Recitations for ORIE 6300: Mathematical Programming I

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1 Recitation 1: Linear Images of Polyhedra

Today's recitation provides a proof for the following fact that was stated in class:

Proposition 1.1 (Linear maps preserve polyhedrality)**.** *Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polyhedral set, which is a set defined by a finite number of linear inequalities. Consider $A \in \mathbb{R}^{m \times n}$. Then, the set*

$$\{Ax \mid x \in \mathcal{P}\}$$

is a polyhedral set.

Before proceeding with the proof, note that assuming $\mathcal{P}$ is defined by linear inequalities is without loss of generality; indeed, we have the following equivalence:

$$a_i^T x = b_i \Leftrightarrow \begin{cases} a_i^T x \leq b_i, \\ a_i^T x \geq b_i \end{cases},$$

which allows us to convert any linear equality into two inequalities. The proof is **constructive**, i.e. we can identify an explicit way to start from the inequalities defining $\mathcal{P}$ and construct linear inequalities that define $\{Ax \mid x \in \mathcal{P}\}$.

The next step will make our lives much easier when working towards the proof of Proposition 1.1:

Claim 1. It suffices to prove Proposition 1.1 for projection matrices $P \in \mathbb{R}^{(n-1)\times n}$, which eliminate one coordinate when applied to a vector $x \in \mathbb{R}^n$.

Proof of Claim. Let's take a look at an example of such a matrix:

$$P = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n-1 \times n}.$$

This projection "eliminates" the variable x_n from a point x.

Now let us see why the Claim holds. First, we prove by induction that we can eliminate as many variables as we want and keep the set polyhedral, as long as we've proved the base step.

- Base step: the set $\{Px, x \in \mathcal{P}\}$, where P is matrix that eliminates one of the coordinates and $\mathcal{P}$ is a polyhedron, is itself polyhedral.
- Now, assume we've proved that eliminating k variables from the set $\{x \in \mathcal{P}\}$ results in a polyhedral set.
- Since we have proved the result for k variables, we have a set in $\mathbb{R}^{n-k}$ which is polyhedral. Then, by the base step, we can apply a projection again to result in a set that is polyhedral and in $\mathbb{R}^{n-k-1}$, i.e. we have eliminated one more variable.

Now, consider the case of an arbitrary linear map A: if you examine the following set closely, you will get a clear idea of where we're heading next:

$$\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \mid x \in \mathcal{P}, y = Ax\} \triangleq \mathcal{P}' \tag{1.1}$$

First, let's convince ourselves that the set in (1.1) is polyhedral. If $\mathcal{P}$ was to be described by the inequalities $\{x \mid Cx \le d\}$, we can rewrite $\mathcal{P}'$ as

$$\mathcal{P}' = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{n+m} \mid \begin{bmatrix} C & \underline{0}_{m\times m} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le d,\ \begin{bmatrix} A & -I_{m\times m} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0\}$$

Now, appealing to our inductive argument, we can readily eliminate the first n variables in (1.1), giving us precisely the set $\{Ax \mid x \in \mathcal{P}\}$. □

A numerical example in $\mathbb{R}^2$ Consider a very simple polyhedral constraint, shown below:

$$\mathcal{P} = \begin{cases} x_1 + x_2 & \ge 3 \\ 2x_1 - x_2 & \le 5 \\ -x_1 + 2x_2 & \le 3 \end{cases}$$

The polyhedron and its projection to the variable x_1 are shown in 1. Let's see what we can

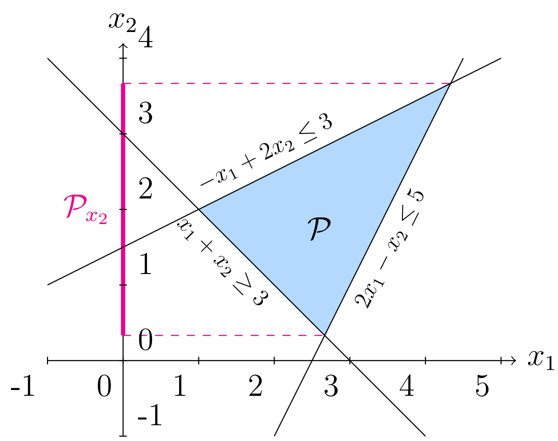

Figure 1: A polyhedron $\mathcal{P} \subseteq \mathbb{R}^2$ and its projection to x_2, $\mathcal{P}_{x_2}$

eliminate by hand: if we seek to bring x_1 to the LHS of the constraints shown above, we obtain

$$\begin{cases} x_1 \ge 3 - x_2, \\ 2x_1 \le 5 + x_2, \\ -x_1 \le 3 - 2x_2 \end{cases}$$

Dividing the second inequality by 2 and multiplying the last one by -1, gives us

$$\begin{cases} x_1 \ge 3 - x_2, \\ x_1 \le \frac{5}{2} + \frac{x_2}{2}, \\ x_1 \ge 2x_2 - 3 \end{cases}$$

Now, combine the second inequality with the first and third ones to eliminate x_1 and obtain

$$\frac{1}{3} \leq x_2 \leq \frac{11}{3},$$

which is exactly the projection shown in 1.

An example in $\mathbb{R}^3$ Let's do the same for a polyhedral set in 3 dimensions that is easy to visualize. Consider the following polyhedron:

$$\mathcal{P} := \{x \in \mathbb{R}^3 \mid x \geq 0,\ x_1 + x_2 + x_3 \leq 1\} \tag{1.2}$$

Suppose we want to find the set $C = \{Px \mid x \in \mathcal{P}\}$, where P is the projection matrix that eliminates the variable x_1. If we follow the same strategy to eliminate x_1, we get

$$x_1 \leq 1 - x_2 - x_3,\ x_1 \geq 0$$

which gives us

$$1 - x_2 - x_3 \geq 0 \Rightarrow x_2 + x_3 \leq 1.$$

It is easy to verify visually that the resulting set is exactly what we would expect to obtain if we projected $\mathcal{P}$ to its last 2 coordinates, as shown in (2)

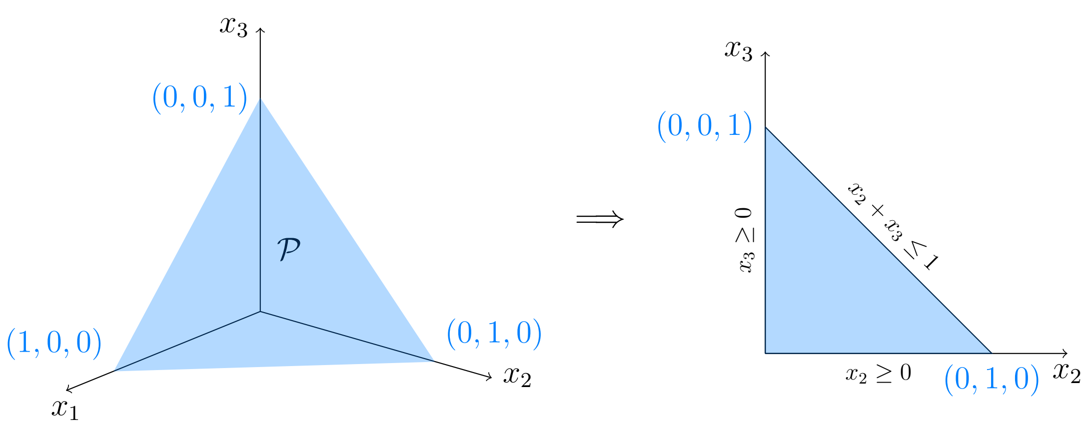

Figure 2: Eliminating x_1 from (1.2)

Proof of Proposition 1.1. Equipped with Claim 1, let us see how to prove the desired result for the case where we want to eliminate x_n. Like before, assume $\mathcal{P}$ is described by linear inequalities only. Additionally, let us index those inequalities, like below:

$$\begin{cases} a_1^T x \leq b_1, \\ a_2^T x \leq b_2, \\ \quad \vdots \\ a_m^T x \leq b_m \end{cases}.$$

Denote $I_0, I_+, I_- \subseteq [m]$ the index sets where the coefficient of x_n is 0, positive and negative, respectively. We maintain the the inequalities in I_0 as they were, since x_n is not present there. For the remaining constraints, multiply the left and right hand sides so as to make the coefficient of x_n equal to 1. Denote by c_i the scaled vectors of coefficients for the first $n-1$ variables, d_i for the scaled constants, and:

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}.$$

Then, we can write

$$\begin{cases} c_i^T \bar{x} \le d_i, & i \in I_0 \\ c_i^T \bar{x} + x_n \le d_i, & i \in I_+ \\ c_i^T \bar{x} + x_n \ge d_i, & i \in I_- \end{cases} \Leftrightarrow \begin{cases} c_i^T \bar{x} \le d_i, & i \in I_0 \\ x_n \le d_i - c_i^T \bar{x}, & i \in I_+ \\ x_n \ge d_i - c_i^T \bar{x}, & i \in I_- \end{cases}$$

For now, assume that both sets I_+, I_- are nonempty. Since $x_n \le d_i - c_i^T \bar{x}$ for all $i \in I_+$ and also $x_n \ge d_i - c_i^T \bar{x}$ for all $i \in I_-$, we can combine the inequalities to obtain

$$\begin{cases} 0 \le d_i - c_i^T \bar{x}, & i \in I_0 \\ d_j - c_j^T \bar{x} \le x_n \le d_i - c_i^T \bar{x}, & \forall (i,j) \in I_+ \times I_- \end{cases}$$

A point $\bar{x}$ is in the projection if it satisfies the above inequalities for some number x_n. This number exists if and only if

$$d_j - c_j^T \bar{x} \le d_i - c_i^T \bar{x}, \ \forall (i,j) \in I_+ \times I_-$$

which is a set of linear inequalities which do not involve x_n, hence a polyhedron in $\mathbb{R}^{n-1}$.

What if one of the two sets I_+, I_- is empty? In that case, the inequalities in the nonempty set are redundant. To see why, assume without loss of generality that the set I_- is empty. Then, the system becomes

$$\begin{cases} 0 \le d_i - c_i^T \bar{x}, & i \in I_0 \\ d_j - c_j^T \bar{x} \le x_n, & i \in I_+ \end{cases}$$

Letting x_n approach $+\infty$, we end up with a vector $\bar{x}$ that, additionally to the constraints in I_0, trivially satisfies all of the constraints in I_+. This completes the proof. $\square$

Remarks on the complexity of describing a polyhedron So far we have shown that sets $\{Ax \mid x \in \mathcal{P}\}$ are polyhedral. However, it may not be more efficient to describe this set by a collection of inequalities.

To be precise, suppose $\mathcal{P}$ can be described using n inequalities. Then our proof of Proposition 1.1 shows the polyhedron given by projecting one coordinate away can be described using at most $(n/2)^2$ inequalities (when $|I_-| = |I_+| = n/2$). Repeating this argument to remove d coordinates may result in needing an exponential number of inequalities. Hence even though $\{Ax \mid x \in \mathcal{P}\}$ is a polyhedron, it may not always be to your benefit to put it in that form.

A concrete example of this complexity difference appears in the first homework assignment for the ℓ_1-ball.