Globally convergent decomposition algorithm for risk parity problem in portfolio selection

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1 Introduction

2 Preliminary background

Let us consider the following optimization problem:

$$\min_{x,y} \quad f(x,y) \tag{1a}$$

$$\text{s.t.} \quad l \leq x \leq u \tag{1b}$$

$$\mathbf{1}^T x = 1 \tag{1c}$$

$$x \geq 0 \tag{1d}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a *which are the hypothesis on f?*, $l, u \in \mathbb{R}^n$ with $l < u$ and $\mathbf{1} \in \mathbb{R}^n$ is the identity vector. We indicate by $\mathcal{F}$ the feasible set of Problem (1), namely

$$\mathcal{F} = \{x \in \mathbb{R}^n : \mathbf{1}^T x = 1, l \leq x \leq u, x \geq 0\}. \tag{2}$$

Since the constraints of Problem (1) are linear, we have that a feasible point (x, y) is a stationary point of Problem (1) if and only if the Karush-Kuhn-Tucker (KKT) conditions are satisfied.

Proposition 2.1 (Optimality conditions (Necessary)) *Let $(x^*, y^*) \in \mathbb{R}^{n+1}$, with $x^* \in \mathcal{F}$, a local optimum for Problem (1). Then there are two multipliers $\lambda^* \in \mathbb{R}$, $\mu^* \in \mathbb{R}^n$ satisfying*

$$\frac{\partial f(x^*, y^*)}{\partial y} = 0 \tag{3a}$$

$$\frac{\partial f(x^*, y^*)}{\partial x_i} + \lambda^* - \mu_i^* = 0 \tag{3b}$$

$$\mu_i^* x_i^* = 0 \tag{3c}$$

$$\mu_i^* \geq 0 \tag{3d}$$

From Proposition (2.1) it follows that:

Corollary 2.2 *If* $(x^*, y^*) \in \mathbb{R}^{n+1}$ *is a local optimum for Problem (1), then*

$$x_j^* > 0 \quad \Rightarrow \quad \frac{\partial f(x^*, y^*)}{\partial x_j} \leq \frac{\partial f(x^*, y^*)}{\partial x_i} \quad \forall i \in \{1, .., n\} \tag{4}$$

Given a feasible point (x, y), we define two indexes $i^*, j^* \in \{1, .., n\}$ in the following way:

$$x_{i^*} < u_{i^*} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial x_{i^*}} \leq \frac{\partial f(x,y)}{\partial x_h} \quad h \text{ s.t. } x_h < u_h \tag{5a}$$

$$x_{j^*} > l_{j^*} \quad \text{and} \quad \frac{\partial f(x,y)}{\partial x_{j^*}} \geq \frac{\partial f(x,y)}{\partial x_h} \quad h \text{ s.t. } x_h > l_h \tag{5b}$$

Now we define a direction $d^{i^*,j^*} \in \mathbb{R}^n$ with only two non-zero components such that:

$$d_h^{i^*,j^*} = \begin{cases} 1, & h = i^* \\ -1, & h = j^* \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Proposition 2.3 *Let* (x, y) *a feasible point for Problem (1). Then the direction* d^{i^*,j^*} *is a descent direction for* $f(x, y)$.

Proof From Eq.(5) and Eq.(6) we can write:

$$\nabla_x f(x,y)^T d^{i^*,j^*} = \frac{\partial f(x,y)}{\partial x_i^*} - \frac{\partial f(x,y)}{\partial x_j^*} \leq 0 \tag{7}$$

Manca da dimostrare il minore secco.

2.1 Armijo-Type Line Search Algorithm

In this section, we describe the well-known Armijo-type line search along a feasible direction [1]. The procedure will be used in the decomposition method presented in the next section. Let $d^{(k)}$ be a feasible direction at $(x^{(k)}, y^{(k)})$ with $x^{(k)} \in \mathcal{F}$. We denote by $\beta_{\mathcal{F}}^{(k)}$ the maximum feasible steplength along $d^{(k)}$, namely $\beta_{\mathcal{F}}^{(k)}$ satisfies

$$l \leq x + \beta d^{(k)} \leq u \quad \text{for every} \quad \beta \in [0, \beta_{\mathcal{F}}^{(k)}]$$

We have at least an index $i \in \{1, .., n\}$ such that

$$x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = l_i \qquad \text{or} \qquad x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = u_i$$

3 A decomposition framework

4 Convergence analysis

5 Computational experiments