Method of Lines

a) Discretize in space first

"upwinding" $u_x(x_i, t) \longrightarrow \dfrac{u(x_i, t) - u(x_{i-1}, t)}{h}$

"downwinding" $u_x(x_i, t) \longrightarrow \dfrac{u(x_{i+1}, t) - u(x_i, t)}{h}$

b) Now a system of ODEs $\vec{u} = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix}$

M.O.L

$$u_i(t) \approx u(x_i, t)$$

$$\vec{u}_t + \frac{a}{h} D \vec{u} = 0 \qquad D = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

$$-u_{i-1}(t) + u_i(t)$$

$$\underbrace{-u_0(t)}_{0} + u_1(t)$$

b) Now a system of ODEs $\vec{u} = \begin{bmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{bmatrix}$

$$u_i(t) \approx u(x_i, t)$$

$$\vec{u}_t + \frac{a}{h} D \vec{u} = 0 \qquad D = \begin{bmatrix} 1 & & & \\ -1 & \ddots & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{bmatrix}$$

Forward Euler

$$\frac{\vec{u}(t_{j+1}) - \vec{u}(t_j)}{k} + \frac{a}{h} D \vec{u} = 0$$

$$\vec{u}(t_{j+1}) = \vec{u}(t_j) - \underbrace{\frac{ka}{h}}\, D\, \vec{u}(t_j)$$

$$\frac{ka}{h} = \lambda$$

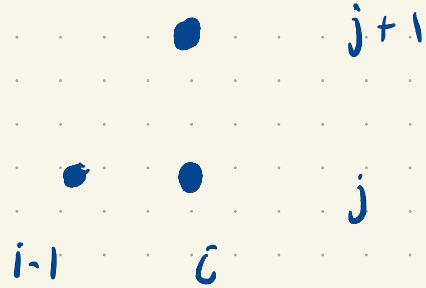
$$= \underbrace{(I - \lambda D)}_{A}\, \vec{u}(t_j)$$

$$A = \begin{bmatrix} 1-\lambda & & & 0 \\ \lambda & 1-\lambda & & \\ & \ddots & \ddots & \\ 0 & & \lambda & 1-\lambda \end{bmatrix}$$

DEMO

Numerical Domain of Dependence

$$u_{i,j+1} = (1-\lambda)\, u_{i,j} + \lambda\, u_{i-1,j} \qquad (u_{0,j} = 0)$$

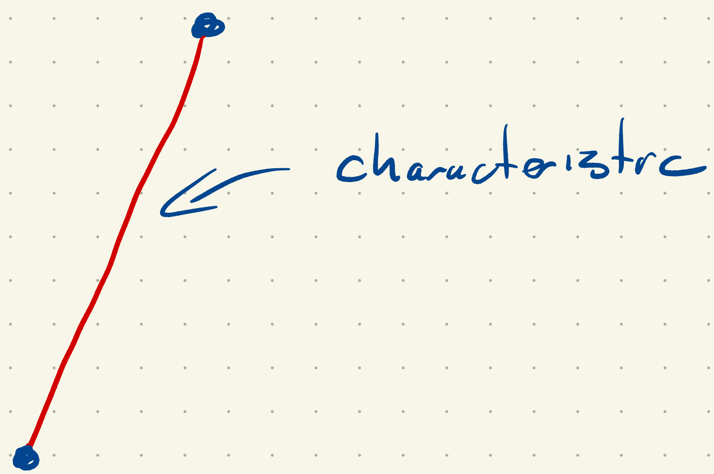


Numerical Domain of Dependence

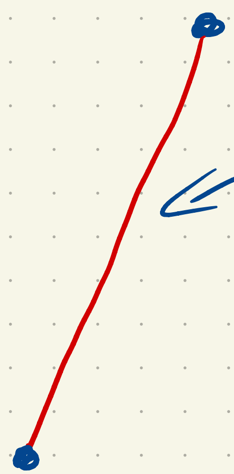
$$u_{i,j+1} = (1-\lambda)\, u_{i,j} + \lambda\, u_{i-1,j} \qquad (u_{0,j} = 0)$$

True Domain of dependence

$$u_t + a u_x = f(x,t,u)$$



True Domain of dependence



characteristic

(even for $u_t + a u_x = f(x,t)$

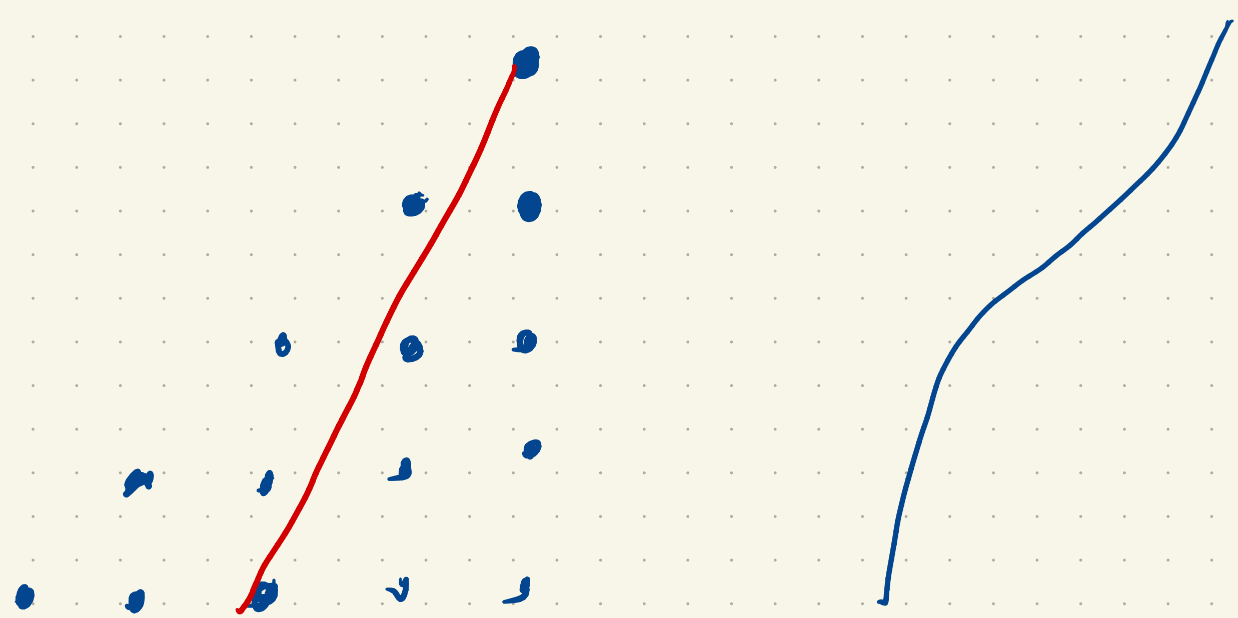
$u_t + a u_x = f(x,t,u)$)

CFL Condition

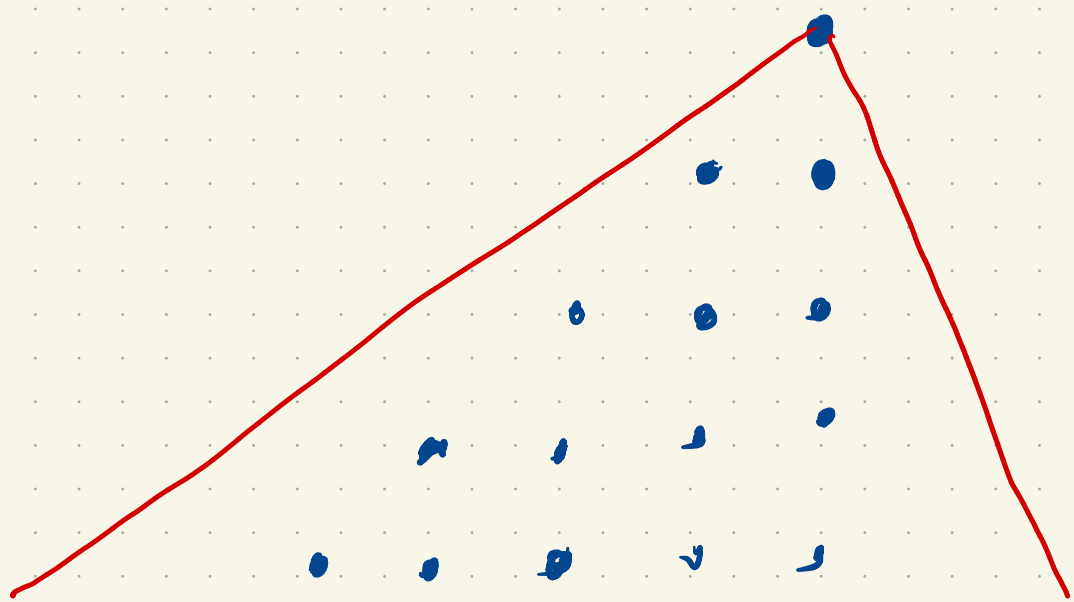
$$u_t + v u_x = 0$$

Continuous DoD $\subseteq$ Numerical DoD

[Courant, Friedrichs, Levy] '28

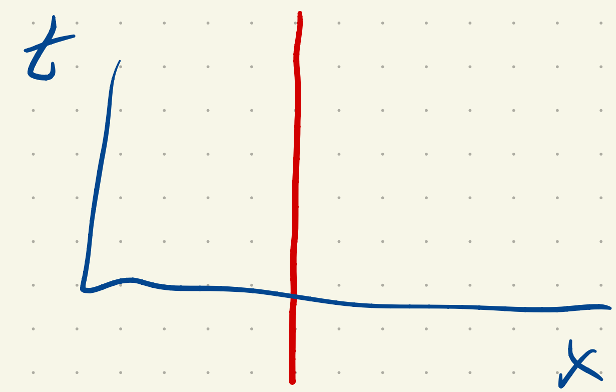


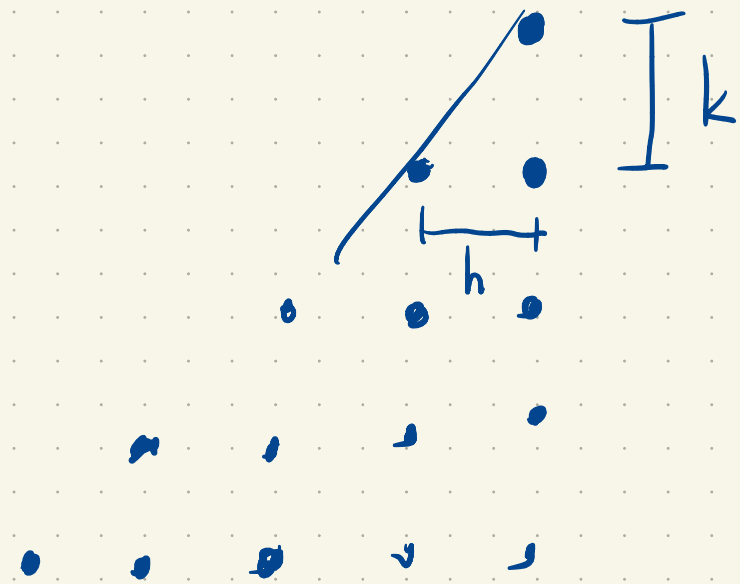
Failure of CFL



[No hope of convergence!]

Grid Speed



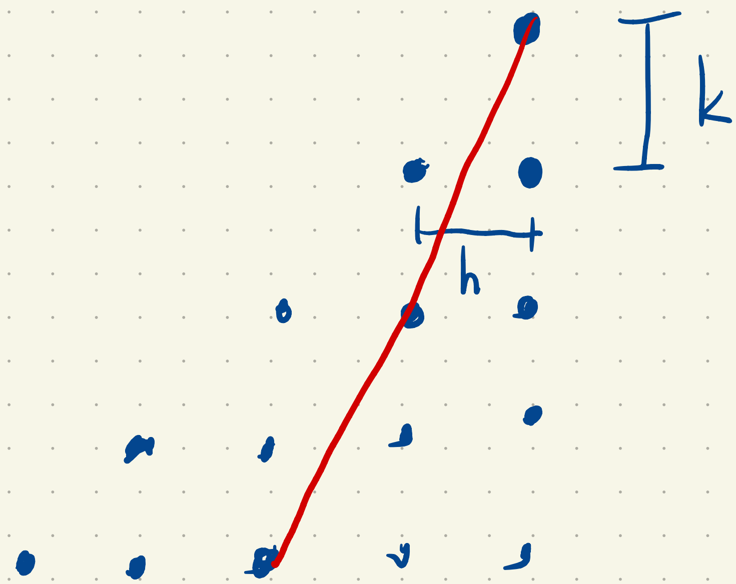


velocity: $\frac{h}{k}$

a

$$\frac{h}{k} > a$$

Grid Speed



CFL:

$$\frac{h}{k} \geq a \geq 0$$

[Grid transmits info as fast as a does]

Connection to λ

$$\lambda = \frac{ka}{n}$$

$$0 \le a \le \frac{n}{k} \iff 0 \le \frac{ka}{n} \le 1$$

$$0 \le \lambda \le 1$$

Connection to λ

$$\lambda = \frac{ka}{h}$$

$$0 \leq a \leq \frac{h}{k} \iff 0 \leq \frac{ka}{h} \leq 1$$

$$0 \leq \lambda \leq 1$$

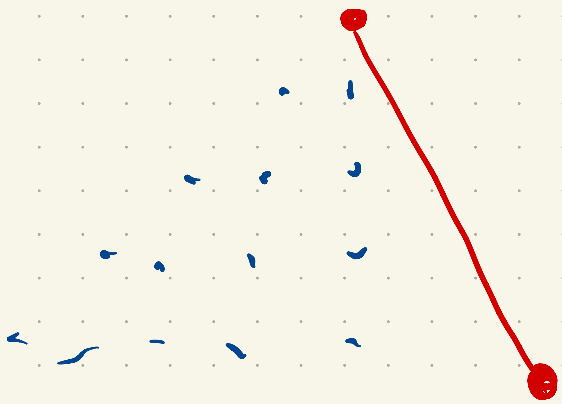
$x_0 = 0$

$T = 1, \quad x_1 = 1, \quad a = 1$

$k = \frac{T}{M} = \frac{1}{M}, \quad h = \frac{1}{N},$

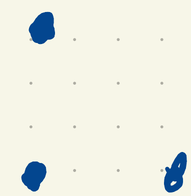
$$\lambda = \frac{N}{M} \qquad N \leq M$$

What if $a < 0$?

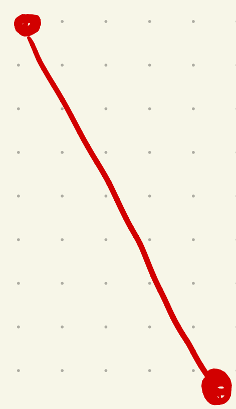


$$u_x(x_i) \approx \frac{-u(x_i) + u(x_{i+1})}{h}$$

instead.

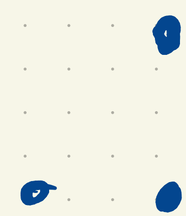


What if $a < 0$?

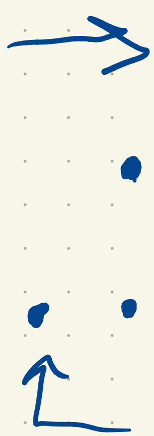


$$u_x(x_i) \approx \frac{u(x_i) - u(x_{i+1})}{h}$$

instead.



$a > 0$



upwind

$a < 0$