DESIGN OPTIMIZATION

1.)

$$\min_{x_1, x_2} (x_1+1)^2 + (x_2-2)^2$$

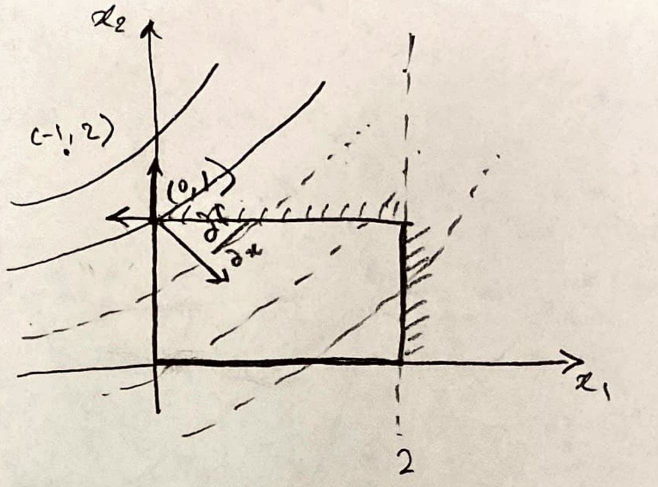
s.t $x_1 - 2 \le 0$, $-x_1 \le 0$

$x_2 - 1 \le 0$, $-x_2 \le 0$

$\mu_1 > 0$ $-x_1 \le 0$, $-x_2 \le 0$ $\mu_2 = 0$

$x_1 \le 2$, $\boxed{x_2 \le 1}$

$\mu_2 = 0$ $\quad \mu_4 > 0$



$$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(-x_1) + \mu_2(-x_2) + \mu_3(x_1-2) + \mu_4(x_2-1)$$

$$\frac{dL}{dx} = \begin{bmatrix} 2(x_1+1) - \mu_1 + \mu_3 \\ 2(x_2-2) - \mu_2 + \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Enumerate over $\mu_1 \sim \mu_4$

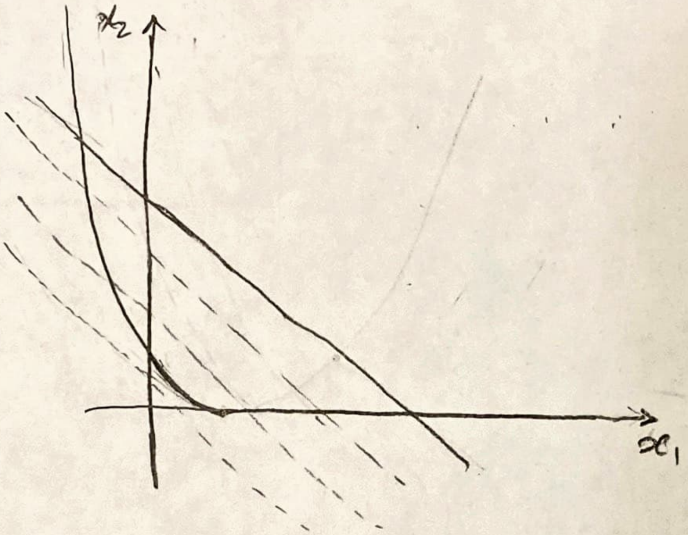
Here pass $\mu_1 > 0,\ \mu_4 > 0,\ \mu_2 = \mu_3 = 0$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \\ 2x_1 + 2 - \mu_1 = 0 \\ 2(x_2-2) + \mu_4 = 0 \end{cases} \Rightarrow \begin{cases} \mu_1 = 2 > 0 \\ \mu_4 = 2 > 0 \end{cases}$$

2.)

$\min_{x_1, x_2} \; f = -x_1$

s.t $g_1 = x_2 - (1-x_1)^3 \leq 0 \qquad x_2 \geq 0$



$$\frac{\partial g_1}{\partial x} = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} @ [1, 0]$$

$$\frac{\partial g_2}{\partial x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$L = -x_1 + \mu_1 \left(x_2 - (1-x_1)^3\right) + \mu_2 (x_2)$$

$$\frac{dL}{dx} = \begin{bmatrix} -1 + 3\mu_1 (1-x_1)^2 \\ \mu_1 - \mu_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

① $\mu_1 = \mu_2 > 0$

$\Rightarrow x_2 = 0, x_1 = 1$

$\Rightarrow -1 + 3 \cdot \mu_1 \cdot 0 \neq 0$

② $\mu_1 = \mu_2 = 0$

$\Rightarrow -1 \neq 0$

Thus you cannot find the solution based on the optimality conditions.

3.) $\max_{x_1,x_2,x_3} \quad f = x_1x_2 + x_2x_3 + x_1x_3$ (3)

s.t $\quad h = x_1 + x_2 + x_3 - 3 = 0$

Reduced gradient

$$\min_{x_1,x_2,x_3} f = -(x_1x_2 + x_2x_3 + x_1x_3)$$

$n = 3 \qquad dof = n - m$

$m = 1 \qquad\quad = 3 - 1$

$\qquad\qquad\quad = 2$

$$d = [d_1, d_2] = [x_1, x_2]$$

$$s = x_3$$

$$\Rightarrow \frac{dh}{ds} = \frac{dh}{dx_3} = 1$$

$$\Rightarrow \frac{ds}{dd} = -\left(\frac{dh}{ds}\right)^{-1} \frac{dh}{dd} = -1 \cdot \begin{bmatrix} \frac{dh}{dx_1} \\ \frac{dh}{dx_2} \end{bmatrix} = -1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\frac{df}{dd} = \begin{bmatrix} \frac{df}{dx_1} \\ \frac{df}{dx_2} \end{bmatrix} = \begin{bmatrix} \frac{df}{dx_1} + \frac{df}{dx_3} \cancel{\frac{dx_3}{dx_1}}^{-1} \\ \frac{df}{dx_2} + \frac{df}{dx_3} \cancel{\frac{dx_3}{dx_2}}_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{df}{dx_1} = -x_2 - x_3, \quad \frac{df}{dx_2} = -x_1 - x_3, \quad \frac{df}{dx_3} = -x_2 - x_1$$

$$\Rightarrow -x_2 - x_3 - (-x_2 - x_1) = 0 \; \text{---(i)}$$

$$-x_1 - x_3 - (-x_2 - x_1) = 0 \; \text{---(ii)}$$

$\Rightarrow$ from (i): $-\cancel{x_2} - x_3 + \cancel{x_2} + x_1 = 0$

$$\Rightarrow x_1 = x_3$$

(ii): $-\cancel{x_1} - x_3 + x_2 + \cancel{x_1} = 0$

$$\Rightarrow x_2 = x_3$$

from $h = x_1 + x_2 + x_3 - 3 = 0$

$\Rightarrow \quad 3x_3 = 3$

$x_3 = 1$

$\Rightarrow x_1 = x_2 = x_3 = 1$

Lagrange multipliers

$$L(x_1, x_2, x_3) = -(x_1x_2 + x_2x_3 + x_1x_3) + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{dL}{dx} = \begin{bmatrix} -x_2 - x_3 + \lambda \\ -x_1 - x_3 + \lambda \\ -x_2 - x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{dL}{d\lambda} = x_1 + x_2 + x_3 - 3 = 0$$

$\Rightarrow \quad -x_2 - x_3 + \lambda = 0$

$\Rightarrow \lambda = x_2 + x_3$

$-x_1 - x_3 + \lambda = 0$

$\lambda = x_1 + x_3$

$x_1 + \cancel{x_3} = x_2 + \cancel{x_3}$

$\Rightarrow x_1 = x_2$

$\lambda = x_1 + x_2$

$\cancel{x_2} + x_3 = x_1 + \cancel{x_2}$

$\Rightarrow x_1 = x_3$

from $\frac{dL}{d\lambda} = 0$

$3x_1 = 3$

$\Rightarrow x_1 = 1$

$\therefore x_1 = x_2 = x_3 = 1$

4.) $\max_{x_1, x_2} \quad f = 2x_1 + bx_2$

s.t $\quad g_1 = x_1^2 + x_2^2 - 5 \le 0$

$\quad g_2 = x_1 - x_2 - 2 \le 0$

min $\quad f = -2x_1 - bx_2$

s.t $\quad g_1 = x_1^2 + x_2^2 - 5 \le 0$

$\quad g_2 = x_1 - x_2 - 2 \le 0$

If optimum is located at $x_1 = 1,\ x_2 = 2$

$g_1 = 1^2 + 2^2 - 5 = 0$ ... (active)

$g_2 = 1 - 2 - 2 = -3$ ... (inactive)

$n = 2$ $\quad$ dof $= n - m$

$m = 1$ $\quad = 2 - 1$

$\quad = 1$

$d = x_2$

$s = x_1$

$\Rightarrow \dfrac{dg}{ds} = \dfrac{dg}{dx_1} = 2x_1$

$$\frac{ds}{dd} = -\left(\frac{dg}{ds}\right)^{-1} \frac{dg}{dd} = -(2x_1)^{-1} \cdot (2x_2)$$

$$\frac{df}{dd} = \frac{df}{dx_2} = \frac{df}{dx_2} + \frac{df}{dx_1}\frac{dx_1}{dx_2} = 0$$

$$\frac{df}{dx_1} = -2, \quad \frac{df}{dx_2} = -b$$

$$\Rightarrow -b + (-2) \cdot (-2x_1)^{-1} (2x_2) = 0$$

$$\Rightarrow b = \frac{-\cancel{2}(2 \cdot x_2)}{-\cancel{2} \cdot x_1} = \frac{2 \cdot 2}{1} = 4$$