**Lemma 0.1.** *We have $2^x \geq x+1$ for any real $x \geq 1$.*

*Proof.* □

**Lemma 0.2.** *We have $2^x \geq x/2+1$ for any real $x \geq 0$.*

*Proof.* □

**Theorem 0.3** (Fundamental theorem of arithmetic)**.** *Every integer $n \geq 1$ factors uniquely into a product of primes $n = p_1^{a_1} \cdots p_k^{a_k}$.*

*Proof.* □

**Definition 0.4** (Divisor function)**.** $\tau(n)$ denotes the number of divisors of an integer $n \geq 1$.

**Lemma 0.5.** *If $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of an integer, then the divisor function evaluated at $n$ equals $\tau(n) = (a_1+1)\cdots(a_k+1)$.*

*Proof.* □

**Lemma 0.6.** *If $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of an integer, then*

$$\frac{\tau(n)}{n^{\varepsilon}} = \prod_{i \leq k} \frac{a_i+1}{p_i^{a_i\varepsilon}}. \tag{1}$$

*Proof.* Use definition 0.4, lemma 0.5. □

**Lemma 0.7.** *Let $p_i, a_i \geq 1$ and $\varepsilon > 0$. If $p_i^{\varepsilon} \geq 2$ then $\frac{a_i+1}{p_i^{a_i\varepsilon}} \leq \frac{a_i+1}{2^{a_i}} \leq 1$.*

*Proof.* Use lemma 0.1 with $x = a_i$. □

**Lemma 0.8.** *Let $p_i, a_i \geq 1$ and $\varepsilon > 0$. If $p_i^{\varepsilon} < 2$ then $\frac{a_i+1}{p_i^{a_i\varepsilon}} \leq 2/\varepsilon$.*

*Proof.* Use lemma 0.2 with $x = \varepsilon$. □

**Lemma 0.9.** *If $\varepsilon > 0$, $a_1, \ldots, a_k \geq 1$ are integers, and $p_1, \ldots, p_k$ are primes, then*

$$\prod_{i \leq k} \frac{a_i+1}{p_i^{a_i\varepsilon}} = \prod_{i \leq k,\, p_i^{\varepsilon} \geq 2} \frac{a_i+1}{p_i^{a_i\varepsilon}} \prod_{i \leq k,\, p_i^{\varepsilon} < 2} \frac{a_i+1}{p_i^{a_i\varepsilon}}$$

*Proof.* Uses no previous result □

**Lemma 0.10.** *If* $\varepsilon > 0$, $a_1, \ldots, a_k \geq 1$ *are integers, and* $p_1, \ldots, p_k$ *are primes, then*

$$\prod_{i \leq k,\, p_i^{\varepsilon} \geq 2} \frac{a_i + 1}{p_i^{a_i \varepsilon}} \leq 1$$

*Proof.* Uses lemma 0.7 ☐

**Lemma 0.11.** *If* $\varepsilon > 0$, $a_1, \ldots, a_k \geq 1$ *are integers, and* $p_1, \ldots, p_k$ *are primes, then*

$$\prod_{i \leq k,\, p_i^{\varepsilon} < 2} \frac{a_i + 1}{p_i^{a_i \varepsilon}} \leq \prod_{i \leq k,\, p_i^{\varepsilon} < 2} \frac{2}{\varepsilon}$$

*Proof.* Uses lemma 0.8 ☐

**Lemma 0.12.** *If* $\varepsilon > 0$, $a_1, \ldots, a_k \geq 1$ *are integers, and* $p_1, \ldots, p_k$ *are primes, then*

$$\prod_{i \leq k,\, p_i^{\varepsilon} < 2} \frac{2}{\varepsilon} \leq (2/\varepsilon)^{2^{1/\varepsilon}}$$

*Proof.* Uses no previous result ☐

**Lemma 0.13.** *If* $\varepsilon > 0$, $a_1, \ldots, a_k \geq 1$ *are integers, and* $p_1, \ldots, p_k$ *are primes, then*

$$\prod_{i \leq k} \frac{a_i + 1}{p_i^{a_i \varepsilon}} \leq (2/\varepsilon)^{2^{1/\varepsilon}}$$

*Proof.* Uses lemmas 0.9, 0.10, 0.11, 0.12 ☐

**Lemma 0.14.** *Let* $\varepsilon > 0$. *Then* $(2/\varepsilon)^{2^{1/\varepsilon}} \leq O_{\varepsilon}(1)$.

*Proof.* Uses definition of $O_{\varepsilon}(1)$. ☐

**Lemma 0.15.** *Let* $\varepsilon > 0$ *and* $n \geq 1$. *Then* $\frac{\tau(n)}{n^{\varepsilon}} \leq O_{\varepsilon}(1)$

*Proof.* Uses lemma 0.6, 0.13, 0.14 ☐

**Lemma 0.16.** *Let* $\varepsilon > 0$ *and* $n \geq 1$. *Then* $\tau(n) \leq O_{\varepsilon}(n^{\varepsilon})$.

*Proof.* Uses lemma 0.15 ☐

**Lemma 0.17.** *We have* $\tau(n) \leq O_{\varepsilon}(n^{\varepsilon})$ *for any integer* $n \geq 1$ *and any* $\varepsilon > 0$.

*Proof.* Uses lemma 0.16 ☐

**Theorem 0.18** (Divisor bound)**.** *We have $\tau(n) \leq n^{o(1)}$ for any integer $n \geq 1$.*

*Proof.* Uses lemma 0.17 and definition of $o(1)$. $\square$

**Definition 0.19** (Radical)**.** For an integer $n \geq 1$ define the radical $\operatorname{rad}(n) = \prod_{p|n} p$.

**Lemma 0.20.** *If $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of an integer, then $\operatorname{rad}(n) = p_1 \cdots p_k$.*

*Proof.* Uses definition 0.19. $\square$

**Lemma 0.21.** *$\operatorname{rad}(n)$ is a multiplicative function. That is, if $a, b \geq 1$ are integers with $\gcd(a, b) = 1$ then $\operatorname{rad}(ab) = \operatorname{rad}(a) \operatorname{rad}(b)$.*

*Proof.* Uses lemma 0.20 and theorem 0.3. $\square$

**Lemma 0.22.** *If $a, b \geq 1$ are integers with $\gcd(a, bc) = 1$ and $\gcd(b, c) = 1$ then $\operatorname{rad}(abc) = \operatorname{rad}(a) \operatorname{rad}(b) \operatorname{rad}(c)$.*

*Proof.* Uses lemma 0.21 with $\{a, bc\}$, and again uses lemma 0.21 with $\{b, c\}$ $\square$

**Lemma 0.23.** *If $a, b \geq 1$ are integers with $\gcd(a, bc) = 1$ and $\gcd(b, c) = 1$ then $\operatorname{rad}(ab) \operatorname{rad}(ac) \operatorname{rad}(bc) = (\operatorname{rad}(abc))^2$.*

*Proof.* Uses lemma 0.22, and then uses lemma 0.21 three times, with $\{a, b\}$ and $\{a, c\}$ and $\{b, c\}$. $\square$

**Lemma 0.24.** *Let $p_1 < \cdots < p_k$ be distinct primes, and denote the product $r = p_1 \cdots p_k$. If an integer $n \geq 1$ satisfies $\operatorname{rad}(n) = r$, then $n = p_1^{a_1} \cdots p_k^{a_k}$ for some integers $a_1, \ldots, a_k \geq 1$.*

*Proof.* Uses theorem 0.3 with $n$, then uses lemma 0.20. $\square$

**Definition 0.25.** Let $N, r \geq 1$. Define $\mathcal{R}(r, N) := \Big\{ n \leq N : \operatorname{rad}(n) = r \Big\}$. When context is clear, we may simply write $\mathcal{R} = \mathcal{R}(r, N)$.

**Lemma 0.26.** *Let $p_1 < \cdots < p_k$ be distinct primes, and denote the product $r = p_1 \cdots p_k$. Then*

$$\mathcal{R} \subset \Big\{ p_1^{a_1} \cdots p_k^{a_k} : a_1, \ldots, a_k \geq 1 \Big\}.$$

*Proof.* Uses lemma 0.20 and theorem 0.3. $\square$

**Lemma 0.27.** *For any set $\mathcal{S} \subset \mathbb{Z}$, we have $|\mathcal{S}| = \sum_{n \in S} 1$.*

*Proof.* Uses no previous result $\square$

**Lemma 0.28.** *If $\varepsilon > 0$ and $1 \leq n \leq N$, then $\frac{1}{n^\varepsilon} \geq \frac{1}{N^\varepsilon}$.*

*Proof.* Uses no previous result $\square$

**Lemma 0.29.** *For any $\varepsilon > 0$ and $N, r \geq 1$, we have*

$$\sum_{n \in \mathcal{R}} \frac{1}{n^\varepsilon} \geq \frac{|\mathcal{R}|}{N^\varepsilon},$$

*Proof.* Uses lemma 0.27 with $\mathcal{S} = \mathcal{R}$, and lemma 0.28. $\square$

**Lemma 0.30.** *Let $\varepsilon > 0$ and $N \geq 1$. If $r = p_1 p_2$, then $\mathcal{R} = \mathcal{R}(r, N)$ satisfies*

$$\sum_{n \in \mathcal{R}} \frac{1}{n^\varepsilon} \leq \sum_{a_1 \geq 1} \sum_{a_2 \geq 1} \frac{1}{(p_1^{a_1} p_2^{a_2})^\varepsilon}$$

*Proof.* Uses lemma 0.26. $\square$

**Lemma 0.31.** *Let $\varepsilon > 0$ and $N \geq 1$. If $r = p_1 \cdots p_k$, then $\mathcal{R} = \mathcal{R}(r, N)$ satisfies*

$$\sum_{n \in \mathcal{R}} \frac{1}{n^\varepsilon} \leq \sum_{a_1, \ldots, a_k \geq 1} \frac{1}{(p_1^{a_1} \cdots p_k^{a_k})^\varepsilon}$$

*Proof.* Uses lemma 0.26. $\square$

**Lemma 0.32.** *Let $\varepsilon > 0$ and $N \geq 1$. Then*

$$\sum_{a_1 \geq 1} \sum_{a_2 \geq 1} \frac{1}{(p_1^{a_1} p_2^{a_2})^\varepsilon} = \Big( \sum_{a_1 \geq 1} \frac{1}{p_1^{\varepsilon a_1}} \Big) \Big( \sum_{a_2 \geq 1} \frac{1}{p_2^{\varepsilon a_2}} \Big)$$

*Proof.* Uses no previous result $\square$

**Lemma 0.33.** *Let $\varepsilon > 0$ and $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\sum_{a_1, \ldots, a_k \geq 1} \frac{1}{(p_1^{a_1} \cdots p_k^{a_k})^\varepsilon} = \prod_{i \leq k} \Big( \sum_{a_i \geq 1} \frac{1}{p_i^{\varepsilon a_i}} \Big)$$

*Proof.* Uses no previous result $\square$

**Theorem 0.34** (Geometric series)**.** *For any $0 < x < 1$ we have $\sum_{a \geq 1} x^a = \frac{x}{x-1}$.*

*Proof.* Uses no previous result □

**Lemma 0.35.** *Let $\varepsilon > 0$ and let $p_i \geq 2$ be a prime. Then*

$$\sum_{a_i \geq 1} \frac{1}{p_i^{\varepsilon a_i}} = \frac{1}{p_i^{\varepsilon} - 1}.$$

*Proof.* Uses lemma 0.34 □

**Lemma 0.36.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \leq k} \Big( \sum_{a_i \geq 1} \frac{1}{p_i^{\varepsilon a_i}} \Big) = \prod_{i \leq k} \frac{1}{p_i^{\varepsilon} - 1}$$

*Proof.* Uses lemma 0.35 □

**Lemma 0.37.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \leq k} \frac{1}{p_i^{\varepsilon} - 1} = \prod_{i \leq k, p_i^{\varepsilon} \geq 2} \frac{1}{p_i^{\varepsilon} - 1} \prod_{i \leq k, p_i^{\varepsilon} < 2} \frac{1}{p_i^{\varepsilon} - 1}$$

*Proof.* Uses no previous result □

**Lemma 0.38.** *Let $p_i, a_i \geq 1$ and $\varepsilon > 0$. If $p_i^{\varepsilon} \geq 2$ then $\frac{1}{p_i^{\varepsilon} - 1} \leq 1$.*

*Proof.* Uses no previous result □

**Lemma 0.39.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \leq k, p_i^{\varepsilon} \geq 2} \frac{1}{p_i^{\varepsilon} - 1} \leq 1.$$

*Proof.* Uses lemma 0.38 □

**Lemma 0.40.** *Let $\varepsilon > 0$. Then $\frac{1}{2^{\varepsilon} - 1} \leq \frac{2}{\varepsilon}$.*

*Proof.* Uses lemma 0.2 □

**Lemma 0.41.** *Let $p_i \geq 2$, $a_i \geq 1$ and $\varepsilon > 0$. If $p_i^{\varepsilon} < 2$ then $\frac{1}{p_i^{\varepsilon} - 1} \leq \frac{2}{\varepsilon}$.*

*Proof.* Uses $p_i \geq 2$ and lemma 0.40 □

**Lemma 0.42.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \leq k, p_i^{\varepsilon} < 2} \frac{1}{p_i^{\varepsilon} - 1} \leq \prod_{i \leq k, p_i^{\varepsilon} < 2} \frac{2}{\varepsilon}$$

*Proof.* Uses lemma 0.41 □

**Lemma 0.43.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \le k,\, p_i^\varepsilon < 2} \frac{2}{\varepsilon} \le (2/\varepsilon)^{2^{1/\varepsilon}}.$$

*Proof.* Uses no previous results □

**Lemma 0.44.** *Let $\varepsilon > 0$ and let $p_1 < \cdots < p_k$ be distinct primes. Then*

$$\prod_{i \le k} \frac{1}{p_i^\varepsilon - 1} \le (2/\varepsilon)^{2^{1/\varepsilon}}.$$

*Proof.* Uses lemmas 0.37, 0.39, 0.42, 0.43 □

**Lemma 0.45.** *Let $\varepsilon > 0$ and $N \ge 1$. Then $(2/\varepsilon)^{2^{1/\varepsilon}} \le O_\varepsilon(1)$.*

*Proof.* Uses lemma 0.14. □

**Lemma 0.46.** *For any $\varepsilon > 0$ and $N, r \ge 1$, we have*

$$\frac{|\mathcal{R}|}{N^\varepsilon} \le O_\varepsilon(1).$$

*Proof.* Uses lemmas 0.29, 0.31, 0.33, 0.36, 0.44, 0.45. □

**Lemma 0.47.** *Let $\varepsilon > 0$ and $1 \le r \le N$. Then we have*

$$|\mathcal{R}(r, N)| \le O_\varepsilon(N^\varepsilon)$$

*Proof.* Uses lemma 0.46 □

**Theorem 0.48.** *Let $1 \le r \le N$. Then we have*

$$|\mathcal{R}(r, N)| \le N^{o(1)}.$$

*Proof.* Uses lemma 0.47 and definition of $o(1)$ □

**Lemma 0.49.** *Let $N \ge 1$ and $0 < \lambda < 1$. We have*

$$\left|\left\{n \le N : \mathrm{rad}(n) \le N^\lambda\right\}\right| = \sum_{1 \le r \le N^\lambda} \sum_{\substack{1 \le n \le N \\ \mathrm{rad}(n) = r}} 1$$

*Proof.* Uses no previous results □

**Lemma 0.50.** *Let* $N \geq 1$ *and* $0 < \lambda < 1$. *We have*

$$\left|\left\{n \leq N : \mathrm{rad}(n) \leq N^{\lambda}\right\}\right| = \sum_{1 \leq r \leq N^{\lambda}} |\mathcal{R}(r, N)|$$

*Proof.* Uses lemma 0.49 and definition 0.25. □

**Lemma 0.51.** *Let* $N \geq 1$ *and* $0 < \lambda < 1$. *We have*

$$\left|\left\{n \leq N : \mathrm{rad}(n) \leq N^{\lambda}\right\}\right| \leq \sum_{1 \leq r \leq N^{\lambda}} N^{o(1)}$$

*Proof.* Uses lemma 0.50 and theorem 0.48. □

**Theorem 0.52.** *Let* $N \geq 1$ *and* $0 < \lambda < 1$. *We have*

$$\left|\left\{n \leq N : \mathrm{rad}(n) \leq N^{\lambda}\right\}\right| \;\leq\; N^{\lambda + o(1)}.$$

*Proof.* Uses lemma 0.51 □

**Definition 0.53** (Exceptional set)**.** Let $N \geq 1$ and $\varepsilon > 0$. Define the exceptional set

$$\mathcal{E}(N) \;=\; \Big\{(a, b, c) \in \{1, \ldots, N\}^3 \;:\; \gcd(a, b) = 1, a+b = c,\ \mathrm{rad}(abc) \;<\; c^{1-\varepsilon}\Big\}.$$

**Lemma 0.54.** *Let* $a, b \geq 1$ *be integers with* $\gcd(a, bc) = 1$ *and* $\gcd(b, c) = 1$. *If* $\mathrm{rad}(ab)$, $\mathrm{rad}(ac)$, $\mathrm{rad}(bc)$ *all exceed* $c^{\frac{2}{3}(1-\varepsilon)}$, *then*

$$c^{2-2\varepsilon} \leq \mathrm{rad}(ab)\,\mathrm{rad}(ac)\,\mathrm{rad}(bc) = (\mathrm{rad}(abc))^2.$$

*Proof.* Uses lemma 0.23 □

**Lemma 0.55.** *Let* $a, b \geq 1$ *be integers with* $\gcd(a, bc) = 1$ *and* $\gcd(b, c) = 1$. *If* $\mathrm{rad}(ab)$, $\mathrm{rad}(ac)$, $\mathrm{rad}(bc)$ *all exceed* $c^{\frac{2}{3}(1-\varepsilon)}$, *then* $c^{1-\varepsilon} \leq \mathrm{rad}(abc)$.

*Proof.* Uses lemma 0.54 □

**Lemma 0.56.** *Let* $(a, b, c) \in \mathcal{E}(N)$. *Then either* $\mathrm{rad}(ab) < c^{\frac{2}{3}(1-\varepsilon)}$ *or* $\mathrm{rad}(ac) < c^{\frac{2}{3}(1-\varepsilon)}$ *or* $\mathrm{rad}(bc) < c^{\frac{2}{3}(1-\varepsilon)}$.

*Proof.* Uses lemma 0.55 in contrapositive form □

**Lemma 0.57.** *Let* $(a, b, c) \in \mathcal{E}(N)$. *Then there exist* $x, y \in \{a, b, c\}$ *with* $x \neq y$ *and* $\mathrm{rad}(xy) \leq N^{2/3-\varepsilon}$.

*Proof.* Uses lemma 0.56 □

**Lemma 0.58.** *We have*

$$|\mathcal{E}(N)| = \sum\sum\sum_{\substack{1\le a,b,c\le N\\ \gcd(a,b)=1\\ \mathrm{rad}(abc)\ <\ c^{1-\varepsilon}\\ a+b=c}} 1.$$

*Proof.* Uses no previous results □

**Lemma 0.59.** *Let $\varepsilon > 0$ and $N \geq 1$. We have*

$$\sum\sum\sum_{\substack{1\le a,b,c\le N\\ \gcd(a,b)=1\\ \mathrm{rad}(abc)\ <\ c^{1-\varepsilon}\\ a+b=c}} 1 \ \le \ \sum_{r\le N^{2/3-\varepsilon}} \sum\sum\sum_{\substack{1\le a,b,c\le N\\ \gcd(a,b)=1\\ r=\mathrm{rad}(abc)\\ a+b=c}} 1$$

*Proof.* Uses no previous results □

**Lemma 0.60.** *Let $\varepsilon > 0$ and $r, N \geq 1$. We have*

$$\sum\sum\sum_{\substack{1\le a,b,c\le N\\ \gcd(a,b)=1\\ r=\mathrm{rad}(abc)\\ a+b=c}} 1 \ \le \ 3 \sum\sum_{\substack{1\le x,y\le N\\ \gcd(x,y)=1\\ r=\mathrm{rad}(xy)}} 1$$

*Proof.* Uses lemma 0.57 □

**Lemma 0.61.** *Let $\varepsilon > 0$ and $r, N \geq 1$. We have*

$$\sum\sum_{\substack{1\le x,y\le N\\ \gcd(x,y)=1\\ r=\mathrm{rad}(xy)}} 1 \ \le \ \sum_{\substack{1\le n\le N^2\\ r=\mathrm{rad}(n)}} \tau(n).$$

*Proof.* Uses definition 0.4 with $n = xy$ □

**Lemma 0.62.** *We have*

$$\sum_{\substack{1\le n\le N^2\\ r=\mathrm{rad}(n)}} \tau(n) \le \sum_{\substack{1\le n\le N^2\\ r=\mathrm{rad}(n)}} N^{o(1)}$$

*Proof.* Uses theorem 0.18, and that $n \leq N$ □

**Lemma 0.63.** *We have*

$$3 \sum_{r \le N^{2/3-\varepsilon}} \sum\sum_{\substack{1 \le x,y \le N \\ \gcd(x,y)=1 \\ r = \mathrm{rad}(xy)}} 1 \;\le\; \sum_{r \le N^{2/3-\varepsilon}} \sum_{\substack{1 \le n \le N^2 \\ r = \mathrm{rad}(n)}} N^{o(1)}$$

*Proof.* Uses lemmas 0.61, 0.62 □

**Lemma 0.64.** *We have*

$$\sum_{r \le N^{2/3-\varepsilon}} \sum_{\substack{1 \le n \le N^2 \\ r = \mathrm{rad}(n)}} 1 = \left|\left\{n \le N^2 : \mathrm{rad}(n) \le N^{2/3-\varepsilon}\right\}\right|.$$

*Proof.* Uses no previous results □

**Lemma 0.65.** *Let $N \ge 1$ and $\varepsilon > 0$. We have*

$$|\mathcal{E}(N)| \;\le\; N^{o(1)} \cdot \left|\left\{n \le N^2 : \mathrm{rad}(n) \le N^{2/3-\varepsilon}\right\}\right|$$

*Proof.* Uses lemmas 0.58, 0.59, 0.60, 0.63, 0.64 □

**Lemma 0.66.** *Let $N \ge 1$ and $\varepsilon > 0$. We have*

$$\left|\left\{n \le N^2 : \mathrm{rad}(n) \le N^{2/3-\varepsilon}\right\}\right| \;\le\; N^{2/3-\varepsilon+o(1)}$$

*Proof.* Uses definition 0.25, theorem 0.52 with $N^2$ and $\lambda = 2/3 - \varepsilon$ □

**Theorem 0.67.** *We have $|\mathcal{E}(N)| \le O(N^{2/3})$.*

*Proof.* Uses lemmas 0.65, 0.66 □