

1. Text, 4.20
2. Text, 4.5
3. Consider the following method for solving the advection equation $u_t + au_x = 0$, where a is constant:

$$u_{i,j+1} = u_{i,j-1} + \frac{ak}{h}(u_{i-1,j} + u_{i+1,j})$$

This is the leapfrog method.

- a) Determine the order of accuracy of this method (in both space and time). The answer will be in the form $\tau(x, t) = O(k^p) + O(h^q)$; determine p, q .
- b) This scheme can be derived by applying the method of lines. Do this as follows.
 - i) First, discretize in space using centered differences to obtain a system of ODEs

$$U'(t) = AU(t)$$

for some matrix A . What are the entries of the matrix A assuming $x \in [0, 1]$ is the space domain and periodic boundary conditions are used ($u(0, t) = u(1, t)$)?

- ii) Then use an ODE discretization method to derive the whole scheme. What rule from Table 1.1 on page 7 should be applied at this step to derive the leapfrog scheme?
- c) What are the eigenvalues of the matrix A from the previous derivation?
- d) Compute the region of absolute stability for the ODE discretization method used in part b). Then discuss the stability of the leap frog method using this information and your computation from part c).
- e) Implement this leapfrog method on the following periodic boundary condition problem: $x \in [0, 1]$, $a = 0.5$, $T = 10$, $u(x, 0) = \sin(6\pi x)$. To make the implementation work you will have to compute the first step by some other scheme; describe and justify what you do.
- f) What is the exact solution to the problem in part e)? Use

$$h = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$$

and $k = h$ and show a log-log convergence plot using the infinity norm for the error. What $O(h^p)$ do you expect for the rate of convergence, and what do you measure?

4. Text, 3.7, Redo. You may find the following idea helpful. For the Neumann condition you know that $u'(0) = \alpha$. The second-order approximation for u_{xx} at $x = 0$ is

$$u_{xx}(0) = \frac{u_{-1} - 2u_0 + u_1}{h^2} + O(h^2)$$

The issue is that you don't know u_{-1} . But you also know that

$$\alpha = u_x(0) = \frac{u_{-1} - u_1}{2h} + O(h^2)$$

So you can either make u_{-1} a true unknown by adding this equation to the mix, or you substitute back into the equation for the second derivative at $x = 0$.

In addition, implement your $O(h^2) + O(k^2)$ version of the discretization and test on the domain $0 \leq x \leq 1$, $0 \leq t \leq 0.1$ with

$$\begin{aligned} u_0(x) &= \cos(\pi x/2) + (1 - x) \\ \alpha &= -1 \end{aligned} \tag{1}$$

Verify you obtain the desired order of accuracy.