Diagonal Dominance:

A tridiagonal matrix is diagonal dominant

$$\text{if} \quad |a_k| \geq |b_k| + |c_k| \quad \forall k$$

and strictly so if the inequality holds strictly.

$$0 \cdot 0 \; b_k \; a_k \; c_k \; 0 \cdots 0$$

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Thm: Strictly diagonally dominant tridiagonal! matrices are invertible, and the above algorithm works.

Rough Motivation:

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Rough Motivation:

- I is invertible
- $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is invertible if each $\lambda_i > 0$
- $\det(A) \neq 0$ is an open condition
- perturbations of are also invertible

Sadly

$$\begin{bmatrix} -2 & 1 & 0 & \cdots & & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ \cdots & & & & 1 & -2 \end{bmatrix}$$

Diagonally dominant, but not strictly.

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Strict cannot be relaxed: $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

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$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

We cannot drop struct without adding something else.

We cannot drop strict without adding something else.

Thm: If A is tridiagonal and diagonal dominant and

1) $c_k \neq 0$, $1 \leq k \leq N-1$

2) $|a_N| > |b_N|$ (strict at last equation)

then A is invertible and the above algorithm converges.

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2) $|a_N| > |b_N|$

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$c_k = 1$ ✓

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$c_k = 1$ ✓ $\quad |-2| > |1|$ ✓

Step 1: $w = a_1$, $d_1 = \dfrac{c_1}{w}$, $r_1 = \dfrac{f}{w}$

Step k: $w = a_k - b_k d_{k-1}$, $d_k = \dfrac{c_k}{w}$, $r_k = \dfrac{f_k - b_k r_{k-1}}{w}$

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Aim: show $|d_j| \leq 1$ for all j.

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Aim: show $|d_j| \leq 1$ for all j.

If so:
$$
\begin{aligned}
|w| &= |a_k - b_k d_{k-1}| \\
&\geq |a_k| - |b_k||d_{k-1}| \\
&\geq |a_k| - |b_k| \\
&\geq |c_k|
\end{aligned}
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If $|d_{k-1}| \leq 1$ then $|w| > 0$ <u>and</u>

$$|d_k| = \frac{|c_k|}{|w|} \leq 1.$$

Base case:

$$w = a_1, \quad d_1 = \frac{c_1}{w} = \frac{c_1}{a_1}$$

$$|c_1| \leq |a_1| \text{ by diagonal dominance!}$$

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Final case:

$$w = a_N - b_N d_{N-1}$$

$$|w| \geq |a_N| - |b_N|\,|d_{N-1}|$$

$$\geq |a_N| - |b_N| > 0.$$

Exercise: Modify the above to show strictly diagonally dominant matrices are invertible!

Our system

$u'' + p u' + q u = f$

$b_k = 1 - \frac{p_k h}{2}$ $\qquad a_k = -2 + q_k$ $\qquad c_k = 1 + \frac{p_k h}{2}$

Need

- $|a_k| \geq |b_k| + |c_k|$
- $c_k > 0$
- $|a_N| > |b_N|$

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

If $\left|\frac{p_k h}{2}\right| \leq 1$, $|b_k| = b_k$ $|c_k| = c_k$.

Our system

$$|a_k| \geq |b_k| + |c_k| = 2$$

$$b_k = 1 - \frac{P_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{P_k h}{2}$$

If $\left|\frac{P_k h}{2}\right| \leq 1$, $|b_k| = b_k$ $|c_k| = c_k$.

$$b_k + c_k = 2.$$

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

If $\left|\frac{p_k h}{2}\right| \leq 1$, $\quad |b_k| = b_k \quad |c_k| = c_k$.

$$b_k + c_k = 2.$$

So $\quad |a_k| \geq |b_k| + |c_k| \quad$ if

Our system

$u'' + pu' + qu = f$

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

If $\left|\frac{p_k h}{2}\right| \leq 1$, $|b_k| = b_k$ $|c_k| = c_k$.

$$b_k + c_k = 2.$$

So $|a_k| \geq |b_k| + |c_k|$ if

- $\left|\frac{p_k h}{2}\right| \leq 1$
- $q \leq 0$

Our system

$$\left|\frac{p_k h}{2}\right| \leq 1, \quad q \leq 0$$

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

Need

- $c_k \neq 0$

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

Need • $c_k > 0$

$$\left| \frac{p_k h}{2} \right| < 1 \quad \text{ensures this}$$

strict now!

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

Need

- $|a_N| > |b_N|$

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

Need $\bullet$ $|a_N| > |b_N|$ $\qquad \left|\frac{p_N h}{2}\right| < 1$

$$-2 + q_N \quad \text{vs} \quad \underbrace{1 - \frac{p_N h}{2}}$$

$$0 < \uparrow < 2$$

Our system

$$b_k = 1 - \frac{p_k h}{2} \qquad a_k = -2 + q_k \qquad c_k = 1 + \frac{p_k h}{2}$$

Need • $|a_N| > |b_N|$

$$-2 + q_k \quad \text{vs} \quad 1 - \frac{p_k h}{2}$$

$q \le 0$ and $\left|\frac{p_k h}{2}\right| < 1$ ensures this, too

Summary:

$$Lu = f$$

If

- $q \leq 0$
- $\left|\frac{p_k h}{2}\right| < 1$

then $L = \begin{bmatrix} a_1 & c_1 & & \\ b_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{N-1} \\ & & b_N & a_N \end{bmatrix}$ is invertible.

Summary:

If

- $q \le 0$
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convergence condition (discrete system approximates cts system)

then $L = \begin{bmatrix} a_1 & c_1 & & \\ b_2 & \ddots & \ddots & \\ & \ddots & \ddots & c_{N-1} \\ & & b_N & a_N \end{bmatrix}$ is invertible.

Summary: $\boxed{\begin{array}{c} u'' + pu' + qu = f \\ u|_{0,\ell} = 0 \end{array}}$ $\quad \mathcal{L}u = f$

PDE condition

If

- $q \leq 0$
- $\left|\dfrac{p_k h}{2}\right| < 1$

convergence condition (discrete system approximates cts system)

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Why $q \le 0$?

$[0, \pi]$

$\sin(kx)$

$-k^2 \sin(kx)$

$u_{xx} + q u$

negative eigenvalues

if positive, can create a zero eigenvalue.

Why $q \le 0$?

$$\underbrace{(\partial_x^2 + 1)}_{L} \sin(x) = 0$$

There is no solution of

$$u_{xx} + u = \sin(x) \quad \text{on } [0,\pi]$$

$$u\big|_{x=0} = u\big|_{x=\pi} = 0.$$

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$$\int_0^\pi u\left[\partial_x^2 \sin x + \sin(x)\right]dx$$

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$$\downarrow$$

$$\int_0^\pi u\left[\partial_x^2 \sin x + \sin(x)\right] dx = 0$$

Convergence

[consistency + stability $\Rightarrow$ convergence]

Convergence

$$\left[\text{consistency} + \text{stability} \Rightarrow \text{convergence}\right]$$

Consistency: $\tau = O(h^2)$ ✓

Convergence

[consistency + stability => convergence]

Consistency: $\tau = O(h^2)$ ✓

$u^h - U^h$

Stability?

$$L^h U^h = f^h$$

$$L^h u^h = f^h + \tau$$

Convergence

$$\|L^h E^h\|_p \le \|L^h\|_p \|E^h\|_p$$

[consistency + stability => convergence]

Consistency: $\tau = O(h^2)$ ✓

Stability?

$$L^h U^h = f^h$$

$$L^h u^h = f^h + \tau$$

$$L^h \underbrace{(u^h - U^h)}_{E^h} = \tau$$