# Countability / Cardinality

Two sets $A$ and $B$ have the same cardinality if there is a bijection $f: A \to B$. $A \sim B$

(exercise: $\sim$ is an equivalence relation)

A set is finite if it is empty or if it has the cardinality of $\{1, 2, \dots, n\}$ for some $n$.

A set is infinite if it is not finite.

A set is countably infinite if it has the cardinality of $\mathbb{N}$

[Exercise: $\mathbb{N}$ is infinite]

A set that is <u>infinite</u> but not countably infinite is uncountable.

A set is <u>countable</u> (<u>at most</u> <u>countable</u>) if it is not uncountable (i.e. it is either finite or countably infinite).

Lemma: If $\underbrace{A}_{\{1,2,\dots,n\}}$ is finite and $B \subseteq A$ then $B$ is finite.

Cor: If $B \subseteq A$ and $B$ is infinite then $A$ is infinite

Contrapositive!

Lemma: If $A$ is infinite and $A \subseteq \mathbb{N}$ then $A$ is countably infinite.

Pf: Let $A_1 = A$.

Let $a_1$ be the least element of $A_1$. (Well Ordering Property)

Let $A_2 = A_1 \setminus \{a_1\}$. Then $A_2$ is ~~infinite~~ nonempty.

Let $a_2$ be the least element of $A_2$.

Continuing inductively we can construct a sequence

$a_1 < a_2 < a_3 < \ldots$

in $A$. We claim the sequence exhausts $A$.

$$[k \longmapsto a_k]$$

Suppose $c \in A$. Since $a_k \geq k$ for all $k$, $a_c \geq c$.

Now: $c \in A = A_{c+1} \cup \{a_1, a_2, \dots, a_c\}$.

But $c \notin A_{c+1}$ so $c = a_k$ for some $k = 1, 2, \dots c$.

The map $k \longmapsto a_k$ is then obviously a bijection.

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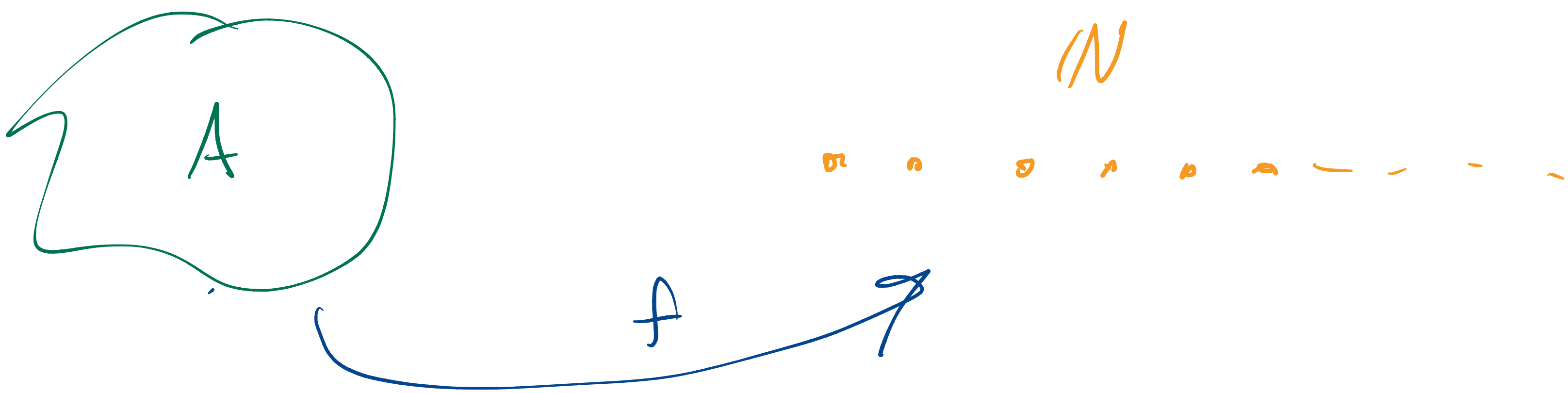
Cor: If $A \subseteq \mathbb{N}$ then $A$ is (at most) countable.

Cor: $A$ is countable $\Longleftrightarrow$ $A \sim B \subseteq \mathbb{N}$.

Cor: If $B \subseteq A$ and $A$ is countable, then $B$ is countable.

Cor: If $f: A \to \mathbb{N}$ is an injection then $A$ is countable.




Cor: If $f: A \to B$ is a surjection and $A$ is countable then $B$ is countable.

Pf: WLOG we can assume $A \subseteq \mathbb{N}$.

For each $b \in B$ let $g(b) = \min\left( f^{-1}(\{b\}) \right)$

(Remark: $f^{-1}(\{b\}) \neq \emptyset$ by surjectivity of $f$).

Observe $f(g(b)) = b$ and hence $g$ is injective.

So $g : B \to \mathbb{N}$ is injective and $B$ is countable.

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$$\underbrace{id}_{inj} = f \circ \overset{inj}{g}$$

$b_1, b_2$ $\qquad$ $g(b_1) = g(b_2)$

Injective: If $g(b_1) = g(b_2) \Rightarrow b_1 = b_2$.

$$f(g(b_1)) = f(g(b_2))$$
$$id(b_1) = id(b_2)$$
$$b_1 = b_2$$

Cor: If $f: A \twoheadrightarrow B$ is a surjection and $B$ is uncountable then $A$ is uncountable.

Pf: contrapositive

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Cor: If $A \subseteq B$ and $A$ is uncountable then $B$ is uncountable.

(contrapositive)

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Cor: A nonempty set $A$ is countable iff there is a surjection $\mathbb{N} \to A$.

Countable sets: $\mathbb{N}$

$\mathbb{N} \times \mathbb{N}$

$$\begin{array}{cccc} 1 & 2 & 4 & 7 \\ 3 & 5 & 8 & \\ 6 & 9 & & \\ 10 & & & \end{array}$$

$$\mathbb{Q}_+ = \{ q \in \mathbb{Q} : q > 0 \}$$

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$$

$$f(a,b) = a/b$$

$A, B$ countable $\Rightarrow$ $A \cup B$ is countable

$f_1 : \mathbb{N} \to A$
$f_2 : \mathbb{N} \to B$ , sorjectios $(A, B \neq \emptyset)$

$f : \mathbb{N} \times \{0,1\} \longrightarrow A \cup B$

$f(n, k) = f_k(n)$ is a sorjection

$\mathbb{N} \times \{0,1\} \subseteq \mathbb{N} \times \mathbb{N}$ and is hence countable.

$f : \mathbb{N} \times \{0,1\} \to A \cup B$ is a sorjection from a countable set.

$\bigcup_{k=1}^{\infty} A_k$ $\qquad A_k$ countable.