$$\varphi_n = \varphi_n^+ - \varphi_n^-$$

$\varphi_n \to f$ pointwise

$\left.\begin{array}{l}\varphi_n^+ \to f_+ \\ \varphi_n^- \to f_-\end{array}\right]$ p.w.

$$|\varphi_n| = \varphi_n^+ + \varphi_n^- \leq f_+ + f_- = |f|$$

$$= \varphi_n^+ + \varphi_n^- \leq \varphi_{n+1}^+ + \varphi_{n+1}^- = |\varphi_{n+1}|$$

(and uniformly ~~on~~ on any set where f is bounded)

Measurable functions are nearly continuous functions

$[a,b]$

$E \subseteq [a,b]$

Lemma: Given a measurable set E and $\varepsilon > 0$ there is a step function φ s.t.

$$m(\{\varphi \neq \chi_E\}) < \varepsilon$$

Let $\varepsilon > 0$.

Pf: Since $m(E) < \infty$ there is a finite union of intervals A such that $m(E \Delta A) < \varepsilon$. WLOG $A \subseteq [a,b]$.

Then $\varphi = \chi_A$ is a step function and

$$\{ \varphi \neq \chi_E \} = E \Delta A.$$

Cor: Let ψ be a simple function on $[a,b]$. Given $\varepsilon > 0$ there is a step function φ with $m(\{ \psi \neq \varphi \}) < \varepsilon$.

Pf: We can write $\psi = \sum_{k=1}^{n} a_k \chi_{E_k}$ where $E_k \subseteq [a,b]$ is measurable. For each k, let φ_k be a step function with $m(\{\varphi_k \neq \chi_{E_k}\}) < \frac{\varepsilon}{n}$.

Let $\varphi = \sum_{k=1}^{n} a_k \varphi_k$, so φ is a step function.

If for some x $\varphi_k(x) = \chi_{E_k}(x)$ for all k

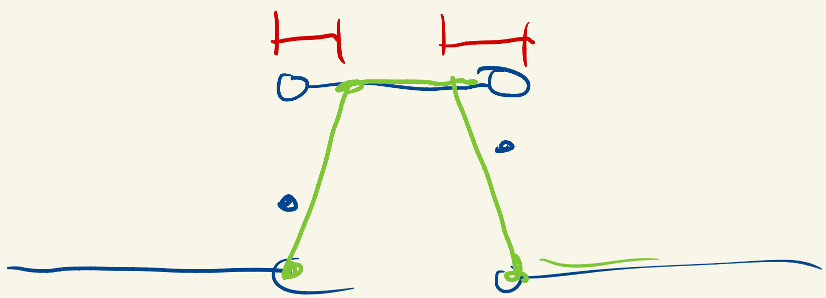
then $\varphi(x) = \psi(x)$. Hence

$$\{\varphi \neq \psi\} \subseteq \bigcup_k \{\varphi_k \neq \chi_{E_k}\}.$$

But then $m(\{\varphi \neq \psi\}) \leq \sum_k m(\{\varphi_k \neq \chi_{E_k}\})$

$$< \sum_k \varepsilon/n = \varepsilon.$$

Lemma: Given a step function ϕ on $[a,b]$ and $\varepsilon > 0$ there is a continuous function g on $[a,b]$ where $m(\{g \neq \phi\}) < \varepsilon$.



Lemma: Suppose $f : [a,b] \to \mathbb{R}$ is measurable and finite a.e. Given $\varepsilon > 0$ there exists $K > 0$ such that $m(\{|f| \leq K\}) < \varepsilon$.

Pf: Let $A_n = \{ |f| \geq n \}$. So each A_n is measurable and $A_{n+1} \subseteq A_n$. Since $[a,b]$ has finite measure continuity from above implies

$$m(A_n) \longrightarrow m\left(\bigcap_n A_n \right) = m(\{ |f| = \infty \})$$

$$= 0.$$

So, given $\varepsilon > 0$ there exists n so that $m(A_n) < \varepsilon$. That is,

$$m(\{ |f| \geq n \}) < \varepsilon.$$

Thm (Borel)

Suppose $f: [a,b] \to \overline{\mathbb{R}}$ is measurable and finite a.e. Given $\varepsilon > 0$ there is a continuous function $g: [a,b] \to \mathbb{R}$ with $m(\{|f-g| \geq \varepsilon\}) < \varepsilon$.

Pf: Pick K with $m(\underbrace{\{|f| \geq K\}}_{F_1}) < \varepsilon/3$.

Let $\hat{f} = \max(-K, \min(f, K))$, so $\hat{f}$ is measurable and $m(\underbrace{\{\hat{f} \neq f\}}_{\subseteq F_1}) < \varepsilon/3$. Let ψ be a simple function such that $|\hat{f} - \psi| < \varepsilon$ everywhere. (via the Basic Construction)

Let φ be a step function with $m(\overbrace{\{\varphi \neq \psi\}}^{F_2}) < \varepsilon/3$.

Let g be a continuous function with $m(\underbrace{\{g \neq \varphi\}}_{F_3}) < \varepsilon/3$.

If at some point x, $x \notin (F_1 \cup F_2 \cup F_3)$

then $|f| \leq K$ so $f(x) = \hat{f}(x)$ and

$|\psi(x) - \hat{f}(x)| < \varepsilon$ and $\psi(x) = \varphi(x)$ and $\varphi(x) = g(x)$.

So $|g(x) - f(x)| < \varepsilon$.

Hence $\{|g-f| \geq \varepsilon\} \subseteq \bigcup_{i=1}^{3} F_i$ and

$$m(\{|g-f| \geq \varepsilon\}) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \qquad \square$$

See also: Luzin's Thm

If $f: \mathbb{R} \to \mathbb{R}$ is measurable then for all $\varepsilon > 0$ there is a set E, $m(E) < \varepsilon$ and $f|_{\mathbb{R}\setminus E}$ is continuous on $\mathbb{R} \setminus E$.

$\chi_{\mathbb{Q}}$ $\qquad E = \mathbb{Q}$

$$\chi_{\mathbb{Q}}\big|_{\mathbb{R}\setminus\mathbb{Q}} = 0$$