Analysis of Algorithms

Self-Balancing Search Trees

References to binary search trees, red-black trees, and AVL trees refer to typical implementations, unless otherwise specified.

The terms *niece* and *nephew* come from web page: *The best red-black tree pseudocode ever*.

If more than one question appears correct, choose the more specific answer, unless otherwise instructed.

Concept: *rotations in a BST*

1. Which of the following is true for rotations in a BST:

   (A) the number of leaf nodes always decreases
   (B) BST-ordering is always preserved
   (C) the tree always becomes more balanced
   (D) the number of leaf nodes always increases

2. Consider a right-rotation of a node n upwards in a BST. The former right child of n, if it exists:

   (A) remains the right child of n
   (B) becomes the niece or nephew of n
   (C) becomes the sibling of n
   (D) becomes the left child of n
   (E) becomes the left child of the former parent of n
   (F) becomes the right child of the former parent of n

3. Consider a right-rotation of a node n upwards in a BST. The former left child of n, if it exists:

   (A) becomes the left child of the former parent of n
   (B) remains the left child of n
   (C) becomes the right child of n
   (D) becomes the right child of the former parent of n
   (E) becomes the niece or nephew of n
   (F) becomes the sibling of n

4. Consider a right-rotation of a node n upwards in a BST. The former parent of n, assuming it exists:

   (A) becomes the niece or nephew of n
   (B) becomes the right child of n
   (C) becomes the left child of n
   (D) remains the parent of n
   (E) becomes the sibling of n

5. Consider a right-rotation of a node n upwards in a BST. The former sibling of n, assuming it exists:

   (A) remains the sibling of n
   (B) becomes the right child of n
   (C) becomes a grandchild of n
   (D) becomes the niece or nephew of n
   (E) becomes the left child of n

Concept: *red-black trees*

6. The number of rotations that occur after an insertion into a red-black tree is:

   (A) $\Theta(1)$
   (B) $\Theta(n)$
   (C) $\Theta(n \log n)$
   (D) $\Theta(\log n)$

7. The maximum number of rotations that occur after an insertion into a red-black tree is:

   (A) 2
   (B) 1
   (C) $\sim \log n$
   (D) 3

8. The maximum number of rotations that occur after a deletion from a red-black tree is:

   (A) $\Theta(1)$
   (B) $\Theta(\log n)$
   (C) $\Theta(n)$
   (D) $\Theta(n \log n)$

9. The maximum number of rotations that occur after a deletion from a red-black tree is:

   (A) 3
   (B) $\sim \log n$
   (C) 2
   (D) 1

10. Consider a node n in a red-black tree and all paths from n to a leaf. Which of the following is a constraint on these trees?

    (A) the number of black nodes on each path is the same
    (B) each path must start with a black node
    (C) the number of red nodes on each path is the same
    (D) the number of nodes (red or black) on each path is the same

11. Consider a node n in a red-black tree and all paths from n to a leaf. Which of the following is a constraint on these trees?

    (A) no black node can have a red parent
    (B) no red node can have a red parent
    (C) no black node can have a black parent
    (D) no red node can have a black parent

12. Consider a black interior node n in a red-black tree and any path from n to a leaf. Which of the following is a constraint on these trees, where R is the number of red nodes and B is the number of black nodes?

    (A) $B < R$
    (B) $B \leq R$
    (C) $B \leq R + 1$
    (D) $R \leq B + 1$
    (E) $R = B$
    (F) $R \leq B$

13. Consider a red node n in a red-black tree and the length of the shortest possible path from n to a leaf, S, and the length of the longest possible path from n to a leaf, L. Which of the following is a constraint on these trees?

    (A) $L = 2S$
    (B) $L = 2S + 2$
    (C) $L = 2S - 2$
    (D) $L = 2S + 1$
    (E) $L = 2S - 1$

14. Inserting a value in a red-black tree and a regular BST, respectively, takes time:

    (A) $\Theta(1)$ and $\Theta(\log n)$
    (B) $\Theta(n)$ and $\Theta(n)$
    (C) $\Theta(\log n)$ and $\Theta(\log n)$
    (D) $\Theta(\log n)$ and $\Theta(n)$

15. Suppose one wished to allow more red nodes in a red-black tree, but still wished this new tree to have the same asymptotic behavior as before. One could allow more red nodes on any path to a leaf as long as:

    (A) the number of red nodes between any two black nodes is bounded by a constant.
    (B) the number of black nodes between any two red nodes is bounded by a constant.
    (C) no black node could have a red parent.
    (D) no red node could have a red sibling.

16. **T** or **F**: A black node in a red-black tree can have one child.

17. **T** or **F**: A red node in a red-black tree can have one child.

18. **T** or **F**: A red node in a red-black tree can have a red parent.

19. **T** or **F**: A black node in a red-black tree can have a black parent.

Assume the algorithm posted at `http://beastie.cs.ua.edu/red-black/` is used as a basis for implementation. Assume the default color of a newly inserted node is red. Count a double rotation as two rotations.

20. Which insertion to the following red-black tree would cause a single rotation?

    (A) 13
    (B) 4
    (C) 0
    (D) 8

21. Which insertion to the following red-black tree would cause a double rotation?

(A) 9
(B) 0
(C) 6
(D) 3

22. Which insertion to the following red-black tree would cause no rotations?

(A) 5
(B) 25
(C) 0
(D) All of the other answers are correct

23. After inserting a 13 into an empty BST, how many rotations and recolorings occurred, respectively?

(A) 1 / 0
(B) 0 / 1
(C) 1 / 1
(D) 0 / 0

24. Continuing with the previous question, after inserting 17, how many new rotations and new recolorings occurred, respectively?

(A) 0 / 1
(B) 1 / 0
(C) 1 / 1
(D) 0 / 0

25. Continuing with the previous question, after inserting 24, how many new rotations and new recolorings occurred, respectively?

(A) 0 / 1
(B) 1 / 1
(C) 2 / 2
(D) 2 / 1
(E) 0 / 2
(F) 1 / 2
(G) 2 / 0
(H) 1 / 0

26. Continuing with the previous question, after inserting 15, how many new rotations and new recolorings occurred, respectively?

(A) 1 / 3
(B) 0 / 4
(C) 1 / 1
(D) 0 / 2
(E) 0 / 3
(F) 1 / 2
(G) 1 / 4
(H) 0 / 1

27. Continuing with the previous question, after inserting 14, how many new rotations and new recolorings occurred, respectively?

(A) 1 / 1
(B) 0 / 1
(C) 1 / 2
(D) 0 / 2
(E) 1 / 0
(F) 2 / 0
(G) 2 / 1
(H) 2 / 2

28. Choose the order of insertion such that a red-black tree performs a total of seven recolorings and one rotation.

(A) 4 11 3 5
(B) 11 5 4 3
(C) 5 4 11 3
(D) 5 3 4 11

29. Choose the order of insertion such that a red-black tree performs a total of eight recolorings and no rotations.

(A) 4 15 17 18 20 11
(B) 11 18 15 4 20 17
(C) 11 4 17 15 18 20
(D) 15 11 4 18 20 17

30. Choose the order of insertion such that a red-black tree performs a total of seven recolorings and two rotations.

(A) 11 15 0 14 3 16
(B) 11 3 16 0 14 15
(C) 14 15 3 16 0 11
(D) 3 16 0 11 14 15

31. Choose the order of insertion such that a red-black tree performs a total of nine recolorings and two rotations.

(A) 14 15 3 16 0 11
(B) 3 16 0 11 14 15
(C) 0 3 11 14 15 16
(D) 11 15 0 14 3 16

32. Choose the order of insertion such that a red-black tree performs a total of ten recolorings and one rotation.

(A) 17 20 13 8 19 15
(B) 15 17 20 8 13 19
(C) 13 20 15 8 17 19
(D) 13 8 15 17 19 20

33. Choose an order of insertion for seven consecutive integers such that a red-black tree performs no rotations for any of the insertions.

(A) 3 2 5 0 1 4 6
(B) 3 2 6 0 1 4 5
(C) 3 2 6 1 0 5 4
(D) 3 2 5 1 0 4 6
(E) 3 1 5 2 0 4 6

34. Choose an order of insertion for seven consecutive integers such that a red-black tree performs one rotation for one of the insertions and no rotations for the other insertions.

(A) 3 2 5 0 1 4 6
(B) 3 2 5 1 0 4 6
(C) 3 2 6 0 1 4 5
(D) 3 2 6 1 0 5 4
(E) 3 1 5 2 0 4 6

35. Choose an order of insertion for seven consecutive integers such that a red-black tree performs two rotations for one of the insertions and no rotations for the other insertions.

(A) 3 2 6 1 0 5 4
(B) 3 2 5 1 0 4 6
(C) 3 2 5 0 1 4 6
(D) 3 1 5 2 0 4 6
(E) 3 2 6 0 1 4 5

36. Choose an order of insertion for seven consecutive integers such that a red-black tree performs one rotation for two of the insertions and no rotations for the other insertions.

(A) 3 2 5 0 1 4 6
(B) 3 2 5 1 0 4 6
(C) 3 1 5 2 0 4 6
(D) 3 2 6 0 1 4 5
(E) 3 2 6 1 0 5 4

37. Choose an order of insertion for seven consecutive integers such that a red-black tree performs no rotations for any of the insertions *and* yields the most unbalanced tree.

38. **T** or **F**: Inserting the following numbers, in the order given, into an empty BST:

```
0 4 3 8 1 2 6 5 9 7
```

yields a tree whose shape is consistent with a red-black tree.

39. Consider inserting the following numbers, in the order given, into an empty BST and then coloring the root black and the other nodes such that no red node has a red parent: each node:

```
0 4 3 8 1 2 6 5 9 7
```

What is the minimum / maximum number of red nodes possible?

(A) 3 / 5
(B) 0 / 5
(C) 0 / 4
(D) 3 / 3
(E) the correct answer is not listed
(F) 0 / 3
(G) 3 / 4
(H) 0 / 6

40. Consider an node with a single child in a red-black tree. If that node has a sibling and you wish to maximize the number of descendants the sibling has, what color is the sibling and how many descendants does it have? Do not include the null children. Choose the best answer.

(A) either black or red / 2
(B) either black or red / 4
(C) red / 2
(D) the correct answer is not listed
(E) red / 6
(F) black / 6
(G) black / 2
(H) either black or red / 5

41. Consider inserting the following numbers, in the order given, into an empty red-black tree:

```
5 3 8 1 7 6 4
```

How many rotations and how many node recolorings are performed? Don't forget what happens on the initial insertion.

(A) 1 / 7
(B) 1 / 6
(C) 0 / 5
(D) 1 / 5
(E) 0 / 7
(F) 0 / 6
(G) the correct answer is not listed

42. Consider inserting the following numbers, in the order given, into an empty BST:

```
0 4 3 8 1 2 6 5 9 7
```

What is the minimum number of rotations that would yield a tree with a shape consistent with a red-black tree?

(A) 1
(B) 3
(C) the correct answer is not listed
(D) 4
(E) 2
(F) 5
(G) 0

43. Consider inserting the following numbers, in the order given, into an empty red-black tree:

```
0 4 3 8 1 2 6 5 9 7
```

After which insertion value does the red-black tree make its first rotation?

(A) the correct answer is not listed
(B) 3
(C) 8
(D) 2
(E) 4
(F) 1
(G) 6
(H) 0

44. Consider inserting the following numbers, in the order given, into an empty red-black tree:

```
0 4 3 8 1 2 6 5 9 7
```

Which values, when inserted, cause a rotation?

(A) 6 7
(B) 3 2 6
(C) 3 2 7
(D) the correct answer is not listed
(E) 2 6 7
(F) none

45. Consider inserting the following numbers, in the order given, into an empty red-black tree:

```
0 4 3 8 1 2 6 5 9 7
```

Which values, when inserted, cause a double rotaion?

(A) 2 6
(B) 2 7
(C) 6
(D) 7
(E) 3 6
(F) 3 7
(G) the correct answer is not listed

Concept: *AVL trees*

46. Consider a node n in an AVL tree and the height of the left subtree of n, LH and the height of the right subtree of n, RH. Assuming $LH > RH$, which of the following is a constraint on these trees?

    (A) $LH - RH = 2$
    (B) $LH - RH = 1$
    (C) $LH - RH < 2$
    (D) $LH - RH < 3$
    (E) $LH - RH = 0$

47. The number of rotations that occur after an insertion into an AVL tree is, in the worst case:

    (A) $\Theta(1)$
    (B) $\Theta(n)$
    (C) $\Theta(n \log n)$
    (D) $\Theta(\log \log n)$
    (E) $\Theta(\log n)$

48. The number of rotations that occur after an insertion into an AVL tree is, in the worst case:

    (A) 1
    (B) $\Theta(n)$
    (C) $\Theta(\log n)$
    (D) 2

49. The number of rotations that occur after an deletion in an AVL tree is, in the worst case:

    (A) $\Theta(n)$
    (B) $\Theta(\log n)$
    (C) $\Theta(1)$
    (D) $\Theta(n \log n)$
    (E) $\Theta(\log \log n)$

50. The number of rotations that occur after an deletion into an AVL tree is, in the worst case:

    (A) 2
    (B) $\Theta(\log n)$
    (C) $\Theta(n)$
    (D) 1

51. The minimum number of rotations that occur after an deletion into an AVL tree is:

    (A) 2
    (B) $\Theta(n)$
    (C) $\Theta(\log n)$
    (D) 0
    (E) 1

52. **T** or **F**: Inserting the following numbers, in the order given, into an empty BST:

    ```
    0 4 3 8 1 2 6 5 9 7
    ```

    yields a tree whose shape is consistent with an AVL tree.

53. Consider inserting the following numbers, in the order given, into an empty BST and then computing the balance factors of each node:

    ```
    0 4 3 8 1 2 6 5 9 7
    ```

    What nodes are out of balance, with respect to AVL balance factors?

    (A) 0 3
    (B) 0 1 3 8
    (C) the correct answer is not listed
    (D) 0 4 3
    (E) 0 3 8
    (F) 2 5 6 7 9
    (G) 0 1 2 3

54. Consider an node with a single child in an AVL tree. If that node has a sibling, what is the least / most number of descendants the sibling can have?

    (A) 0 / 4
    (B) the correct answer is not listed
    (C) 0 / 6
    (D) 1 / 6
    (E) 0 / 5
    (F) 1 / 5
    (G) 1 / 7
    (H) 0 / 7
    (I) 1 / 4

55. Consider inserting the following numbers, in the order given, into an empty BST and then computing the balance factors of each node:

```
0 4 3 8 1 2 6 5 9 7
```

Performing a level order traversal of the tree in which balance factors are displayed, which of the following sequences of balance factors is consistent with the above tree? Assume the height of a null child is zero. Do not include the null children in the output.

(A) the correct answer is not listed
(B) -1 0 1 1 -1 0 0 0 0 0
(C) -9 -2 2 2 -1 0 0 0 0 0
(D) 1 0 1 1 1 0 0 0 0 0
(E) -4 0 2 1 -1 0 0 0 0 0

56. Consider inserting the following numbers, in the order given, into an empty BST and then computing the heights of each node:

```
0 4 3 8 1 2 6 5 9 7
```

Performing a level order traversal of the tree in which node heights are displayed, which of the following sequences of heights is consistent with the above tree? Assume the height of a null child is zero. Do not include the null children in the output.

(A) the correct answer is not listed
(B) 4 4 3 3 2 2 1 1 0 0
(C) 5 5 4 4 3 3 2 2 1 1
(D) 4 3 2 2 1 1 1 0 0 0
(E) 5 4 3 3 2 2 2 1 1 1

57. Consider inserting the following numbers, in the order given, into an empty BST:

```
0 4 3 8 1 2 6 5 9 7
```

What is the minimum number of rotations that would yield a tree with balance factors consistent with an AVL tree?

(A) 3
(B) 0
(C) the correct answer is not listed
(D) 4
(E) 2
(F) 5
(G) 1

58. Consider inserting the following numbers, in the order given, into an empty AVL tree:

```
0 4 3 8 1 2 6 5 9 7
```

After which insertion value causes the AVL tree's first rotation?

(A) 8
(B) 6
(C) 1
(D) 3
(E) 5
(F) the correct answer is not listed
(G) 2
(H) 7
(I) 9

59. Consider inserting the following numbers, in the order given, into an empty AVL tree:

```
0 4 3 8 1 2 6 5 9 7
```

Which values, when inserted, cause rotations?

(A) 7 6
(B) 3 7 2
(C) the correct answer is not listed
(D) 3 7 2 6
(E) 7 2 6

60. Consider inserting the following numbers, in the order given, into an empty AVL tree:

```
0 4 3 8 1 2 6 5 9 7
```

Which values, when inserted, cause a double rotaion?

(A) 6
(B) 2 6
(C) 3
(D) 7
(E) 2
(F) 3 7
(G) the correct answer is not listed

61. Choose an order of insertion for seven consecutive integers such that an AVL tree performs no rotations for any of the insertions.

62. Choose an order of insertion for seven consecutive integers such that an AVL tree performs one rotation for one of the insertions and no rotations for the other insertions.

63. Choose an order of insertion for seven consecutive integers such that an AVL tree performs two rotations for one of the insertions and no rotations for the other insertions.

64. Choose an order of insertion for seven consecutive integers such that an AVL tree performs one rotation each for two of the insertions and no rotations for the other insertions.

65. Choose an order of insertion for seven consecutive integers such that an AVL tree performs no rotations for any of the insertions *and* yields the most unbalanced tree.

66. What is the balance factor of this tree?

(A) 2
(B) a single node has no balance factor
(C) 1
(D) 0

67. What is the balance factor of the root of the following tree?

(A) 0
(B) -1
(C) 1
(D) 2

68. **T** or **F**: If the balance factor of the root of a binary search tree is zero, then all the balance factors of all the other nodes in the tree must be 0, 1, or -1.

69. Is the following tree a valid AVL tree; and if so, how many nodes will have their balance factors changed if the value 11 is added to the tree? Do not count the newly added node.

(A) yes, 1
(B) no
(C) yes, 2
(D) yes, 4
(E) yes, 6
(F) yes, 0
(G) yes, 5
(H) yes, 3