Preamble: There is a total of **49** points on this exam; not every problem is equally weighted. Please write out your proofs and your statements of theorems and definitions as clearly as you can. In grading, however, some consideration will be made for the exam's time constraints. Good luck!

1. [12 points]

Define (or state, as appropriate) the following

a) $\limsup x_n = L$

b) A totally bounded metric space.

c) The Extreme Value Theorem (most general version).

d) The Weierstrass Approximation Theorem.

e) A bounded linear map.

f) The function f on X is uniformly continuous

2. [7 points]

Give characterizations of the following concepts.

a) Two characterizations of a totally bounded set.

b) Three characterizations of a compact set.

c) Two characterizations of a dense subset of a metric space.

3. [6 points]

Let (x_n) be a Cauchy sequence in a metric space X. Suppose it has a convergent subsequence x_{n_k} converging to x. Carefully show that $x_n \to x$.

4. [6 points]

Suppose $f_n : X \to \mathbb{R}$ is a sequence of functions defined on a metric space X that converges uniformly to a limit f. Suppose $x \in X$ and f_n is continuous at x for each n. Show that f is continuous at x.

5. [6 points]

Give an example of the following:

a) A bounded sequence of functions in $C[0,1]$ with no convergent subsequence. I.e., no subsequence converges in $C[0,1]$.

b) A sequence of functions in $C[0,1]$ that converges in the L^1 norm to zero, but not uniformly.

c) A sequence in $(C[0,1], L^1)$ that is Cauchy but has no convergent subsequence.

6. [6 points]

a) **[3 points]**

Give an example of a sequence of functions f_n on $[0, 2\pi]$ that converges uniformly to zero, but such that for at least one $x \in [0, 2\pi]$, $f_n'(x) \not\to f'(x)$.

b) **[3 points]**

State the theorem from class that concerns uniform convergence and differentiation. (i.e. the theorem that gives conditions that ensures $f_n' \to f'$)

7. [6 points]

Prove that the series $\sum \frac{1}{1+k^2x^2}$ converges uniformly for $x \in [1,2]$.