What if $u_0(x)$ is merely Riemann integrable?

We can still define

$$b_k = \int_0^1 u_0(x)\, e_k(x)\, dx \quad \rightarrow \text{Fourier sine coeffs.}$$

$$\sqrt{2}\, b_k = c_k$$

$$\sum_{k=1}^{\infty} r^k = \frac{1}{1-r}$$

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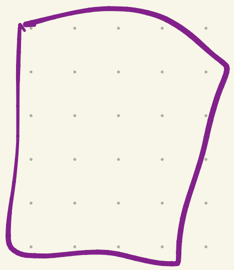
To what extent is

$$u_0(x) = \sum_{k=1}^{\infty} b_k e_k(x)$$

what does this mean?



$\mathbb{R}^s$ $\quad \|\cdot\|$ $\quad \vec{x}_n \to \vec{x}$

$$u_0(x) = \sum_{k=1}^{\infty} b_k e_k(x)$$

$$\sum_{k=1}^{n} b_k e_k(x) = s_n(x)$$

$$s_n \longrightarrow u_0$$

more than one notion of this

$$u_0(x) = \sum_{k=1}^{\infty} b_k e_k(x)$$

what does this mean?

$$\lim_{n\to\infty} \left\| u_0 - \sum_{k=1}^{n} b_k e_k \right\|_2 \to 0$$

$$\|f\|_2 = \left[\int_0^1 |f(x)|^2 dx\right]^{1/2}$$

$\hookrightarrow L^2$ norm

e.g. $u_0(x) = 1$

$$b_k = \overbrace{\sqrt{2}\int_0^1 \sin(k\pi x)\,dx}^{e_k}$$

$$= \sqrt{2}\,\frac{1}{k\pi}\Big(-\cos(k\pi x)\Big)\Big|_0^1$$

$$= \frac{\sqrt{2}}{k\pi}\left[(-1)^{k+1} + 1\right]$$

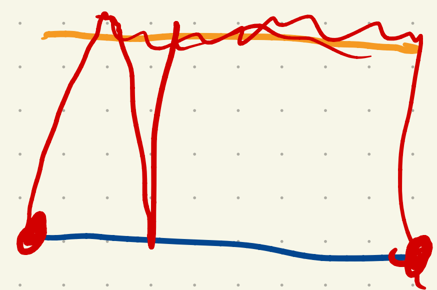
$$= \begin{cases} \dfrac{2\sqrt{2}}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

$$b_k \to 0$$

$$1 \stackrel{?}{=} \sum_{k=1}^{\infty} \sqrt{2}\, b_k \sin(k\pi x)$$

$$b_k = O\left(\frac{1}{k}\right)$$

$$b_k = \begin{cases} \frac{2\sqrt{2}}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$



$S_n(0) = 0$ for all n

$S_n(1) = 0$ ...

$$c_n = \sqrt{2}\, b_n = \begin{cases} \frac{4}{k\pi} & k \text{ add} \\ 0 & k \text{ even} \end{cases}$$

$$1 \stackrel{?}{=} \sum_{j=0}^{\infty} \frac{4}{(2j+1)\pi} \sin\big((2j+1)\pi x\big)$$

Not true at $x = 0$, right?

$$c_n = \sqrt{2}\, b_n = \begin{cases} \frac{4}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

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Not true at $x = 0$, right?

Nevertheless, $\left\| 1 - \sum_{j=0}^{\infty} c_k \sin(k\pi x) \right\|_2 \longrightarrow 0.$

↑ weaker than uniform convergence.

If c_k are determined by $\sqrt{2} \int u_0(x) e_k(x) dx$

To what extent does

$$u(x,t) = \sum_{k=1}^{\infty} e^{-k\pi^2 t} c_k \sin(k\pi x)$$

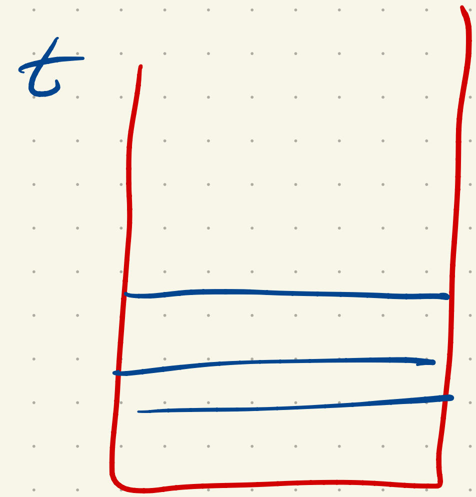
satisfy

1. $u_t = u_{xx}$
2. $u(0,t) = 0, \quad u(1,t) = 0$
3. $u(x,0) = u_0(x)$

① holds if $t>0$

② holds if $t>0$

in place of ③



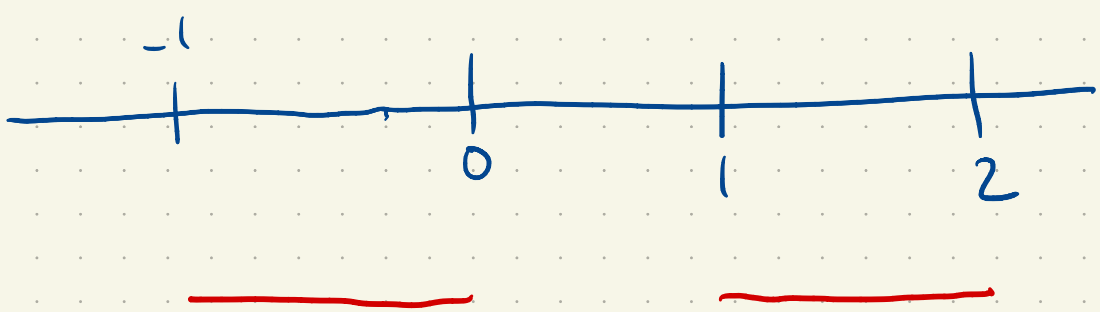
$$w[t](x) = u(x,t).$$

$$\| u_0 - w[t] \|_2 \longrightarrow 0$$

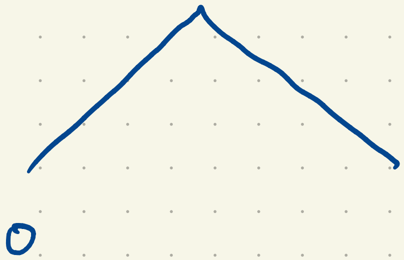
as $t \to 0$.

Key Fact

The rate of decay of the Fourier sine coefficients tells you about smoothness of the function. (or at least its odd periodic extension)



etc . . .

E.g. $f(x) =$ 

$$f(x) = \begin{cases} 2x & 0 < x \le \frac{1}{2} \\ 2(1-x) & x \ge \frac{1}{2} \end{cases}$$

$$\int_0^1 f(x) \sin(k\pi x)\, dx = \int_0^{1/2} 2x \sin(k\pi x)\, dx + \int_{1/2}^1 2(1-x) \sin(k\pi x)\, dx$$

$$= \int_0^1 2x \frac{d}{dx}\left(\frac{-\cos(k\pi x)}{k\pi}\right) + \int_{1/2}^1 2(1-x) \frac{d}{dx}\left[\frac{-\cos(k\pi x)}{k\pi}\right]$$

$$-\frac{1}{k\pi}\int_0^1 2x \frac{d}{dx}\cos(k\pi x)\,dx = \frac{2}{k\pi}\int_0^{1/2}\cos(k\pi x) + -\frac{1}{k\pi}\, 2x\cos(k\pi x)\Big|_0^{\frac{1}{2}}$$

$$= \frac{2}{k\pi}\left[\frac{\sin\left(k\pi\frac{1}{2}\right)}{(k\pi)} - \frac{1}{k\pi}\cos\left(\frac{k\pi}{2}\right)\right]$$

$$-\frac{1}{k\pi}\int_0^1 2x \frac{d}{dx}\cos(k\pi x)\,dx = \frac{2}{k\pi}\int_0^{1/2}\cos(k\pi x) + -\frac{1}{k\pi}2x\cos(k\pi x)\Big|_0^{1/2}$$

$$= \frac{2}{k\pi}\left[\frac{\sin(k\pi\frac{1}{2})}{(k\pi)} - \frac{1}{k\pi}\cos\left(\frac{k\pi}{2}\right)\right]$$

$$-\frac{1}{k\pi}\int_{1/2}^1 2(1-x)\frac{d}{dx}\cos(k\pi x)\,dx = -\frac{2}{k\pi}\int_{1/2}^1\cos(k\pi x)\,dx - \frac{2}{k\pi}(1-x)\cos(k\pi x)\Big|_{1/2}^1$$

$$= -\frac{2}{(k\pi)^2}\sin(k\pi x)\Big|_{1/2}^1 + \frac{2}{k\pi}\cos(k\pi)$$

$$= \frac{2}{(k\pi)^2}\sin(k\pi/2) + \frac{2}{(k\pi)^2} + \frac{2}{k\pi}\cos(k\pi)$$

$$c_k = \frac{8}{(k\pi)^2} \sin\left(k\pi/2\right)$$

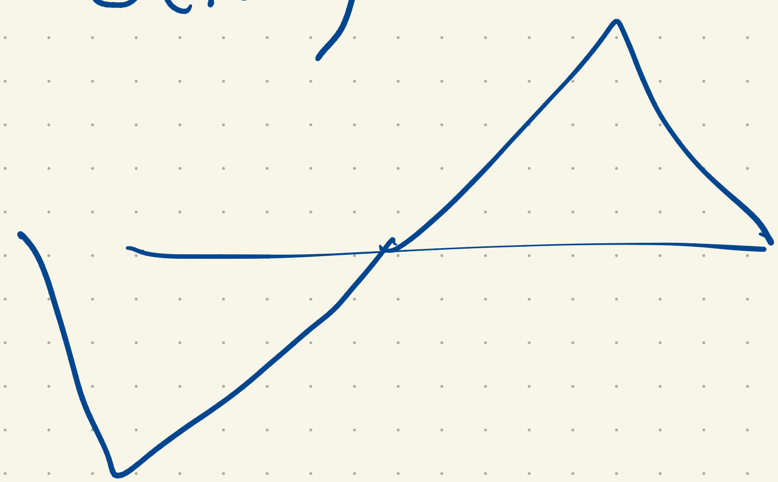
$$\left(1, 0, -1, 0, \ldots\right)$$

$$c_k = \frac{8}{(k\pi)^2} \sin(k\pi/2)$$

$$\sin(k\pi/2) \longrightarrow (1, 0, -1, 0, \ldots)$$

We saw: not quite C^0 $\quad O(k^{-1})$

not quite C^1 $\quad O(k^{-2})$



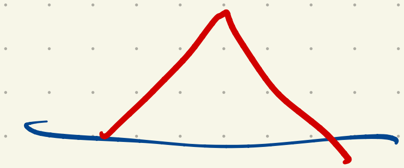
If f is C^1, $f(0)=0$, $f(1)=0$

$$\int_0^1 \sin(k\pi x) f(x)\,dx = -\frac{1}{k\pi}\int_0^1 \left[\frac{d}{dx}\cos(k\pi x)\right] f(x)\,dx$$

$$= \frac{1}{k\pi}\int_0^1 \cos(k\pi x) f'(x)\,dx$$

$$- \cos(k\pi x) f(x)\Big|_0^1$$

$$= O(k^{-1})$$



Prop: If f is in $C^{\ell}[0,1]$

$f^{(j)}(0) = -f^{(j)}(1)$ j even $j \le \ell$

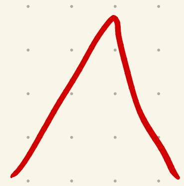
Then c_k is $O(k^{-\ell})$

Prop: If c_k is $O\left(k^{-[\ell+1+\varepsilon]}\right)$ $\quad \varepsilon > 0$

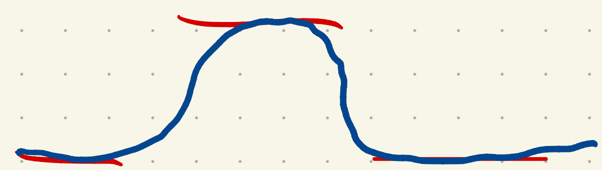
Then $\displaystyle\sum_{k=1}^{\infty} a_k \sin(k\pi x)$ converges pointwise to some $f(x)$,

$f \in C^{\ell}[0,1]$ and

$$\| s_n^{(j)} - f^{(j)} \|_{\infty} \longrightarrow 0, \qquad 0 \le j \le \ell.$$



$O(k^{-2})$



Prop: If c_k is $O\left(k^{-[\ell+1+\varepsilon]}\right)$ $\varepsilon > 0$

Then $\sum_{k=1}^{\infty} a_k \sin(k\pi x)$ converges pointwise to some $f(x)$

$f \in C^{\ell}[0,1]$ and

$\| s_n^{(j)} - f^{(j)} \|_{\infty} \to 0, \quad 0 \le j \le \ell.$

$O(k^{-1})$ $\ell = 0$? no Not guaranteed to be C^0

$O(k^{-2})$ $\ell = 1$? no Not guaranteed to be C^1

Upshot:

If u_0 is not smooth, its Fourier sine coefficients has a heavy tail. $O(k^{-1})$

The smoother u_0 is (with periodicity restrictions) the more rapidly the coefficients decay.

Crank-Nicholson vs. Spectrum

$u' = \lambda u$

CN = Trapezoid Rule

$$u_{k+1} = u_k + \frac{h}{2}\left(\lambda u_k + \lambda u_{k+1}\right)$$

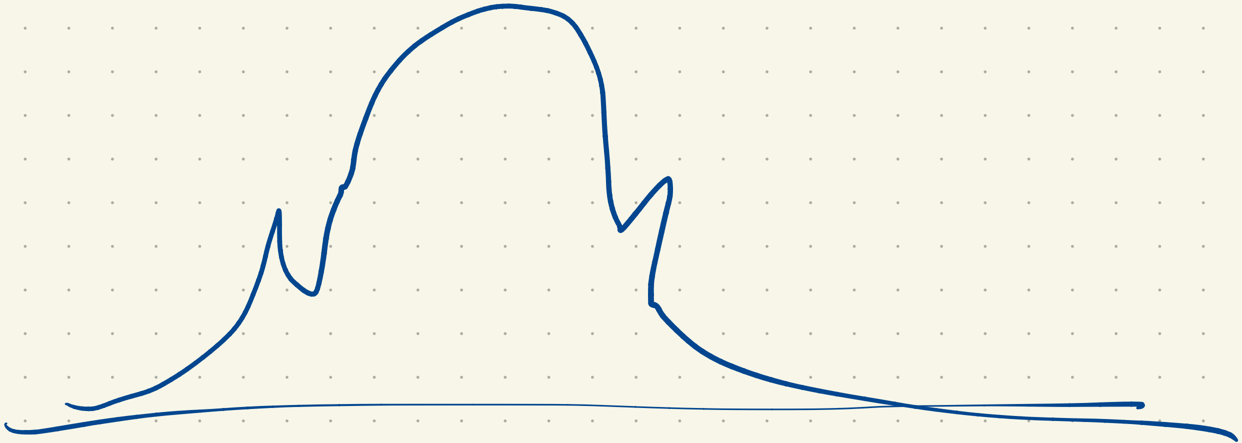
$$u_{k+1} = \frac{1 + h\lambda/2}{1 - h\lambda/2}\, u_k$$

$$= \frac{2+z}{2-z}\, u_k$$

A stable

$$\left|\frac{2+z}{2-z}\right| < 1 \quad \text{if } \operatorname{Re} z < 0$$

$$\to -1 \text{ as } z \to \infty$$



$$e^{-k^2\pi^2 t}$$

$$\sum_{k=1}^{n} b_k \sin(k\pi x)$$

$$\left(\frac{t+z}{t-z}\right)$$

$h,\quad -k^2\pi$

(M, N)

(h, k)

$$\boxed{\frac{k}{h^2} = \lambda}$$

$M \quad M+1$

$N \quad M$

$$\frac{\frac{1}{10}}{\frac{4}{100}}$$

0.02

$\frac{10}{4}$

0.002

$\frac{100}{4} = 25$

$$\left|\frac{2+z}{2-z}\right| \leq 1 \quad \text{if} \quad \operatorname{Re} z \leq 0$$

But if z is real

$$\lim_{z\to-\infty} \frac{2+z}{2-z} = 1$$

High decay modes are damped by a factor that $\to 1$ as $z \to \infty$.

Fast decay modes for the continuous system decay more and more slowly.

Compare with BE

$$u_{k+1} = u_k + \lambda h u_{k+1}$$

$$u_{k+1} = \frac{1}{1-z} u_k \qquad \left|\frac{1}{1-z}\right| \to 0$$

as $z \to -\infty$

Def: A $\underset{\text{one step}}{\wedge}$ numerical method for ODEs

is L stable if it is A stable and

if $|R(z)| \to 0$ as $|z| \to \infty$

where $u_{n+1} = R(z)\, u_n$ applied to

$u' = \lambda u$, $z = \lambda h$.