Globally convergent decomposition algorithm for risk parity problem in portfolio selection

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1 Introduction

2 Preliminary background

Let us consider the following optimization problem:

$$
\begin{aligned}
\min_{x,y} \quad & f(x,y) && \text{(1a)}\\
\text{s.t.} \quad & l \le x \le u && \text{(1b)}\\
& \mathbf{1}^T x = 1 && \text{(1c)}
\end{aligned}
$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is a *TODO: which are the hypothesis on f?*, $l, u \in \mathbb{R}^n$ with $l < u$ and $\mathbf{1} \in \mathbb{R}^n$ is the identity vector. We indicate by $\mathcal{F}$ the feasible set of Problem (1), namely

$$\mathcal{F} = \{x \in \mathbb{R}^n : \mathbf{1}^T x = 1, l \le x \le u\}. \tag{2}$$

Since the constraints of Problem (1) are linear, we have that a feasible point (x, y) is a stationary point of Problem (1) if and only if the Karush-Kuhn-Tucker (KKT) conditions are satisfied.

Proposition 2.1 (TODO: Optimality conditions (Necessary)) *Let $(x^*, y^*) \in \mathbb{R}^{n+1}$, with $x^* \in \mathcal{F}$, a local optimum for Problem (1). Then there are two multipliers $\lambda^* \in \mathbb{R}$, $\mu^* \in \mathbb{R}^n$ satisfying*

$$
\begin{aligned}
\frac{\partial f(x^*, y^*)}{\partial y} &= 0 && \text{(3a)}\\
\frac{\partial f(x^*, y^*)}{\partial x_i} + \lambda^* - \mu_i^* &= 0 && \text{(3b)}\\
\mu_i^* x_i^* &= 0 && \text{(3c)}\\
\mu_i^* &\ge 0 && \text{(3d)}
\end{aligned}
$$

From Proposition (2.1) it follows that:

Corollary 2.2 *If $(x^*, y^*) \in \mathbb{R}^{n+1}$ is a local optimum for Problem (1), then*

$$x_j^* > 0 \quad \Rightarrow \quad \frac{\partial f(x^*, y^*)}{\partial x_j} \leq \frac{\partial f(x^*, y^*)}{\partial x_i} \quad \forall i \in \{1, .., n\} \tag{4}$$

Given a feasible point (x, y), we define two indexes $i^*, j^* \in \{1, .., n\}$ in the following way:

$$x_{i^*} < u_{i^*} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial x_{i^*}} \leq \frac{\partial f(x, y)}{\partial x_h} \quad h \text{ s.t. } x_h < u_h \tag{5a}$$

$$x_{j^*} > l_{j^*} \quad \text{and} \quad \frac{\partial f(x, y)}{\partial x_{j^*}} \geq \frac{\partial f(x, y)}{\partial x_h} \quad h \text{ s.t. } x_h > l_h \tag{5b}$$

Now we define a direction $d^{i^*,j^*} \in \mathbb{R}^n$ with only two non-zero components such that:

$$d_h^{i^*,j^*} = \begin{cases} 1, & h = i^* \\ -1, & h = j^* \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Proposition 2.3 *Let (x, y) a feasible point for Problem (1). Then the direction d^{i^*,j^*} is a descent direction for $f(x, y)$.*

Proof From Eq.(5) and Eq.(6) we can write:

$$\nabla_x f(x, y)^T d^{i^*,j^*} = \frac{\partial f(x, y)}{\partial x_i^*} - \frac{\partial f(x, y)}{\partial x_j^*} \leq 0 \tag{7}$$

TODO: Manca da dimostrare il minore secco.

2.1 Armijo-Type Line Search Algorithm

In this section, we describe the well-known Armijo-type line search along a feasible direction. The procedure will be used in the decomposition method presented in the next section. Let $d^{(k)}$ be a feasible direction at $(x^{(k)}, y^{(k)})$ with $x^{(k)} \in \mathcal{F}$. We denote by $\beta_{\mathcal{F}}^{(k)}$ the maximum feasible steplength along $d^{(k)}$, namely $\beta_{\mathcal{F}}^{(k)}$ satisfies

$$l \leq x + \beta d^{(k)} \leq u \quad \text{for every} \quad \beta \in [0, \beta_{\mathcal{F}}^{(k)}]$$

We have at least an index $i \in \{1, .., n\}$ such that

$$x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = l_i \qquad \text{or} \qquad x_i^{(k)} + \beta_{\mathcal{F}}^{(k)} d_i^{(k)} = u_i$$

Let β_u be a positive scalar and set

$$\beta^{(k)} = \min\{\beta_{\mathcal{F}}^{(k)}, \beta_u\} \tag{8}$$

An Armijo-type line search algorithm is described below.

Algorithm 1: Armijo-Type Line Search

Data: Given $\alpha > 0$, $\delta \in (0,1)$, $\gamma \in (0, 1/2)$ and the initial stepsize $\alpha^{(k)} = \min\{\beta^{(k)}, \alpha\}$

Set $\lambda = \alpha^{(k)}$

while $f(x^{(k)} + \lambda d^{(k)}) > f(x^{(k)}) + \gamma\lambda\nabla_x f(x^{(k)})^T d^{(k)}$ **do**

  Set $\lambda = \delta\lambda$

2.2 Exact Line Search

When we move along the direction d^{i^*,j^*}, defined in (6), we modify only 2 variables (x_{i^*}, x_{j^*}) leaving the others unchanged. Thus, we can see our $f(x, y)$ as a function of two components, i.e. we can rewrite Problem (1) as

$$\min_{x_{i^*}, x_{j^*}} \quad f(x_{i^*}, x_{j^*}) \tag{9a}$$

$$\text{s.t.} \quad l_{i^*} \leq x_{i^*} \leq u_{i^*} \tag{9b}$$

$$l_{j^*} \leq x_{j^*} \leq u_{j^*} \tag{9c}$$

$$x_{i^*} + x_{j^*} = \underbrace{1 - \sum_{h \neq i^*, j^*} x_h}_{c} \tag{9d}$$

Thanks to the last constraint, we can substitute $x_{i^*} = c - x_{j^*}$ and then we obtain

$$\min_{x_{j^*}} \quad f(x_{j^*}) \tag{10a}$$

$$\text{s.t.} \quad x_{i^*} = c - x_{j^*} \tag{10b}$$

$$ll_{j^*} = \max\{l_{j^*}, c - u_{i^*}\} \leq x_{j^*} \leq \min\{u_{j^*}, c - l_{i^*}\} = uu_{j^*} \tag{10c}$$

Because the domain is $I = [ll_{j^*}, uu_{j^*}]$, if $f(x_{j^*})$ is continuous in I, then f has a minimum in I. If $f(x_{j^*})$ is differentiable in I we can compute $f'(x_{j^*})$. Let $R = \{r \mid f'(r) = 0, r \in I\}$ be set set of the real feasible roots of f'. Each $r \in R$ can be a local maximum, minimum or flex; if $R = \{\emptyset\}$, then the minimum of f is on the extreme points of I.
Let $r* = \arg\min_{r \in R} f(r)$, then the optimal step α^* along the direction d^{i^*,j^*} is

$$\alpha^* = x_{j^*} - r^* > 0 \tag{11}$$

TODO: problema di notazione, x_{j^*} rappresenta il valore della componente j^* del vettore x prima della ricerca di linea

3 A decomposition framework

$$y^{(k+1)} = \arg\min_{y} f(x^{(k)}, y) \tag{12}$$

Algorithm 2: Decomposition Algorithm

```
Data: Given the initial feasible guess (x^(0), y^(0))
Set k = 0
while (not convergence) do
    Compute y^(k+1)
    Compute ∇_x f(x^(k), y^(k+1))
    Choose indexes i*(k), j*(k) using the Gauss-Southwell rules
    Choose a step α^(k) along the direction d^{i*(k),j*(k)}
    Set x^(k+1) = x^(k) + α^(k) d^{i*(k),j*(k)}
    Set k = k + 1
```

Data: Given the initial feasible guess $(x^{(0)}, y^{(0)})$

Set $k = 0$

while *(not convergence)* **do**

- Compute $y^{(k+1)}$
- Compute $\nabla_x f(x^{(k)}, y^{(k+1)})$
- Choose indexes $i^*(k), j^*(k)$ using the Gauss-Southwell rules
- Choose a step $\alpha^{(k)}$ along the direction $d^{i^*(k),j^*(k)}$
- Set $x^{(k+1)} = x^{(k)} + \alpha^{(k)} d^{i^*(k),j^*(k)}$
- Set $k = k + 1$

4 Convergence analysis

5 Computational experiments