# The Knaster-Tarski theorem

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# The statement

### Theorem (Knaster-Tarski)

*Let* $(L, \sqsubseteq)$ *be a complete lattice, and* $f : L \to L$ *be an order-preserving function. Let* $P \subseteq L$ *be the set of fixpoints of* $f$*. Then* $(P, \sqsubseteq)$ *forms a complete lattice.*

# Basic definitions

## Definition (Bottom and top element)

A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write $\bot$ and $\top$ for a bottom element and a top element, respectively.

$$\forall x \in L, \bot \sqsubseteq x$$

$$\forall x \in L, x \sqsubseteq \top$$

## Definition (Infimum and supremum)

Let $(L, \sqsubseteq)$ be a poset and $A$ be a subset of $L$. Then $\alpha$ is an infimum (meet) of $A$ if $\alpha$ is a lower bound of $A$ and $l \sqsubseteq \alpha$ for every lower bound $l$ of $A$. Similary, $\beta$ is a supremum (join) of $A$ if $\beta$ is an upper bound of $A$ and $\beta \sqsubseteq u$ for every upperbound $u$ of $A$.

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

## Basic definitions

### Definition (Complete lattice)

A poset $(L, \sqsubseteq)$ is a complete lattice if every subset of $L$ has an infimum and a supremum. We write $\bigsqcap A$ and $\bigsqcup A$ for the infimum and the supremum of $A$, respectively.

### Example (Power set lattice)

Given a set $A$, $(\mathcal{P}(A), \subseteq)$ forms a complete lattice.

**Lemma**

*Complete lattice has the bottom and the top element, and they can be represented as follows.*

$$\bot = \bigsqcap L = \bigsqcup \varnothing$$
$$\top = \bigsqcap \varnothing = \bigsqcup L$$

**Lemma**

*Let $(L, \sqsubseteq)$ be a poset where every subset of $L$ has a supremum. Then $L$ is a complete lattice.*

**Proof.**

Let $A$ be a subset of $L$. Define $D$ as the set of lower bounds of $A$, and $\alpha = \bigsqcup D$. Then $\alpha$ is the infimum of $A$. $\square$

### Definition (Fixed point)

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfixed point of $f$ when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.

# Knaster-Tarski theorem

### Lemma

*Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Then $f$ has the least fixpoint $\mu = \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigsqcup\{x \in L \mid x \sqsubseteq f(x)\}$.*

### Proof.

Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigsqcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of $f$. Since $x$ is arbitrary, $f(\nu)$ is an upper bound of $D$, and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of $f$, $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of $\sqsubseteq$. The same argument can be used to show that $\mu$ is the least fixpoint. $\square$

# Knaster-Tarski theorem

### Theorem (Knaster-Tarski)

*Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of $f$. Then $(P, \sqsubseteq)$ forms a complete lattice.*

### Proof.

Let $W$ be a subset of $P$. Define $D$ as the set of upper bounds of $W$. Then $D$ is a complete sublattice of $L$ with the bottom element $\bigsqcup W$ and a supremum $\bigsqcup W \sqcup \bigsqcup A$ for all $A \subseteq D$.
Suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of $f$. Since $w$ and $x$ are arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$.
Consider a restriction of $f$ from $D$ to $D$. Then $f$ has the least fixpoint in $D$. Call it $\alpha$. Then it is evident that $\alpha$ is the supremum of $W$ in $P$. $\square$

## Inductively defined set

$$\frac{}{\mathsf{nil} \in \mathsf{Tree}} \qquad \frac{t_0 \in \mathsf{Tree} \quad t_1 \in \mathsf{Tree}}{\mathsf{bin}(t_0, t_1) \in \mathsf{Tree}}$$

Figure: An inductive definition of binary tree

# Inductively defined set

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Figure: An inductive definition of binary tree

Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$$

$$\hat{t} \in T = \mathbf{2}^{\mathbf{2}^*}$$

Define $f : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows.

$$\mathsf{nil}(\alpha) = 0$$

$$\mathsf{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

# Inductively defined set

$f$ is an order-preserving function.

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

By using the Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of $f$.

$$\mathsf{Tree} = \boldsymbol{\mu}(f)$$

$$\mathsf{Tree} = f(\mathsf{Tree})$$

$$f(X) \subseteq X \to \mathsf{Tree} \subseteq X$$

# Inductively defined relation

$$\frac{}{\mathsf{nil} \lesssim_{\mathsf{Tree}} t} \quad \frac{t_0 \lesssim_{\mathsf{Tree}} t_0' \quad t_1 \lesssim_{\mathsf{Tree}} t_1'}{\mathsf{bin}(t_0, t_1) \lesssim_{\mathsf{Tree}} \mathsf{bin}(t_0', t_1')}$$

Figure: Inductive definition of tree relation

Consider $(\mathcal{P}(\mathsf{Tree} \times \mathsf{Tree}), \subseteq)$. We can define following partial order on Tree.

$$g_{\mathsf{Tree}}(R) = \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

$$(\lesssim_{\mathsf{Tree}}) = \boldsymbol{\mu}(g_{\mathsf{Tree}})$$

$$(\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}})$$

$$g_{\mathsf{Tree}}(R) \subseteq R \rightarrow (\lesssim_{\mathsf{Tree}}) \subseteq R$$

# Inductive proof

The following proof is an example of inductive proof using Tarski's principle.

**Example (Reflexivity of $\lesssim_{\mathsf{Tree}}$)**

Let $X = \{t \mid t \lesssim_{\mathsf{Tree}} t\}$. We have to show that Tree $\subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of $f$ and $X$, we have to show $(t,t) \in (\lesssim_{\mathsf{Tree}})$ whenever

$$t \in \{\mathsf{nil}\} \cup \{\mathsf{bin}(t_0, t_1) \mid t_0 \lesssim_{\mathsf{Tree}} t_0, t_1 \lesssim_{\mathsf{Tree}} t_1\}$$

(i) When $t = \mathsf{nil}$,

$$(\mathsf{nil}, \mathsf{nil}) \in (\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}}) = \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{\dots\}$$

(ii) When $t = \mathsf{bin}(t_0, t_1)$ for some $t_0, t_1$ s.t. $t_0 \lesssim_{\mathsf{Tree}} t_0$ and $t_1 \lesssim_{\mathsf{Tree}} t_1$,

$$(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0, t_1)) \in (\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}}) = \\ \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid t_0 \lesssim_{\mathsf{Tree}} t_0', t_1 \lesssim_{\mathsf{Tree}} t_1'\}$$

# Coinductively defined set and relation

The greatest fixpoint of $f$ is a set of non-wellfounded trees.

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

$$\mathsf{CoTree} = \boldsymbol{\nu}(f)$$

$$\mathsf{CoTree} = f(\mathsf{Tree})$$

$$X \subseteq f(X) \to X \subseteq \mathsf{CoTree}$$

Define a partial order of CoTree.

$$g_{\mathsf{CoTree}}(R) = \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\mathsf{CoTree}}) = \boldsymbol{\nu}(g_{\mathsf{CoTree}})$$

$$(\lesssim_{\mathsf{CoTree}}) = g_{\mathsf{CoTree}}(\lesssim_{\mathsf{CoTree}})$$

$$R \subseteq g_{\mathsf{CoTree}}(R) \to R \subseteq (\lesssim_{\mathsf{CoTree}})$$

# Coinductive proof

## Example (Transitivity of $\lesssim_{\mathsf{CoTree}}$)

Let $R = \{(t, t'') \mid \exists t', t \lesssim_{\mathsf{CoTree}} t', t' \lesssim_{\mathsf{CoTree}} t''\}$. We will show that $R \subseteq (\lesssim_{\mathsf{CoTree}})$. By using Tarski's principle, it is enough to show $R \subseteq g_{\mathsf{CoTree}}(R)$. Let $(t, t'') \in R$. Then there is $t'$ s.t. $t \lesssim_{\mathsf{CoTree}} t'$ and $t' \lesssim_{\mathsf{CoTree}} t''$. By unfolding the definition of $g_{\mathsf{CoTree}}$, we have to show

$$(t, t'') \in \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

By using $(\lesssim_{\mathsf{CoTree}}) = g_{\mathsf{CoTree}}(\lesssim_{\mathsf{CoTree}})$, we know that

$$\begin{aligned}(t, t'), (t', t'') \in \{&(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \\ &\cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid t_0 \lesssim_{\mathsf{CoTree}} t_0', t_1 \lesssim_{\mathsf{CoTree}} t_1'\}\end{aligned}$$

Then there are two possible cases.
(i) When $t = \mathsf{nil}$,

$$(\mathsf{nil}, t'') \in \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\}$$

(ii) When $t = \mathsf{bin}(t_0, t_1)$, $t' = \mathsf{bin}(t_0', t_1')$, and $t'' = \mathsf{bin}(t_0'', t_1'')$ for some $t_0, t_1, t_0', t_1', t_0'', t_1''$ s.t. $t_0 \lesssim_{\mathsf{CoTree}} t_0'$, $t_1 \lesssim_{\mathsf{CoTree}} t_1'$, and $t_0'' \lesssim_{\mathsf{CoTree}} t_1''$,

$$(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0'', t_1'')) \in \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$