Product topology.

Consider two spaces X, Y

$X \times Y \to$ ordered pairs (x, y).

(really a map σ from $\{0, 1\}$ to $X \cup Y$, $\sigma(0) \in X$, $\sigma(1) \in Y$).

We want a natural topology on $X \times Y$.

Are there any natural associated maps?

Yes: $\pi_X : X \times Y \to X$ (projections)
$\pi_Y : X \times Y \to Y$

We want these to be continuous.

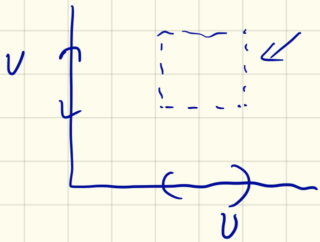
So there need to be enough open sets on $X \times Y$ to make this so.

The natural choice is to take the coarsest topology that does this.

Let $\mathcal{A} = \{ \pi_X^{-1}(U) : U \subseteq X \text{ is open} \} \cup \{ \pi_Y^{-1}(V) : V \subseteq Y \text{ is open} \}$

In the spirit of problem 2-12, this is a subbasis on $X \times Y$ and generates a topology on $X \times Y$, the coarsest topology on $X \times Y$ that contains $\mathcal{A}$. [p 2-12 adds in X, but it's already there]

A basis for this topology is finite intersections, which have the form $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$.



$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$

This topology is known as the <u>product topology</u>.

For finitely many spaces $X_1, \dots, X_n$, $\to \prod_{k=1}^{n} X_k = X_1 \times \dots \times X_n$.

subbasis is $\mathcal{A} = \{ \pi_i^{-1}(U) : U \subseteq X_i, i \in \{1, \dots, n\} \}$.

The basis is the intersection of such sets,

$$\pi_1^{-1}(U_1) \cap \dots \cap \pi_n^{-1}(U_n) = U_1 \times \dots \times U_n.$$

($U_i \subseteq X_i$ is open).

Now $\mathbb{R}^n$ already has a topology (metric).

But $\mathbb{R}^n = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n\text{-times}}$.

So now it has two topologies.

Are they the same? $\to$ certainly standard is finer, since projections are cts.

It would be nice to have a tool that detects this.

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[dr, dotted, "f_i"'] & \prod_{k=1}^{n} X_k \arrow[d, "\pi_i"] \\ & X_i \end{tikzcd}$$

Claim: f is cts iff each f_i is.

Note $f(z) = (f_1(z), \dots, f_n(z))$

The f_i's are exactly the coordinate functions.

If f is cts, then certainly each f_i is (comp of cts!)

For the converse, suppose each f_i is cts.

Consider a subbasic open set $\pi_i^{-1}(U)$.

Then $f^{-1}(\pi_i^{-1}(U)) = f_i^{-1}(U)$, which is open.

So f is continuous.

This claim is The Characteristic Property of the Product Top.

Functions <u>into</u> product spaces are cts iff their coordinate functions are.

Related to problem 2-12

A subbasis in X is a subset $\mathcal{S} \subseteq \mathcal{P}(X)$ such that $\bigcup_{S \in \mathcal{S}} S = X$.

From $\mathcal{S}$ we construct $\mathcal{B} = \{ S_1 \cap \cdots \cap S_n : S_k \in \mathcal{S} \}$.

Is $\mathcal{B}$ a pre-basis? 1) Since $\mathcal{S} \subseteq \mathcal{B}$,

$$X = \bigcup_{S \in \mathcal{S}} S \subseteq \bigcup_{B \in \mathcal{B}} B.$$

2) It is obvious that $\mathcal{B}$ is closed under finite intersections.

So $\mathcal{B}$ generates a topology. It's the coarsest topology that contains $\mathcal{B}$.

But any topology containing $\mathcal{S}$ must contain $\mathcal{B}$.

So it's also the coarsest containing $\mathcal{S}$.

[Text: drops $\bigcup_{S \in \mathcal{S}} S = X$ but adds X to the mix. Often $X \in \mathcal{S}$ explicitly, in which case it's obvious the two constructions are same]

If Y has a topology generated by a subbasis $\mathcal{S}$, then $f: X \to Y$ is cts iff $f^{-1}(S)$ is open in X $\forall$ $S \in \mathcal{S}$.

One direction is obvious.

If $B \in \mathcal{B}$ $\quad B = S_1 \cap \cdots \cap S_n$

$$f^{-1}(B) = \bigcap_{k=1}^{n} f^{-1}(S_k) \text{ , open in } X.$$

Prop Suppose τ is a topology on $X = \prod_{k=1}^{n} X_k$ that satisfies the char property.

Then τ is the product topology.

Pf: Let $X_{(r)}$, $X_{(p)}$ be random, product top spaces.

I claim each $\pi_i^{(r)}: X_{(r)} \to X_i$ is cts.

$$\begin{tikzcd} X_{(r)} \arrow[r, "id"] \arrow[dr, "\pi_i"'] & X_{(r)} \arrow[d, "\pi_i^{(r)}"] \\ & X_i \end{tikzcd}$$

id is cts, so π_i is,

Now consider $id_{rp}: X_p \to X_r$, $\pi_i^{(r)}$ $\pi_i^{(p)}$

$$\begin{tikzcd} X_{(r)} \arrow[r, "id_{(r,p)}"] \arrow[dr, "\pi_i^{(r)}"'] & X_{(p)} \arrow[d, "\pi_i^{(p)}"] \\ & X_i \end{tikzcd}$$

Each $\pi_i^{(r)}$ is cts by above.

So $id_{(r,p)}$ is cts.

Same argument shows $id_{(p,r)}$ is cts.

So topologies are same.

You have repeatedly used the fact that $\mathbb{R}^n$ satisfies the Char property. So its topology is the product topology.

Let's double check.

Lemma: $\{z_n\}$ $z_n \to z$ iff $\pi_i(z_n) \to \pi_i(z)$.

Pf: Since the projections are cts