Two lines in the hyperbolic plane are parallel if they meet at a point at infinity. Two non-intersecting lines that are not parallel are called hyper-parallel.

Part I: Getting to know triangles.

1. An asymptotic triangle has three sides, all parallel to each other. Draw one of these in the Poincaré ball model. Then draw two of them in the half-plane model. Of these last two, one should have all three ideal points on $\mathbb{R}$ and one should have one ideal point not on $\mathbb{R}$.

2. Draw pictures to show that given two non-colinear rays starting from a common vertex, there is a unique line that is parallel to the two rays. Because there is a special point (the vertex of the two rays) you may find it convenient to do this in the ball model.

This configuration (a triangle with two points at infinity) is known as a 2/3 asymptotic triangle.

3. A 1/3 asymptotic triangle has two vertices in $\mathbb{H}^2$ and one vertex at infinity. Draw a picture of one of these in the ball model and then a picture of one of these in the half-plane model.

Part II: Asymptotic triangles have finite area. (!)

4. Show that the area of a 1/3 asymptotic triangle is finite. Hint: Use the half plane model, and put the vertex at infinity somewhere useful. You'll need the formula for area in terms of an integral, but don't work too hard: if A is a subset of B then the area of A is less than or equal to the area of B.

5. Show that the area of a 2/3 asymptotic triangle is finite. Hint: put the finite vertex somewhere helpful in the ball model and decompose your triangle.

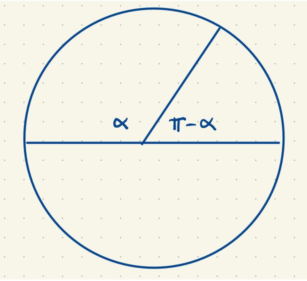
6. Use the half plane model to show that every asymptotic triangle can be decomposed into two 2/3 asymptotic triangles. Conclude that every asymptotic triangle has finite area.

Part III: Asymptotic triangles are all congruent to each other.

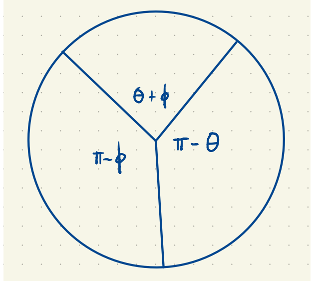
7. On your next homework you will show that given three **ideal** points p_1, p_2 and p_3 in the upper half plane model, there is a hyperbolic transformation that takes these points to 0, 1 and ∞ (though perhaps not in this order). Assuming this, show that all asymptotic triangles are congruent. We'll denote the area of an asymptotical triangle by I.

Part IV: You can compute the area of a 2/3 asymptotic triangle from its exterior angle.

8. Let $T(\alpha)$ be the area of a 2/3 asymptotic triangle with **exterior** angle α. Show that $T(\alpha) + T(\pi - \alpha) = I$. Hint:



9. Show that $T(\theta) + T(\phi) + T(\pi - (\theta + \phi)) = I$ for any angles $\theta, \phi \in (0, \pi)$. Hint: A triangle with exterior angle θ has interior angle $\pi - \theta$. Also, hint:



10. Combine the last two results to show that $T(\theta) + T(\phi) = T(\theta + \phi)$; that is, T is additive. It can be shown that A is also increasing (come back to this if you have extra time). A standard result from analysis states that increasing additive function is necessarily linear, and we conclude that $T(\theta) = c\theta$ for some constant c. Show that in fact $c = I/\pi$. Hint: Problem 8.

Part V: The area of a triangle is determined by its angles.

11. In the diagram below the inner triangle has area A. Use it and your results thus far to show

$$A = \frac{I}{\pi}\left[\pi - (\alpha + \beta + \gamma)\right].$$