# The Knaster-Tarski theorem

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## 1 Basic definitions

**Definition 1.1** (Bottom and top element). *A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write $\bot$ and $\top$ for a bottom element and a top element, respectively.*

$$\forall x \in L, \bot \sqsubseteq x$$
$$\forall x \in L, x \sqsubseteq \top$$

**Definition 1.2** (Infimum and supremum). *Let $(L, \sqsubseteq)$ be a poset and $A$ be a subset of $L$. Then $\alpha$ is an infimum (meet) of $A$ if $\alpha$ is a lower bound of $A$ and $l \sqsubseteq \alpha$ for every lower bound $l$ of $A$. Similary, $\beta$ is a supremum (join) of $A$ if $\beta$ is an upper bound of $A$ and $\beta \sqsubseteq u$ for every upperbound $u$ of $A$.*

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

**Definition 1.3** (Complete lattice). *A poset $(L, \sqsubseteq)$ is a complete lattice if every subset of $L$ has an infimum and a supremum. We write $\bigsqcap A$ and $\bigsqcup A$ for the infimum and the supremum of $A$, respectively.*

**Example 1.4** (Power set lattice). *Given a set $A$, $(\mathcal{P}(A), \subseteq)$ forms a complete lattice.*

**Lemma 1.5.** *Complete lattice has the bottom and the top element, and they can be represented as follows.*

$$\bot = \bigsqcap L = \bigsqcup \varnothing$$
$$\top = \bigsqcap \varnothing = \bigsqcup L$$

**Lemma 1.6.** *Let $(L, \sqsubseteq)$ be a poset where every subset of $L$ has a supremum. Then $L$ is a complete lattice.*

*Proof.* Let $A$ be a subset of $L$. Define $D$ as the set of lower bounds of $A$, and $\alpha = \bigsqcup D$. Then $\alpha$ is the infimum of $A$. $\qquad\square$

**Definition 1.7** (Fixed point). *Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfixed point of $f$ when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.*

$$\frac{}{\mathrm{nil} \in \mathrm{Tree}} \qquad \frac{t_0 \in \mathrm{Tree} \quad t_1 \in \mathrm{Tree}}{\mathrm{bin}(t_0, t_1) \in \mathrm{Tree}}$$

$$\frac{}{\mathrm{nil} \lesssim_{\mathrm{Tree}} t} \qquad \frac{t_0 \lesssim_{\mathrm{Tree}} t'_0 \quad t_1 \lesssim_{\mathrm{Tree}} t'_1}{\mathrm{bin}(t_0, t_1) \lesssim_{\mathrm{Tree}} \mathrm{bin}(t'_0, t'_1)}$$

Figure 1: An inductive definition of binary tree and a relation about it

## 2 Knaster-Tarski theorem

**Lemma 2.1.** *Let* $(L, \sqsubseteq)$ *be a complete lattice, and* $f : L \to L$ *be an order-preserving function. Then* $f$ *has the least fixpoint* $\mu = \bigsqcap \{x \in L \mid f(x) \sqsubseteq x\}$ *and the greatest fixpoint* $\nu = \bigsqcup \{x \in L \mid x \sqsubseteq f(x)\}$.

*Proof.* Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigsqcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of $f$. Since $x$ is arbitrary, $f(\nu)$ is an upper bound of $D$, and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of $f$, $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of $\sqsubseteq$. The same argument can be used to show that $\mu$ is the least fixpoint. $\qquad \square$

**Theorem 2.2** (Knaster-Tarski)**.** *Let* $(L, \sqsubseteq)$ *be a complete lattice, and* $f : L \to L$ *be an order-preserving function. Let* $P \subseteq L$ *be the set of fixpoints of* $f$. *Then* $(P, \sqsubseteq)$ *forms a complete lattice.*

*Proof.* Let $W$ be a subset of $P$. Define $D$ as the set of upper bounds of $W$. Then $D$ is a induced subposet of $L$ that is complete with the bottom element $\bigsqcup W$ and a supremum $\bigsqcup W \sqcup \bigsqcup A$ for all $A \subseteq D$.

Suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of $f$. Since $w$ and $x$ are arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$.

Consider a restriction of $f$ from $D$ to $D$. Then $f$ has the least fixpoint in $D$. Call it $\alpha$. Then it is evident that $\alpha$ is the supremum of $W$ in $P$. $\qquad \square$

## 3 Inductive definition and proof

As an example, we will define a set of binary tree and a binary relation on it by using the Knaster-Tarski theorem. Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \dots\}$$

$$\hat{t} \in T = \mathbf{2}^{\mathbf{2}^*}$$

Define $f : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows.

$$\mathrm{nil}(\alpha) = 0$$

$$\mathrm{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\mathrm{nil}\} \cup \{\mathrm{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

Then $f$ is an order-preserving function.

By using the Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of $f$.

$$\mathrm{Tree} = \boldsymbol{\mu}(f)$$

$$\mathrm{Tree} = f(\mathrm{Tree})$$

$$f(X) \subseteq X \rightarrow \mathrm{Tree} \subseteq X$$

Consider $(\mathcal{P}(\mathrm{Tree} \times \mathrm{Tree}), \subseteq)$. We can inductively define following partial order on Tree.

$$g_{\mathrm{Tree}}(R) = \{(\mathrm{nil}, t) \mid t \in \mathrm{Tree}\} \cup \{(\mathrm{bin}(t_0, t_1), \mathrm{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

$$(\lesssim_{\mathrm{Tree}}) = \boldsymbol{\mu}(g_{\mathrm{Tree}})$$

$$(\lesssim_{\mathrm{Tree}}) = g_{\mathrm{Tree}}(\lesssim_{\mathrm{Tree}})$$

$$g_{\mathrm{Tree}}(R) \subseteq R \rightarrow (\lesssim_{\mathrm{Tree}}) \subseteq R$$

The following is an example of inductive proof using Tarski's principle.

**Example 3.1** (Reflexivity of $\lesssim_{\mathrm{Tree}}$)**.** *Let $X = \{t \mid t \lesssim_{\mathrm{Tree}} t\}$. We have to show that $\mathrm{Tree} \subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of $f$ and $X$, we have to show $(t, t) \in (\lesssim_{\mathrm{Tree}})$ whenever*

$$t \in \{nil\} \cup \{bin(t_0, t_1) \mid t_0 \lesssim_{\mathit{Tree}} t_0, t_1 \lesssim_{\mathit{Tree}} t_1\}$$

*(i) When $t = nil$,*

$$(nil, nil) \in (\lesssim_{\mathit{Tree}}) = g_{\mathit{Tree}}(\lesssim_{\mathit{Tree}}) = \{(nil, t) \mid t \in \mathit{Tree}\} \cup \{\ldots\}$$

*(ii) When $t = bin(t_0, t_1)$ for some $t_0, t_1$ s.t. $t_0 \lesssim_{\mathit{Tree}} t_0$ and $t_1 \lesssim_{\mathit{Tree}} t_1$,*

$$(bin(t_0, t_1), bin(t_0, t_1)) \in (\lesssim_{\mathit{Tree}}) = g_{\mathit{Tree}}(\lesssim_{\mathit{Tree}}) =$$
$$\{(nil, t) \mid t \in \mathit{Tree}\} \cup \{(bin(t_0, t_1), bin(t_0', t_1')) \mid t_0 \lesssim_{\mathit{Tree}} t_0', t_1 \lesssim_{\mathit{Tree}} t_1'\}$$

# 4 Coinductive definition and proof

We defined the set of binary tree as the least fixpoint of some order preserving function $f$. On the other hand, the greatest fixpoint of the same function gives a set of non-wellfounded trees (tree of possibly infinite height).

$$f(X) = \{\mathrm{nil}\} \cup \{\mathrm{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

$$\mathrm{CoTree} = \boldsymbol{\nu}(f)$$

$$\mathrm{CoTree} = f(\mathrm{Tree})$$

$$X \subseteq f(X) \to X \subseteq \mathrm{CoTree}$$

Also, we can coinductively define a partial order of CoTree.

$$g_{\mathrm{CoTree}}(R) = \{(\mathrm{nil}, t) \mid t \in \mathrm{CoTree}\} \cup \{(\mathrm{bin}(t_0, t_1), \mathrm{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

$$(\lesssim_{\mathrm{CoTree}}) = \boldsymbol{\nu}(g_{\mathrm{CoTree}})$$

$$(\lesssim_{\mathrm{CoTree}}) = g_{\mathrm{CoTree}}(\lesssim_{\mathrm{CoTree}})$$

$$R \subseteq g_{\mathrm{CoTree}}(R) \to R \subseteq (\lesssim_{\mathrm{CoTree}})$$

The following is an example of coinductive proof.

**Example 4.1** (Transitivity of $\lesssim_{\mathrm{CoTree}}$)**.** *Let $R = \{(t, t'') \mid \exists t', t \lesssim_{CoTree} t', t' \lesssim_{CoTree} t''\}$. We will show that $R \subseteq (\lesssim_{CoTree})$. By using Tarski's principle, it is enough to show $R \subseteq g_{CoTree}(R)$. Let $(t, t'') \in R$. Then there is $t'$ s.t. $t \lesssim_{CoTree} t'$ and $t' \lesssim_{CoTree} t''$. By unfolding the definition of $g_{CoTree}$, we have to show*

$$(t, t'') \in \{(nil, t) \mid t \in CoTree\} \cup \{(bin(t_0, t_1), bin(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

*By using $(\lesssim_{CoTree}) = g_{CoTree}(\lesssim_{CoTree})$, we know that*

$$(t, t'), (t', t'') \in \{(nil, t) \mid t \in CoTree\}$$
$$\cup \{(bin(t_0, t_1), bin(t_0', t_1')) \mid t_0 \lesssim_{CoTree} t_0', t_1 \lesssim_{CoTree} t_1'\}$$

*Then there are two possible cases.*
*(i) When $t = nil$,*

$$(nil, t'') \in \{(nil, t) \mid t \in CoTree\}$$

*(ii) When $t = bin(t_0, t_1)$, $t' = bin(t_0', t_1')$, and $t'' = bin(t_0'', t_1'')$ for some $t_0, t_1, t_0', t_1', t_0'', t_1''$ s.t. $t_0 \lesssim_{CoTree} t_0'$, $t_1 \lesssim_{CoTree} t_1'$, and $t_0'' \lesssim_{CoTree} t_1''$,*

$$(bin(t_0, t_1), bin(t_0'', t_1'')) \in \{(bin(t_0, t_1), bin(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$