1. Text, 4.12

Solution, part a:
The solution is $u(x,t) = g(x - at)e^{-bt}$.

Solution, part b:
The upwind solution is a minor modification of the standard upwind method:

$$u_{i,j+1} = \lambda u_{i-1,j} + (1-\lambda)u_{i,j} - bku_{i,j}$$

Solution, part c:
These are all identical to the standard upwind method.

Solution, part d:
See worksheet.

The speed of the bump looks good in both cases. For λ near 1 we see an excellent match between solution including the damping. For $\lambda \approx 0.5$ there is, like standard upwinding, significant damping and spreading of the solution.

2. Text, 4.17

Solution, part a:
Following the technique used to derive Lax-Wendroff, we find A, B and C satisfy

$$\begin{aligned} A + B + C &= 1 \\ B + 2C &= \lambda \\ B + 4C &= \lambda^2 \end{aligned}$$

From these,

$$\begin{aligned} A &= 1 - (\lambda/2)(3-\lambda) \\ B &= \lambda(2-\lambda) \\ C &= (\lambda^2 - \lambda)/2 \end{aligned}$$

The order of the method is $O(h^2) + O(k^2)$ like Lax-Wendroff.

Solution, part b:
The grid speed is twice that of Lax-Wendroff and hence the CFL condition is $a \le 2h/k$ or $\lambda \le 2$.

To compute stability we substitue a tentative solution of the form $\kappa^j e^{Irx_i}$ into the numerical method and obtain

$$\kappa = 1 - \lambda(2-\lambda)(e^{i\theta} - 1) + \lambda(\lambda - 1/2)(e^{-2i\theta} - 1)$$

with $\theta = rh$. A tedious computation using the trigonometric double angle formulas shows

$$|\kappa|^2 = 1 - 4\lambda(2-\lambda)(\lambda-1)^2 \sin^4(\theta/2).$$

For $0 \le \lambda \le 2$ we conclude that $|\kappa|^2 \le 1$.

Solution, part c:
See worksheet.

Solution, part d:

No damping. Oscillation from dispersion like Lax Wendroff. Oscillations are in advance of the solution, look larger in scale, but smaller in extent.

3. Consider the equation

$$u_t + au_x - bu_{xxx} = 0$$

where $x \in \mathbb{R}$. Suppose at $t = 0$, $u(x,t) = e^{ikx}$. Find a solution of the differential equation. Describe the solution as a traveling wave. What is the speed of the wave? How does the speed change as we change the spatial frequency k?

Solution:
We guess a solution of the form $e^{-i\omega t + ikx}$. Subsituting we find

$$-\omega + ak + bk^3 = 0$$

Thus the solution is

$$e^{ik(x-\omega/kt)}.$$

Since

$$\omega = k(a + bk^2)$$

we conclude that the solution is

$$e^{ik(x-vt)}$$

with $v = a + bk^2$. This is a plane wave with speed $a + bk^2$. As the wave number k increases, the speed of the wave increases also.