

Knaster-Tarski theorem

문순원

December 1, 2023

Definition (Bottom and top element)

A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write \perp and \top for a bottom element and a top element, respectively.

$$\forall x \in L, \perp \sqsubseteq x$$

$$\forall x \in L, x \sqsubseteq \top$$

Definition (Infimum and supremum)

Let (L, \sqsubseteq) be a poset and A be a subset of L . Then α is an infimum (meet) of A if α is a lower bound of A and $l \sqsubseteq \alpha$ for every lower bound l of A . Similarly, β is a supremum (join) of A if β is an upper bound of A and $\beta \sqsubseteq u$ for every upperbound u of A .

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

Definition (Complete lattice)

A poset (L, \sqsubseteq) is a complete lattice if every subset of L has an infimum and a supremum. We write $\sqcap A$ and $\sqcup A$ for the infimum and the supremum of A , respectively.

Example (Power set lattice)

Given a set A , $(\mathcal{P}(A), \sqsubseteq)$ forms a complete lattice.

Lemma

Complete lattice has the bottom and the top element, and they can be represented as follows.

$$\perp = \sqcap L = \sqcup \emptyset$$

$$\top = \sqcap \emptyset = \sqcup L$$

Remark

Existence of supremum is enough for a poset to be a complete lattice. Let (L, \sqsubseteq) be a poset where every subset of L has a supremum, and let A be a subset of L . Define $D = \{l \mid l \text{ is a lower bound of } A\}$ and $\alpha = \sqcup D$. Then α is the infimum of A .

Definition (Fixed point)

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfix point when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.

Lemma

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$ be an order-preserving function. Then f has the least fixpoint $\mu = \bigcap \{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigcup \{x \in L \mid x \sqsubseteq f(x)\}$.

Proof.

Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of f . Since x is arbitrary, $f(\nu)$ is an upper bound of D , and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of f , $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of \sqsubseteq . The same argument can be used to show that μ is the least fixpoint. □

Theorem (Knaster-Tarski)

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f . Then (P, \sqsubseteq) forms a complete lattice.

Proof.

Let W be a subset of P . Define D as the set of upper bounds of W , and suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of f . Since w is arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$.

Consider a restriction of f from D to D . Then f has the least fixpoint in D . Call it α . Then it is evident that α is the supremum of W in P . □

$$\frac{}{\text{nil} \in \text{Tree}} \quad \frac{t_0 \in \text{Tree} \quad t_1 \in \text{Tree}}{\text{bin}(t_0, t_1) \in \text{Tree}}$$

Figure: An inductive definition of binary tree

$$\frac{}{\text{nil} \in \text{Tree}} \quad \frac{t_0 \in \text{Tree} \quad t_1 \in \text{Tree}}{\text{bin}(t_0, t_1) \in \text{Tree}}$$

Figure: An inductive definition of binary tree

Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \dots\}$$

$$\hat{t} \in T = \mathbf{2}^{2^*}$$

Define $f : \mathcal{P}(T) \rightarrow \mathcal{P}(T)$ as follows.

$$\text{nil}(\alpha) = 0$$

$$\text{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(t_0, t_1) \mid t_0, t_1 \in X\}$$

f is an order-preserving function.

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(t_0, t_1) \mid t_0, t_1 \in X\}$$

By using Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of f .

$$\text{Tree} = \mu(f)$$

$$\text{Tree} = f(\text{Tree})$$

$$f(X) \subseteq X \Rightarrow \text{Tree} \subseteq X$$

$$\frac{}{\text{nil} \lesssim t} \quad \frac{t_0 \lesssim t'_0 \quad t_1 \lesssim t'_1}{\text{bin}(t_0, t_1) \lesssim \text{bin}(t'_0, t'_1)}$$

Figure: Inductive definition of tree relation

Consider $(\mathcal{P}(\text{Tree} \times \text{Tree}), \subseteq)$. We can define following partial order on Tree.

$$f_{\lesssim}(R) = \{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim) = \mu(f_{\lesssim})$$

$$(\lesssim) = f_{\lesssim}(\lesssim)$$

$$f_{\lesssim}(R) \subseteq R \Rightarrow (\lesssim) \subseteq R$$

The following proof is an example of inductive proof using Tarski's principle.

Example (Reflexivity of \lesssim)

Let $X = \{t \mid t \lesssim t\}$. We have to show $\text{Tree} \subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of f and X , we have to show $(t, t) \in (\lesssim)$ whenever

$$t \in \{\text{nil}\} \cup \{\text{bin}(t_0, t_1) \mid t_0 \lesssim t_0, t_1 \lesssim t_1\}$$

(i) When $t = \text{nil}$,

$$(\text{nil}, \text{nil}) \in (\lesssim) = f_{\lesssim}(\lesssim) = \{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{\dots\}$$

(ii) When $t = \text{bin}(t_0, t_1)$ for some t_0, t_1 s.t. $t_0 \lesssim t_0$ and $t_1 \lesssim t_1$,

$$\begin{aligned} (\text{bin}(t_0, t_1), \text{bin}(t_0, t_1)) \in (\lesssim) &= f_{\lesssim}(\lesssim) = \\ &\{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid t_0 \lesssim t'_0, t_1 \lesssim t'_1\} \end{aligned}$$