Version Spaces + Candidate Elimination

Lecture Outline:

- Quick Review of Concept Learning and General-to-Specific Ordering
- Version Spaces
- The Candidate Elimination Algorithm
- Inductive Bias

Reading:

Chapter 2 of Mitchell

Version Spaces

- One limitation of the **FIND-S** algorithm is that it outputs just one hypothesis consistent with the training data – there might be many.

  To overcome this, introduce notion of **version space** and algorithms to compute it.

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- Note difference between definitions of *consistent* and *satisfies*:
  - an example x *satisfies* hypothesis h when $h(x) = 1$, regardless of whether x is $+$ve or $-$ve example of target concept
  - an example x is *consistent* with hypothesis h iff $h(x) = c(x)$

The LIST-THEN-ELIMINATE Algorithm

- Can represent version space by listing all members.

- Leads to **List-Then-Eliminate** concept learning algorithm:

> 1. $VersionSpace \leftarrow$ a list containing every hypothesis in H
> 2. For each training example, $\langle x, c(x) \rangle$
>
>   remove from $VersionSpace$ any hypothesis h for which $h(x) \neq c(x)$
> 3. Output the list of hypotheses in $VersionSpace$

- **List-Then-Eliminate** works in principle, so long as version space is finite.

- However, since it requires exhaustive enumeration of all hypotheses in practice it is not feasible.

- Is there a more compact way to represent version spaces?

The CANDIDATE-ELIMINATION Algorithm

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  - represents version space by its most general and most specific members

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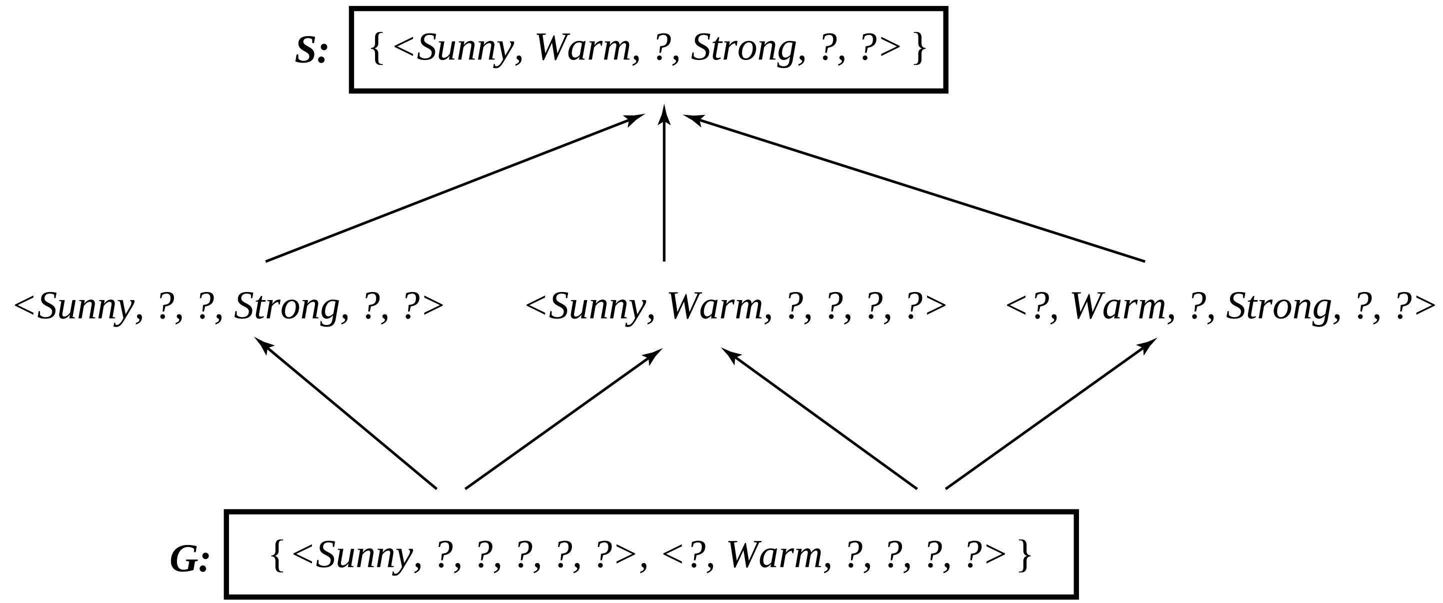

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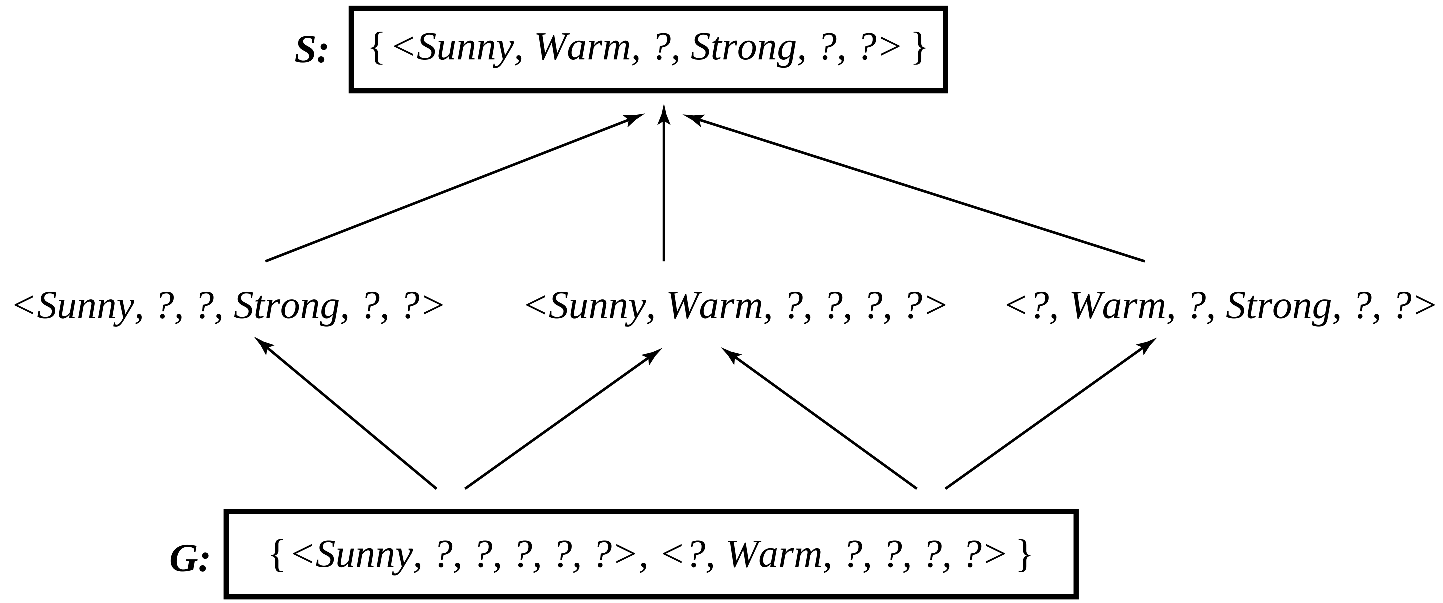

- The **Candidate-Elimination** algorithm represents the version space by recording only the most general members (G) and its most specific members (S)
  - other intermediate members in general-to-specific ordering can be generated as needed

The CANDIDATE-ELIMINATION Algorithm (cont)

- The **General boundary**, G, of version space $VS_{H,D}$ is the set of its maximally general members

- The **Specific boundary**, S, of version space $VS_{H,D}$ is the set of its maximally specific members

The CANDIDATE-ELIMINATION Algorithm (cont)

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- **Version Space Representation Theorem**
  Every member of the version space lies between these boundaries

$$VS_{H,D} = \{h \in H | (\exists s \in S)(\exists g \in G)(g \geq_g h \geq_g s)\}$$

  where $x \geq_g y$ means x is more general or equal to y
  (see Mitchell, p. 32, for proof)

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- Intuitively, **Candidate-Elimination** algorithm proceeds by
  - initialising G and S to the maximally general and maximally specific hypotheses in H
  - considering each training example in turn and
    - using positive examples to drive the maximally specific boundary up
    - using negative examples to drive the maximally general boundary down

The CANDIDATE-ELIMINATION Algorithm (cont)

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- If d is a positive example
  - Remove from G any hypothesis inconsistent with d

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For each training example d, do

- If d is a positive example
  - Remove from G any hypothesis inconsistent with d
  - For each hypothesis s in S that is not consistent with d

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  - Remove from G any hypothesis inconsistent with d
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    - Add to S all minimal generalizations h of s such that

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      1. h is consistent with d, and
      2. some member of G is more general than h

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      1. h is consistent with d, and
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    - Remove from S any hypothesis that is more general than another hypothesis in S
- If d is a negative example
  - Remove from S any hypothesis inconsistent with d
  - For each hypothesis g in G that is not consistent with d
    - Remove g from G
    - Add to G all minimal specializations h of g such that
      1. h is consistent with d, and
      2. some member of S is more specific than h
    - Remove from G any hypothesis that is less general than another hypothesis in G

The CANDIDATE-ELIMINATION Algorithm: Example

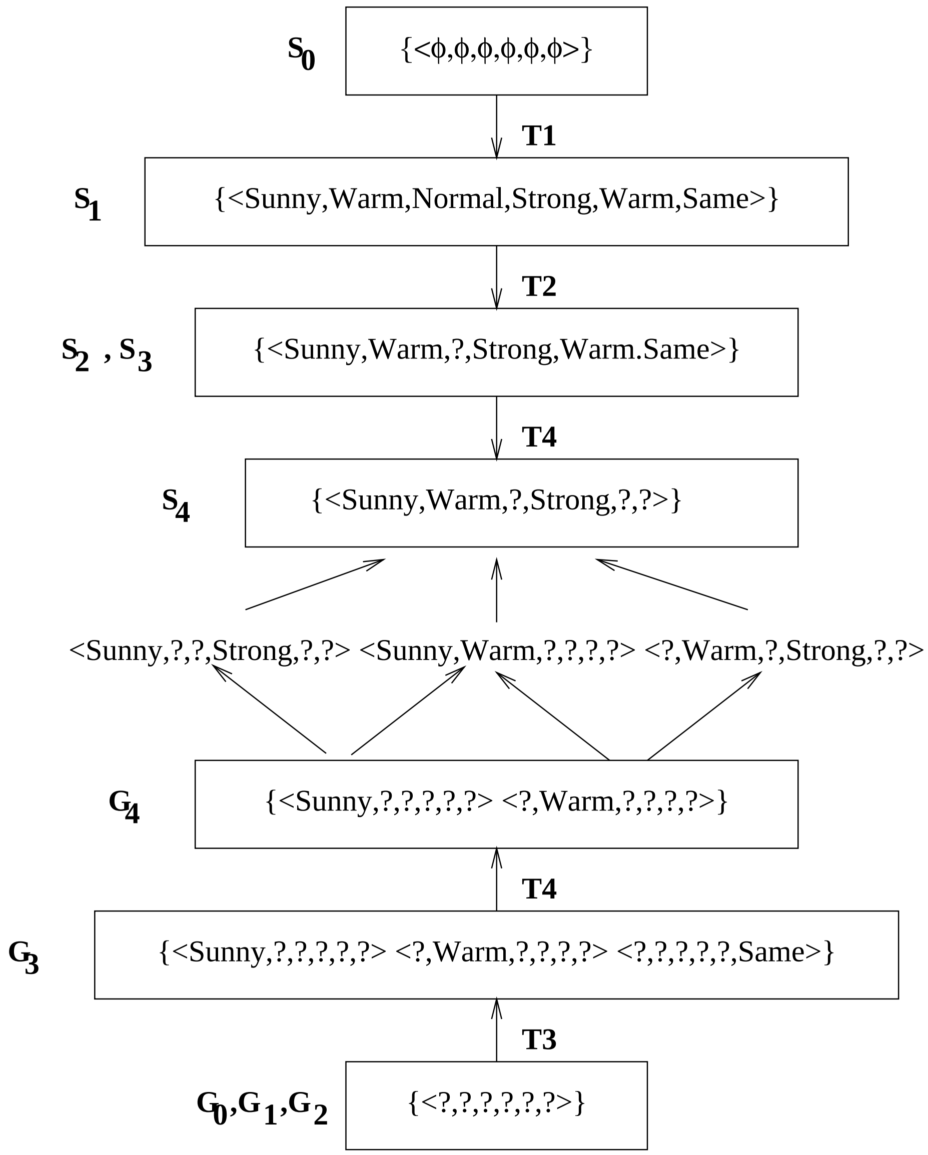
Training Examples:

T1: $\langle Sunny, Warm, Normal, Strong, Warm, Same \rangle, Yes$

T2: $\langle Sunny, Warm, High, Strong, Warm, Same \rangle, Yes$

T3: $\langle Rainy, Cold, High, Strong, Warm, Change \rangle, No$

T4: $\langle Sunny, Warm, High, Strong, Cool, Change \rangle, Yes$



The CANDIDATE-ELIMINATION Algorithm: Remarks

- Version space learned by **Candidate-Elimination** algorithm will converge towards correct hypothesis provided:
  - no errors in training examples
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- If algorithm can request next training example (e.g. from teacher) can increase speed of convergence by requesting examples that split the version space
  - E.g. T5: $\langle Sunny, Warm, Normal, Light, Warm, Same \rangle$ satisfies 3 hypotheses in previous example
    - If T5 positive, S generalised, 3 hypotheses eliminated
    - If T5 negative, G specialised, 3 hypotheses eliminated
  - Optimal query strategy is to request examples that exactly split version space – converge in $\lceil log_2 |VS| \rceil$ steps. However, this is not always possible.

The CANDIDATE-ELIMINATION Algorithm: Remarks (cont)

- When using (i.e **not** training) a classifier that has not completely converged, new examples may be
  1. classed as positive by all $h \in VS$
  2. classed as negative by all $h \in VS$
  3. classed as positive by some, and negative by other, $h \in VS$

  Cases 1 and 2 are unproblematic. In case 3. may want to consider proportion of positive vs. negative classifications (but then *a priori* probabilities of hypotheses are relevant)

Inductive Bias

- As noted, version space learned by **Candidate-Elimination** algorithm will converge towards correct hypothesis provided:
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- Consider the training data

Example	Sky	Temp	Humid	Wind	Water	Forecast	EnjoySport
1	Sunny	Warm	Normal	Strong	Warm	Same	Yes
2	Cloudy	Warm	Normal	Strong	Warm	Same	Yes
3	Rainy	Warm	Normal	Strong	Warm	Same	No

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- No hypotheses consistent with 3 examples.

  Most specific hypothesis consistent with Ex 1 and 2 *and representable in H*:

$$\langle ?, Warm, Normal, Strong, Warm, Same \rangle$$

But this is inconsistent with Ex 3.

Inductive Bias (cont)

- Need more expressive hypothesis representation language.

  E.g. allow disjunctive or negative attribute values:

$$
\begin{aligned}
Sky &= Sunny \vee Cloudy \\
Sky &\neq Rainy
\end{aligned}
$$

An Unbiased Learner

- What about ensuring **every** concept can be represented in H?
  - Since concepts are subsets of instance space X, want H to be able to represent any set in power set of X
    - $*$ for *EnjoySport* there were 96 possible instances
      so, power set contains $2^{96} \approx 10^{28}$ possible target concepts
    - $*$ recall biased conjunctive hypothesis space can represent only 973 of these

- Can do this by allowing hypotheses that are arbitrary conjunctions, disjunctions and negations of our earlier hypotheses
  - New problem: concept learning algorithm cannot generalise beyond observed examples!
    - $*$ S boundary $=$ disjunction of positive examples – exactly covers observed positive examples
    - $*$ G boundary $=$ negation of disjunction of negative examples – exactly rules out observed negative examples

An Unbiased Learner

- Capacity of **Candidate-Elimination** to generalise lies in its implicit assumption of **bias** – that target concept can be represented as a conjunction of attribute values

- Fundamental property of inductive inference:

  a learner that makes no a priori assumptions regarding the identity of the target concept has no rational basis for classifying any unseen instances

  I.e. **bias-free learning is futile**

Inductive Bias, More Formally

- Since all inductive learning involves bias, useful to characterise learning approaches by the type of bias they employ

- Consider
  - concept learning algorithm L
  - instances X, target concept c
  - training examples $D_c = \{\langle x, c(x) \rangle\}$
  - let $L(x_i, D_c)$ denote the classification, positive or negative, assigned to the instance x_i by L after training on data D_c.

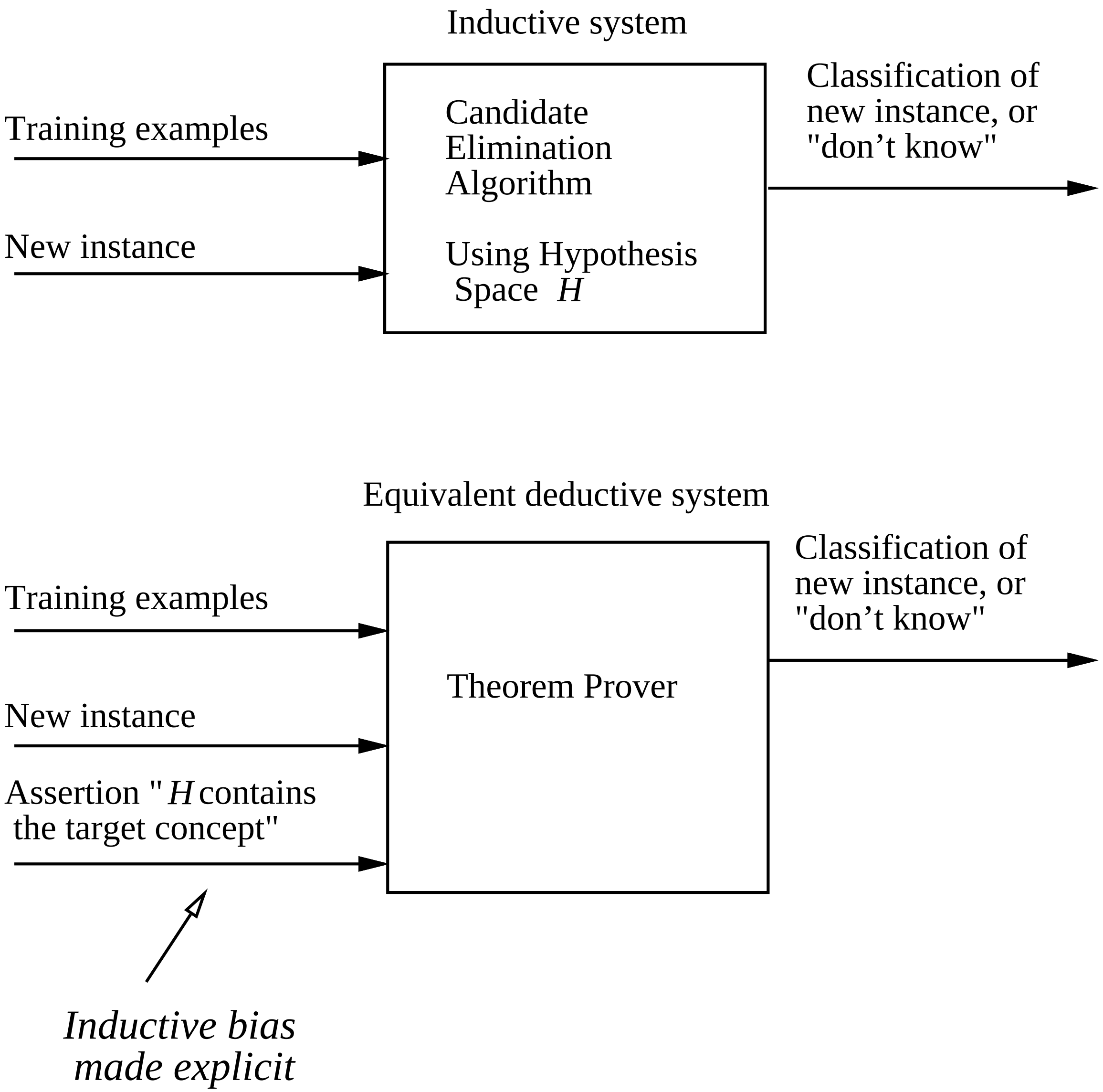
Definition:

The **inductive bias** of L is any minimal set of assertions B such that for any target concept c and corresponding training examples D_c

$$(\forall x_i \in X)[(B \wedge D_c \wedge x_i) \vdash L(x_i, D_c)]$$

where $A \vdash B$ means A logically entails B

Modelling Inductive Systems by Deductive Systems



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- In fact, all inductive learning supposes some *a priori* assumptions about the nature of the target concept, or else there is no basis for generalisation beyond observed examples: **bias-free learning is futile**.