1. Text, 4.20

Solution, part a:

Let $U(r,s) = u(r+as,s)$. Then

$$\begin{aligned} U_r &= u_x \\ U_s &= au_x + u_t. \end{aligned} \tag{1}$$

Using the ODE satisfied by u,

$$U_s + bU = f(x(r,s),t(r,s)).$$

We treat r as a constant in the above. Multiplying both sides by e^{bs} we find

$$\frac{d}{ds}(e^{bs}U) = e^{bs} f(x(r,s),t(r,s)).$$

Now $t(r,s) = s$ and $x(r,s) = r + as$. Thus

$$e^{bs}U = U(0,r) + \int_0^s e^{b\sigma} f(r+a\sigma,\sigma)\,d\sigma.$$

So

$$U(r,s) = U(r,0)e^{-bs} + \int_0^s e^{b(\sigma-s)} f(r+a\sigma,\sigma)\,d\sigma.$$

Now $U(r,0) = u(r,0) = g(r)$. Moreover, $r = x - as = x - at$. We conclude

$$u(x,t) = U(r(x,t),s(x,t)) = g(x-at)e^{-bt} + \int_0^t e^{b(\sigma-t)} f(x+a(\sigma-t),\sigma)\,d\sigma.$$

This is the desried formula.

Solution, part b:

The characteristic curves are defined by

$$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= e^{\alpha x} \end{aligned} \tag{2}$$

These have solutions

$$\begin{aligned} t &= s + t_0 \\ x &= -\frac{1}{\alpha}\ln(\alpha(s_0 - s)). \end{aligned} \tag{3}$$

We take $s = t$ and use r for the constant of integration in x. So

$$\begin{aligned} t &= s \\ x &= -\frac{1}{\alpha}\ln(\alpha(r-s)) \end{aligned} \tag{4}$$

Now let

$$U(r,s) = u(x(r,s),s)$$

where $x(r,s)$ is given by the above. Then

$$\frac{\partial U}{\partial s} = u_x \frac{\partial x}{\partial_s} + u_t.$$

Now

$$\frac{\partial x}{\partial s} = \frac{1}{\alpha(r-s)} = e^{\alpha x}.$$

Thus

$$\frac{\partial U}{\partial s} = 0$$

and U is a function of r alone. Moreover,

$$U(r,0) = u(x(r,0),0) = g(-\frac{1}{\alpha}\ln(\alpha r)).$$

Thus

$$u(x(r,s),s) = g(-\frac{1}{\alpha}\ln(\alpha r))$$

Since

$$r = t + 1/\alpha e^{-\alpha x}$$

we conclude

$$u(x,t) = g(-1/\frac{1}{\alpha}\ln(e^{-\alpha x} + \alpha t)).$$

Note that if $\alpha > 0$ then the argument to the natural logarithm is guaranteeded to be non-negative for $x \in \mathbb{R}$ and $t \geq 0$. and the solution is well defined. When $\alpha < 0$, the solution fails to exist for algebraic reasons. The real issue is that the characteristic curves do not intersect the line $t = 0$.

2. Text, 4.5

Solution, part a:

Solution, part b:
Starting from

$$u_t + au_x = 0$$

we integrate in space $x_{i-1} \leq x \leq x_i$ and time $t_j \leq t \leq t_{j+1}$ and apply the Fundamental Theorem of Calculus to conclude

$$\int_{x_{i-1}}^{x_i} (u(\xi, t_{j+1}) - u(\xi, t_j))\, d\xi + a\int_{t_j}^{t_{j+1}} (u(x_i, \tau) - u_{x_{i-1},\tau})\, d\tau = 0.$$

Using the Trapezoidal rule we conclude

$$\frac{h}{2}\left[u(x_i,t_{j+1}) - u(x_i,t_j) + u(x_{i-1},t_{j+1}) - u(x_{i-1},t_j)\right] + O(h^3) - \frac{ak}{2}\left[u(x_i,t_{j+1}) - u(x_{i-1},t_{j+1} + u(x_i,t_j) - u(x_{i-1},t_j\right.$$

Dividing both sides by h, using the identity $ak/h = \lambda$, and rearranging terms we conclude

$$(1+\lambda)u(x_i,t_{j+1}) + (1-\lambda)u(x_{i-1},t_{j+1}) = (1-\lambda)u(x_i,t_j) + (1+\lambda)u(x_{i-1},t_j) + O(h^2) + \lambda O(k^2).$$

Dropping the higher order terms then derives the scheme.

Solution, part c:
The method always satisfies the CFL condition as the numerical domain of dependence of the point (x_i, t_j) includes all grid points to the left and below, and hence the characteristic regardless of the value of a.

As for stability, we substitue $\kappa^j e^{Irx_i}$ into the formula to find

$$\kappa = \frac{(1-\lambda) + (1+\lambda)e^{-Irh}}{(1+\lambda) + (1-\lambda)e^{-Irh}}$$

Letting $z = e^{-Irh}$ and $w = (1-\lambda) + (1+\lambda)z$ we see

$$\kappa = \frac{1}{z}\frac{w}{\bar{w}}.$$

Since z has norm 1, so does κ and the method is stable.

Solution, part d:
Although the scheme is implicit, if a boundary condition is known at (x_0, t) we can compute explicitly a value at (x_1, t_{j+1}) from the values at (x_0, t_j), (x_0, t_{j+1}) and (x_1, t_j). Now that values are known at (x_1, t_j) and (x_1, t_{j+1}) we can repeat the argument to compute the value at (x_2, t_{j+1}) and so forth.

3. Consider the following method for solving the advection equation $u_t + au_x = 0$, where a is constant:

$$u_{i,j+1} = u_{i,j-1} + \frac{ak}{h}(u_{i-1,j} + u_{i+1,j})$$

This is the leapfrog method.

a) Determine the order of accuracy of this method (in both space and time). The answer will be in the form $\tau(x,t) = O(k^p) + O(h^q)$; determine p, q.

b) This scheme can be derived by applying the method of lines. Do this as follows.

i) First, discretize in space using centered differences to obtain a system of ODEs

$$U'(t) = AU(t)$$

for some matrix A. What are the entries of the matrix A assuming $x \in [0,1]$ is the space domain and periodic boundary conditions are used ($u(0,t) = u(1,t)$?

ii) Then use an ODE discretization method to derive the whole scheme. What rule from Table 1.1 on page 7 should be applied at this step to derive the leapfrog scheme?

c) What are the eigenvalues of the matrix A from the previous derivation?

d) Compute the region of absolute stability for the ODE discretization method used in part b). Then discuss the stability of the leap frog method using this information and your computation from part c).

e) Implement this leapfrog method on the following periodic boundary condition problem: $x \in [0,1]$, $a = 0.5$, $T = 10$, $u(x,0) = \sin(6\pi x)$. To make the implementation work you will have to compute the first step by some other scheme; describe and justify what you do.

f) What is the exact solution to the problem in part e)? Use

$$h = 0.1, 0.05, 0.02, 0.01, 0.005, 0.002$$

and $k = h$ and show a log-log convergence plot using the infinity norm for the error. What $O(h^p)$ do you expect for the rate of convergence, and what do you measure?

Solution:

Solution, part a:
Starting with a solution of $u_t + au_x = 0$ and the equations

$$\begin{aligned}
u(x,t+k) &= u + u_t k + \frac{1}{2}u_{tt}k^2 + O(k^3) \\
u(x,t-k) &= u - u_t k + \frac{1}{2}u_{tt}k^2 + O(k^3) \\
u(x+h,t) &= u - u_x h + \frac{1}{2}u_{xx}h^2 + O(h^3) \\
u(x-h,t) &= u - u_x h + \frac{1}{2}u_{xx}h^2 + O(h^3)
\end{aligned}$$

we substitute into the numerical method to conclude the local truncation error is

$$u_t + au_x + O(k^2) + O(h^2) = O(k^2) + O(h^2).$$

Solution, part b:
Discritizing in space using centered differences, the equation becomes

$$U' = AU$$

with

$$A = \frac{a}{2h}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \cdots \\ 0 & \cdots & 0 & -1 & 0 & 1 \\ 1 & 0 & \cdots & 0 & -1 & 0 \end{pmatrix}$$

Now using centered differences for the time derivative we obtain

$$U_{j+1} = U_{j-1} + \frac{ak}{h} A U_j$$

The discritization scheme is exactly this equation written row-wise.

Solution, part c:

Multiplying $\hat{U}_i = e^{Irx_i}$ by A we find this is an eigenvector with eigenvalue μ so long as

$$\frac{ak}{h}(-e^{-Irh} + e^{Irh}) = \mu$$

and so long as the periodicity condition is respected: $e^{Ir\ell} = 1$ where ℓ is the space domain length. Thus $r = 2\pi j/\ell$ for some integer j and the eigenvalues are

$$\frac{a}{h} I \sin(2\pi jh/\ell).$$

In particular they are pure imaginary and lie between $-a/h$ and a/h.

Solution, part d:
Recall that to compute the region of absolute stability for the midpoint method we look at the method applied to the ODE $u' = \lambda u$. Thus we have

$$u_{j+1} = u_{j-1} + \lambda k u_j$$

Letting $\lambda k = z$ we obtain a characteristic polynomial

$$\rho^2 - 2z\rho - 1 = 0.$$

The low order coefficient shows that the roots satsify $\rho_1 \rho_2 = 1$. Thus if one root has size less than one and the other has size larger than one, the method is unstable. We therefore require roots of unit size with product -1; these have the form $e^{I\theta}$ and $-e^{-I\theta}$ for some angle θ. Hence z has the form

$$2z = e^{I\theta} - e^{-I\theta} = 2I\cos(\theta)$$

and $z = I\cos(\theta)$. That is z is pure imaginary with $|z| \leq 1$.

From the previous problem we know that the eigenvalues are indeed pure imaginary and we simply require $|a/h|k \leq 1$, exactly the CFL condition.

Solution, part e:

4. Text, 3.7, Redo. You may find the following idea helpful. For the Neumann condition you know that $u'(0) = \alpha$. The second-order approximation for u_{xx} at $x = 0$ is

$$u_x x(0) = \frac{u_{-1} - 2u_0 + u_1}{h^2} + O(h^2)$$

The issue is that you don't know u_{-1}. But you also know that

$$\alpha = u_x(0) = \frac{u_{-1} - u_1}{2h} + O(h^2)$$

So you can either make u_{-1} a true unknown by adding this equation to the mix, or you substitute back into the equation for the second derivative at $x = 0$.

In addition, implement your $O(h^2) + O(k^2)$ version of the discritization and test on the domain $0 \le x \le 1$, $0 \le t \le 0.1$ with

$$\begin{aligned} u_0(x) &= \cos(\pi x/2) + (1 - x) \\ \alpha &= -1 \end{aligned} \tag{5}$$

Verify you obtain the desired order of accuracy.

Solution:

Pending.