1. Text, 4.12

   Solution, part a:
   The solution is $u(x,t) = g(x - at)e^{-bt}$.

   Solution, part b:
   The upwind solution is a minor modification of the standard upwind method:

   $$u_{i,j+1} = \lambda u_{i-1,j} + (1 - \lambda)u_{i,j} - bku_{i,j}$$

   Solution, part c:
   The numerical domain of dependence and the CFL condition are identical to the standard upwind method.

   To consider stability, we substitute $u_{i,j} = \kappa^j e^{Irx_i}$ into the numerical method we find

   $$\kappa = \lambda e^{-Irh} + (1 - \lambda - bk)$$

   We then find
   $$|\kappa|^2 = (1 - bk)^2 + 4\lambda(\lambda - 1 + bk)\sin^2(rh/2)$$

   When $\lambda \geq 1$ we find
   $$|\kappa|^2 \geq (1 - bk)^2 + 4bk\sin^2(rh/2).$$

   Since $\sin^2(rh/2)$ might be as large as 1 we obtain from a short compuation $|\kappa|^2 \geq (1+bk)^2$ which is incompatable with stability. Thus we require $\lambda < 1$.

   Further analysis is a little tricky. When $\lambda = 0$, for example, we obtain $0 \leq bk \leq 2$.

   Solution, part d:
   See worksheet.

   The speed of the bump looks good in both cases. For λ near 1 we see an excellent match between solution including the damping. For $\lambda \approx 0.5$ there is, like standard upwinding, significant damping and spreading of the solution.

2. Text, 4.17

   Solution, part a:
   Following the technique used to derive Lax-Wendroff, we find A, B and C satisfy

   $$A + B + C = 1$$
   $$B + 2C = \lambda$$
   $$B + 4C = \lambda^2$$

   From these,

   $$A = 1 - (\lambda/2)(3 - \lambda)$$
   $$B = \lambda(2 - \lambda)$$
   $$C = (\lambda^2 - \lambda)/2$$

The order of the method is $O(h^2) + O(k^2)$ like Lax-Wendroff.

Solution, part b:
The grid speed is twice that of Lax-Wendroff and hence the CFL condition is $a \le 2h/k$ or $\lambda \le 2$.

To compute stability we substitue a tentative solution of the form $\kappa^j e^{Irx_i}$ into the numerical method and obtain

$$\kappa = 1 - \lambda(2 - \lambda)(e^{i\theta} - 1) + \lambda(\lambda - 1/2)(e^{-2i\theta} - 1)$$

with $\theta = rh$. A tedious computation using the trigonometric double angle formulas shows
$$|\kappa|^2 = 1 - 4\lambda(2 - \lambda)(\lambda - 1)^2 \sin^4(\theta/2).$$

For $0 \le \lambda \le 2$ we conclude that $|\kappa|^2 \le 1$.

Solution, part c:
See worksheet.

Solution, part d:

No damping. Oscillation from dispersion like Lax Wendroff. Oscillations are in advance of the solution, look larger in scale, but smaller in extent.

3. Consider the equation
$$u_t + au_x - bu_{xxx} = 0$$
where $x \in \mathbb{R}$. Suppose at $t = 0$, $u(x, t) = e^{ikx}$. Find a solution of the differential equation. Describe the solution as a traveling wave. What is the speed of the wave? How does the speed change as we change the spatial frequency k?

Solution:
We guess a solution of the form $e^{-i\omega t + ikx}$. Subsituting we find

$$-\omega + ak + bk^3 = 0$$

Thus the solution is
$$e^{ik(x-\omega/kt)}.$$

Since
$$\omega = k(a + bk^2)$$
we conclude that the solution is
$$e^{ik(x-vt)}$$

with $v = a + bk^2$. This is a plane wave with speed $a + bk^2$. As the wave number k increases, the speed of the wave increases also.