$\log(x)$ centered at $x=1$

$\log(1+x)$ - - - $x=0$ $\quad x \in [-1/2, 1/2]$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots - \frac{x^{10}}{10} + \underbrace{\log^{(11)}(1+c)\frac{x^{11}}{11!}}_{R_D}$$

$$\log^{(11)}(1+c) = 10!\,(1+c)^{-11}$$

$x>0$ $\quad$ c is between 0 and x $\Rightarrow c>0$

$$R_D = \frac{10!}{11!}\left(\frac{x}{1+c}\right)^{11}$$

$$|R_D| < \frac{1}{11}\left(\frac{1}{2}\right)^{11} \qquad \frac{x}{1+c} \leq x \leq \frac{1}{2}$$

$$\approx 4 \times 10^{-5}$$

A cheap estimate for $x < 0$

$x < c < 0$

is unfortunately

$$|R_D| < \frac{1}{11} \cdot \left|(-1)^{4}\right| = \frac{1}{11}$$

ODEs (refs needed!)

$$\left.\begin{aligned} u' &= f(t, u) \\ u(t_0) &= u_0 \end{aligned}\right] \text{ IVP}$$

initial value problem

u is $\mathbb{R}$-valued

e.g.

$$\left.\begin{aligned} u' &= g(t) \\ u' &= \lambda u + g(t) \\ u' &= \lambda(t) u + g(t) \end{aligned}\right] \text{ linear ODEs}$$

$$u' = \lambda u (1 - u) \longleftarrow \text{nonlinear}$$

Given an IVP is there a solution?

how many?

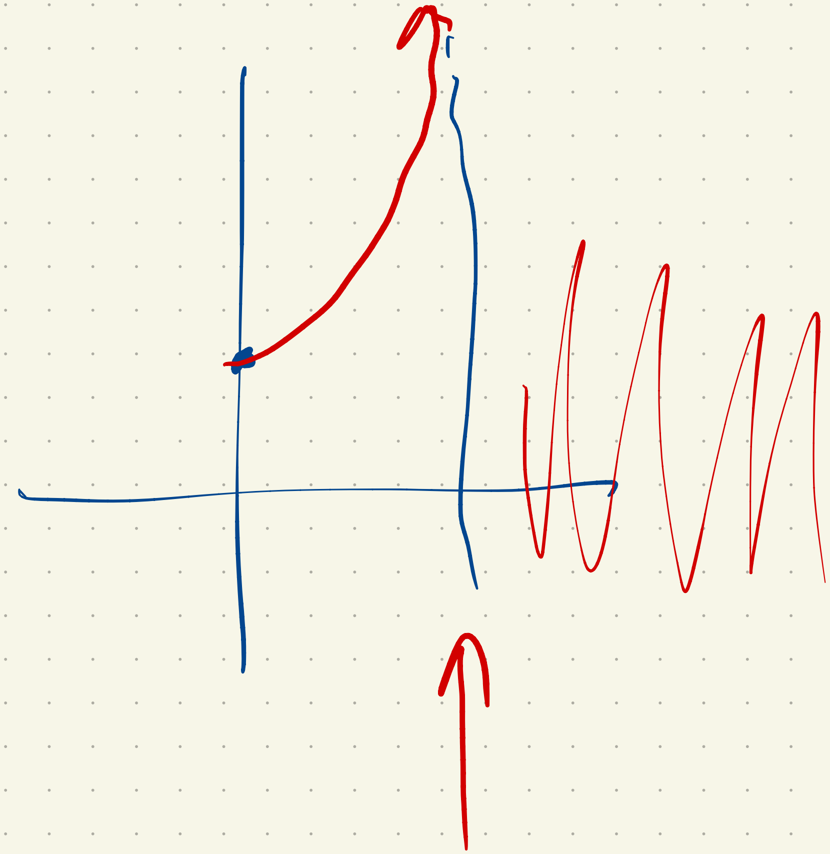
1) differentiable, 2) continuous + Lipshitz in u

Thm: If f is "nice" then there is an $\varepsilon > 0$ and a unique function u on $(t_0 - \varepsilon, t_0 + \varepsilon)$ such that $u' = f(t,u)$ and $u(t_0) = u_0$.

Why is there an ε here?

Consider
$$u' = u^2$$
$$u(0) = 1$$

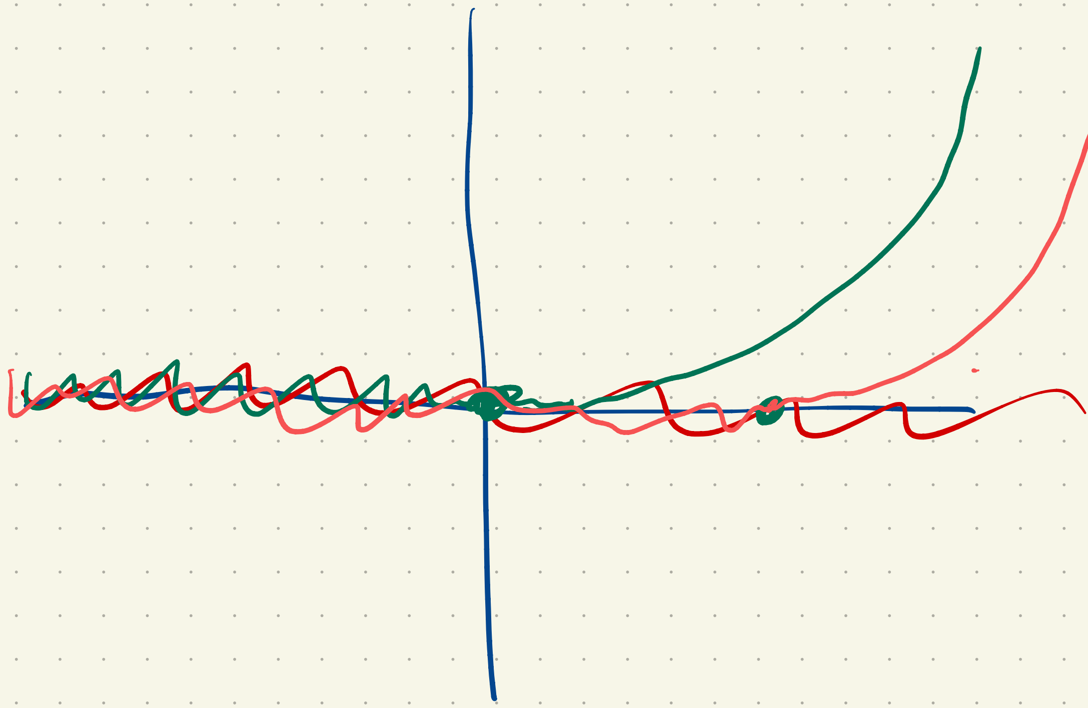
$$u(t) = \frac{1}{1-t}$$



$$u' = u^{1/3}$$
$$u(0) = 0$$

solution $u(t) = 0$

$$u(t) = \begin{cases} \left(\frac{2}{3}t\right)^{3/2} & t > 0 \\ 0 & t \le 0 \end{cases}$$



Systems $\quad \vec{u}\,' = \vec{f}(t, \vec{u})$

I'll drop the arrows.

What about higher order?

$$u'' + cu' + ku = 0$$

New variable $v = u'$

$$u' = v$$

$$v' + cv + ku = 0$$

$$u' = v$$

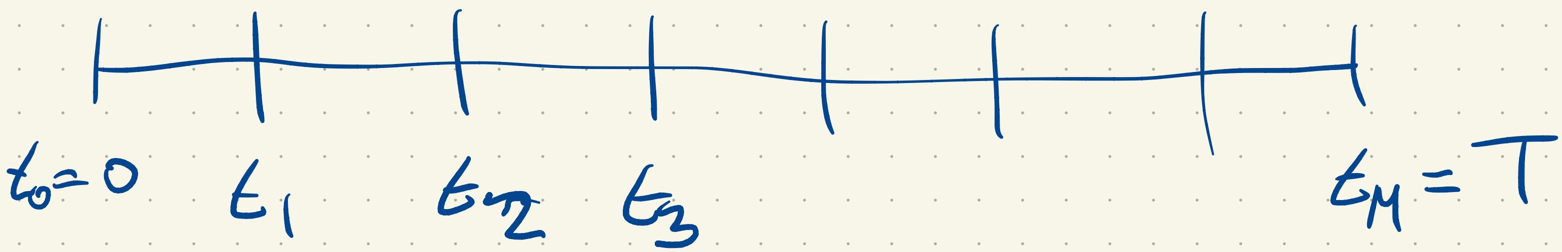
$$v' = -ku - cv$$

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

Euler's Method

$$\left.\begin{aligned} u' &= f(t,u) \\ u(t_0) &= u_0 \end{aligned}\right] \quad (*)$$

replace w/ a discrete approx



M subintervals of equal width $\quad h = \frac{T}{M}$

Suppose $u(t)$ is a solution of $(*)$

$$u(t_{i+1}) = u(t_i + h) = u(t_i) + u'(t_i)h + \frac{1}{2}u''(\eta_i)h^2$$

between t_i and t_{i+1}

Rewrite:

$$\frac{u(t_{i+1}) - u(t_i)}{h} = u'(t_i) + \frac{1}{2}u''(\eta_i)h$$

$$u'(t_i) = \underbrace{\frac{u(t_{i+1}) - u(t_i)}{h}}_{\text{discrete approx of derivative}} \underbrace{- \frac{1}{2}u''(\eta_i)h}_{\text{local truncation error}}$$

τ_i

$$u'(t_i) = D_i + \tau_i$$

Strategy: we will find $u_0, u_1, u_2, \ldots, u_n$ that approximate $u(t_i)$

$$u' = f(t,u)$$

$$\downarrow$$

$$\frac{u(t_{i+1}) - u(t_i)}{h} + \tau_i = f(t_i, u(t_i))$$

Now drop τ_i and replace $u(t_i)$ with u_i

$$\frac{u_{i+1} - u_i}{h} = f(t_i, u_i)$$

Or,

$$u_{i+1} = u_i + h f(t_i, u_i)$$

$$u_0 = u_0$$

Let's apply this to

$$u' = 10 u(1-u)$$
$$u(0) = 1/100$$

Exercise: exact solution is

$$u(t) = \frac{1}{1 + 99e^{-10t}}$$

To compare numerical solution with the exact solution we need a notion of error.

$$u_i - u(t_i) = e_i \quad (\text{Mistake at } t = t_i)$$

$$\frac{1}{\sqrt{M}} \sqrt{e_1^2 + e_2^2 + \cdots + e_M^2} = \|\vec{e}\|_2$$

$$\|e\|_\infty = \max_{0 \le i \le M} |e_i|$$