A classical solution of

$$-u_{xx} = f$$

satisfies

$$\int_0^l u_x \phi_x - f\phi = 0 \qquad \phi \text{ is a test function}$$

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~~f~~ ϕ is a test function

$[\text{or } C^1; \ \phi|_0^\ell = 0]$

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$$-u_{xx} = f$$

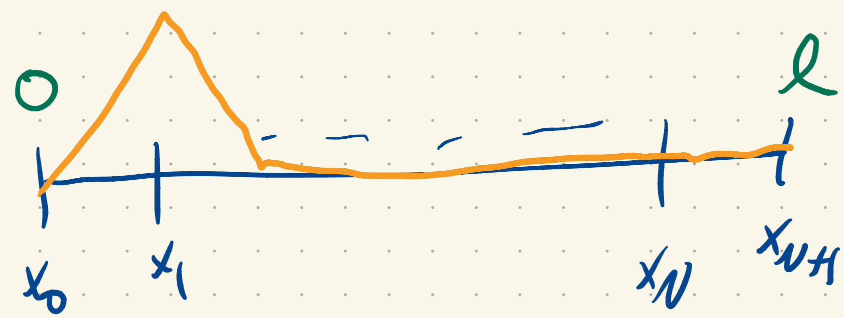
satisfies

$$\int_0^\ell u_x \phi_x - f\phi = 0 \qquad \forall\ \phi \text{ is a test function}$$

$[\text{or } C^1,\ \phi|_0^\ell = 0]$

H_0^1 $[\text{or } \underset{C^0,}{\wedge} \text{ piecewise linear!}]$

Modified problem:



$$\mathcal{V}_h = \text{span}\{\varphi_1, \ldots, \varphi_N\}$$

True solution $\displaystyle\int u_x \phi_x - u\phi = 0 \quad \forall$ test functions ϕ (including piecewise linear!)

Modified problem:

$$\mathcal{V}_h = \text{span}\{\varphi_1, \dots, \varphi_N\}$$

True solution: $\int u_x \phi_x - u\phi = 0 \quad \forall$ test functions ϕ

Approx: Find $u^h \in \mathcal{V}_h$ with

$$\int u^h_x \phi_x - u^h \phi = 0 \quad \forall \phi \in \mathcal{V}_h$$

$$\int u^h_x \phi_x - f\phi = 0 \quad \forall \phi \in \mathcal{V}_h$$

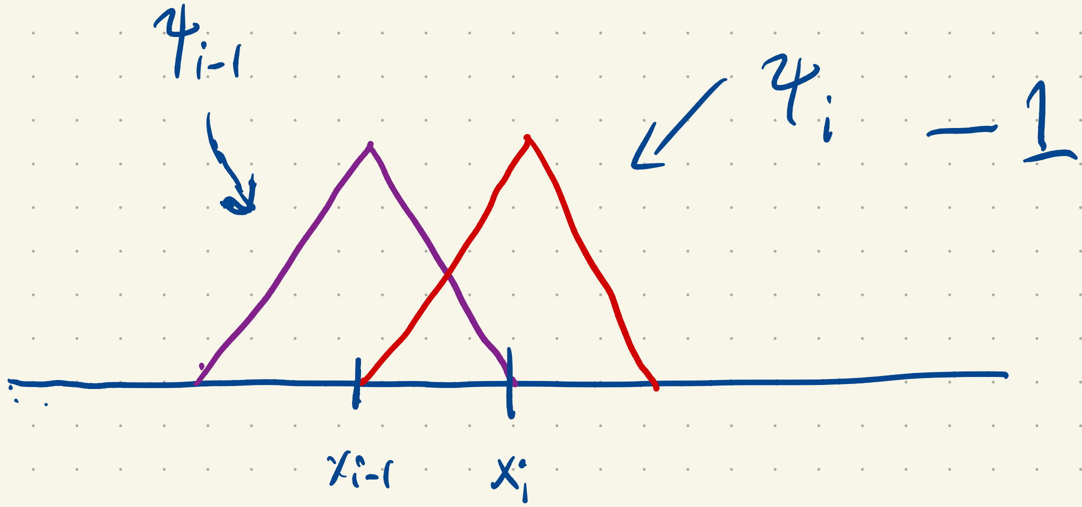
$$u^h = \sum_{j=1}^{N} \overset{\text{number}}{\downarrow}{u_j}\, \psi_j$$

$$\int u^h_x \phi_x - f\phi = 0 \quad \forall \phi \in \mathcal{V}_h$$

$$u^h = \sum_{j=1}^{N} \overset{\substack{\text{number}\\ \downarrow}}{u_j} \psi_j$$

$$\int \sum_{j=1}^{N} u_j \left[\partial_x \psi_i \, \partial_x \psi_j \right] = \int f \psi_i \qquad i = 1, \dots, N$$

$$A_{ij} = \int \partial_x \psi_i \, \partial_x \psi_j$$



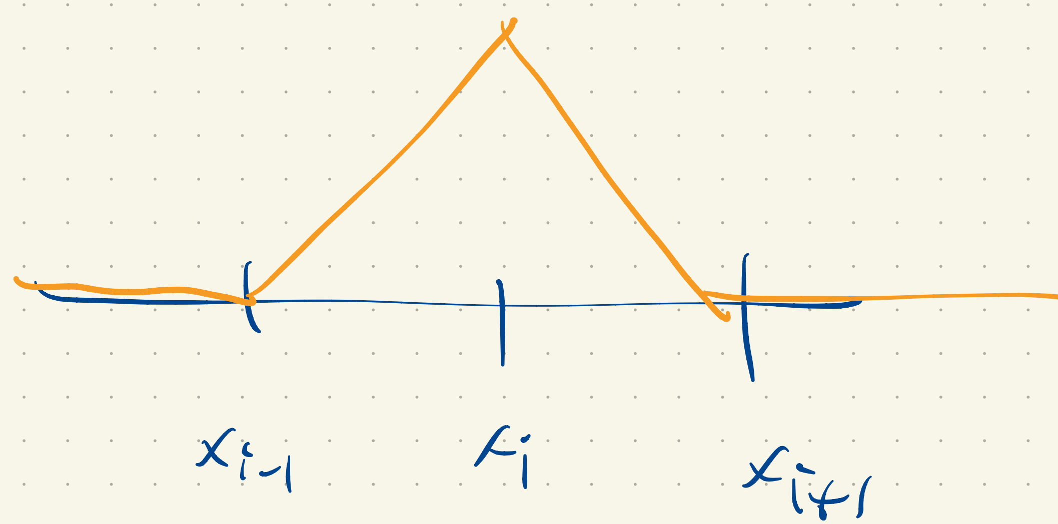
$$\int \underbrace{\partial_x \psi_i \, \partial_x \psi_i}_{-1/h^2} = ?$$

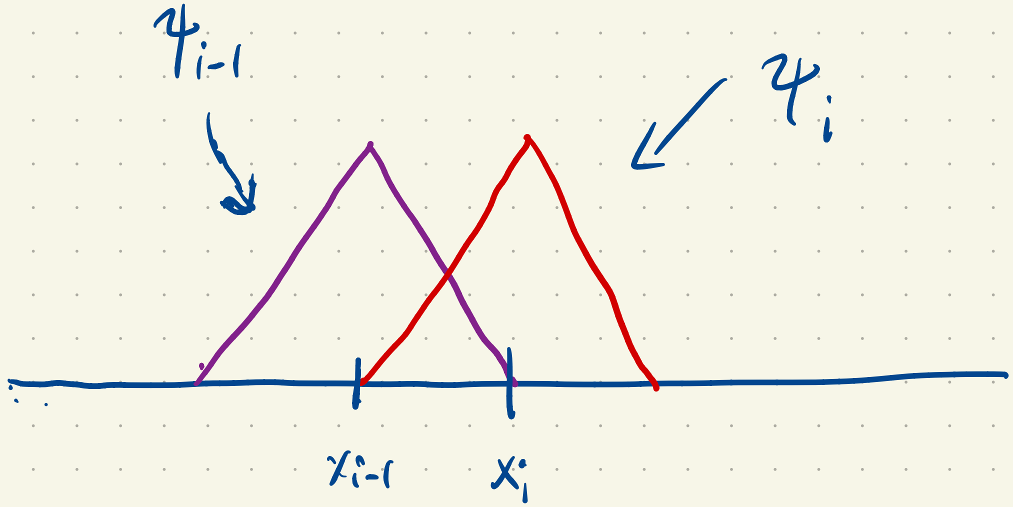
$$\partial_x \psi_i$$

$$\int \partial_x \psi_{i-1} \, \partial_x \psi_i = \frac{-1}{h}$$

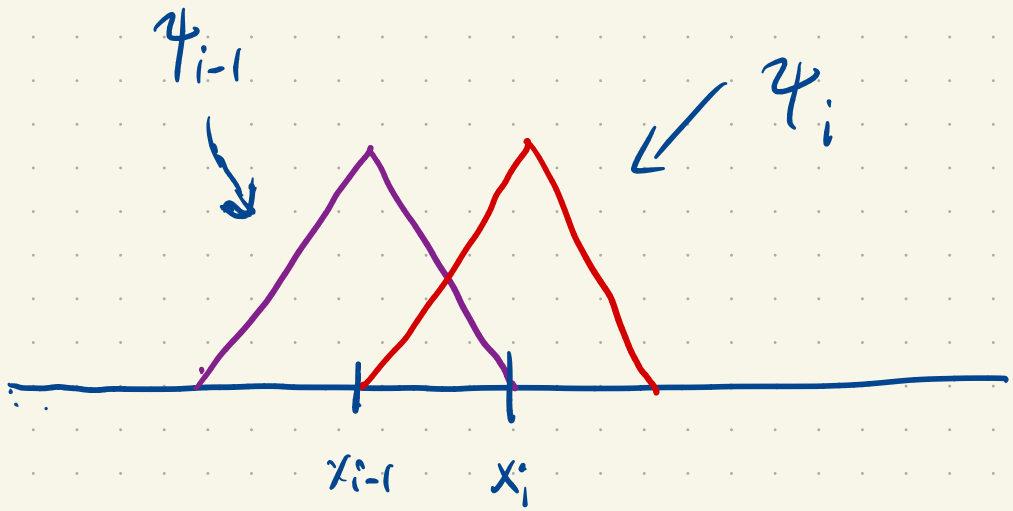
$$\int_{x_{i-1}}^{x_{i+1}} \partial_x \psi_i \, \partial_x \psi_i$$

$$\frac{2}{h}$$





$$\int_{x_{i-1}}^{x_i} \partial_x \varphi_{i-1} \, \partial_x \varphi_i = -\frac{1}{h^2} h = -\frac{1}{h}$$



$$\int_{x_{i-1}}^{x_i} \partial_x \varphi_{i-1} \, \partial_x \varphi_i = -\frac{1}{h^2} h = -\frac{1}{h}$$

$$\int_{x_{i-1}}^{x_{i+1}} \partial_x \varphi_i \, \partial_x \varphi_i = \frac{1}{h^2} \cdot 2h = \frac{2}{h}$$

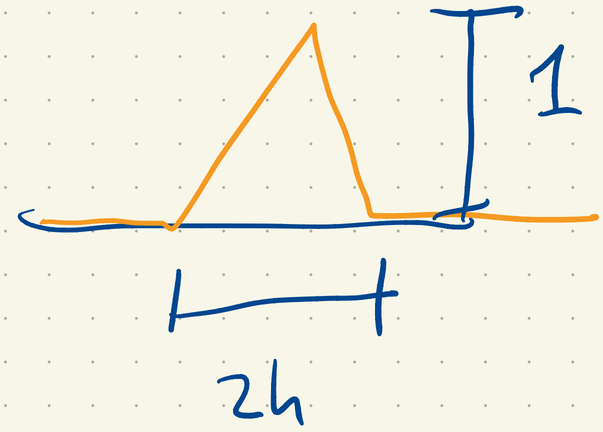
$$\int \partial_x \varphi_i \, \partial_x \varphi_j : \quad \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & 0 & \cdots & \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ & & & & & \\ \cdots & & & 0 & -1 & 2 \end{bmatrix} = A$$

stiffness matrix

$$\sum_{j=1}^{n} A_{ij} u_j = \underbrace{\int_{x_{i-1}}^{x_{i+1}} f \varphi_i}$$

approximate by numerical quadrature

$$\int_{x_{i-1}}^{x_{i+1}} f\,\varphi_i \approx f(x_i) \int_{x_{i-1}}^{x_{i+1}} \varphi_i$$



$$\int_{x_{i-1}}^{x_{i+1}} f\,\varphi_i \approx f(x_i) \int_{x_{i-1}}^{x_{i+1}} \varphi_i = f(x_i)\, \frac{1}{2} \cdot 2h \cdot 1 = h\, f(x_i)$$

$$\int_{x_{i-1}}^{x_{i+1}} f\,\varphi_i \approx f(x_i)\int_{x_{i-1}}^{x_{i+1}} \varphi_i = f(x_i)\,\frac{1}{2}\cdot 2h\cdot 1 = h f(x_i)$$

$$\left(= h f(x_i) + O(h^3)\right)$$

$$\int_{x_{i-1}}^{x_{i+1}} f\,\varphi_i \approx f(x_i)\int_{x_{i-1}}^{x_{i+1}} \varphi_i = f(x_i)\,\frac{1}{2}\cdot 2h\cdot 1 = h f(x_i)$$

$$\left(= h f(x_i) + O(h^3)\right)$$

Or interpolate $\sum_{j=1}^{n} f(x_j)\varphi_j$

$$\int \varphi_i \varphi_j \, dx = \begin{cases} \frac{2}{3}h & i=j \\ \frac{1}{6}h & |i-j|=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_0^1 x(1-x)\,dx$$

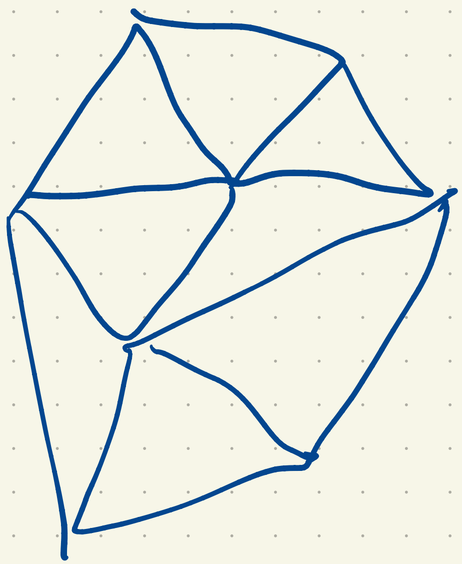
$$\int_0^1 x^2\,dx$$

A

$\int f \varphi_i$

So solving a modification of

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdots & & \\ -1 & 2 & -1 & 0 & \cdots & \\ & & & & & \\ & & & & & \cdot \end{bmatrix}\vec{u} = h\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}$$



$$\frac{1}{h^2}\begin{bmatrix} \ast \end{bmatrix}\vec{u} = \vec{f} + O(h^2)$$

Convergence?

H_0^1 norm: $\left[\int_0^\ell (V_x)^2\right]^{1/2}$ (this is a norm).

$-u_{xx} = f$

$\int u_x \phi_x - f\phi = 0 \qquad \forall \phi \in H_0^1$

H^1_0 norm: $\left[\int_0^\ell (v_x)^2\right]^{1/2}$ (this is a norm).

H^1

$$\int u_x \phi_x - f\phi = 0 \qquad \forall \phi \in H^1_0$$

$$\int u^h_x \phi_x - f\phi = 0 \qquad \forall \phi \in \mathcal{V}_h \subseteq H^1_0$$

$u^h - u$

$$\int_0^\ell e^h_x \phi_x = 0 \quad \forall \phi \in \mathcal{V}_h.$$

H^1_0 norm: $\left[\int_0^\ell (v_x)^2\right]^{1/2}$ (this is a norm).

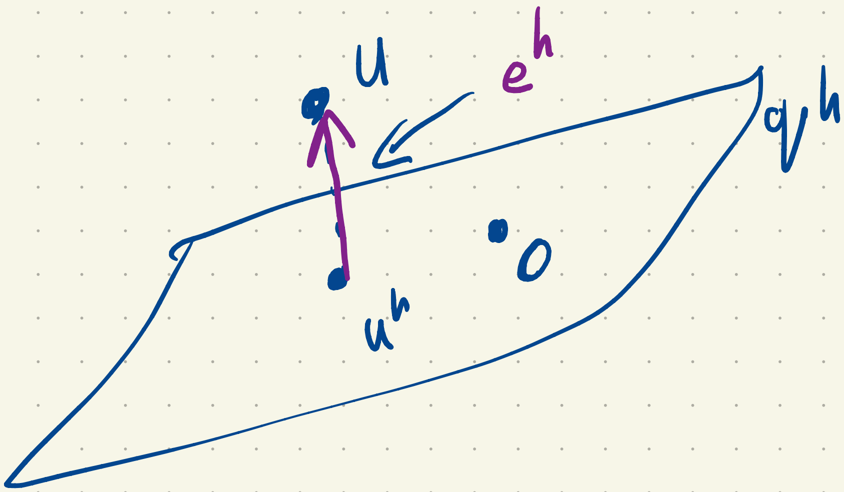
$$\int u_x \phi_x - f\phi = 0 \qquad \forall \phi \in H^1_0$$

$$\int u^h_x \phi_x - f\phi = 0 \qquad \forall \phi \in \mathcal{V}_h \subseteq H^1_0$$

$$\int e^h_x \phi_x = 0 \qquad \forall \phi \in \mathcal{V}_h.$$

[The error solves an equation!]

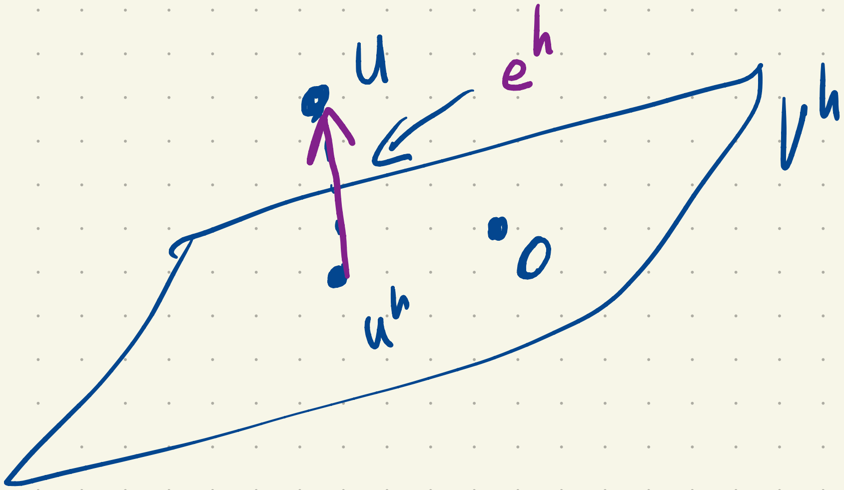
u^h is closest (in $\mathcal{V}_h$) to u!



$$\|u^h - u\|_{H^1_0}$$

$$\left[\int_0^\ell \partial_x(u^h - u)\,\partial_x(u_h - u)\right]^{1/2}$$

u^h is closest (in $\mathcal{V}_h$) to u!



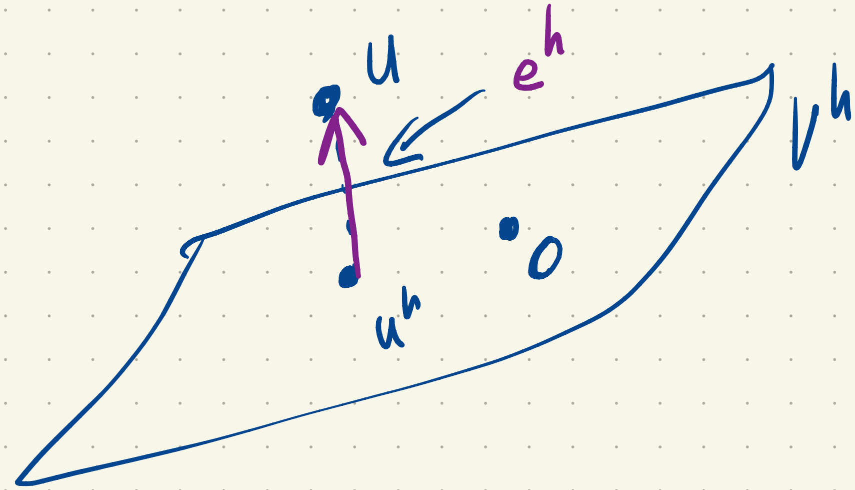
$e^h \perp V^h$ $\qquad H^1_0$

$$\int \partial_x v \, \partial_x w$$

$$\int \partial_x e^h \, \partial_x w = 0$$

$$\forall w \in \mathcal{V}^h$$

u^h is closest (in $\mathcal{V}_h$) to u!



$e^h \perp V^h$

$$\int e^h_x \, \phi_x = 0 \quad \forall \phi \in \mathcal{V}_h$$

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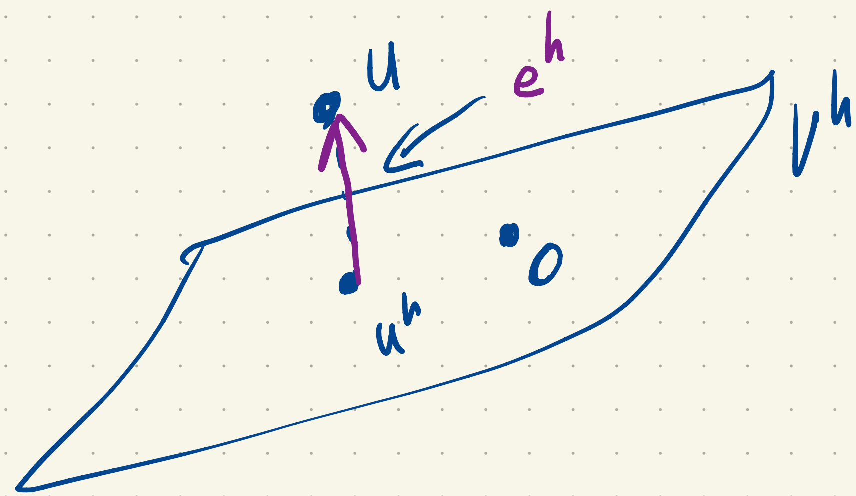
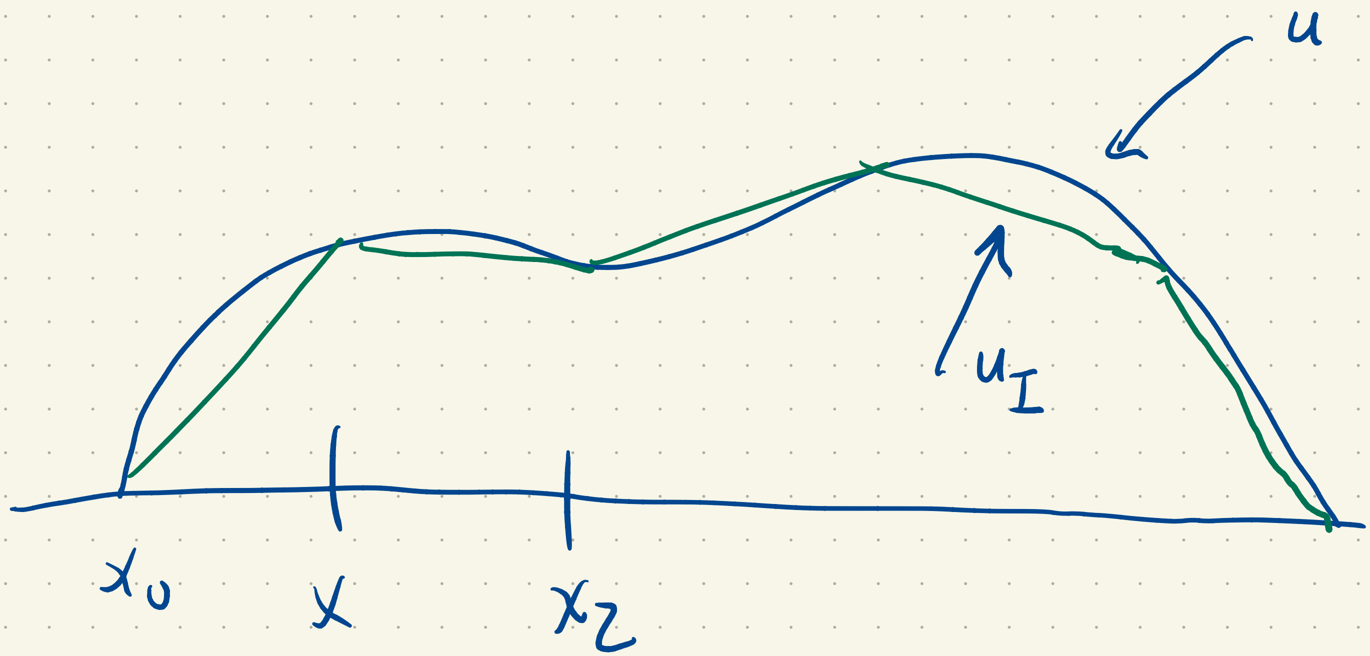


$e^h \perp V^h$

$$\int e^h_x \phi_x = 0 \quad \forall \phi \in \mathcal{V}_h$$

So we can estimate the error by finding <u>anybody</u> in V^h who is close to u.

Reasonable choice: the linear interpolant:



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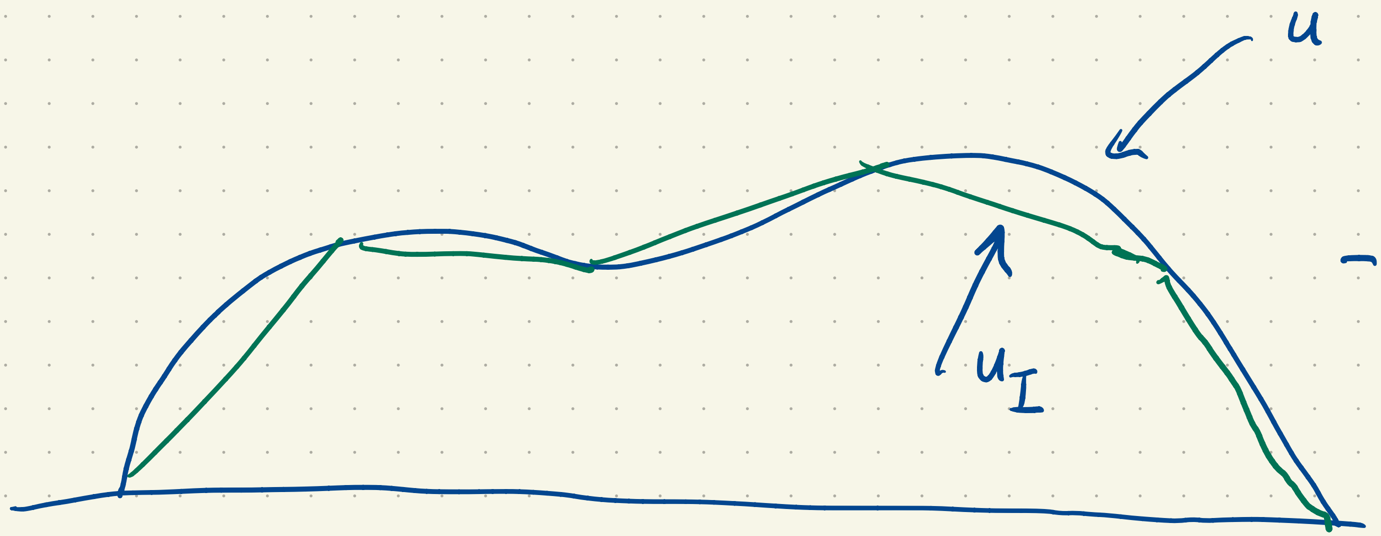
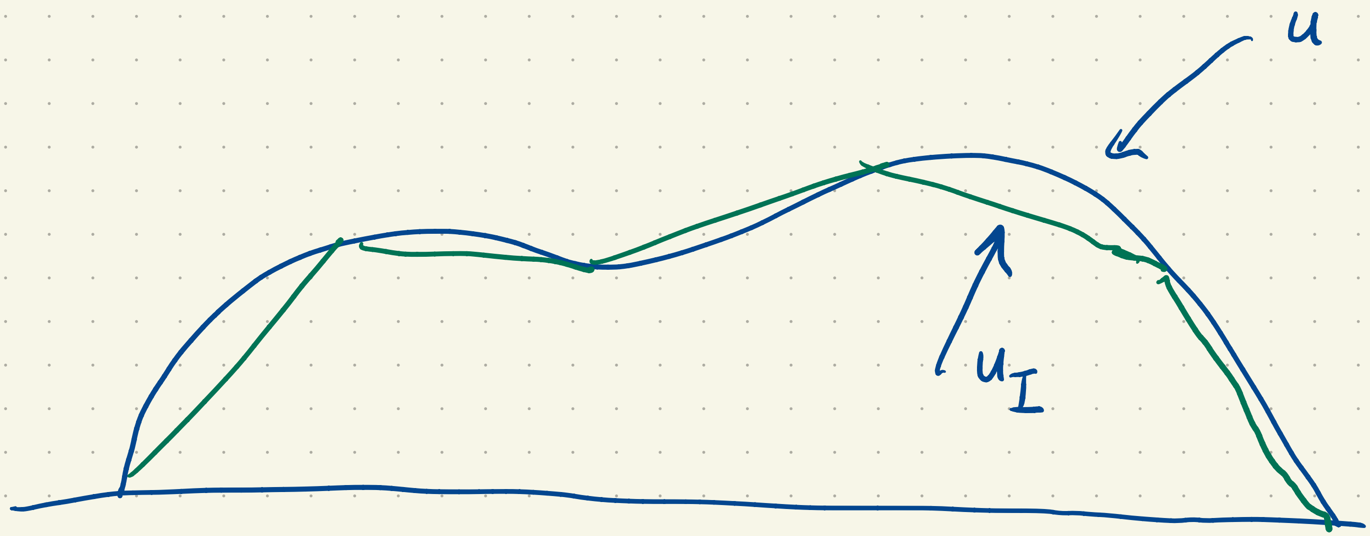


$-u'' = f$

Facts: $\|u_I - u\|_{H^1_0} \leq C\, h \|\overset{f}{u''}\|_{L^2}$

$$\left(\left[[u]\,[x]^{-1}\right]^2 [x]\right)^{1/2} \qquad [x]\left(\left([u]\,[x]^{-2}\right)^2 [x]\right)^{1/2}$$

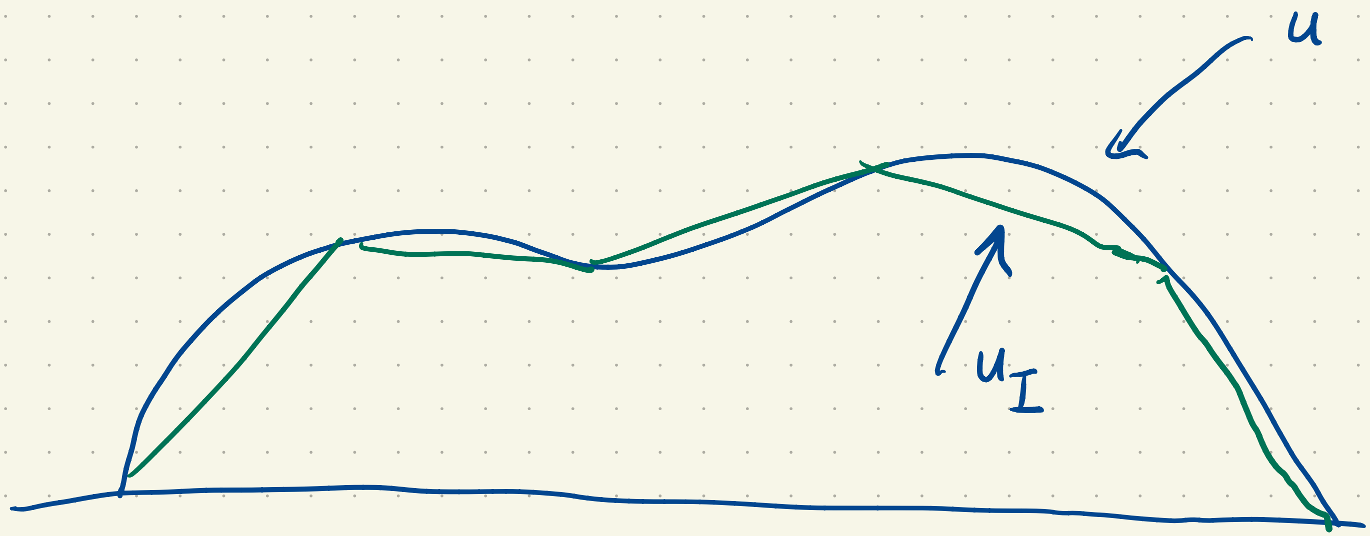
Reasonable choice: the linear interpolant:



Facts: $\| u_I - u \|_{H_0^1} \leq C \, h \| u'' \|_{L^2}$

$$\max |u_I - u| \leq C h^2 \max |u''|$$

Reasonable choice: the linear interpolant:



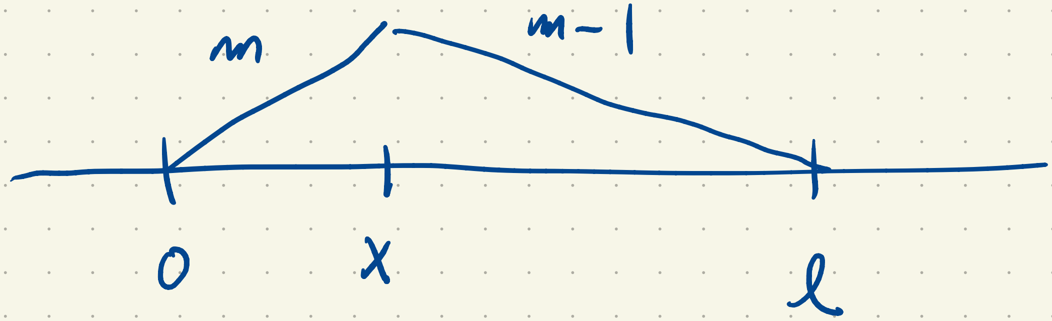
Facts: $\|u_I - u\|_{H_0^1} \le C\, h \|u''\|_{L^2}$ f

$\max |u_I - u| \le C h^2 \max |u''|$ f

$$\|u^h - u\|_{H_0^1} \leq \|u_I - u\|_{H_0^1} \leq C\, h\, \|\overset{f}{u''}\|_{L^2}$$

$$u^h \to u \qquad O(h) \quad \text{in} \quad H_0^1 \ \text{norm}$$

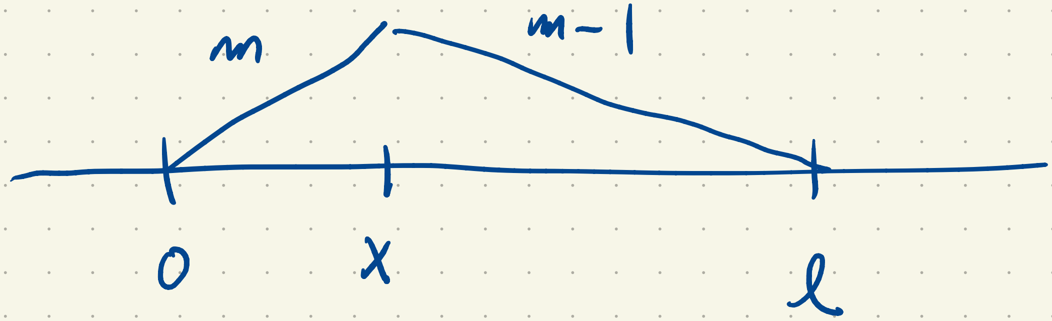
$g_x =$



$$-g_x'' = \delta$$

(Baby Green's Function!)

$u_h \to u \quad O(h)$ in H_0^1

$g_x =$ 

$O(h^2)$ in max norm

H_0^1

$$-g_x'' = \delta$$

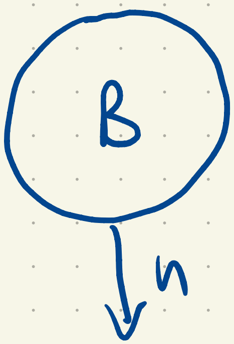
$$(u_h - u)(x_i) = \int (u - u_h)'(s)\, g_{x_i}'(s)\, ds$$

$$= 0 \quad \text{since } g_{x_i} \in V_h.$$

u_h $\underline{\underline{\text{is}}}$ linear interpolant, $\max |u_h - u| \le C h^2 \max |f|$

Elliptic Problems in Higher Dimensions

$\partial_x u$

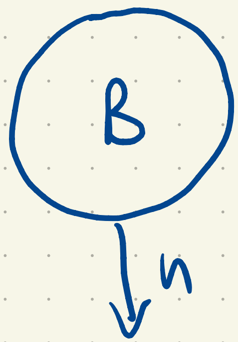


Heat flux is proportional to ∇u,

points in opposite direction.

Flux $= -K \nabla u$.

Elliptic Problems in Higher Dimensions

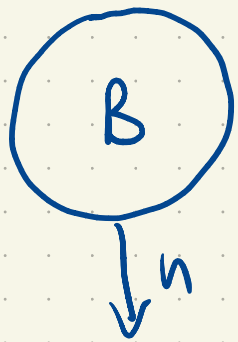


Heat flux is proportional to ∇u, points in opposite direction.

Flux: $-K\nabla u$.

Net flux (leaving): $\displaystyle\int_{\partial B} -K\nabla u\cdot\vec{n} = -\int_{B} \operatorname{div}(K\nabla u)$

Elliptic Problems in Higher Dimensions



Heat flux is proportional to ∇u, points in opposite direction.

Flux: $-K \nabla u$.

Net flux (leaving): $$\int_B -K \nabla u \cdot \vec{n} = -\int_B \operatorname{div}(K \nabla u)$$

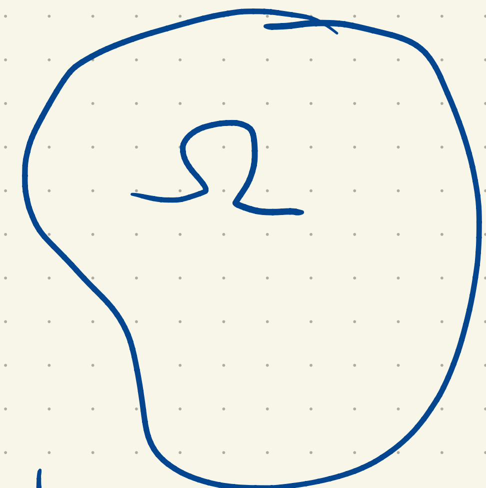
$$\frac{d}{dt} \int_B u = + \int_B \operatorname{div}(K \nabla u)$$

Heat equation $\qquad u_t = \partial_x (K u_x)$

$$u_t = \operatorname{div}(K \nabla u)$$

Steady state: $\quad \operatorname{div}(K \nabla u) = 0$

(K constant: $\quad \Delta u = 0$)



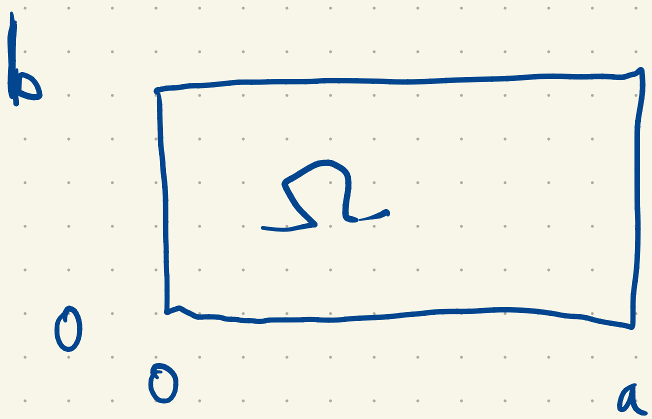
$$\left.\begin{aligned} \Delta u &= 0 \\ u\big|_{\partial\Omega} &= g \end{aligned}\right]$$

Basic Dirichlet problem.

Poisson Equation:

$$-\Delta u = f$$

$$u\big|_{\partial\Omega} = 0$$

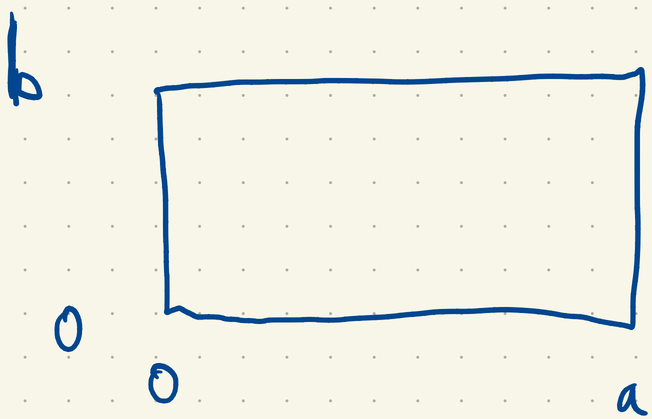


Poisson Equation:

$$-\Delta u = f$$

$$u\big|_{\partial\Omega} = 0$$

$k, l = 1, 2, 3 \ldots$



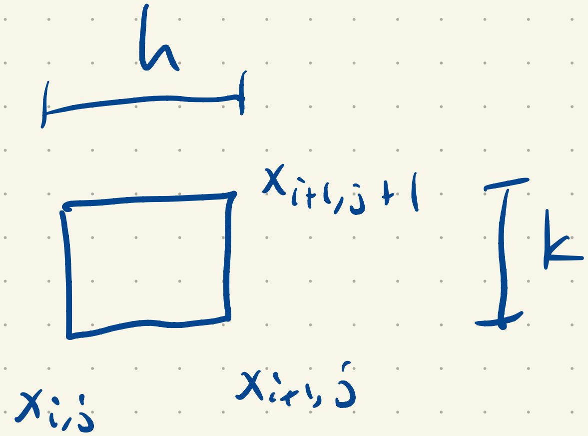
$$-\Delta u = \lambda u$$

$$u = \sin\left(k\pi \frac{x}{a}\right) \sin\left(l\pi \frac{y}{b}\right)$$

$$\lambda = \pi^2 \left[\left(\frac{k}{a}\right)^2 + \left(\frac{l}{b}\right)^2\right]$$

Numerical Strategy:

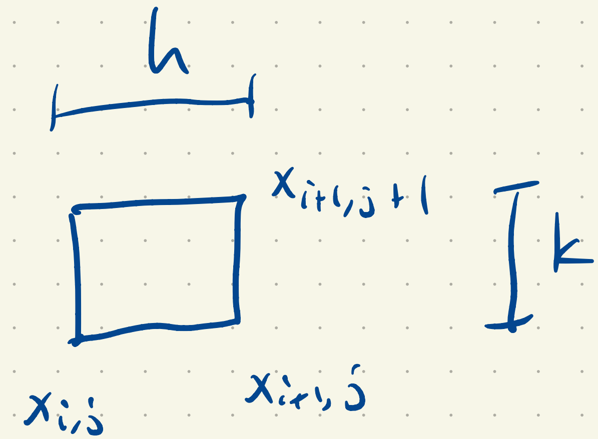
Centered Differences.



$0 \le i \le N+1 \quad 0 \le j \le M+1$

Numerical Strategy:

Centered Differences.



$0 \le i \le N, \quad 0 \le j \le M$

$$-\Delta u \approx \frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h^2} + \frac{-u_{i,j+1} + 2u_{i,j} - u_{i,j-1}}{k^2}$$

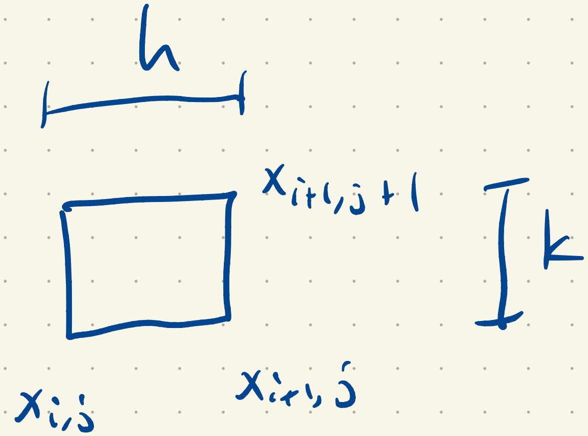
$$-\Delta u = -\partial_x^2 u - \partial_y^2 u = -u_{xx} - u_{yy}$$

$-\Delta u = f$

Numerical Strategy:

$f(x_{ij})$

Centered Differences.



$0 \le i \le N, \quad 0 \le j \le M$

$$-\Delta u \approx \frac{-u_{i+1,j} + 2u_{i,j} - u_{i-1,j}}{h^2} + \frac{-u_{i,j+1} + 2u_{i,j} - u_{i,j-1}}{k^2}$$

$$= \frac{1}{k^2}\left[-\lambda^2\left(u_{i+1,j} + u_{i-1,j}\right) + 2(1+\lambda^2)\,u_{i,j} - \left(u_{i,j-1} + u_{i,j+1}\right)\right]$$

$$\lambda = \frac{k}{h} \qquad \left(\beta = 2\,(1+\lambda^2)\right)$$