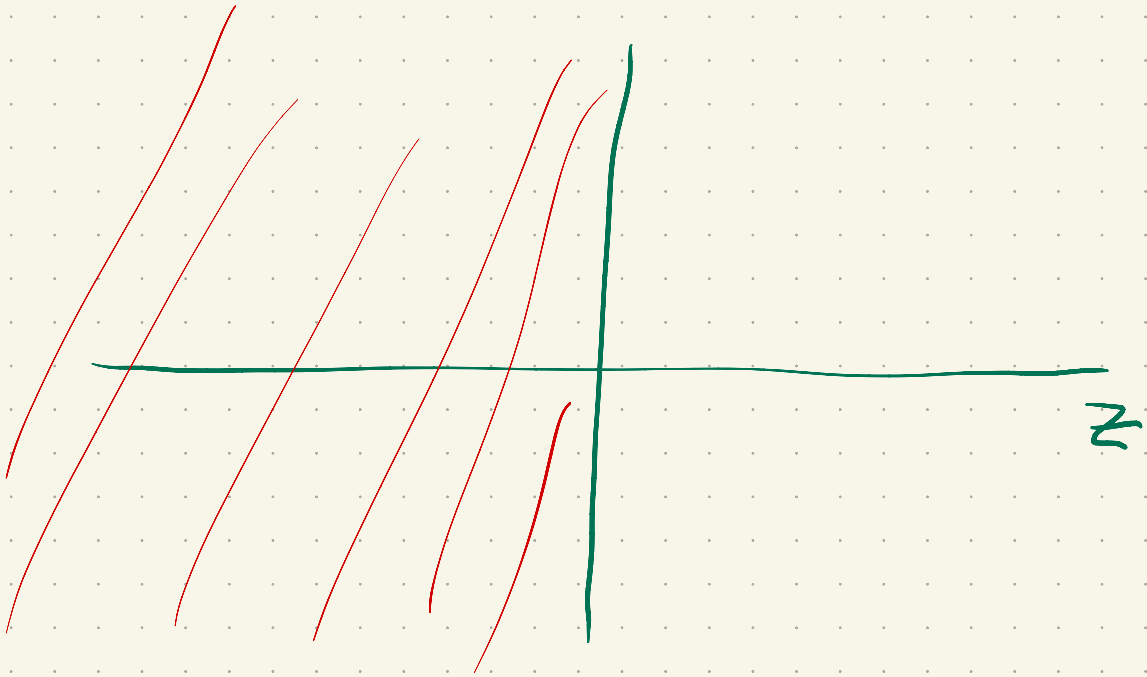
$$|1 + z| \leq 1$$

Region of abs stability.



What's an ideal region of stability

$$u' = \lambda u \qquad z = h\lambda$$



A method is A-stable if its region of abs stability includes the left half plane.

Backwards Euler $\qquad (u' = \lambda u)$

$$u_{k+1} = u_k + \underbrace{h\lambda}\, u_{k+1}$$

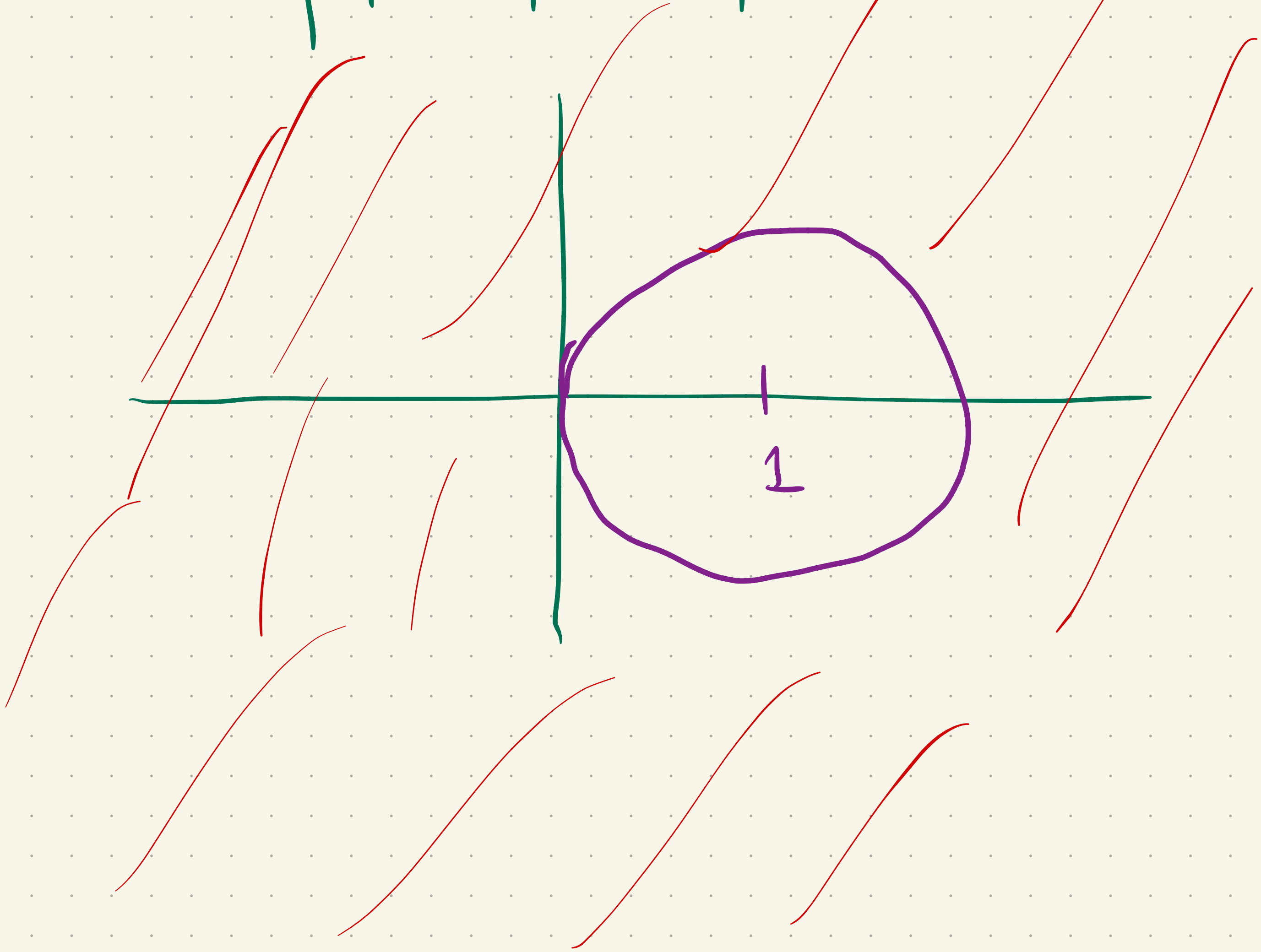
$$u_{k+1}(1 - \lambda h) = u_k$$

$$u_{k+1} = \frac{1}{1-z}\, u_k \qquad u_k = \left(\frac{1}{1-z}\right)^k u_0$$

Now, to get bounded solutions, need

$$\left|\frac{1}{1-z}\right| \leq 1$$

$$|1-z| \geq 1$$



This is an A-stable method.

LMM

$$\alpha_k u_{n+k} + \dots + \alpha_0 u_{n+0} = h\left(\beta_k f_{n+k} + \dots + \beta_0 f_{n+0}\right)$$

$$u' = \lambda u$$

$$f_i = \lambda u_i$$

$$= h\left(\beta_k \lambda u_{n+k} + \dots + \beta_0 \lambda u_{n+0}\right)$$

$$= z\left(\beta_k u_{n+k} + \dots + \beta_0 u_n\right)$$

$$(\alpha_k - z\beta_k) u_{n+k} + \cdots + (\alpha_1 - z\beta_1) u_{n+1} + (\alpha_0 - z\beta_0) u_n$$
$$= 0$$

Characteristic polynomial

$$\underbrace{(\alpha_k - z\beta_k)\rho^k + \cdots + (\alpha_1 - z\beta_1)\rho + (\alpha_0 - z\beta_0)}_{p(\rho)}$$

The method is abs stable for z

if solutions of the recurrence relation remain bounded.

This happens when

1) All roots of char. polynomial satisfy

$$|\rho| \leq 1$$

2) If ρ is a repeated root, $|\rho| < 1$.

polynomial for zero stability

$$p(\rho) = \sigma(\rho) - z\left(\beta_k \rho^k + \cdots + \beta_0\right)$$

Forward Euler:

$$p(\rho) = \rho - 1 - z$$

$$u_{k+1} = u_k + \lambda h\, u_k$$

$$\rho = 1 + z$$

$p(\rho)$

$$\rho - (1+z) = 0$$

$$\rho = 1+z$$

$$|\rho| \le 1$$

$$|1+z| \le 1$$

Backwards Euler

$$u_{k+1} = u_k + \lambda h\, u_{k+1}$$

$$\rho = 1 + z\rho$$

$$\rho(1-z) - 1$$

$$\rho(1-z) - 1 = 0$$

roots $\rho = \dfrac{1}{1-z}$

$$\left|\frac{1}{1-z}\right| \le 1$$

Trapezoidal

$$u_{k+1} = u_k + \frac{h}{2}\left(f_{k+1} + f_k\right)$$

$$u_{k+1} = u_k + \frac{h}{2}\left(\lambda u_{k+1} + \lambda u_k\right)$$

$$u_{k+1} = u_k + \frac{z}{2}\left(u_{k+1} + u_k\right)$$

$$\rho = 1 + \frac{z}{2}\left(\rho + 1\right)$$

$$\overbrace{\left(1 - \frac{z}{2}\right)\rho - \left(1 + \frac{z}{2}\right)} = 0$$

$$\rho = \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}}$$

Boundary of region of abs stab

$$|\rho| = 1 \qquad \frac{\left|1 + \frac{z}{2}\right|^2}{\left|1 - \frac{z}{2}\right|^2} = 1$$

$$\left|1+\frac{z}{2}\right|^2 = \left|1-\frac{z}{2}\right|^2 \qquad |w|^2 = w\bar{w}$$

$$\downarrow$$

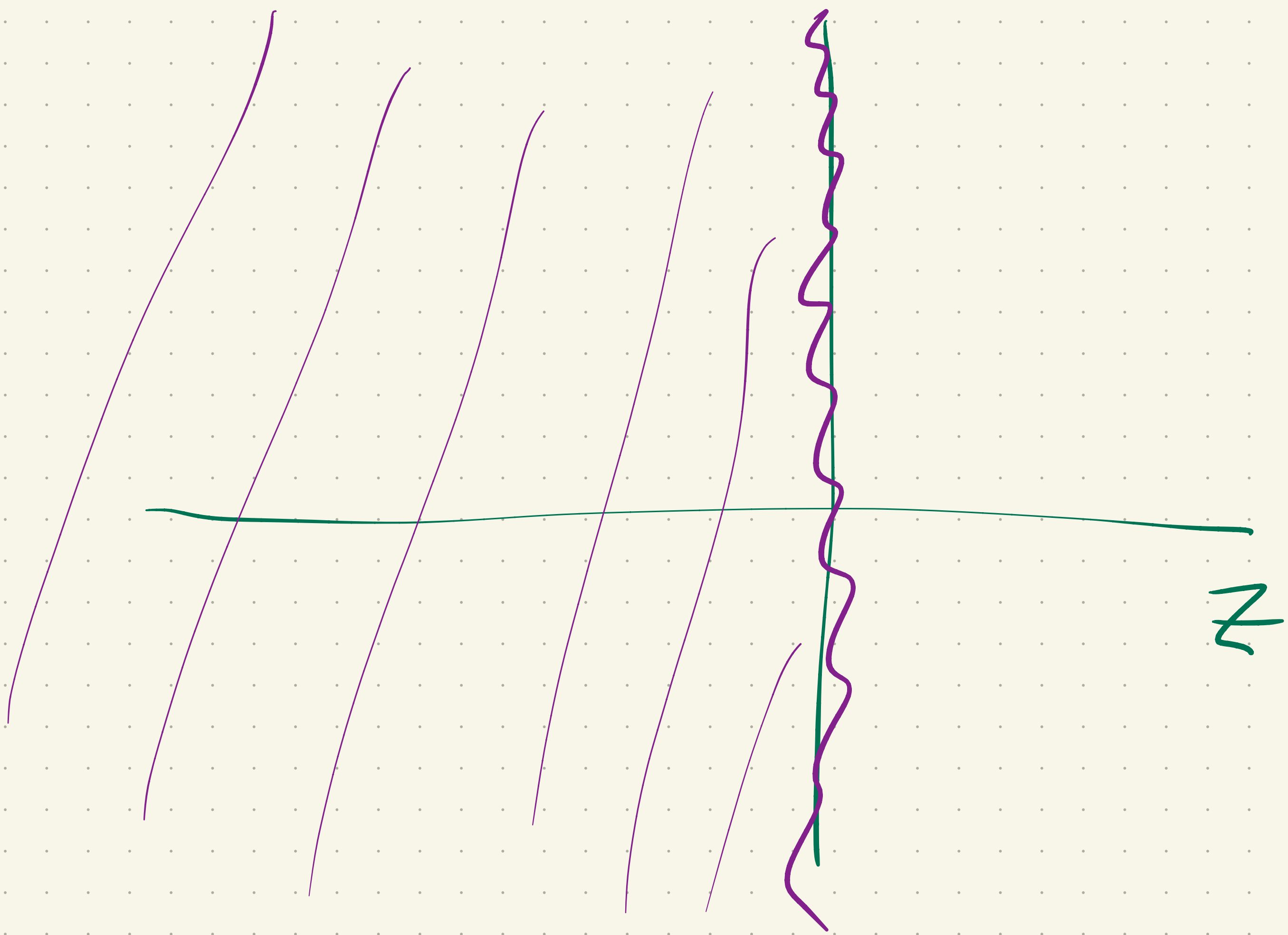
$$\left(1+\frac{z}{2}\right)\left(1+\frac{\bar{z}}{2}\right) = \left(1-\frac{z}{2}\right)\left(1-\frac{\bar{z}}{2}\right)$$

$$1 + \frac{z}{2} + \frac{\bar{z}}{2} + \frac{|z|^2}{4} = 1 - \frac{z}{2} - \frac{\bar{z}}{2} + \frac{|z|^2}{4}$$

$$z + \bar{z} = 0 \qquad \bar{z} = -z$$

$$\uparrow$$

z is imaginary.



$$\rho = \frac{1+\frac{z}{2}}{1-\frac{z}{2}}$$

$$z=1 \Rightarrow \rho = \frac{3/2}{1/2} = 3$$

$$z=-1 \Rightarrow \rho = \frac{1/2}{3/2} = \frac{1}{3}$$

Midpoint:

$$\lambda u_{k+1}$$

$$u_{k+2} = u_k + 2h f_{k+1}$$

$$\rho^2 = 1 + 2z\rho$$

$$\underbrace{\rho^2 - 2z\rho - 1}_{\text{char. poly.}} = 0$$

$$\rho = \frac{2z \pm \sqrt{(4z^2) + 4}}{2}$$

$$\zeta = z \pm \sqrt{z^2+1}$$

For z to be in region of abs. stability both roots must have size exactly 1.