Assuming $\|B^{-1}\|_2 \leq 1$, $\|B^{-1}A\|_2 \leq 1$:

$$B\vec{U}_{j+1} = A\vec{U}_j + k\vec{f}_j$$

$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j + k\tau_j$$

Assuming $\|B^{-1}\|_2 \le 1$, $\|B^{-1}A\|_2 \le 1$:

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$$B\vec{u}_{j+1} = A\vec{u}_j + k\vec{f}_j + k\tau_j \qquad E_j = U_j - u_j$$

$$B E_{j+1} = A E_j - k\tau_j$$

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$$BE_{j+1} = AE_j - k\tau_j$$

$$\vec{E}_{j+1} = B^{-1}A\vec{E}_j - k\,\overbrace{B^{-1}\tau_j}$$

$$\|B^{-1}\tau_j\|_2$$

$$\le \|B^{-1}\|_2 \cdot \|\tau_j\|_2$$

$$\le \|\tau_j\|_2$$

$$\|\vec{E}_{j+1}\|_2 \leq \|B^{-1}A\|_2 \, \|\vec{E}_j\|_2 + k\|B^{-1}\|_2 \|\tau_j\|_2$$

$$\|\vec{E}_{j+1}\|_2 \leq \|\vec{E}_j\|_2 + k\|\tau_j\|_2$$

$$\downarrow$$

$$\underbrace{\max_j \|\vec{E}_j\|_2}_{\|E\|_{2,\infty}} \leq \|E_0\|_2 + T \underbrace{\max_j \|\tau_j\|_2}_{\|\tau\|_{2,\infty}}$$

$$\|E\|_{2,\infty} \le \|\vec{E}_0\|_2 + T\,\|\tau\|_{2,\infty}$$

$$\boxed{\frac{1}{\sqrt{N}}\|E\|_{2,\infty} \le \frac{1}{\sqrt{N}}\|\vec{E}_0\|_2 + T\,\underbrace{\frac{1}{\sqrt{N}}\|\tau\|_{2,\infty}}}$$

$$\le \|\tau\|_{\infty,\infty}$$

$$\to 0$$

Does $\|\tau\|_{2,\infty} \to 0$?

$$\frac{1}{\sqrt{N}} \|\tau\|_{2,\infty} \to 0 \quad \text{instead}$$

$$\|\tau_j\|_2 \leq \sqrt{N} \|\tau_j\|_\infty$$

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But this is weaker: $\tau_j = (N^{1/4}, 0, \ldots, 0)$

$$\frac{1}{\sqrt{N}} \|\tau_j\|_2 = N^{-1/4} \to 0$$

Error can concentrate, but on average $\to 0$

$$\vec{x}_N = \underbrace{(N^{1/4}, 0, \ldots, 0)}_{N}$$

$$N \to \infty \qquad \frac{1}{\sqrt{N}} \|\vec{x}_N\|_2 = N^{-1/4} \longrightarrow 0$$

$$\|\vec{x}_N\|_\infty = N^{1/4} \longrightarrow \infty$$

We have convergence $\|E\|_{2,\infty} \to 0$

if $(2\theta-1)\lambda \leq \frac{1}{2}$ if I show you

$$\|B^{-1}\|_2 \leq 1$$

$$\|B^{-1}A\|_2 \leq 1$$

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To compute these norms we'll show

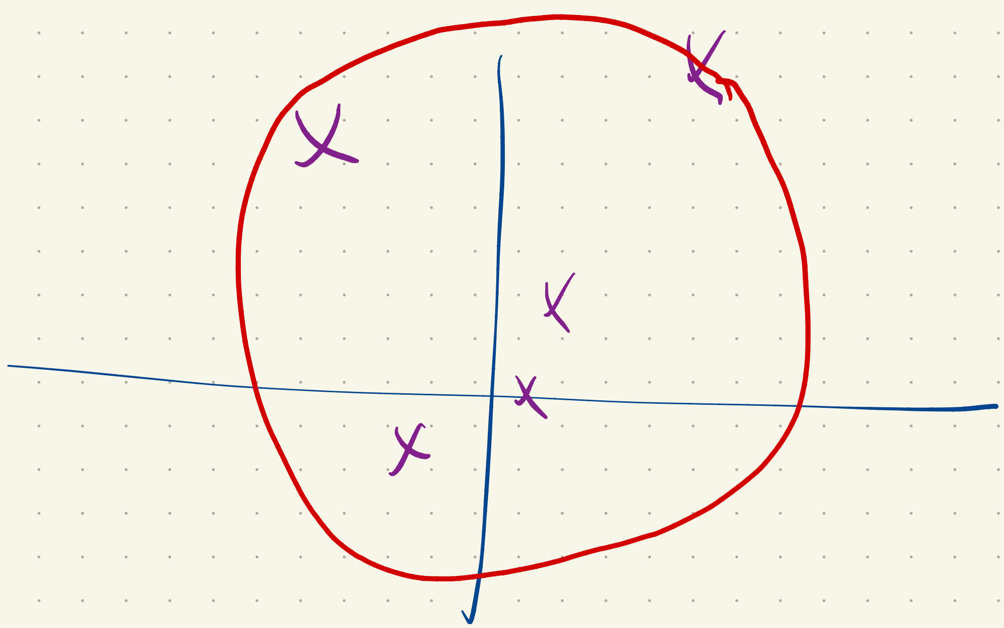
- B^{-1}, $B^{-1}A$ are symmetric

- For a symmetric matrix, can compute $\|\cdot\|_2$ from eigenvalues

Def: The <u>spectral radius</u> of a square matrix A, $\sigma(A)$ is

$\max_i |\lambda_i|$ where $\lambda_1, \ldots, \lambda_k$

are the eigenvalues of A.



Def: The <u>spectral radius</u> of a square matrix A, $\sigma(A)$ is

$$\max_i |\lambda_i| \text{ where } \lambda_1, \dots, \lambda_k$$

are the eigenvalues of A.

Lemma: $\|A\|_p \geq \sigma(A)$

Pf: $|\lambda|$ biggest

$$\|A\|_p \geq \frac{\|Av\|_p}{\|v\|_p} = |\lambda| \frac{\|v\|_p}{\|v\|_p} = |\lambda| = \sigma(A)$$

Claim:

for a symmetric matrix $(A^T = A)$,

$$\|A\|_2 = \sigma(A).$$

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Recall: A symmetric matrix A admits an orthonormal basis of eigenvectors.

$$\vec{v}_1, \dots, \vec{v}_n \qquad \lambda_1, \dots, \lambda_n$$

Claim:

for a symmetric matrix $(A^T = A)$,

$$\|A\|_2 = \sigma(A).$$

Recall: A symmetric matrix A admits an orthonormal basis of eigenvectors.

$$\vec{v}_1, \dots, \vec{v}_n \qquad \lambda_1, \dots, \lambda_n$$

$$P = [\vec{v}_1, \dots, \vec{v}_n] \qquad (P^T P)_{i,j} = \vec{v}_i \cdot \vec{v}_j = \delta_{ij} = I_{ij}$$

P is an <u>orthogonal</u> matrix

$$P = [\vec{v}_1, \dots, \vec{v}_n]$$

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

claim: $A = P \Lambda P^{-1}$

enough to show

$$\underbrace{A \vec{v}_i}_{\lambda_i v_i} = \underbrace{P \Lambda P^{-1} \vec{v}_i}_{\lambda_i v_i} \qquad i = 1, \dots, n$$

$$\vec{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith row}$$

$$\Lambda e_i = \lambda_i e_i$$

$$P \lambda_i e_i$$

$$= \lambda_i (P e_i)$$

$$= \lambda_i v_i$$

$$P e_i = [\vec{v}_1, \dots, \vec{v}_n] \vec{e}_i = \vec{v}_i$$

$$P \vec{e}_i = \vec{v}_i \qquad P^{-1} \vec{v}_i = \vec{e}_i$$

$$P = [\vec{v}_1, \ldots, \vec{v}_n]$$

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

claim: $A = P \Lambda P^{-1}$

enough to show

$$A\vec{v}_i = P \Lambda P^{-1} \vec{v}_i \quad i = 1, \ldots, n$$

$$\begin{aligned} P \Lambda P^{-1} \vec{v}_j &= P \Lambda \vec{e}_j \\ &= P \lambda_j \vec{e}_j \\ &= \vec{v}_j \lambda_j = A \vec{v}_j \end{aligned}$$

$$\vec{e}_j = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} \end{array} \leftarrow j$$

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$$\vec{e}_j = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} & \leftarrow j \end{array}$$

$$A = P \Lambda P^{-1} = P \Lambda P^T$$

A symmetric $\Rightarrow$ $\|A\|_2 = \sigma(A)$

(1) $\|A\|_2 = \|P \Lambda P^{-1}\|_2 \leq \|P\|_2 \|\Lambda\|_2 \|P^T\|_2$

(2) For an orthogonal matrix P, $\|P\|_2 = 1$

$P^{-1} = P^T$

$P^T P = I$

$P P^T = I$

(3) $\|\Lambda\|_2 = \max |\lambda_i| = \sigma(A)$

$\Rightarrow \|A\|_2 \leq \sigma(A)$

$A A^{-1} = A^{-1} A$

Since $\|A\|_p \geq \sigma(A) \Rightarrow \|A\|_2 = \sigma(A)$

(2) For an orthogonal matrix P, $\|P\|_2 = 1$

$$\|Px\|_2^2 = x^T \underbrace{P^T P}_{I} x = x^T x = \|x\|_2^2$$

$$\Rightarrow \frac{\|Px\|_2}{\|x\|_2} = 1 \quad (x \neq 0)$$

$$\Rightarrow \|P\|_2 = 1$$

③ $\|\Lambda\|_2 = \max |\lambda_i| = \sigma(A)$

$x = x_1 e_1 \cdots\cdots x_n e_n$

$\Lambda x = x_1 \lambda_1 e_1 + \cdots + x_n \lambda_n e_n$

$$\|\Lambda x\|_2^2 = |\lambda_1|^2 |x_1|^2 + \cdots\cdots + |\lambda_n|^2 |x_n|^2 \leq \sigma(\Lambda)^2 \|x\|_2^2$$

$\leq \max |\lambda_i|^2$
$= \sigma(\Lambda)^2$

$$\frac{\|\Lambda x\|_2}{\|x\|_2} \leq \sigma(\Lambda) \Rightarrow \|\Lambda\|_2 \leq \sigma(\Lambda).$$

$\hookrightarrow = \sigma(A)$

Converse already: $\sigma(\Lambda) \leq \|\Lambda\|_2$

Back to our claims:

$$\|B^{-1}\|_2 \leq 1$$

$$\|B^{-1}A\|_2 \leq 1$$

a) Compute eigenvectors / values;
show $|\lambda_i| \leq 1$ if $\lambda(2\theta - 1) \leq 1/2$

b) show are symmetric