Math F305 Final Exam Spring 2022

Name: ______________________

Student Id: ______________________

Rules:

You have 120 minutes to complete the exam.

Partial credit will be awarded, but you must show your work.

You may bring a single notecard of notes into the exam.

No calculators, books, or other aids are permitted.

Turn off anything that might go beep during the exam.

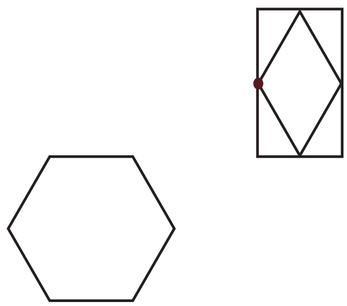
If you need extra space, you can use the back sides of the pages. Please make it obvious when you have done so.

Good luck!

Problem	Possible	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
EC	5	
Total	80	

1. (10 points)

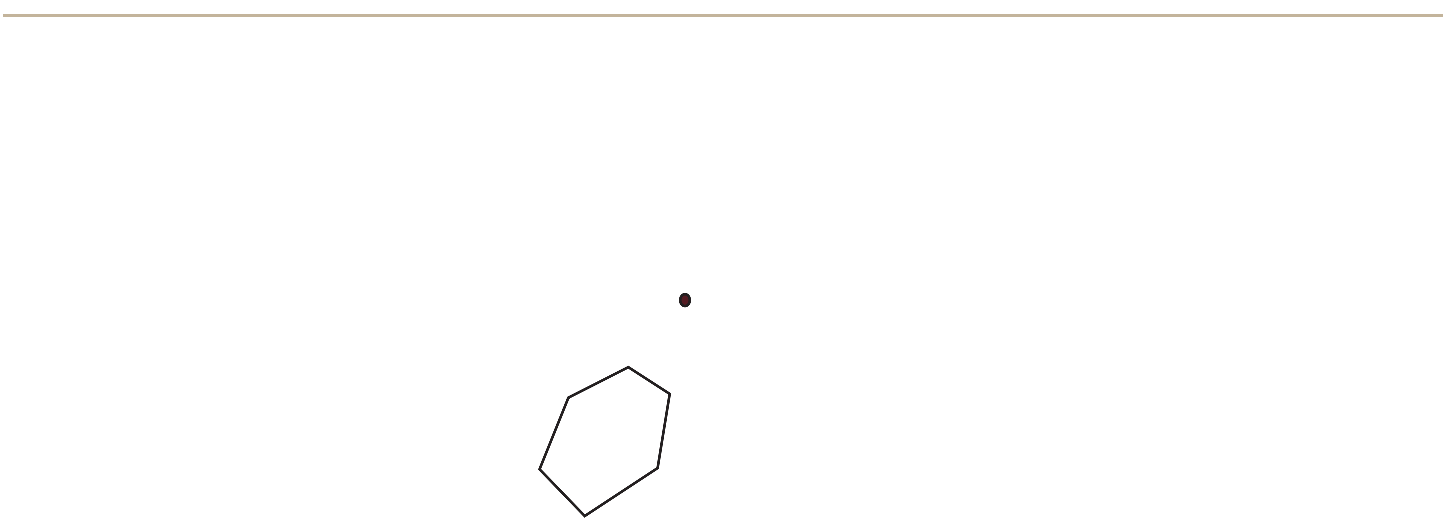
Consider the diagram in the plane below.



Part of this diagram has been drawn in perspective below. The dot in the diagram below corresponds to the dot next to the diamond in the diagram above. The horizontal line is the line at infinity.

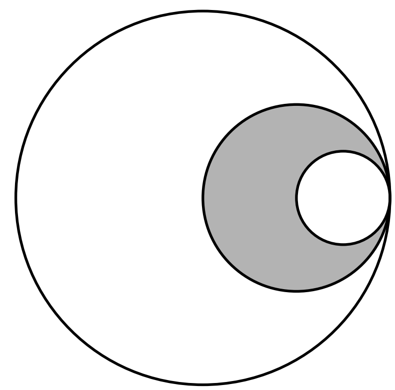
a. (2 points) Explain how if I had not given you the line at infinity you could have reconstructed it using only the hexagon.

b. (8 points) Complete the rest of the diagram using only a straightedge. Please leave all marks that you make. Your diagram may be a bit messy – that's ok. But be sure to circle, hightlight, or otherwise indicated where the rectangle/diamond shape is in your sketch. You may find it helpful to note that each side of the diamond is parallel to one of the sides of the hexagon.



2. (10 points)

In the ball model of hyperbolic geometry, let H_1 be the horocycle with ideal point 1 passing through 0, and let H_2 be the horocycle with ideal point 2 passing through $1/2$. Determine, with rigorous justification, if the area between the two horocycles is finite or infinite.



Hint: This problem should not be a monster computation!

3. (10 points)

A Möbius transformation T takes 0 to 1, 1 to ∞, ∞ to a and a to 0 for some complex number a. Find all possible values of a. Hint: be careful; you might want to write down the Möbius transformations you find.

4. (10 points)

a. (2 points) Let S be the unit sphere. Suppose a triangle on the sphere has interior angles $\theta_1, \theta_2, \theta_3$. What is its area?

b. (6 points) I promise you that the unit sphere can be tiled with 20 congruent equilateral triangles. Determine the interior angle of the triangles.

c. (2 points) How many triangles meet at each vertex?

5. (10 points)

Consider the collection of complex numbers $X = \{1, i, -1, -i\}$. Each $z \in X$ defines a transformation T_z acting on X by $T_z(x) = zx$. Let $\mathcal{S} = \{T_z : z \in X\}$.

a. (4 points) Show that $\mathcal{S}$ is a transformation group.

b. (3 points) True or false (with brief justification): In this geometry, all two point figures are congruent.

b. (3 points) True or false (with brief justification): In this geometry, all three point figures are congruent.

6. (10 points)

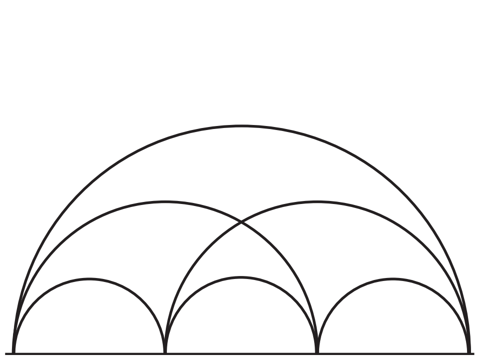
5 points Find the homogeneous coordinates of the projective line $\mathcal{L}$ containing the projective points with homogeneous coordinates $\mathbf{p} = (1, 2, 1)$ and $\mathbf{q} = (0, 1, 2)$.

5 points Consider projective points r and s with homogeneous coordinates $\mathbf{r} = (1, 1, -1)$ and $\mathbf{s} = (1, 1, 2)$ respectively. Which (if any) of r or s lie on $\mathcal{L}$?

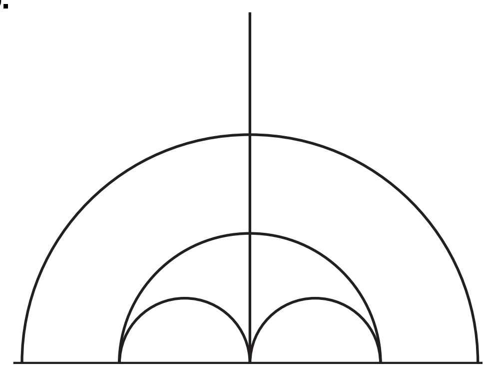
7. (10 points)

There are four pairs of congruent diagrams in our model of the hyperbolic plane below. Find the matching pairs. To indicate a pair matches, write one of the letters **A** through **D** on both diagrams in the pair.

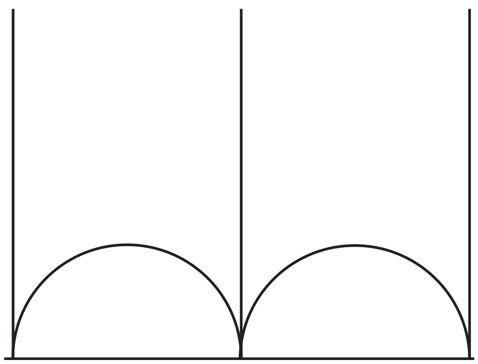
a.



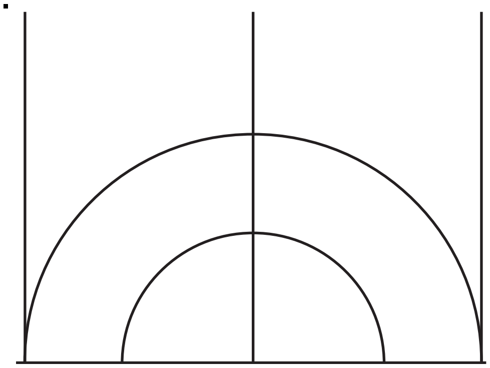
b.



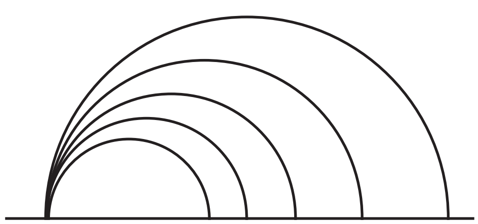
c.



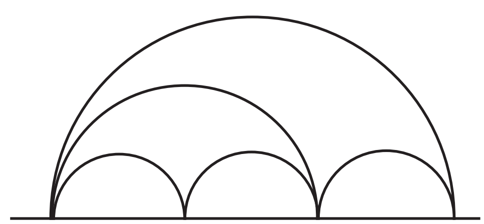
d.



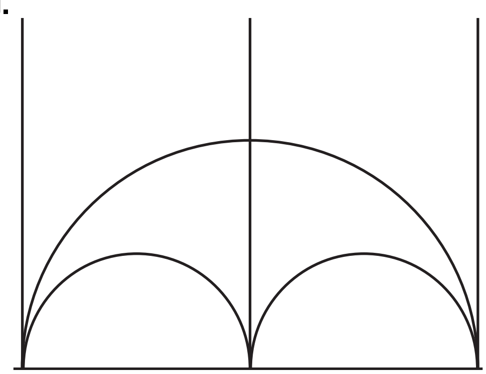
e.



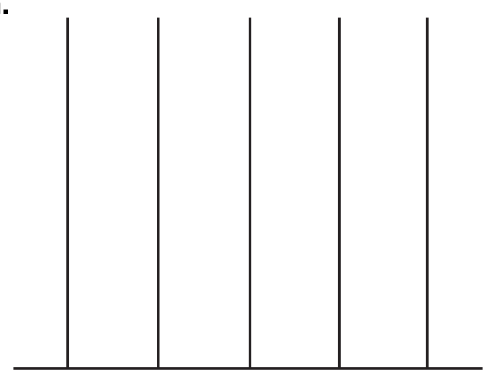
f.



g.



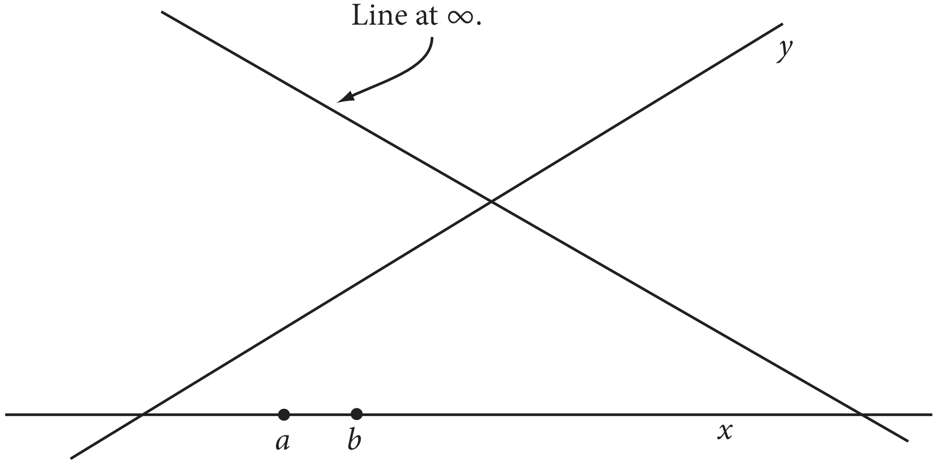
h.



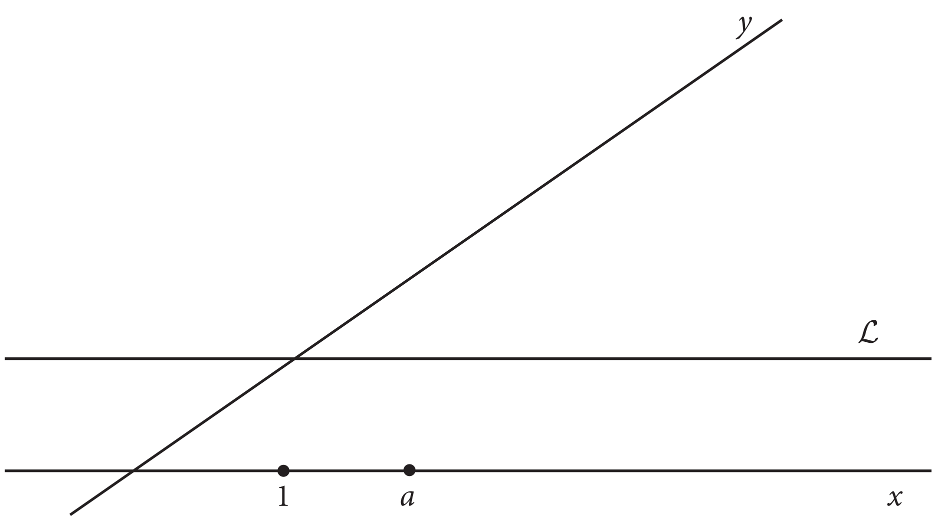
8. (10 points)

a. (2 points) State Desargues Theorem.

b. (4 points) In our work with projective arithmetic we usually drew diagrams with the line at infinity out at infinity. But as we mentioned in class, we can do projective arithmetic with the line at infinity in view as well. Using only a straightedge, complete the diagram below to compute the position of $a + b$.



c. (4 points) Complete the diagram below to construct the length $a^2 - a$. Your diagram should indicate which lines are parallel, but you do **not** need to construct the lines as parallel using straightedge and compass.



9. (Extra credit (5 points))

Show that it is impossible to tile the sphere with 21 equilateral triangles.

Length and Area Formulas

Ball model:

$$L = \int_a^b \frac{2|z'|}{1-r^2}\, dt$$

$$A = \int_\Omega \frac{4r}{(1-r^2)^2}\, dr\, d\theta$$

Upper half plane model:

$$L = \int_a^b \frac{|z'|}{y}\, dt$$

$$A = \int_\Omega \frac{1}{y^2}\, dx\, dy$$