

# Globally convergent decomposition algorithm for risk parity problem in portfolio selection

A.Cassioli, G.Cocchi, F.D'Amato, M.Sciandrone

February 8, 2016

## 1 Introduction

## 2 Preliminary background

Let us consider the following optimization problem:

$$\min_{x,y} f(x,y) \tag{1a}$$

$$\text{s.t.} \quad l \leq x \leq u \tag{1b}$$

$$\mathbf{1}^T x = 1 \tag{1c}$$

$$x \geq 0 \tag{1d}$$

where  $x \in \mathbb{R}^n$ ,  $y \in \mathbb{R}$ ,  $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  is a *TODO: which are the hypothesis on  $f$ ?*,  $l, u \in \mathbb{R}^n$  with  $l < u$  and  $\mathbf{1} \in \mathbb{R}^n$  is the identity vector. We indicate by  $\mathcal{F}$  the feasible set of Problem (1), namely

$$\mathcal{F} = \{x \in \mathbb{R}^n : \mathbf{1}^T x = 1, l \leq x \leq u, x \geq 0\}. \tag{2}$$

Since the constraints of Problem (1) are linear, we have that a feasible point  $(x, y)$  is a stationary point of Problem (1) if and only if the Karush-Kuhn-Tucker (KKT) conditions are satisfied.

**Proposition 2.1 (Optimality conditions (Necessary))** *Let  $(x^*, y^*) \in \mathbb{R}^{n+1}$ , with  $x^* \in \mathcal{F}$ , a local optimum for Problem (1). Then there are two multipliers  $\lambda^* \in \mathbb{R}$ ,  $\mu^* \in \mathbb{R}^n$  satisfying*

$$\frac{\partial f(x^*, y^*)}{\partial y} = 0 \tag{3a}$$

$$\frac{\partial f(x^*, y^*)}{\partial x_i} + \lambda^* - \mu_i^* = 0 \tag{3b}$$

$$\mu_i^* x_i^* = 0 \tag{3c}$$

$$\mu_i^* \geq 0 \tag{3d}$$

From Proposition (2.1) it follows that:

**Corollary 2.2** *If  $(x^*, y^*) \in \mathbb{R}^{n+1}$  is a local optimum for Problem (1), then*

$$x_j^* > 0 \quad \Rightarrow \quad \frac{\partial f(x^*, y^*)}{\partial x_j} \leq \frac{\partial f(x^*, y^*)}{\partial x_i} \quad \forall i \in \{1, \dots, n\} \quad (4)$$

Given a feasible point  $(x, y)$ , we define two indexes  $i^*, j^* \in \{1, \dots, n\}$  in the following way:

$$x_{i^*} < u_{i^*} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial x_{i^*}} \leq \frac{\partial f(x, y)}{\partial x_h} \quad h \text{ s.t. } x_h < u_h \quad (5a)$$

$$x_{j^*} > l_{j^*} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial x_{j^*}} \geq \frac{\partial f(x, y)}{\partial x_h} \quad h \text{ s.t. } x_h > l_h \quad (5b)$$

Now we define a direction  $d^{i^*, j^*} \in \mathbb{R}^n$  with only two non-zero components such that:

$$d_h^{i^*, j^*} = \begin{cases} 1, & h = i^* \\ -1, & h = j^* \\ 0, & \text{otherwise} \end{cases} \quad (6)$$

**Proposition 2.3** *Let  $(x, y)$  a feasible point for Problem (1). Then the direction  $d^{i^*, j^*}$  is a descent direction for  $f(x, y)$ .*

**Proof** From Eq.(5) and Eq.(6) we can write:

$$\nabla_x f(x, y)^T d^{i^*, j^*} = \frac{\partial f(x, y)}{\partial x_{i^*}} - \frac{\partial f(x, y)}{\partial x_{j^*}} \leq 0 \quad (7)$$

*TODO: Manca da dimostrare il minore secco.*

## 2.1 Armijo-Type Line Search Algorithm

In this section, we describe the well-known Armijo-type line search along a feasible direction [1]. The procedure will be used in the decomposition method presented in the next section. Let  $d^{(k)}$  be a feasible direction at  $(x^{(k)}, y^{(k)})$  with  $x^{(k)} \in \mathcal{F}$ . We denote by  $\beta_{\mathcal{F}}^{(k)}$  the maximum feasible steplength along  $d^{(k)}$ , namely  $\beta_{\mathcal{F}}^{(k)}$  satisfies

$$l \leq x + \beta d^{(k)} \leq u \quad \text{for every } \beta \in [0, \beta_{\mathcal{F}}^{(k)}]$$

We have at least an index  $i \in \{1, \dots, n\}$  such that

$$x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = l_i \quad \text{or} \quad x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = u_i$$

Let  $\beta_u$  be a positive scalar and set

$$\beta^{(k)} = \min\{\beta_{\mathcal{F}}^{(k)}, \beta_u\} \quad (8)$$

An Armijo-type line search algorithm is described below.

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**Algorithm 1:** Armijo-Type Line Search

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**Data:** Given  $\alpha > 0$ ,  $\delta \in (0, 1)$ ,  $\gamma \in (0, 1/2)$  and the initial stepsize  
     $\alpha^{(k)} = \min\{\beta^{(k)}, \alpha\}$   
**while**  $f(x^{(k)} + \lambda d^{(k)}) > f(x^{(k)}) + \gamma \lambda \nabla_x f(x^{(k)})^T d^{(k)}$  **do**  
     $\perp$  Set  $\lambda = \delta \lambda$

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### 3 A decomposition framework

### 4 Convergence analysis

### 5 Computational experiments