Nonlinear Problems

$u(0) = u(\ell)$

$u'(0) = u'(\ell)$

$$-u'' + ru + au^5 - bu^{-7} = 0$$

on $[0, \ell]$, periodic.

$a, b > 0$

r constant

Nonlinear Problems

$$-u'' + ru + au^5 - bu^{-7} = 0$$

on $[0, \ell]$, periodic.

$a, b > 0$

r constant

(with $u > 0$)

Nonlinear Problems

$$-u'' + ru + au^5 - bu^{-7} = 0 \quad \text{on } [0, \ell], \text{ periodic.}$$

$a, b > 0$

r constant

(with $u \geq 0$)

We can think of this as $\mathcal{F}(u) = 0$

nonlinear map.

How to find roots?

Nonlinear Problems

$$-u'' + ru + au^5 - bu^{-7} = 0 \quad \text{on } [0, \ell], \text{ periodic.}$$

$a, b > 0$

r constant

(with $u \geq 0$)

We can think of this as $\mathcal{F}(u) = 0$

nonlinear map.

How to find roots? Newton! (in ∞ dimensions)

Discrete version

applied pointwise

$$-A\vec{u} + r\vec{u} + a\vec{u}^5 - b\vec{u}^7 = 0$$

Discrete version

applied pointwise

$$-A\vec{u} + r\vec{u} + a\vec{u}^5 - b\vec{u}^7 = 0$$

This is $\mathcal{F}^h(\vec{u}) = \vec{0}$

Now you really do have Newton's Method.

applied element-wise

$$-A\vec{u} + f(\vec{u}) = 0$$

Newton's Method

Want to solve $\mathcal{F}^h(\vec{u}) = \vec{0}$.

Start with a guess $\vec{u}_0$.

Then seek correction $\delta\vec{u}$

$$\mathcal{F}^h(\vec{u}_0 + \delta\vec{u}) = 0$$

Jacobian

$$\mathcal{F}^h(\vec{u}_0 + \delta\vec{u}) \approx \mathcal{F}^h(\vec{u}_0) + J\,\delta\vec{u}$$

$$\mathcal{F}^h(\vec{u}_0) + J\delta\vec{u} = 0$$

$$J\delta\vec{u} = -\mathcal{F}^h(\vec{u}_0)$$

$$\delta\vec{u} = -J^{-1}\mathcal{F}^h(\vec{u}_0)$$

Newton's Method

1) Start with $\vec{u}_0$

2) Solve $J\,\delta\vec{u} = -\mathcal{F}^h(u_0)$

3) $\vec{u}_1 = \vec{u}_0 + \delta\vec{u}$

Now repeat!

Newton's Method

1) Start with $\vec{u}_0$

2) Solve $J \delta \vec{u} = -\mathcal{F}^h(u_0)$

3) $\vec{u}_1 = \vec{u}_0 + \delta \vec{u}$

Now repeat!

J

$$f'(u_k)\,\delta u = -f(u_k)$$

$$\delta u = -\frac{f(u_k)}{f'(u_k)}$$

$$u_{k+1} = u_k - \frac{f(u_k)}{f'(u_k)}$$

Newton's Method

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} \qquad \begin{matrix} u_i \approx \\ u(x_i) \end{matrix}$$

$$\mathcal{F}^h(u_0 + \delta u) \approx \mathcal{F}^h(u_0) + J\,\delta u = 0$$

$$\mathcal{F}_i^h(\vec{u}) = -\sum_j A_{ij} u_j + f(u_i)$$

$$\mathcal{F}^h : \mathbb{R}^N \to \mathbb{R}^N$$

$$J_{ij} = \frac{\partial}{\partial}$$

$$J = ?$$

$\mathcal{H} : \mathbb{R}^N \to \mathbb{R}^N$ $\qquad$ $u(x)$

$\mathcal{H}(w)$ $\qquad$ $\dfrac{\partial \mathcal{H}_i}{\partial w_j}$

$\mathcal{F}^h(\vec{u})$ $\qquad$ $J = \dfrac{\partial \mathcal{F}^h_i}{\partial u_j}$

Newton's Method

$$\mathcal{F}^h(u_0 + \delta u) \approx \mathcal{F}^h(u_0) + J\,\delta u = 0$$

$$\mathcal{F}^h_i(\vec{u}) = -\sum A_{ij} u_j + f(u_i)$$

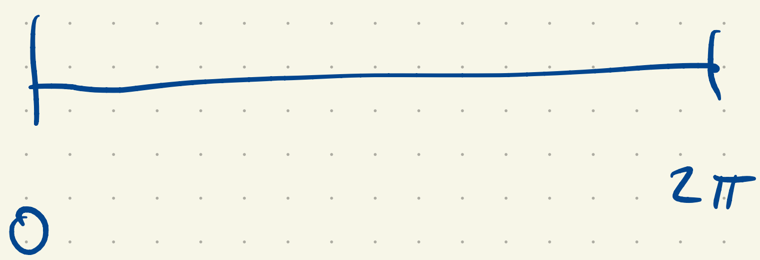
$$J = ?$$

$$J_{ij} = \frac{\partial \mathcal{F}^h_i}{\partial u_j}$$

$$J_{ij} = \frac{\partial F_i}{\partial u_j} = -A_{ij} + f'(u_i)\,\delta_{ij}$$

Can we solve $J\,\delta u = -F^h(u_0)$?

$-\partial_x^2$



$-\Delta$

$\sin(kx)$

$\cos(kx)$

$k \in \mathbb{Z}$

$$J_{ij} = \frac{\partial F_i}{\partial u_j} = -A_{ij} + f'(u_i)\,\delta_{ij}$$

positive operator

energy

$$J_{ij} = \frac{\partial F_i}{\partial u_j} = -A_{ij} + f'(u_i)\,\delta_{ij}$$

positive operator

$f'(x) \geq 0$?

$a u^5 - b u^{-7}$ ✓

$5 a u^4 + 7 b u^{-8}$ ✓

$$J_{ij} = \frac{\partial F_i}{\partial u_j} = -A_{ij} + f'(u_i)\,\delta_{ij}$$

positive operator

$f'(x) \geq 0$?

$a u^5 - b u^{-7}$ ✓

$5 a u^4 + 7 b u^{-8}$ ✓

$J \delta u = -F^n(u)$

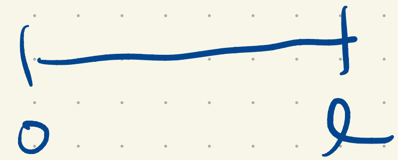
You'll do something like this on final.

(It's posted!)

Finite Element Primer

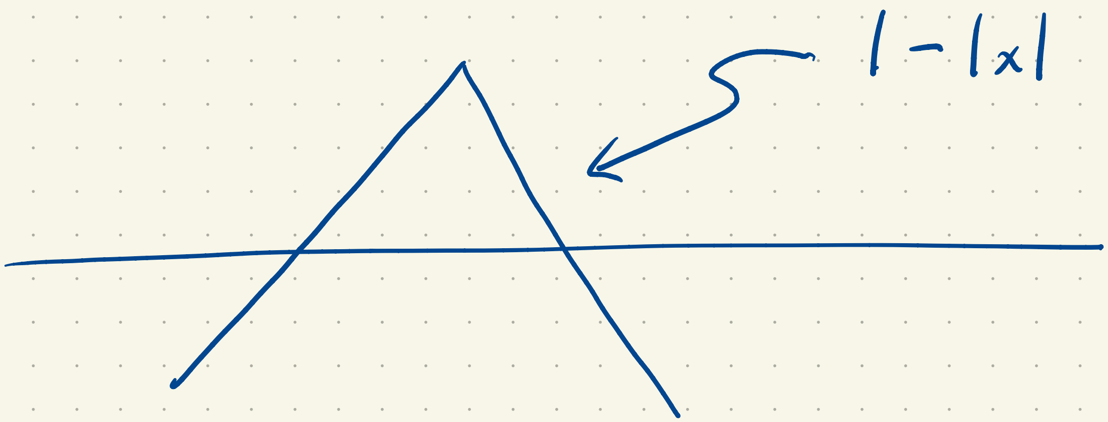
$$u'' + p'u' + qu = f$$

$$-u'' = f$$



$$u\big|_{0,\ell} = 0$$

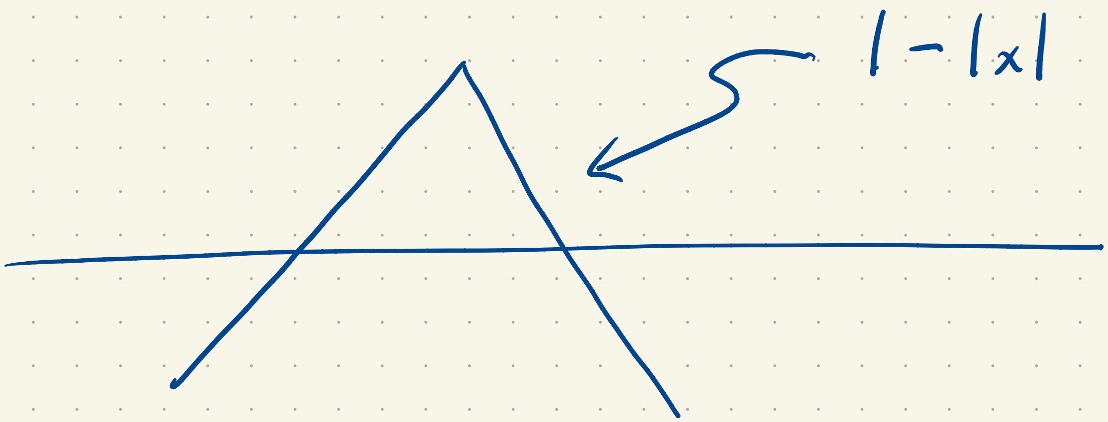
Consider:



Does this function have a derivative?

Finite Element Primer

Consider:



Does this function have a derivative?

Classically, no.

Derivatives by duality:

ϕ smooth, compactly supported. (test function)

$$\int_{\mathbb{R}} f'\phi = -\int_{\mathbb{R}} f\phi' \quad \text{(no boundary terms}$$

$$\int_{-L}^{L} f'\phi = \int_{-L}^{L} -f\phi' + \Big[f(L)\phi(L) - f(\underset{-L}{\uparrow})\,\phi(\underset{-L}{\uparrow}) \Big]$$

$$\int_{\mathbb{R}} f\phi' = -\int_{\mathbb{R}} f'\phi$$

Derivatives by duality:

ϕ smooth, compactly supported. (test function)

$$\int_{\mathbb{R}} f' \phi = - \int f \phi'$$ (no boundary terms

If $f(x) = 1 - |x|$ had a derivative g we'd

want

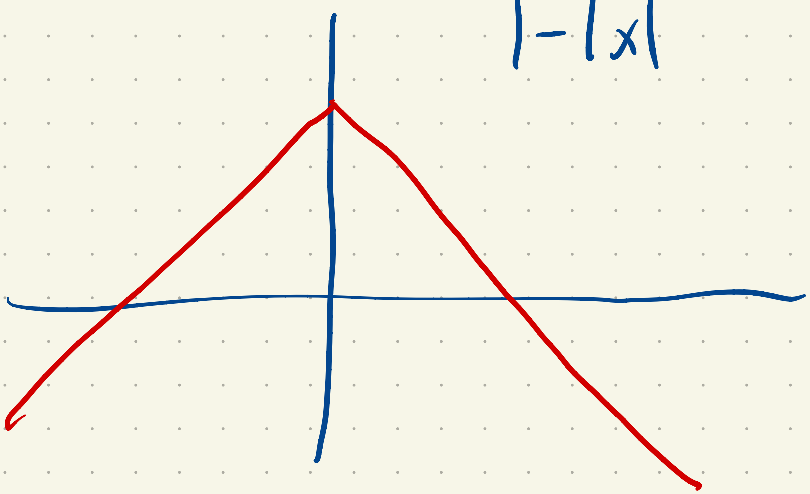
$$\int_{\mathbb{R}} g \phi = - \int f \phi'$$ whenever ϕ is a test function

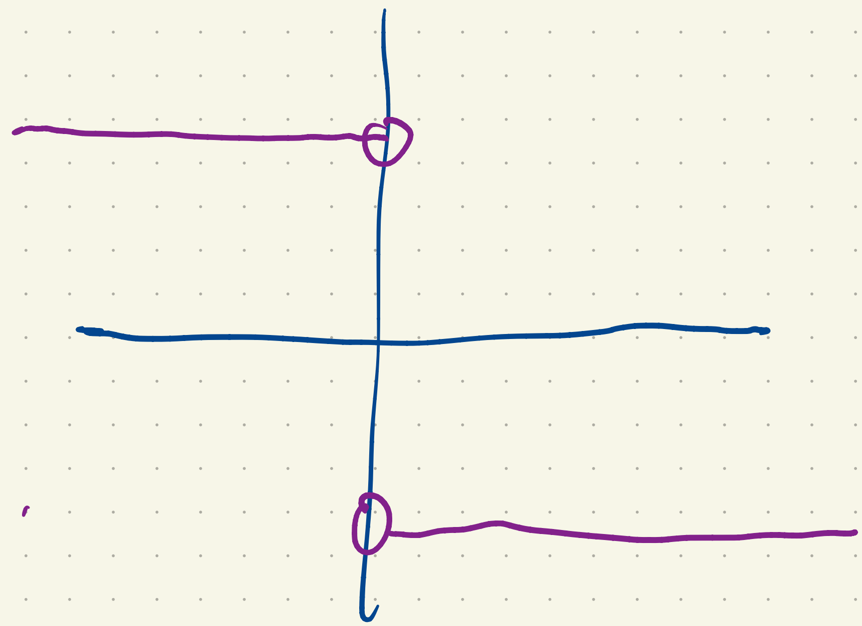
I claim $g(x) = \begin{cases} +1 & x < 0 \\ -1 & x > 0 \end{cases}$ works

(value at 0 irrelevant)

$$\int_{\mathbb{R}} \overbrace{(1-|x|)}^{f} \phi' = \int_{\mathbb{R}} \phi' - \int_{\mathbb{R}} |x| \phi'$$

$1-|x|$





I claim $g(x) = \begin{cases} +1 & x < 0 \\ -1 & x > 0 \end{cases}$ works

(value at 0 irrelevant)

$$\int_{\mathbb{R}} (1 - |x|)\, \phi' = \int_{\mathbb{R}} \phi' - \int_{\mathbb{R}} |x|\, \phi'$$

Support of ϕ: $[-L, L]$

$$\int_{-L}^{L} \phi' = \phi(L) - \phi(-L) = 0.$$

$$x \cdot \phi$$

$$\int_{\mathbb{R}} -|x|\,\phi' = \int_{-L}^{0} x\,\phi' - \int_{0}^{L} x\,\phi'$$

$$= -\Big[\underbrace{\phi(0)\cdot 0}_{} + \overbrace{\phi(-L)\cdot L}^{=0}\Big] + \int_{-L}^{0} -\phi$$

$$-\Big[\underbrace{\phi(L)\cdot L}_{=0} - \overbrace{\phi(0)\cdot 0}^{}\Big] + \int_{0}^{L} \phi$$

$$= \int_{-L}^{0} -1\cdot \phi + \int_{0}^{L} (+1)\cdot\phi$$

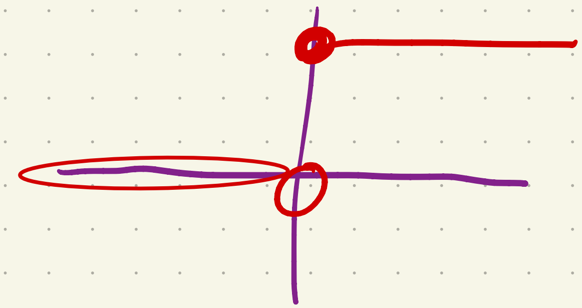
$$= -\int_{-L}^{L} g\cdot\phi \qquad g(x) = \begin{cases} 1 & x<0 \\ -1 & x>0 \end{cases}$$

$$\int f \phi' = -\int g \phi$$

$1 - |x|$

This notion of a derivative is a

<u>distributional</u> derivative.



$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f' = ?$$

This notion of a derivative is a

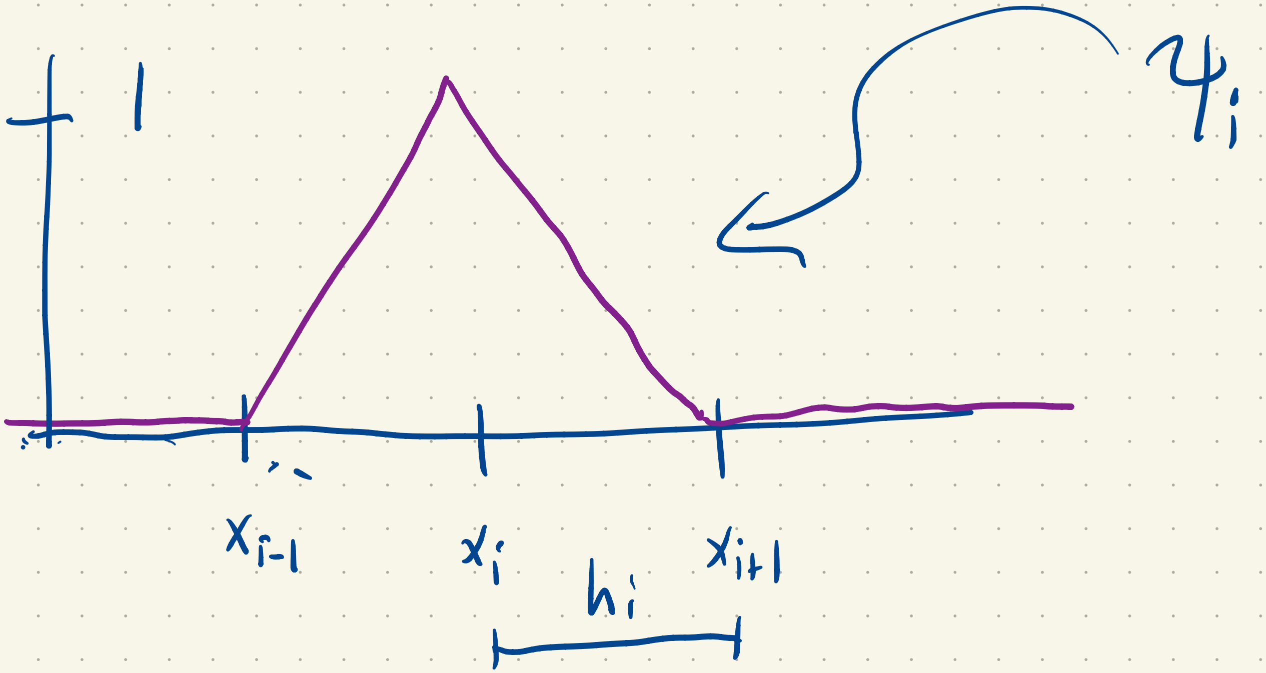
distributional derivative.

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad f' = ?$$

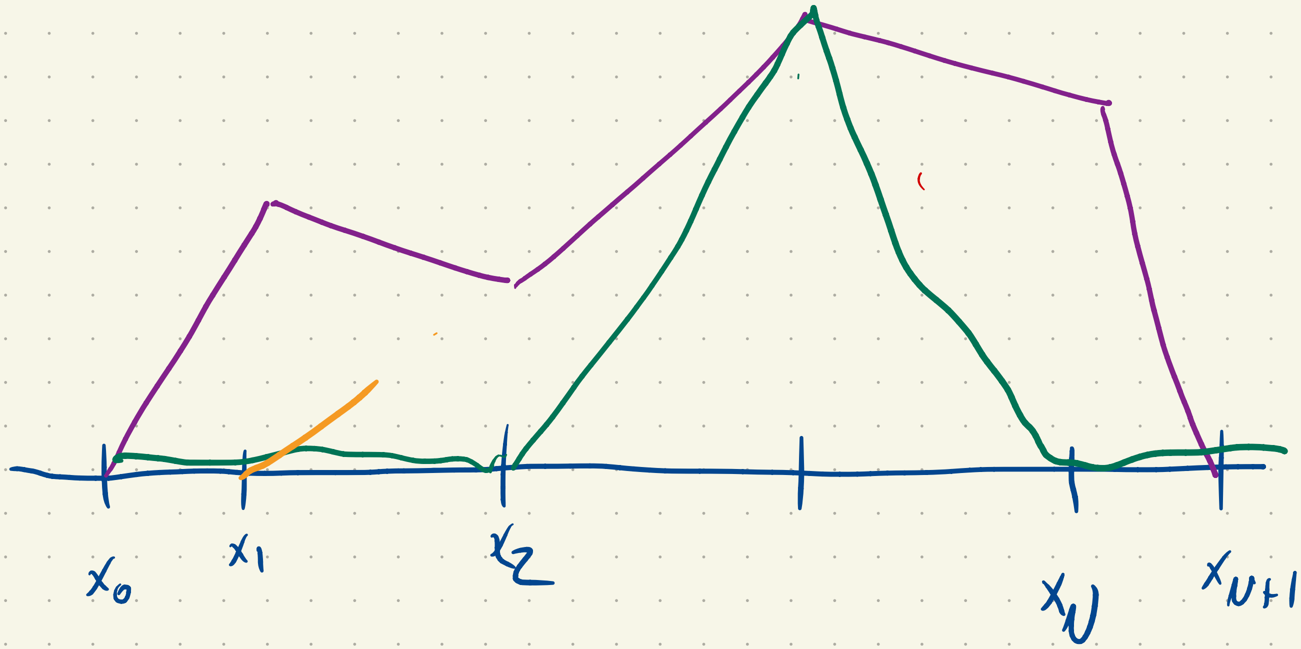
$$-\int_{\mathbb{R}} f \cdot \phi' = -\int_0^L \phi' = -\phi(L) + \phi(0) = \phi(0)$$

There is no function g with $\displaystyle\int_{\mathbb{R}} g\,\phi = \phi(0)$

$f' = \delta$ regardless!



$$\psi_i' = \begin{cases} 1/h_{i-1} & x_{i-1} < x < x_i \\ -1/h_i & x_i < x < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



$$\sum_{k=1}^{N} u_k \varphi_k$$