Knaster-Tarski theorem

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Definition (Bottom and top element)

A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write $\bot$ and $\top$ for a bottom element and a top element, respectively.

$$\forall x \in L, \bot \sqsubseteq x$$

$$\forall x \in L, x \sqsubseteq \top$$

Definition (Infimum and supremum)

Let $(L, \sqsubseteq)$ be a poset and A be a subset of L. Then α is an infimum (meet) of A if α is a lower bound of A and $l \sqsubseteq \alpha$ for every lower bound l of A. Similary, β is a supremum (join) of A if β is an upper bound of A and $\beta \sqsubseteq u$ for every upperbound u of A.

Bottom and top elements, as well as infimum and supremum, are unique if they exists.

Definition (Complete lattice)

A poset $(L, \sqsubseteq)$ is a complete lattice if every subset of L has an infimum and a supremum. We write $\bigwedge A$ and $\bigvee A$ for the infimum and the supremum of A

Lemma

Complete lattice has the bottom and the top element, and they can be represented as follows.

$$\bot = \bigwedge L = \bigvee \emptyset$$

$$\top = \bigwedge \emptyset = \bigvee L$$

Remark

Existence of supremum is enough for a poset to be a complete lattice. Let $(L, \sqsubseteq)$ be a poset whose subsets has a supremum, and A be a subset of L. Define $D = \{l \mid l \text{ is a lower bound of } A\}$ and $\alpha = \bigvee D$. Then α is the infimum of A.

Definition (Fixed point)

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfixed point when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.

Lemma

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Then f has the least fixpoint $\mu = \bigwedge\{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigvee\{x \in L \mid x \sqsubseteq f(x)\}$.

Proof.

Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigvee D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of f. Since x is arbitrary, $f(\nu)$ is an upper bound of D, and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of f, $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of $\sqsubseteq$. The same argument can be used to show that μ is the least fixpoint. $\square$

Theorem (Knaster-Tarski)

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f. Then $(P, \sqsubseteq)$ forms a complete lattice.

Proof.

Let W be a subset of P. Define D as the set of upper bounds of W, and suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of f. Since w is arbitrary, $f(x) \in D$, and thus $f(D) \subseteq D$.
Consider a restriction of f from D to D. Then f has the least fixpoint in D. Call it α. Then it is evident that α is the supremum of W in P. □