

STAT 4830: Numerical optimization for data science and ML

Lecture 4: How to compute gradients in PyTorch

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Manual Gradient Computation

Consider computing this gradient by hand:

$$f(w) = \frac{1}{2} \| \tanh(W_2 \text{ReLU}(W_1 x + b_1) + b_2) - y \|^2$$

annoying

Automatic Differentiation

PyTorch provides:

```
# Define complex function
def f(x, W1, b1, W2, b2):
    h = torch.relu(W1 @ x + b1)
    return 0.5 * torch.sum(
        (torch.tanh(W2 @ h + b2) - y)**2
    )

# Get gradient automatically
f.backward()
```

Key benefits:

1. Automatic gradient computation
2. Handles any differentiable function
3. Memory efficient implementation
4. Scales to large problems

Three Key Ideas

1. **Computational Graph**
2. **Reverse-Mode Differentiation**
3. **Memory-Efficient Implementation**

Outline

1. Computing Gradients

Function → Graph → Gradient

2. Gradient Descent

Gradient → Update → Repeat

3. Neural Networks

Features → Layers → Loss

A Simple Example: Polynomial Function

Let's start with a one-dimensional function:

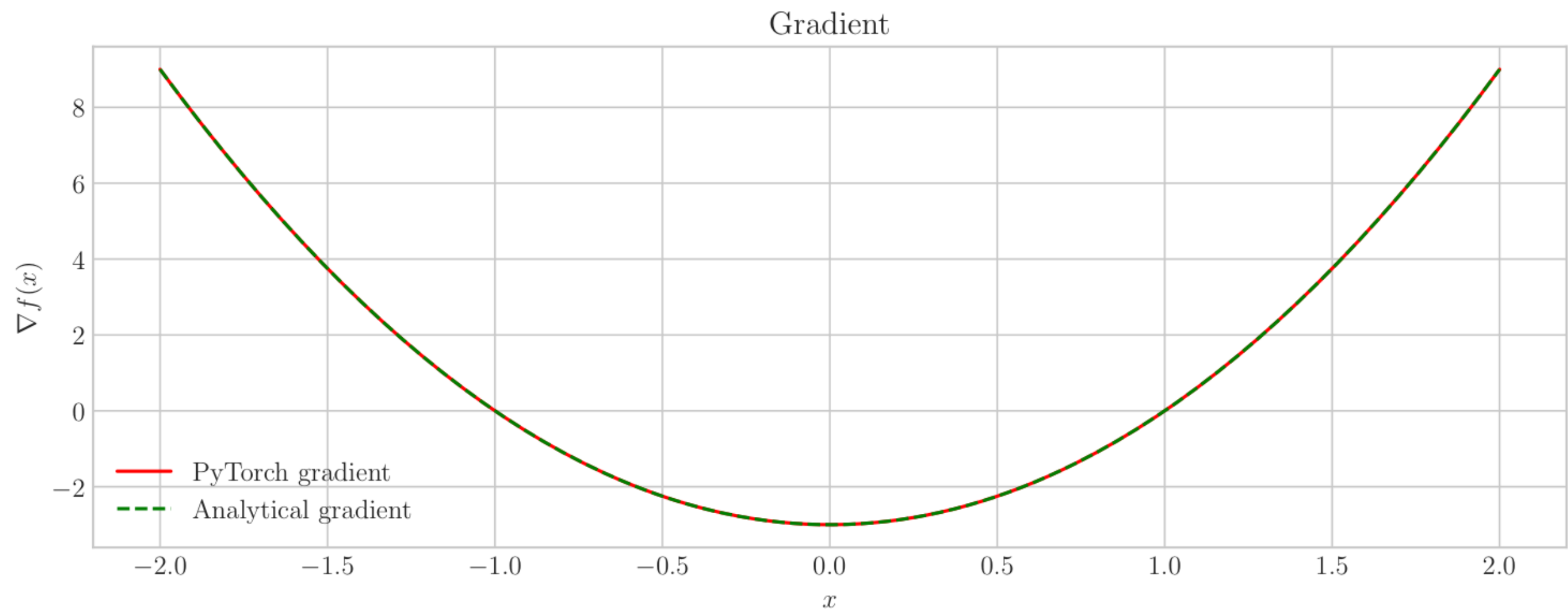
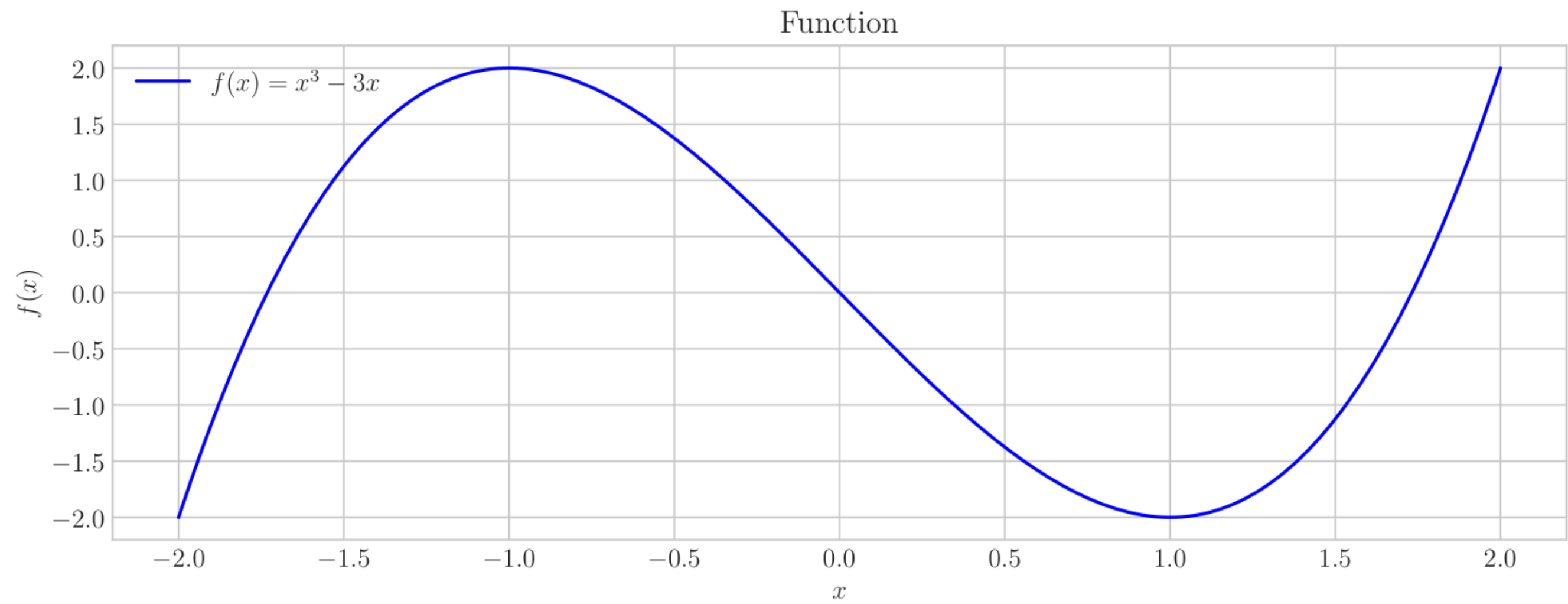
$$f(x) = x^3 - 3x$$

Manual gradient computation:

$$\frac{d}{dx} f(x) = 3x^2 - 3$$

PyTorch automates this:

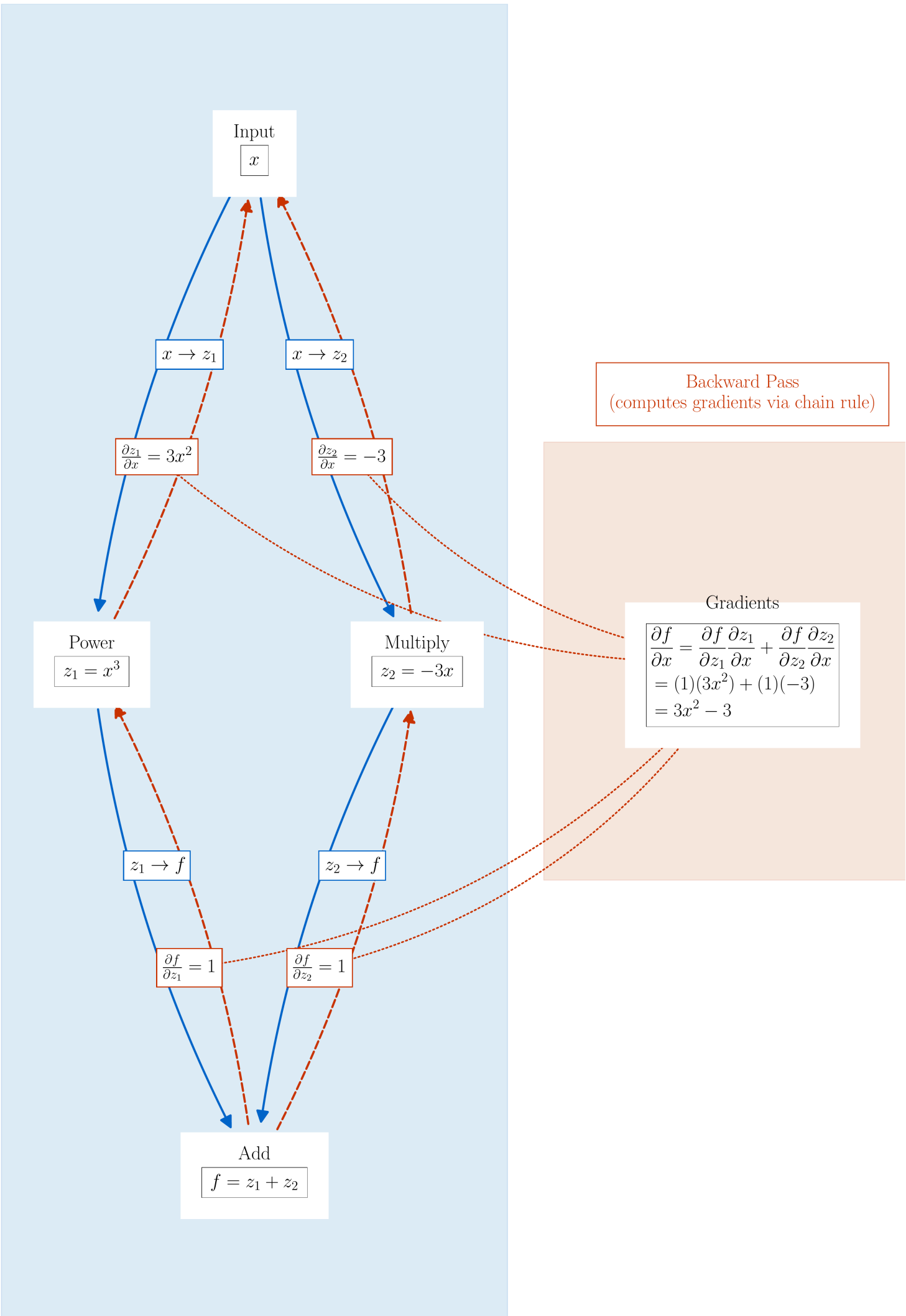
```
x = torch.tensor([1.0], requires_grad=True) # Gradient Tracking
y = x**3 - 3*x # Forward Pass
y.backward() # Backward Pass
print(f"f'(1) = {x.grad}") # Gradient Access
```



How does PyTorch do this?

- **Forward pass:** When you evaluate a function, PyTorch computes a *computational graph* that records all operations like addition, multiplication, powers, etc.
- **Backward pass:** PyTorch traverses the graph in reverse order to compute the gradient, using what is essentially an efficient implementation of the chain rule.

The graph



Building the Computational Graph

Each node in the graph:

- Stores output value from forward pass
- Contains function for local gradients
- Maintains references to inputs

For $f(x) = x^3 - 3x$, we build:

1. Input node storing x
2. Power node computing $z_1 = x^3$
3. Multiply node computing $z_2 = -3x$
4. Add node forming $f = z_1 + z_2$

Computing Gradients: The Process

Starting State:

- Initialize $\frac{\partial f}{\partial f} = 1$ at output
- All other gradients start at 0

Algorithm:

1. Process nodes in reverse order
2. Compute local gradients
3. Multiply by incoming gradient
4. Add to input gradients

backward(): Step by Step

1. Output Node ($f = z_1 + z_2$):

- $\frac{\partial f}{\partial f} = 1$
- $\frac{\partial f}{\partial z_1} = 1, \frac{\partial f}{\partial z_2} = 1$
- Propagate to both input nodes

2. Power Node ($z_1 = x^3$):

- Incoming gradient: 1
- Local gradient: $\frac{\partial z_1}{\partial x} = 3x^2$
- Contribute: $\frac{\partial f}{\partial x} += (1)3x^2$

3. Multiply Node ($z_2 = -3x$):

- Incoming gradient: 1
- Local gradient: $\frac{\partial z_2}{\partial x} = -3$
- Contribute: $\frac{\partial f}{\partial x} += (1)(-3)$

4. Input Node (x):

- Accumulates from both paths
- (-3) from multiply node
- $(3x^2)$ from power node
- Final gradient: $\frac{\partial f}{\partial x} = 3x^2 - 3$

Two Implementation Approaches

1. Using `backward()`

```
# Create graph
x.requires_grad = True
z = g(x)
y = h(z)

# Compute gradients
y.backward()
grad = x.grad # Stored in tensor
```

Best for:

- Training loops
- Multiple gradients
- Memory efficiency

2. Using `autograd.grad()`

```
# Create graph
x.requires_grad = True
z = g(x)
y = h(z)

# Direct computation
grad = torch.autograd.grad(y, x)[0]
```

Best for:

- One-off gradients
- Direct access
- Higher derivatives

Beyond 1d: Least Squares

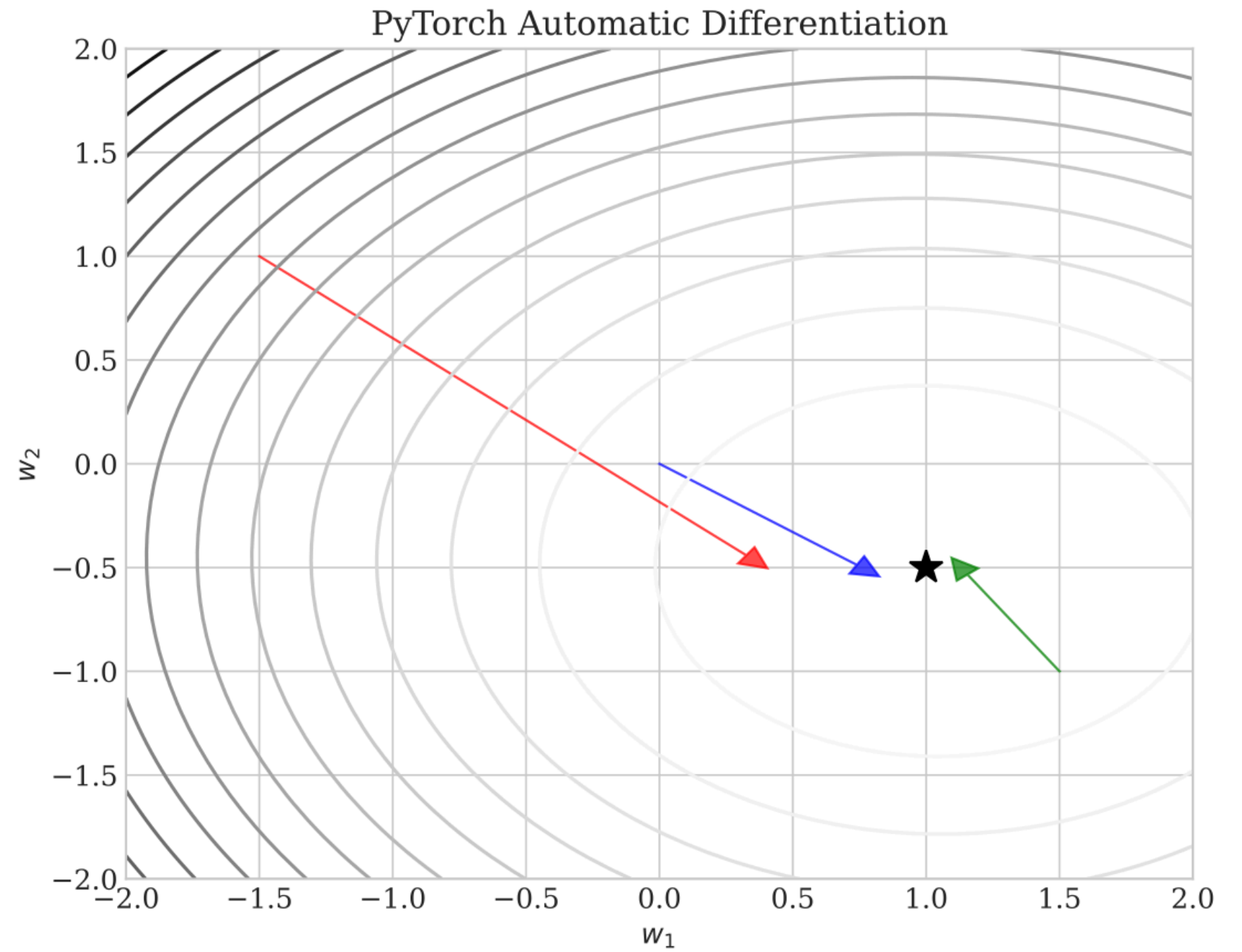
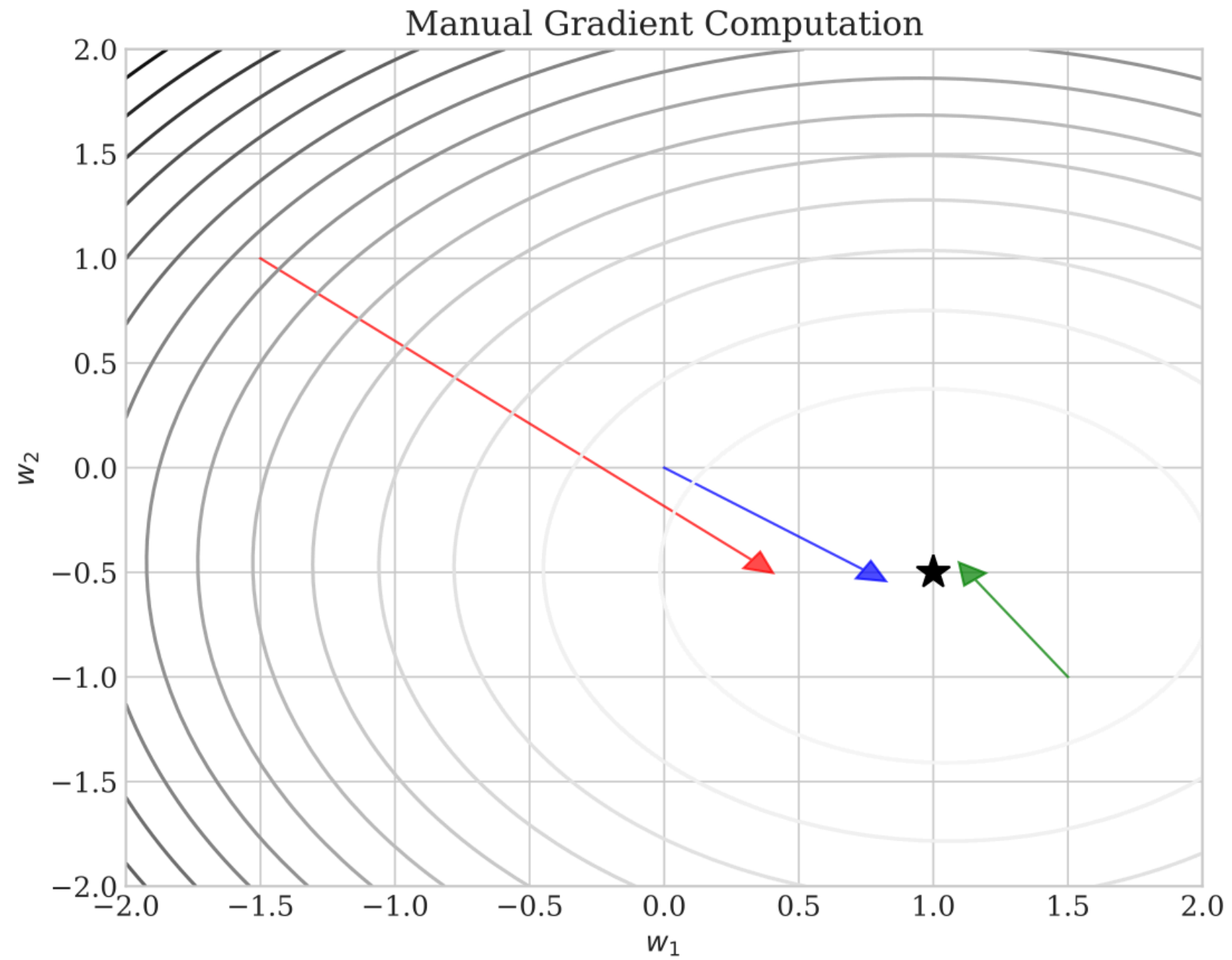
Manual gradient:

$$\nabla f = X^{\top} (Xw - y)$$

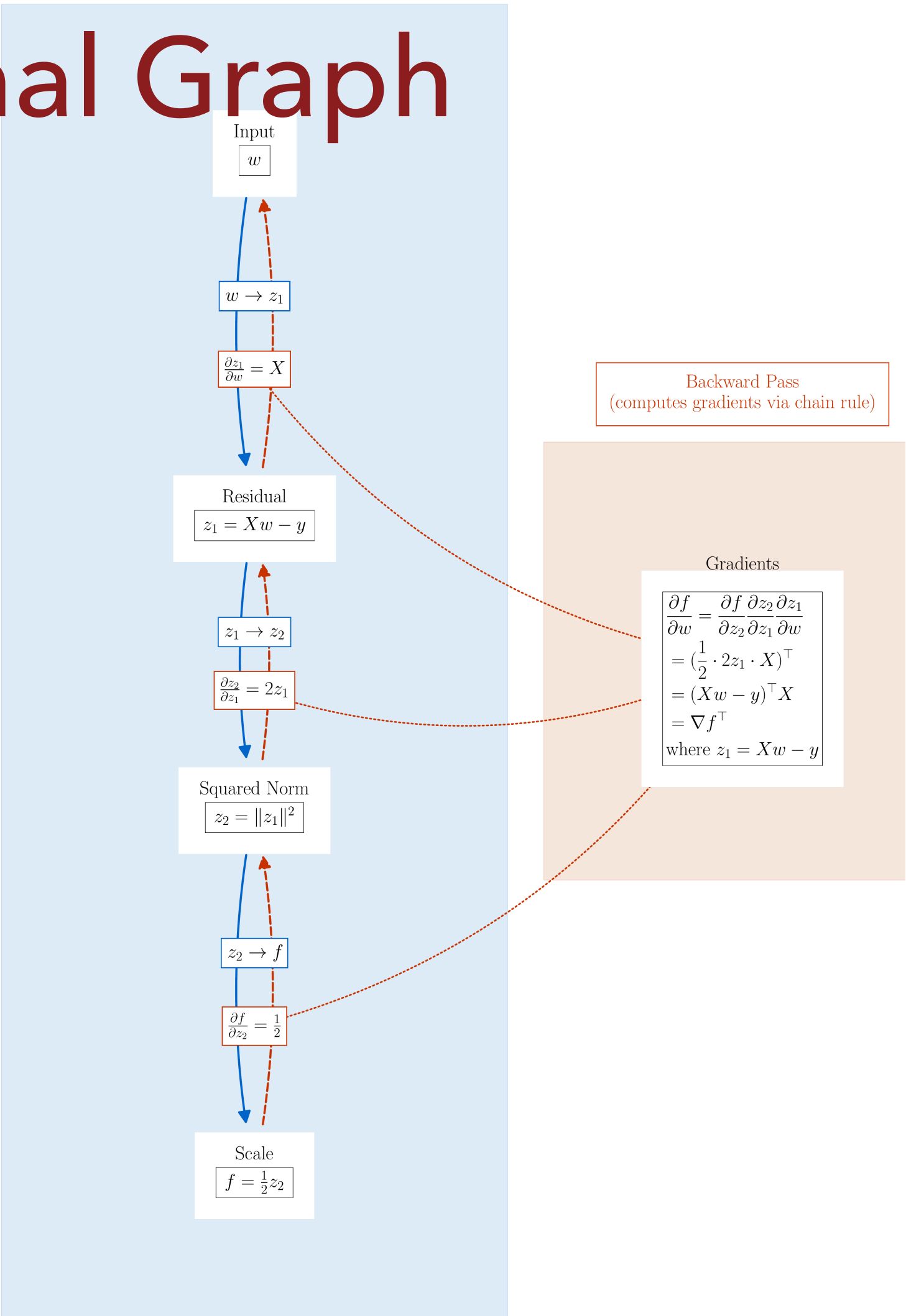
PyTorch gradient:

```
pred = X @ w
loss = 0.5*((pred - y)**2).sum()
loss.backward()
grad = w.grad
```

Agreement between manual and PyTorch



Computational Graph



Building the Least Squares Graph

For $f(w) = \frac{1}{2} \|Xw - y\|^2$, we build:

1. Input nodes storing \mathbf{w}
2. Residual node computing $\mathbf{z}_1 = \mathbf{X}\mathbf{w} - \mathbf{y}$
3. Square norm node computing $z_2 = \|\mathbf{z}_1\|^2$
4. Scale node forming $f = \frac{1}{2} z_2$

Computing Gradients: The Process

Subtlety: Total derivative vs :

Starting State:

- Initialize $\frac{\partial f}{\partial f} = 1$ at output
- All other gradients start at 0

Algorithm:

1. Process nodes in reverse order
2. Compute local gradients
3. Multiply by incoming total derivative
4. Add to input total derivative

Least Squares: backward() Step 1

Output Node ($f = \frac{1}{2} z_2$):

- Incoming gradient: $\frac{\partial f}{\partial f} = 1$ (scalar)
- Total derivative: $\frac{\partial f}{\partial z_2} = \frac{1}{2}$ (scalar)
- Propagate to z_2 node: $\frac{\partial f}{\partial z_2} = \frac{1}{2}$ (1×1 matrix)

Least Squares: backward() Step 2

Square Norm Node ($z_2 = \|\mathbf{z}_1\|^2$):

- Incoming total derivative: $\frac{\partial f}{\partial z_2} = \frac{1}{2}$ (1×1 matrix)
- Local total derivative: $\frac{\partial z_2}{\partial \mathbf{z}_1} = 2\mathbf{z}_1^\top$ ($1 \times n$ matrix)
- Propagate to \mathbf{z}_1 node: $\frac{\partial f}{\partial \mathbf{z}_1} = \frac{\partial f}{\partial z_2} \frac{\partial z_2}{\partial \mathbf{z}_1} = \mathbf{z}_1^\top$ ($1 \times n$ matrix)

Least Squares: backward() Step 3

Residual Node ($\mathbf{z}_1 = \mathbf{X}\mathbf{w} - \mathbf{y}$):

- Incoming total derivative: $\frac{\partial f}{\partial \mathbf{z}_1} = \mathbf{z}_1^\top$ ($1 \times n$ matrix)
- Local total derivative: $\frac{\partial \mathbf{z}_1}{\partial \mathbf{w}} = \mathbf{X}$ ($n \times p$ matrix)
- Total derivative to \mathbf{w} node: $\frac{\partial f}{\partial \mathbf{w}} = \mathbf{z}_1^\top \mathbf{X}$ ($1 \times p$ matrix)

Least Squares: Final Step

Input Node (\mathbf{w}):

- Total derivative: $\frac{\partial f}{\partial \mathbf{w}} = \mathbf{z}_1^\top \mathbf{X}$ ($1 \times p$ matrix)
- Convert to gradient: $\nabla f = \left(\frac{\partial f}{\partial \mathbf{w}}\right)^\top = \mathbf{X}^\top \mathbf{z}_1$ ($p \times 1$ matrix)

Final computation:

$$\nabla f = \left(\frac{\partial z_1}{\partial w} \frac{\partial z_2}{\partial z_1} \frac{\partial f}{\partial z_2} \right)^\top = \mathbf{X}^\top (\mathbf{X}\mathbf{w} - \mathbf{y})$$

Applying Gradient Descent

Minimize least squares loss:

$$f(w) = \frac{1}{2} \|Xw - y\|^2$$

Manual implementation:

```
def manual_gradient(X, y, w):  
    return X.T @ (X @ w - y)  
  
w = torch.zeros(p) # Initialize  
for step in range(max_iters):  
    grad = manual_gradient(X, y, w)  
    w = w - alpha * grad
```

PyTorch Implementation

Same algorithm, automatic gradients:

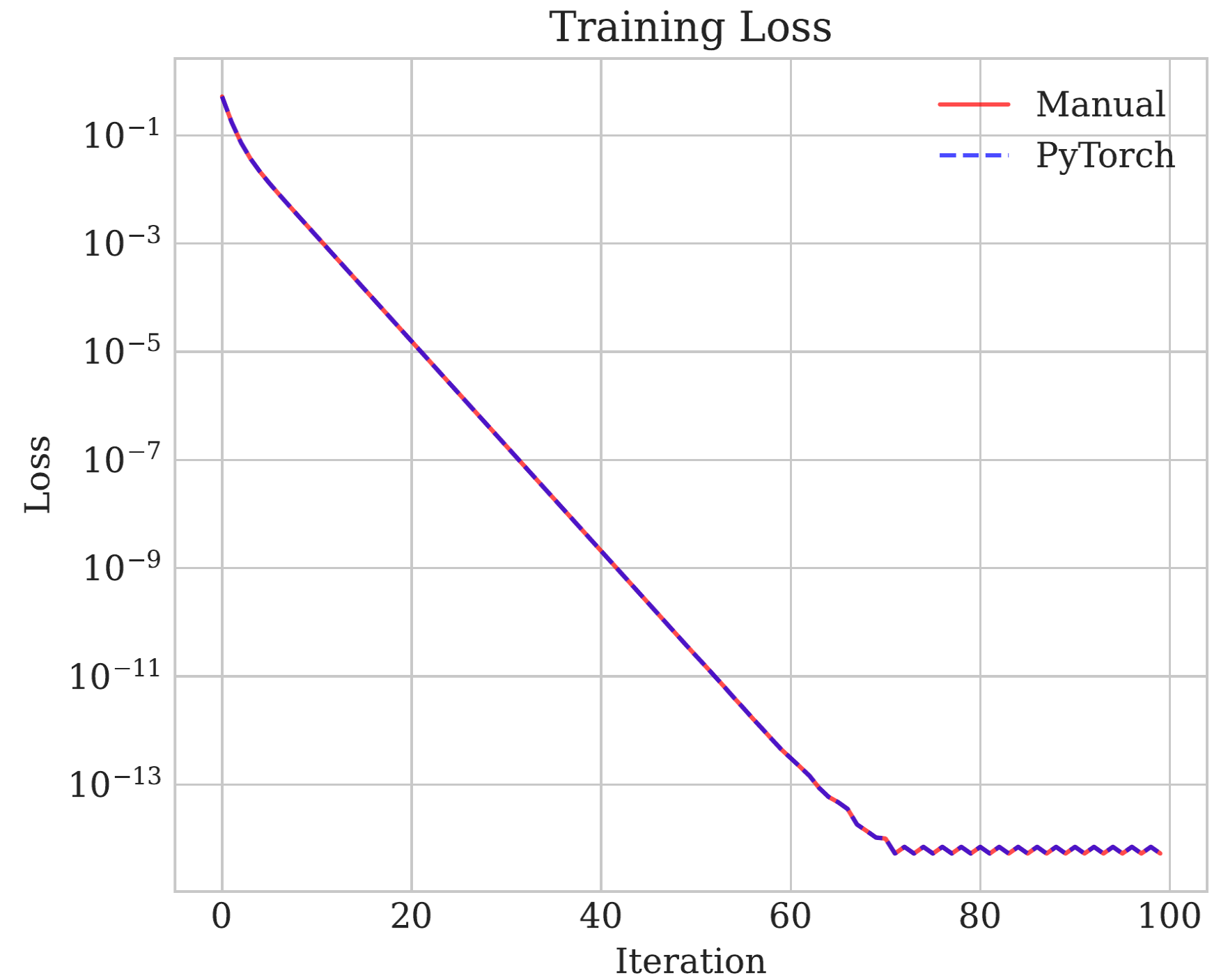
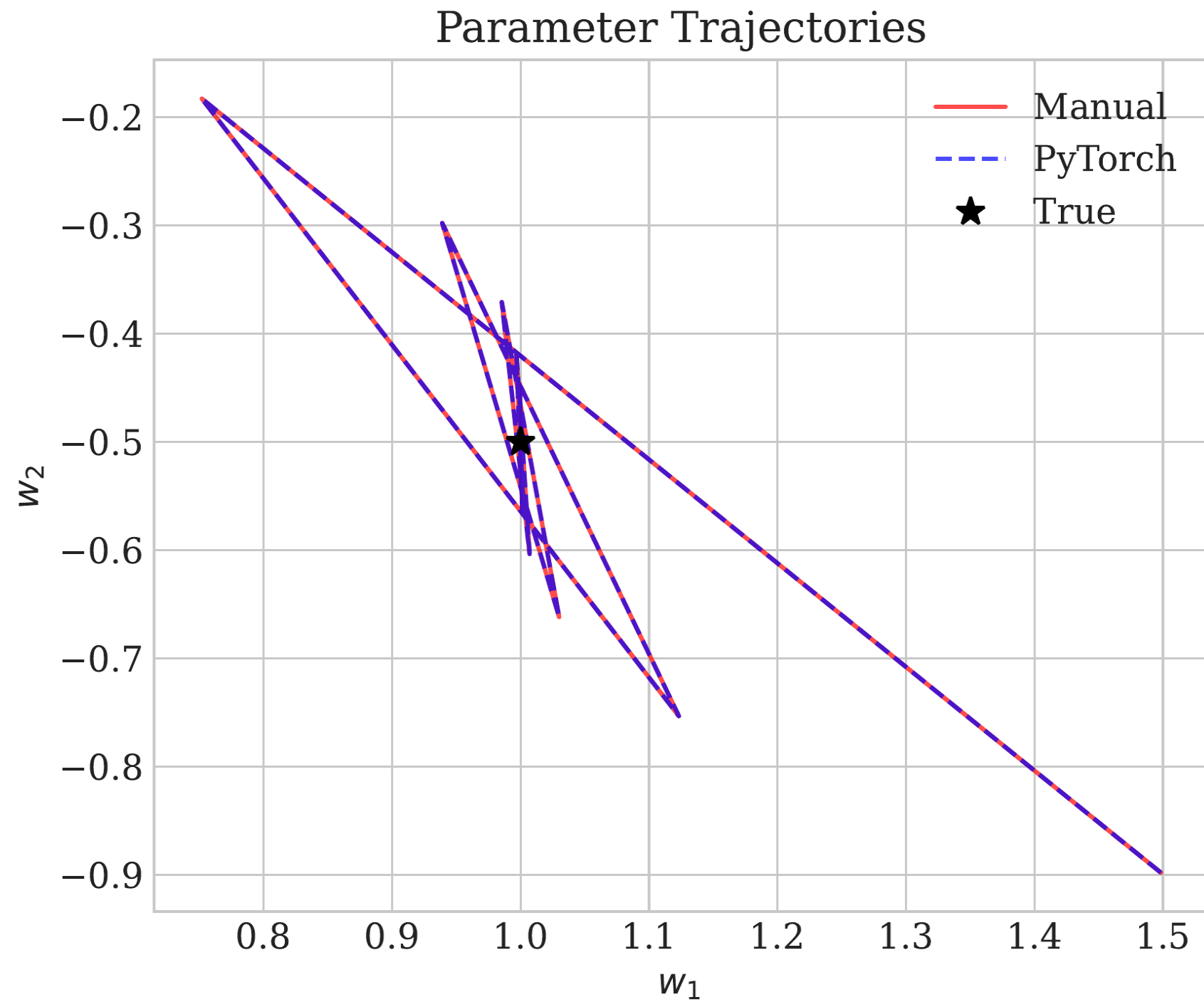
```
w = torch.zeros(p, requires_grad=True) # Initialize weights and require gradients

for step in range(max_iters):
    # Forward pass
    pred = X @ w
    loss = 0.5 * ((pred - y)**2).sum()

    # Backward pass
    loss.backward()

    # Update
    with torch.no_grad(): # Do not modify the computational graph
        w -= alpha * w.grad # update the weights
        w.grad.zero_() # reset the gradient to zero to avoid accumulation
```


Comparison of Approaches



From Linear to Neural Networks

Linear Model:

$$\mathbf{y} = \mathbf{X}\mathbf{w}$$

Neural Network:

$$\mathbf{h} = \tanh(\mathbf{W}_1\mathbf{x} + \mathbf{b}_1)$$

$$P(y = 1 \mid x) = \sigma(\mathbf{w}_2^\top \mathbf{h} + b_2)$$

Key differences:

- Multiple transformations
- Nonlinear activations
- Learnable features

Neural Network Architecture

Layer composition:

Input \rightarrow Linear₁ \rightarrow Tanh \rightarrow Linear₂ \rightarrow Sigmoid
 \mathbb{R}^d $\mathbb{R}^{h \times d}$ \mathbb{R}^h $\mathbb{R}^{1 \times h}$ $[0, 1]$

Dimensions:

- Input: $\mathbf{x} \in \mathbb{R}^d$
- Hidden: $\mathbf{h} \in \mathbb{R}^h$
- Output: $p \in [0, 1]$

Each layer adds:

- Linear transform
- Nonlinearity
- Learnable parameters

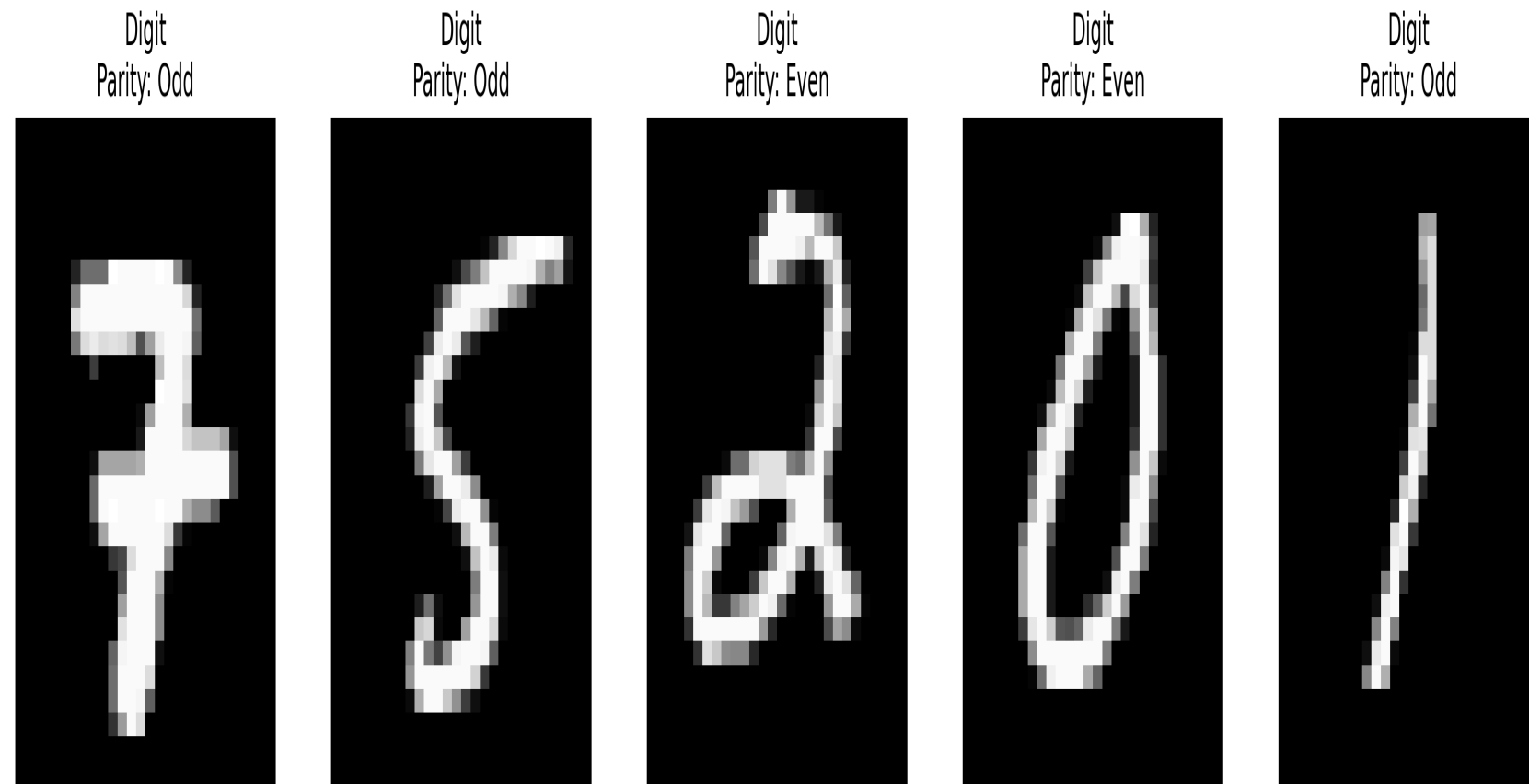
PyTorch Implementation

```
class BinaryClassifier(nn.Module):  
    def __init__(self, d=784, h=32):  
        super().__init__()  
        #  $\mathbb{R}^d \rightarrow \mathbb{R}^h$   
        self.linear1 = nn.Linear(d, h)  
        #  $\mathbb{R}^h \rightarrow \mathbb{R}$   
        self.linear2 = nn.Linear(h, 1)  
  
    def forward(self, x):  
        # Hidden features  
        h = torch.tanh(self.linear1(x))  
        # Probability output  
        return torch.sigmoid(  
            self.linear2(h)  
        )
```

Training Loop

```
def train_step(model, x, y, optimizer):  
    # 1. Forward: compute prediction and loss  
    pred = model(x)  
    loss = criterion(y_pred.squeeze(), y_train)  
  
    # 2. Backward: compute gradients  
    optimizer.zero_grad()  
    loss.backward()  
  
    # 3. Update: apply gradients  
    optimizer.step()  
  
    return loss.item()
```

MNIST Classification: The Task



Dataset:

- 60,000 training images
- 10,000 test images
- 28×28 pixels each
- Binary labels (odd/even)

Preprocessing:

```
transform = transforms.Compose([
    transforms.ToTensor(),
    transforms.Normalize(
        (0.1307,), (0.3081,)
    )
])

# Load data
train_dataset = datasets.MNIST(
    './data',
    train=True,
    transform=transform
)
```

Model Comparison: Architecture

Logistic Regression:

```
class Logistic(nn.Module):  
    def __init__(self):  
        super().__init__()  
        self.linear = nn.Linear(784, 1)  
  
    def forward(self, x):  
        # Single linear layer  
        return torch.sigmoid(  
            self.linear(x.view(-1, 784))  
        )
```

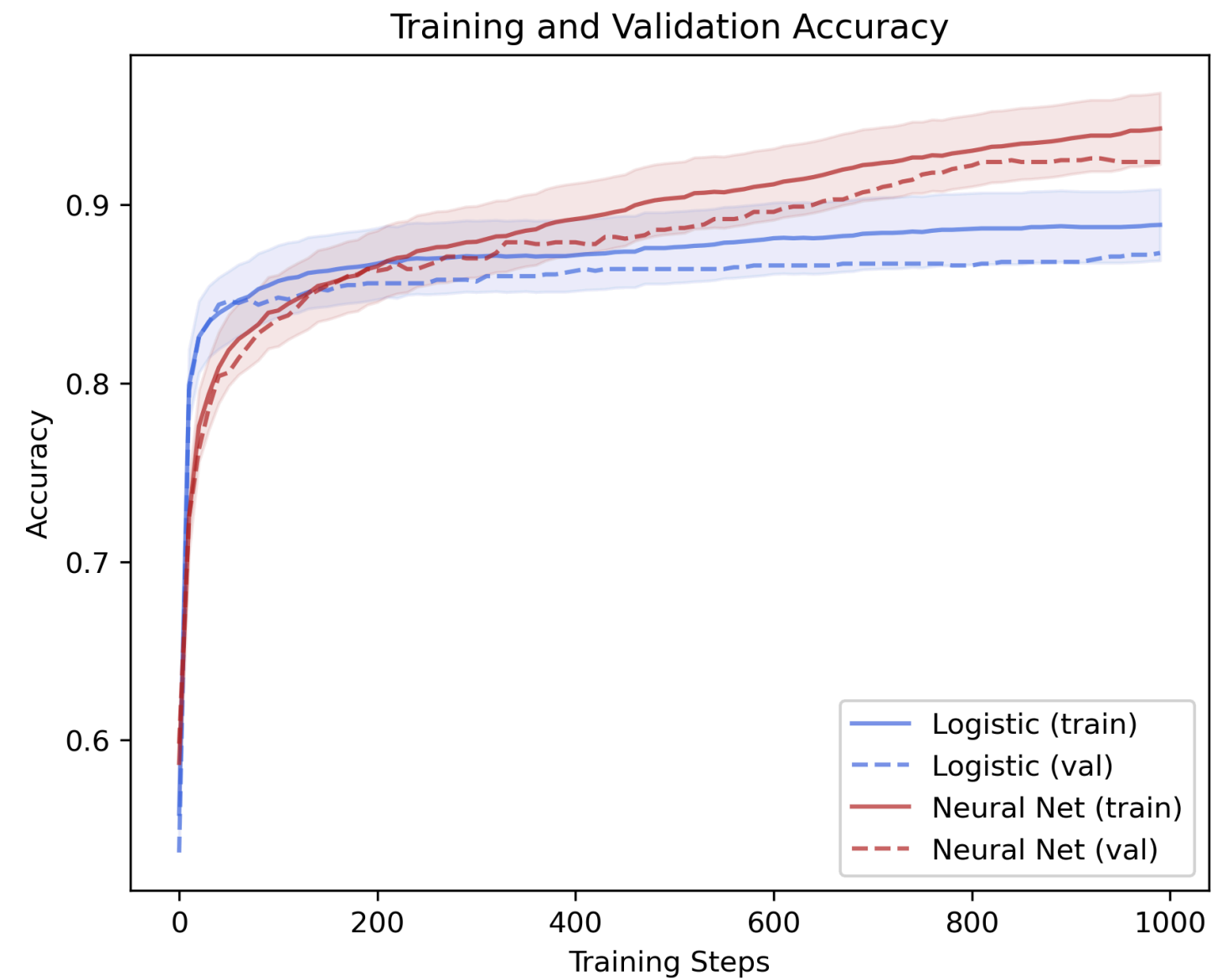
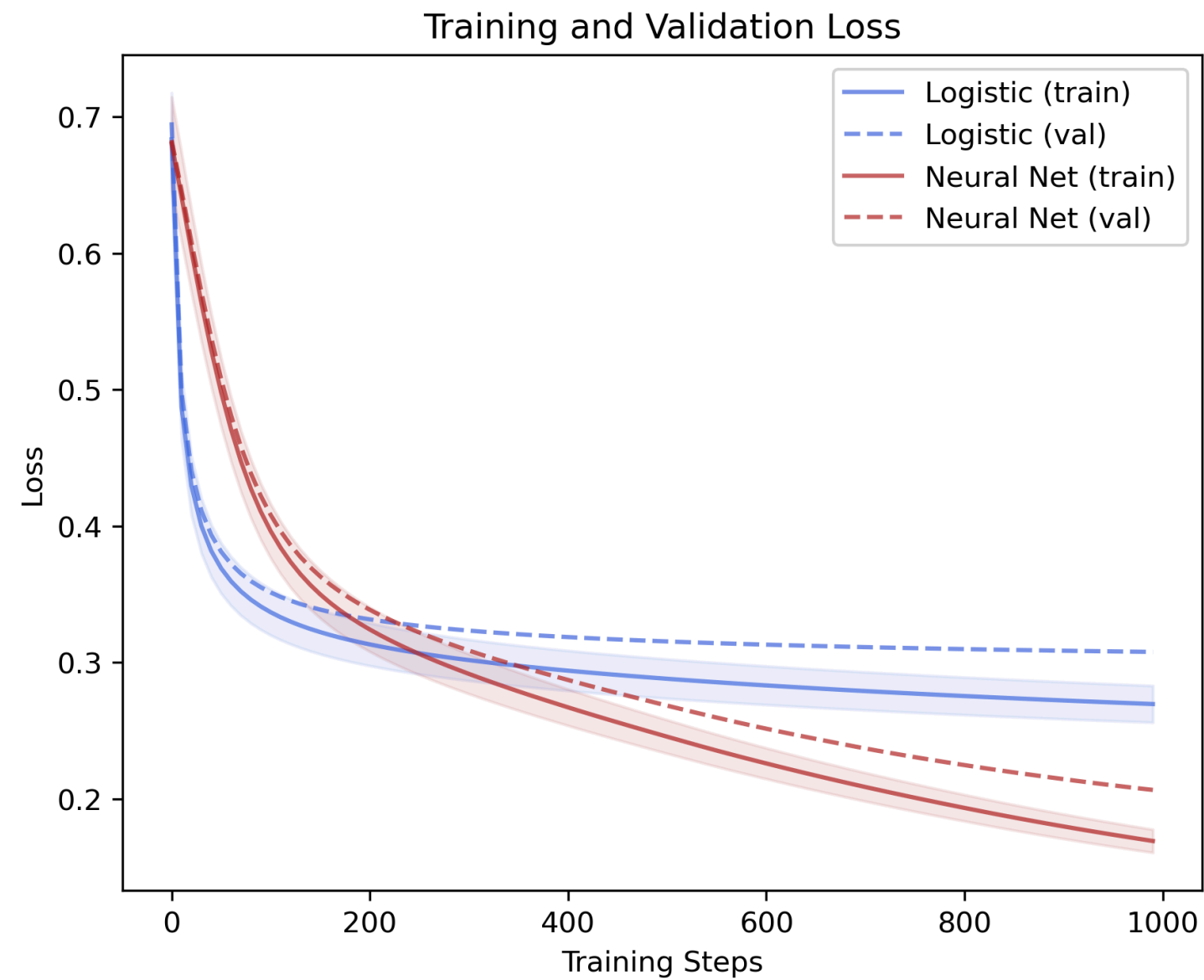
Neural Network:

```
class SimpleNN(nn.Module):  
    def __init__(self):  
        super().__init__()  
        self.fc1 = nn.Linear(784, 32)  
        self.fc2 = nn.Linear(32, 1)  
  
    def forward(self, x):  
        # Hidden layer with ReLU  
        h = torch.relu(  
            self.fc1(x.view(-1, 784))  
        )  
        # Output layer  
        return torch.sigmoid(self.fc2(h))
```

Training Process: Step by Step

```
def train_model(model, X_train, y_train, X_val, y_val, alpha=0.01, n_steps=1000):  
    for step in range(n_steps):  
        # Forward pass  
        y_pred = model(X_train)  
        loss = criterion(y_pred.squeeze(), y_train)  
  
        # Backward pass  
        loss.backward()  
  
        # Update parameters  
        with torch.no_grad(): # Do not modify the computational graph  
            for param in model.parameters():  
                param -= alpha * param.grad # update the parameters  
                param.grad.zero_() # reset the gradient to zero to avoid accumulation
```


Results Analysis



Final Results:

- Logistic: 87.30% accuracy
- Neural Net: 92.40% accuracy

Questions?

- Course website: <https://damek.github.io/STAT-4830/>
- Office hours: Listed on course website
- Email: damek@wharton.upenn.edu