Last class:

convergence: $\|e\|_\infty \to 0$ as $h \to 0$

$u' - f = 0$

consistency: $$\underbrace{\frac{u_{i+1} - u_i}{h} - f(t_i, u_i)}_{} = 0$$

$\to f(t_i, u(t_i)) = u'(t_i)$

substitute $u(t_i)$ for u_i

want this $\to 0$

was $-\tau_i$

$\tau_i = h\tau_i$

$\|\tau\|_\infty \to 0$

introduction to stability

$$u' = \lambda u$$

$$e_{i+1} = (1+\lambda h) e_i + h \tau_i$$

$$e_k = (1+\lambda h)^k e_0 + \sum_{j=1}^{k} h \tau_{j-1} (1+\lambda h)^{k-j}$$

$$\left| (1+\lambda h)^k \right| \leq K \qquad 0 \leq k \leq M$$

ind. of M / h

$$|e_k| \leq K \left[|e_0| + T \|\tau\|_\infty \right]$$

$$K = e^{\lambda T}$$

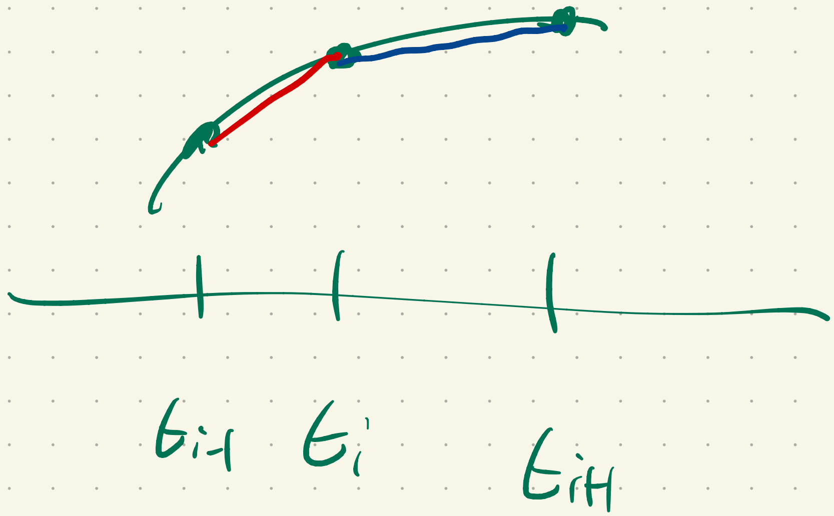
Catalog of Methods

1) Euler's Method

$$u'(t_i) = \frac{u(t_{i+1}) - u(t_i)}{h} \underbrace{- \frac{u''(x_i)\,h}{2}}_{\tau_i}$$

2) Backwards Euler

$$u'(t_i) = \frac{u(t_i) - u(t_{i-1})}{h} + \frac{u''(\eta_i)}{2} h$$



$$\frac{u_i - u_{i-1}}{h} = f(t_i, u_i)$$

$$u_{i+1} = u_i + h f(t_{i+1}, u_{i+1})$$

$u_{i+1} =$

u_i, t_i

$$u_{i+1} - h f(t_{i+1}, u_{i+1}) = u_i$$

Solve for u_{i+1} $\longrightarrow$ implicit method

vs. Euler's method

explicit

Matlab: fzero

Python: scipy.optimize.fsolve

Converges linearly. Why bother?

Stay tuned!

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3) Midpoint (Leapfrog)

$$u'(t_i) = \frac{u(t_{i+1}) - u(t_{i-1})}{2h} + \tau_i$$

on HW $\quad -\frac{u'''(\eta_i)}{6} h^2$

2nd order convergence?

$$u_{i+1} = u_{i-1} + 2h f(t_i, u_i)$$

This requires two previous iterates

In the beginning we have only one

and a bootstrap is required.

$u_0 \longrightarrow u_1$

$(u_0, u_1) \longrightarrow u_2$

$(u_1, u_2) \longrightarrow u_3$

Need e_1 to be $O(h^2)$ to keep 2^{nd} order convergence.

We can use Forward Euler to get e_1

$$e_1 = \underbrace{(1+\lambda h)e_0 + h\underbrace{\tau_D}_{O(h)}}_{O(h^2)}$$

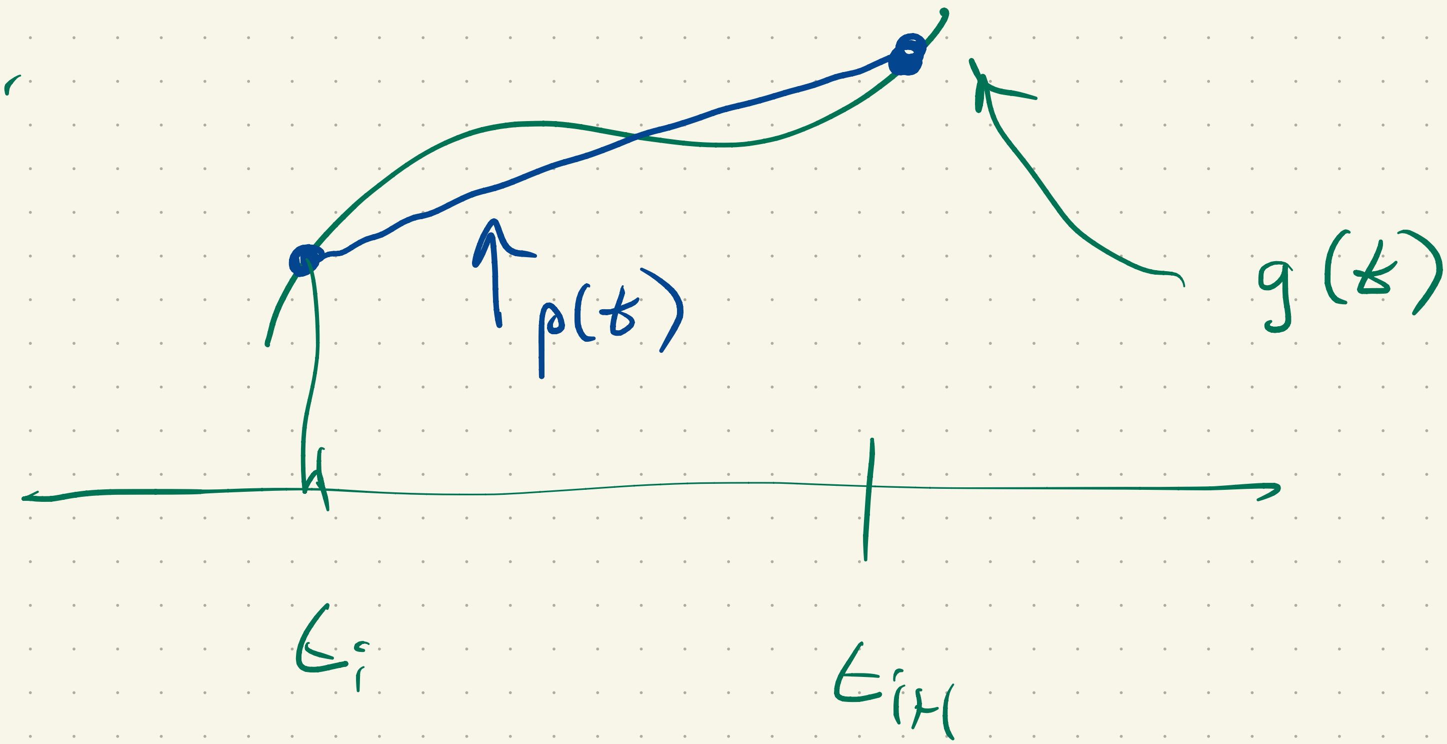
However, we will see Midpoint is unstable.

Methods from Quadrature:

$$u(t_{i+1}) = u(t_i) + \underbrace{\int_{t_i}^{t_{i+1}} u'(s)\,ds}_{\displaystyle\int_{t_i}^{t_{i+1}} f(s,u(s))\,ds}$$

Strategy: replace the integral with something nearby you can actually integrate.

E.g.



$$g(t) = p(t) + e(t)$$

$$|e(t)| \le \frac{1}{8} \max |g''| \, h^2$$

h

$$\int_{t_i}^{t_{i+1}} g(s)\,ds = \int_{t_i}^{t_{i+1}} p(s)\,ds + \underbrace{\int_{t_i}^{t_{i+1}} e(s)\,ds}_{O(h^3)}$$

$$h\,\frac{p(t_{i+1}) + p(t_i)}{2}$$

$$u_{i+1} = u_i + \frac{h}{2}\left[f(t_i, u_i) + f(t_{i+1}, u_{i+1}) \right]$$

Trapezoid

implicit.

Consistent?

$$\frac{u_{i+1} - u_i}{h} - \frac{1}{2}\left[f(t_i, u_i) + f(t_{i+1}, u_{i+1})\right] = 0$$

$$u' = f(t, u)$$

$$\underbrace{\frac{u(t_{i+1}) - u(t_i)}{h} - \frac{1}{2}\left[u'(t_{i+1}) + u'(t_i)\right]}_{\text{LTE}}$$

$$u(t_{i+1}) = u(t_i) + u'(t_i)h + \frac{1}{2}u''(t_i)h^2 + O(h^3)$$

$$\frac{u(t_{i+1}) - u(t_i)}{h} = u'(t_i) + \frac{1}{2}u''(t_i)h + O(h^2)$$

$$u'(t_{i+1}) = u'(t_i) + u''(t_i)h + O(h^2)$$

$$\frac{1}{2}\left(u'(t_{i+1}) + u'(t_i)\right) = \frac{1}{2}\left[2u'(t_i) + u''(t_i)h + O(h^2)\right]$$

$$= u'(t_i) + \frac{1}{2}u''(t_i)h + O(h^2)$$

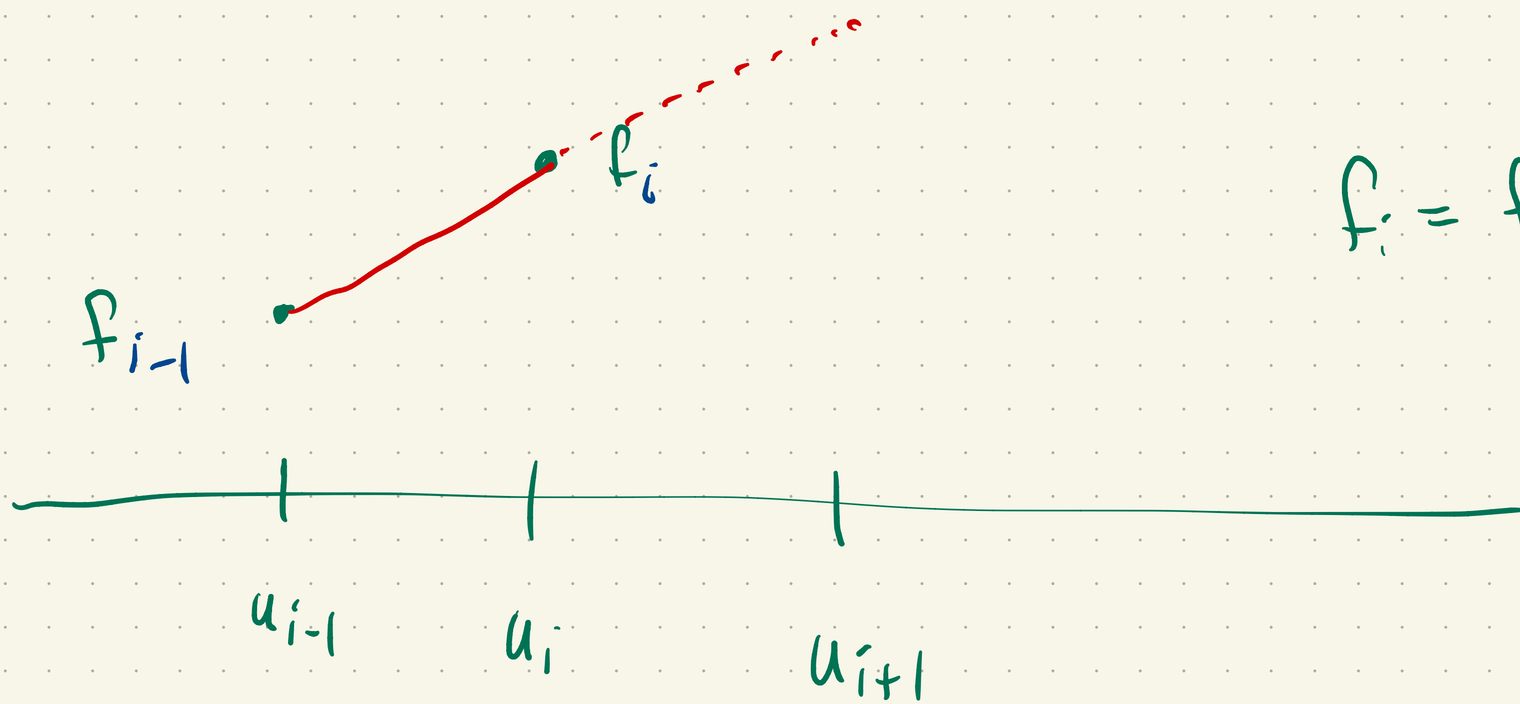
Local truncation error: $O(h^2)$

implicit, 2nd order, stable

Two more families (Adams)

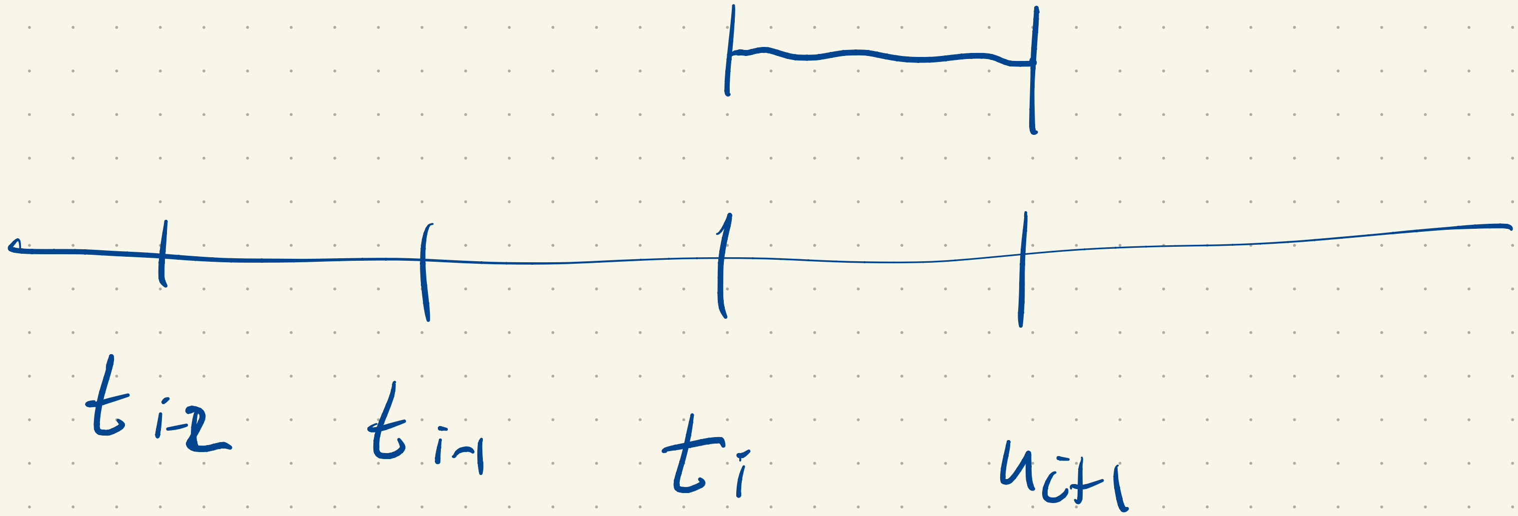
Adams-Bashforth

Adams-Moulton



$$f_i = f(t_i, u_i)$$

$$u(t_{i+1}) = u(t_i) + \int_{t_i}^{t_{i+1}} f(s, u(s))\, ds$$



f_{i-2} f_{i-1} f_i