1. Exercise 3.14149 (Solver: John Gimbel)

If a and b are even integers, then so is $a + b$.

Solution:

Let a and b be even integers. Then there exist integers j and k such that $a = 2j$ and $b = 2k$. But then

$$a + b = 2j + 2k = 2(j + k).$$

Since $j + k \in \mathbb{N}$, $a + b$ is even.

2. Exercise 2.718 (Solver: Jill Faudree)

Let X be a set.

a) An intersection of topologies on X is a topology on X.

b) A union of topologies on X need not be a topology.

Solution (part a):

Let $\{\mathcal{T}_\alpha\}$ be a family of topologies and let $\mathcal{T} = \cap_\alpha \mathcal{T}_\alpha$. Observe that $\varnothing$ and X belong to $\mathcal{T}$ as they belong to each $\mathcal{T}_\alpha$.

Suppose $\{U_\beta\}$ is a family of sets in $\mathcal{T}$ and let $U = \cup_\beta U_\beta$. Fix α and observe that each $U_\beta \in \mathcal{T}_\alpha$. Since $\mathcal{T}_\alpha$ is a topology, $U \in \mathcal{T}_\alpha$. Since α is arbitrary, $U \in \cap \mathcal{T}_\alpha = \mathcal{T}$.

The proof that a finite intersection of sets in $\mathcal{T}$ belongs to $\mathcal{T}$ is essentially similar.

Solution (part b):

Let $X = \{1, 2, 3\}$. Let $\mathcal{T}_1 = \{\varnothing, \{1, X\}\}$ and let $\mathcal{T}_2 = \{\varnothing, \{2\}, X\}$. It is easy to see that these are topologies. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{\varnothing, \{1\}, \{2\}, X\}$. Observe that $\mathcal{T}$ is not closed under taking unions as $\{1\}$ and $\{2\}$ are elements of $\mathcal{T}$ but $\{1, 2\}$ is not.

3. Exercise 9 (Solver: Elizabeth Allman)

Let X be a metric space. Show that the collection of open balls in X forms the basis of a topology.

Solution:

We start with a technical lemma.

Lemma 3.1. *Suppose $B_1 = B_{r_1}(x_1)$ and $B_2 = B_{r_2}(x_2)$ are open balls in X and $x_3 \in B_1 \cap B_2$. Then there is an $r > 0$ such that $B_{r(x_3)} \subseteq B_1 \cap B_2$.*

Proof. Let $r = \min(r_1 - d(x_3, x_1), r_2 - d(x_3, x_2))$ and observe that $r > 0$. Now suppose $z \in B_{r(x_3)}$. The triangle inequality implies

$$
\begin{aligned}
d(x_1, z) &\leq d(x_1, x_3) + d(x_3, z) \\
&< d(x_1, x_3) + r \\
&\leq d(x_1, x_3) + (r_1 - d(x_3, x_1)) \\
&= r_1
\end{aligned}
$$

Hence $z \in B_{\{r_1\}}(x_1) = x_1$. Similarly $z \in B_2$, and hence $B_{r(z)} \subseteq B_1 \cap B_2$. $\square$

Continuing with the solution of the problem, let $\mathcal{B}$ be the collection of open balls in X. Fix $x \in X$ and note that $\cup_{r>0} B_{r(x)} = X$. Hence $\mathcal{B}$ covers X. Moreover, by Lemma 3.1, $\mathcal{B}$ satisfies the refinement property. Hence by the topology construction lemma, $\mathcal{B}$ generates a topology on X, and the open sets are simply the unions of open balls.