Runge-Kutta Methods

Trapezoidal:

- $O(h^2)$
- implicit
- A stable

Midpoint (Leapfrog)

- $O(h^2)$
- explicit
- 2 step (bootstrap)
- negligible region of abs. stab.

R-K methods:

- All single step
- To get higher accuracy

$\underbrace{\text{intermediate steps}}_{\text{stages}}$ are introduced.

1-stage: Euler's method

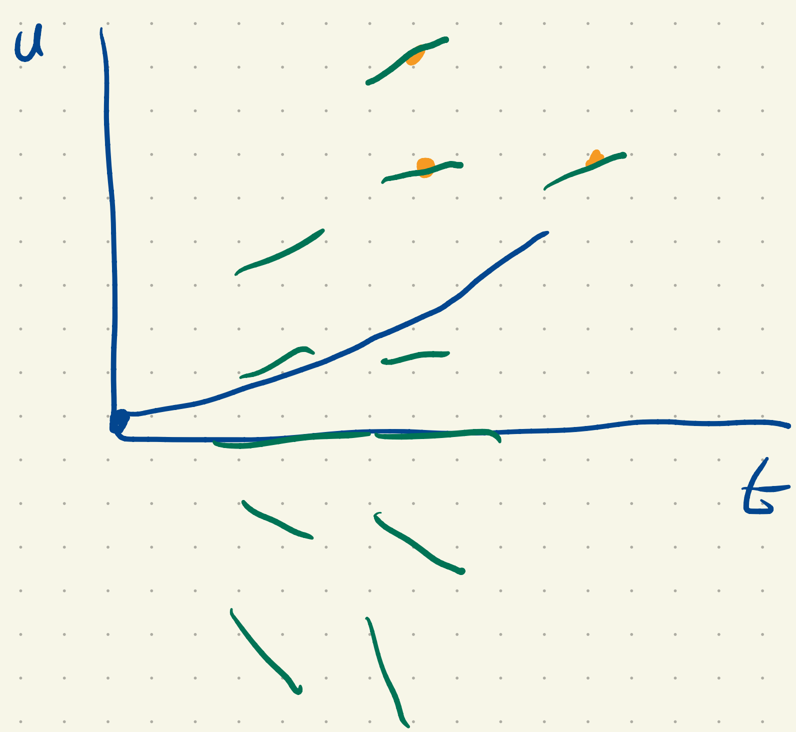
2-stage: Huen's method

Trapezoidal: $u_{k+1} = u_k + \frac{h}{2}\left[f(t_k, u_k) + f(t_k + h, u_{k+1}) \right]$

instead:

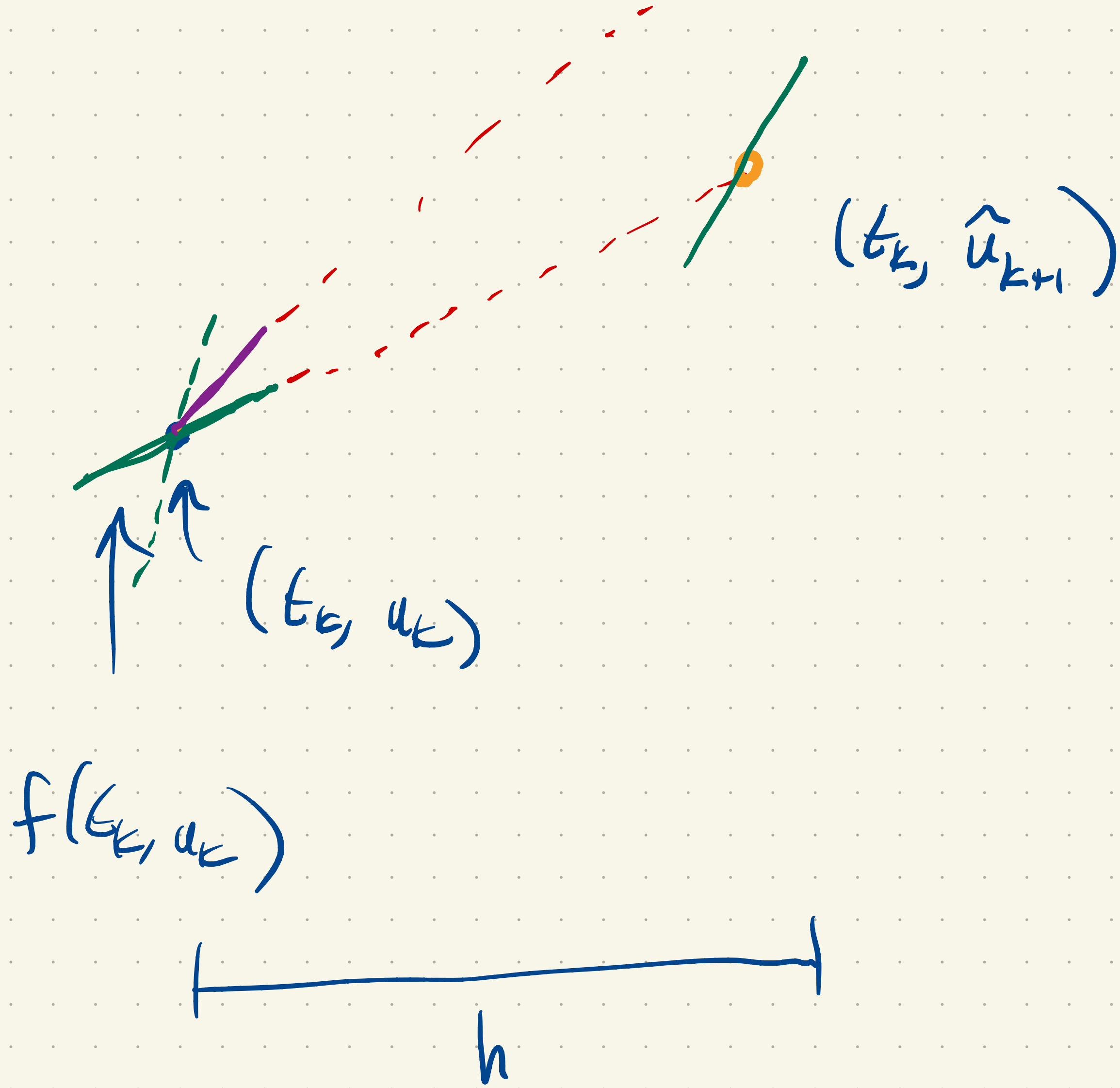
$$\hat{u}_{k+1} = u_k + h\, f(t_k, u_k)$$

$$u_{k+1} = u_k + \frac{h}{2}\left[f(t_k, u_k) + f(t_k + h, \hat{u}_{k+1}) \right]$$



$u' = f(t, u)$

$u' = u$



Claim: Heun's method is $O(h^2)$

Pf:

$$\frac{u_{k+1}-u_k}{h} - \underbrace{\frac{1}{2}\left[f(t_k,u_k) + f(t_k+h,\ \underbrace{u_k + h f(t_k,u_k)}_{\hat{u}_{k+1}})\right]}_{\mathrm{II}} = 0$$

I

subs: $u_k \to u(t_k)$

$$\frac{u(t_k+h)-u(t_k)}{h} = u'(t_k) + \frac{u''(t_k)h}{2} + O(h^2)$$

$$u''(t_k) = \left.\frac{d}{dt}\right|_{t_k} u'(t)$$

$$u' = f(t,u)$$
$$u'' = f_t(t,u) + f_u(t,u)\cdot u'$$

$$u''(t_k) = f_t(t_k, u(t_k)) + f_u(t_k, u(t_k)) \cdot f(t_k, u(t_k))$$

$$u''(t_k) = f_t + f_u f \qquad \left(\text{all at } t_k \atop u(t_k)\right)$$

$$I = f + \frac{h}{2}\left(f_t + f_u f\right) + O(h^2)$$

$$\Big[f(t_k, u_k) + f(t_k + h, u_k + h f(t_k, u_k))$$

$$\underbrace{f(t_k, u(t_k)) + f(t_k + h, u_k + h\, f(t_k, u_k))}_{g(h)}$$

$$g(0) = f + f(t_k, \underset{\uparrow}{u(t_k)}) = 2f$$

$$g'(h) = f_t(t_k + h, u_k + h f(t_k, u_k)) + f_u(t_k + h, u_k + h f(t_k, u_k)) \cdot f(t_k, u_k)$$

$$g'(0) = f_t + f_u \cdot f$$

$$g(h) = 2f + (f_t + f_u f) h + O(h^2)$$

$$\mathrm{II} = f + \frac{h}{2}(f_t + f_u f) + O(h^2)$$

$$\mathrm{I} - \mathrm{II} = O(h^2)$$

Most General 2-stage R-K

$$Y_1 = u_k + h\left[a_{11} f(t_k + c_1 h, Y_1) + a_{12} f(t_k + c_2 h, Y_2)\right]$$

$$Y_2 = u_k + h\left[a_{21} f(t_k + c_1 h, Y_1) + a_{22} f(t_k + c_2 h, Y_2)\right]$$

$$u_{k+1} = u_k + h\left[b_1 f(t_k + c_1 h, Y_1) + b_2 f(t_k + c_2 h, Y_2)\right]$$

Y_j is a guess for $u(t_k + c_j h)$

$$\begin{array}{ccc} c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \\ & b_1 & b_2 \end{array} \qquad \begin{aligned} a_{11} + a_{12} &= c_1 \\ a_{21} + a_{22} &= c_2 \\ b_1 + b_2 &= 1 \end{aligned}$$

Huen:

$a_{11} = a_{12} = 0$, $\quad c_1 = 0$

$a_{21} = 1$, $\quad a_{22} = 0 \quad c_2 = 1$

$b_1 = b_2 = \frac{1}{2}$

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

For an explicit 2-stage method:

$$Y_1 = u_k$$

$$Y_2 = u_k + h\, a_{21} f(t_k + \cancel{c_1 h}, Y_1)$$

$$u_{k+1} = u_k + h\left(b_1 f(t_k + \cancel{c_1 h}, Y_1) + b_2 f(t_k + c_2 h, Y_2)\right)$$

$$c_1 = \underset{0}{a_{11}} + \underset{0}{a_{12}} = 0$$

$$c_2 = a_{21} + a_{22} = a_{21}$$

Two free parameters in a 2-stage R-K method.

$a_{21} = a$

$\hookrightarrow c_2$

$b_1 \qquad (b_2 = 1 - b_1)$

Game: how to pick free parameters to maximize the order of method.

Exercise: 2 stage explicit R-K methods are $O(h)$ unless

$$b_1 + b_2 = 1$$

$$a b_2 = \frac{1}{2}$$

in which case the method is $O(h^2)$

Variations a) $a = 1, \quad b_1 = b_2 = \frac{1}{2}$

Heun

b) $a = \frac{1}{2} \quad b_2 = 1, \quad b_1 = 0$

improved Euler