Design Optimization

1.) Show that the stationary points (zero gradients) of the function, f is a saddle. Find the directions of downslopes away from the saddle.

A:

$$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$g = \nabla.f = \begin{bmatrix} \partial f_{x_1} \\ \partial f_{x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix}$$

Since its a stationary points

$$\nabla f = 0$$

$$\Rightarrow 4x_1 - 4x_2 = 0$$

$$\Rightarrow \cancel{4}x_1 = \cancel{4}x_2$$

$$x_1 = x_2$$

$$-4x_1 + 3x_2 + 1 = 0$$

$$-4x_2 + 3x_2 = -1$$

$$-x_2 = -1$$

$$x_2 = 1$$

$$\Rightarrow [x_1, x_2] = [1, 1]$$

To prove stationary point is a saddle

(i) use eigenvalues

(ii) use principal minors

$$H = \nabla.g = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

(i) using eigenvalues

$$\det\left[\begin{pmatrix} 4 & -4 \\ -4 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] = \lambda^2 - 7\lambda - 4 \qquad \therefore \lambda_1 = \frac{7 + \sqrt{65}}{2}$$

$$\lambda_2 = \frac{7 - \sqrt{65}}{2}$$

2-2

Due to having both positive & negative eigenvalues, we can conclude that the Hessian is negative.

(ii) using principal minors

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}$$

$$D_2 = \begin{vmatrix} 4 & -4 \\ -4 & 3 \end{vmatrix} = 12 - 16 = -4$$

$$D_1 = |4| = 4$$

Since $D_1 > 0$ & $D_2 < 0$, stationary point is a saddle point.

Finding the direction of downslopes away from the saddle using the differential quadratic form

$$\Rightarrow \frac{1}{2}\left(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right) = \frac{1}{2}(x - x_0)^T H_{x_0}(x - x_0)$$

$$= \frac{1}{2}\left(4x_1^2 - 8x_1x_2 + 3x_2^2\right)$$

$$= \frac{1}{2}\left(4x_1^2 - 6x_1x_2 - 2x_1x_2 + 3x_2^2\right) = \frac{1}{2}\left[2x_1(2x_1 - 3x_2) - x_2(2x_1 - 3x_2)\right]$$

$$= \frac{1}{2}\left[(2x_1 - x_2)(2x_1 - 3x_2)\right]$$

$\Rightarrow$ The direction of the downslopes would be provided by the intersection of two lines which are given by the equations below.

$$\Rightarrow x_1 - \frac{1}{2}x_2 < 0 \quad \& \quad x_1 - \frac{3}{2}x_2 < 0$$

2a.) $\min \sqrt{(x_1+1)^2 + x_2^2 + (x_3-1)^2}$

$= \min (x_1+1)^2 + x_2^2 + (x_3-1)^2$

s.t $x_1 + 2x_2 + 3x_3 = 1$

$x_1 = 1 - 2x_2 + 3x_3$

$\min (1-2x_2+3x_3)^2 + x_2^2 + (x_3-1)^2$

$$g = \nabla f = \begin{bmatrix} \partial f_{x_2} \\ \partial f_{x_3} \end{bmatrix} = \begin{bmatrix} 2(1-2x_2+3x_3).-2 + 2x_2 \\ 2(1-2x_2+3x_3).3 + 2(x_3-1) \end{bmatrix}$$

$$= \begin{bmatrix} -4(1-2x_2+3x_3) + 2x_2 \\ 6.(1-2x_2+3x_3) + 2x_3 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 + 8x_2 - 12x_3 + 2x_2 \\ 6 - 12x_2 + 18x_3 + 2x_3 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} 10x_2 - 12x_3 - 4 \\ -12x_2 + 20x_3 + 4 \end{bmatrix}$$

Find stationary points

$\Rightarrow \nabla f = g = 0$

$\Rightarrow 10x_2 - 12x_3 = 4 \quad \cdots (i)$

$-12x_2 + 20x_3 = -4 \cdots (ii)$

$10x_2 = 4 + 12x_3$

$x_2 = \dfrac{4 + 12x_3}{10}$

$-12\left(\dfrac{4 + 12x_3}{10}\right) + 20x_3 = -4$

$\Rightarrow \dfrac{-48 - 144x_3}{10} + 20x_3 = -4$

$(-1)^1 = -1$

$(-1)^2 = 1$

$\Rightarrow -4.8 - 14.4x_3 + 20x_3 = -4$

$5.6x_3 = 0.8$

$x_3 = \frac{0.8}{5.6} = \frac{1}{7}$

$x_2 = \frac{4 + 12(1/7)}{10}$

$= \frac{4}{7}$

$x_1 = 1 - 2\left(\frac{4}{7}\right) + 3\left(\frac{1}{7}\right)$

$= 1 - \frac{8}{7} + \frac{3}{7} = \frac{2}{7}$

$s.p = (2/7, 4/7, 1/7)$

$H = \nabla g = \begin{bmatrix} 10 & -12 \\ -12 & 20 \end{bmatrix}$

$D_1 = |10| = 10$

$D_2 = \begin{vmatrix} 10 & -12 \\ -12 & 20 \end{vmatrix} = 56$

eigenvalues: $(\lambda - 28)(\lambda - 2) = 0$

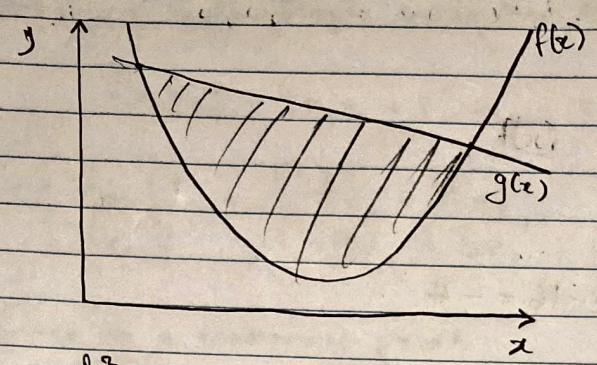
$\Rightarrow \lambda = 28, \lambda = 2$

$\therefore$ H is p-d thus this is a convex problem

3.) Let $f(x)$ & $g(x)$ be two convex functions defined on the convex set $\mathcal{X}$

Ans:

(i)



in R^2 : $a_1x_1 + a_2x_2 = c$

Thus the set is convex on $\mathcal{X}$ of for every $x_1, x_2 \in \mathcal{X}$

$$f(\lambda x_2 + (1-\lambda)x_1) \leq \lambda f(x_2) + (1-\lambda) f(x_1)$$

where $0 \leq \lambda \leq 1$

Thus the graph of the intersection of the convex functions, would be convex as well.

If f_1, f_2 are convex sets then the set $af_1 + bf_2 = \{y \mid y = x_1 + x_2,\ x_1 \in S_1, x_2 \in S_2$ is convex$\}$

$$g(\lambda x_2 + (1-\lambda)x_1) \leq \lambda g(x_2) + (1-\lambda) g(x_1)$$

$n(x) = af(x) + bg(x)$

$\Rightarrow af(\lambda x_2 + (1-\lambda)x_1) + bg(\lambda x_2 + (1-\lambda)x_1)$

$= af\lambda x_2 + af(1-\lambda)x_1 + bg\lambda x_2 + bg(1-\lambda)x_1$

$= (af + bg)\lambda x_2 + (af + bg)(1-\lambda)x_1$

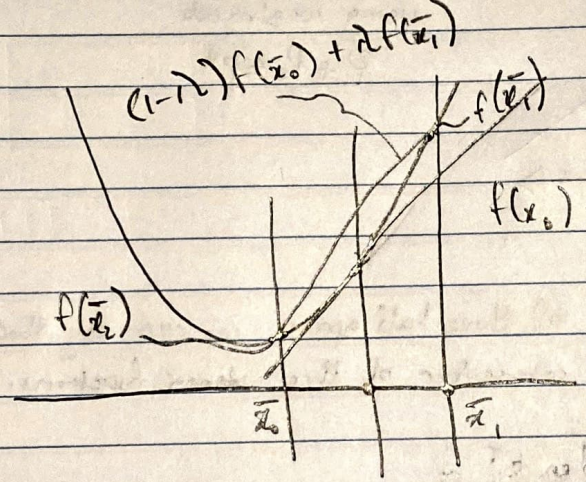
$\Rightarrow n(x)$ is convex as well.

(ii) $g(x) \Rightarrow \lambda g(x_2) + (1-\lambda) g(x_1)$

$\Rightarrow f(g(x)) = f(g(\lambda x_2 + (1-\lambda)x_1)) \leq \lambda f(g(x_2)) + f((1-\lambda) g(x_1))$

which is only valid if $g(x)$ is convex and is already contained on $f(x)$.

4.) $f(x_1) \geq f(x_0) + g_{x_0}^T (x_1 - x_0) \Rightarrow$ convex function



$f(x_0) = g_{x_0}^T (\bar{x}_1 - x_0)$

$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

$f(x_0) + g_{x_0}^T (x - x_0)$

$x_0 = (1-\lambda)\bar{x}_0 + \lambda \bar{x}_1$

5.) Optimization problem formulation

(i) $$\min_{P} \max_{k} \left\{ \left(\sum_{i=1}^{m} a_{ki} P_i - I_k \right)^2 \right\}$$

s.t. $0 \leq P_i \leq P_{max}$

(ii) To check if problem is convex, find the gradient then the Hessian.

$$D = (a^T p - I_k)^2$$

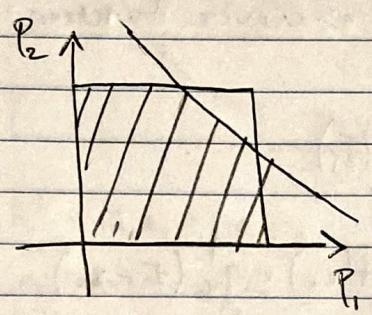
$$\frac{dD}{dP} = g(P) = 2(a_k^T p - I) \cdot a_k$$

$$H = \frac{d^2 D}{dp^2} = 2 \cdot a_k \cdot a_k^T$$

$$H = \sum_{k=1}^{m} 2 a_k a_k^T \quad \text{since } d^T H d \geq 0$$

Since we know that everywhere $H \geq 0$, we can conclude that it is positive semodefinite and thus $D(p)$ is a convex function.

(iii)



using constraint
$P_1 + P_2 \leq p^*$

Since this is an intersection of three half spaces we can say that the feasible domain is convex due to the intersection of three convex functions.

$$d^T H d = \sum_k 2 \cdot d^T a_k a_k^T d$$

When any of the n lamps are used, a_n spans the entire feasible domain and thus does not exist a point where $d \perp a_k$ thus $d^T \cdot a_k \neq 0$.

$\therefore H > 0$ for $1 \leq k \leq n$, thus H is always positive definite. Thus the problem is strictly convex and has a unique solution.

(iv) The feasible domain would be the same as in part (iii) but due to a_k not spanning the entire domain, d could be perpendicular to a_k. Thus $d^T \cdot a_k = 0$ & $H \geq 0$. Thus the problem would be convex and would not have a unique solution since the Hessian is positive semi-definite.