The Knaster-Tarski theorem

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1 Basic definitions

Definition 1.1 (Bottom and top element)**.** *A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write* $\bot$ *and* $\top$ *for a bottom element and a top element, respectively.*

$$\forall x \in L, \bot \sqsubseteq x$$
$$\forall x \in L, x \sqsubseteq \top$$

Definition 1.2 (Infimum and supremum)**.** *Let* $(L, \sqsubseteq)$ *be a poset and* A *be a subset of* L*. Then* α *is an infimum (meet) of* A *if* α *is a lower bound of* A *and* $l \sqsubseteq \alpha$ *for every lower bound* l *of* A*. Similary,* β *is a supremum (join) of* A *if* β *is an upper bound of* A *and* $\beta \sqsubseteq u$ *for every upperbound* u *of* A*.*

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

Definition 1.3 (Complete lattice)**.** *A poset* $(L, \sqsubseteq)$ *is a complete lattice if every subset of* L *has an infimum and a supremum. We write* $\bigsqcap A$ *and* $\bigsqcup A$ *for the infimum and the supremum of* A*, respectively.*

Example 1.4 (Power set lattice)**.** *Given a set* A*,* $(\mathcal{P}(A), \subseteq)$ *forms a complete lattice.*

Lemma 1.5. *Complete lattice has the bottom and the top element, and they can be represented as follows.*

$$\bot = \bigsqcap L = \bigsqcup \varnothing$$
$$\top = \bigsqcap \varnothing = \bigsqcup L$$

Lemma 1.6. *Let* $(L, \sqsubseteq)$ *be a poset where every subset of* L *has a supremum. Then* L *is a complete lattice.*

Proof. Let A be a subset of L. Define D as the set of lower bounds of A, and $\alpha = \bigsqcup D$. Then α is the infimum of A. ☐

Definition 1.7 (Fixed point)**.** *Let* $(L, \sqsubseteq)$ *be a complete lattice, and* $f : L \to L$*. Then* $x \in L$ *is said to be a fixed point, prefixed point, or postfixed point of* f *when it satisfies* $x = f(x)$*,* $f(x) \sqsubseteq x$*,* $x \sqsubseteq f(x)$*, respectively.*

$$\frac{}{\mathrm{nil} \in \mathrm{Tree}} \quad \frac{t_0 \in \mathrm{Tree} \quad t_1 \in \mathrm{Tree}}{\mathrm{bin}(t_0, t_1) \in \mathrm{Tree}}$$

$$\frac{}{\mathrm{nil} \lesssim_{\mathrm{Tree}} t} \quad \frac{t_0 \lesssim_{\mathrm{Tree}} t_0' \quad t_1 \lesssim_{\mathrm{Tree}} t_1'}{\mathrm{bin}(t_0, t_1) \lesssim_{\mathrm{Tree}} \mathrm{bin}(t_0', t_1')}$$

Figure 1: An inductive definition of binary tree and a relation about it

2 Knaster-Tarski theorem

Lemma 2.1. *Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Then f has the least fixpoint $\mu = \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigsqcup\{x \in L \mid x \sqsubseteq f(x)\}$.*

Proof. Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigsqcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of f. Since x is arbitrary, $f(\nu)$ is an upper bound of D, and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of f, $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of $\sqsubseteq$. The same argument can be used to show that μ is the least fixpoint. □

Theorem 2.2 (Knaster-Tarski)**.** *Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f. Then $(P, \sqsubseteq)$ forms a complete lattice.*

Proof. Let W be a subset of P. Define D as the set of upper bounds of W. Then D is a complete sublattice of L with the bottom element $\bigsqcup W$ and a supremum $\bigsqcup W \sqcup \bigsqcup A$ for all $A \subseteq D$.

Suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of f. Since w and x are arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$.

Consider a restriction of f from D to D. Then f has the least fixpoint in D. Call it α. Then it is evident that α is the supremum of W in P. □

3 Inductive definition and proof

As an example, we will define a set of binary tree and a binary relation on it by using the Knaster-Tarski theorem. Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, ...\}$$

$$\hat{t} \in T = \mathbf{2}^{\mathbf{2}^*}$$

Define $f : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows.

$$\mathrm{nil}(\alpha) = 0$$

$$\mathrm{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\mathrm{nil}\} \cup \{\mathrm{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

Then f is an order-preserving function.

By using the Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of f.

$$\mathrm{Tree} = \boldsymbol{\mu}(f)$$

$$\mathrm{Tree} = f(\mathrm{Tree})$$

$$f(X) \subseteq X \to \mathrm{Tree} \subseteq X$$

Consider $(\mathcal{P}(\mathrm{Tree} \times \mathrm{Tree}), \subseteq)$. We can inductively define following partial order on Tree.

$$g_{\mathrm{Tree}}(R) = \{(\mathrm{nil}, t) \mid t \in \mathrm{Tree}\} \cup \{(\mathrm{bin}(t_0, t_1), \mathrm{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\mathrm{Tree}}) = \boldsymbol{\mu}(g_{\mathrm{Tree}})$$

$$(\lesssim_{\mathrm{Tree}}) = g_{\mathrm{Tree}}(\lesssim_{\mathrm{Tree}})$$

$$g_{\mathrm{Tree}}(R) \subseteq R \to (\lesssim_{\mathrm{Tree}}) \subseteq R$$

The following is an example of inductive proof using Tarski's principle.

Example 3.1 (Reflexivity of $\lesssim_{\mathrm{Tree}}$)**.** *Let $X = \{t \mid t \lesssim_{Tree} t\}$. We have to show that $Tree \subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of f and X, we have to show $(t, t) \in (\lesssim_{Tree})$ whenever*

$$t \in \{nil\} \cup \{bin(t_0, t_1) \mid t_0 \lesssim_{Tree} t_0, t_1 \lesssim_{Tree} t_1\}$$

(i) When $t = nil$,

$$(nil, nil) \in (\lesssim_{Tree}) = g_{Tree}(\lesssim_{Tree}) = \{(nil, t) \mid t \in Tree\} \cup \{\ldots\}$$

(ii) When $t = bin(t_0, t_1)$ for some t_0, t_1 s.t. $t_0 \lesssim_{Tree} t_0$ and $t_1 \lesssim_{Tree} t_1$,

$$(bin(t_0, t_1), bin(t_0, t_1)) \in (\lesssim_{Tree}) = g_{Tree}(\lesssim_{Tree}) = \\ \{(nil, t) \mid t \in Tree\} \cup \{(bin(t_0, t_1), bin(t'_0, t'_1)) \mid t_0 \lesssim_{Tree} t'_0, t_1 \lesssim_{Tree} t'_1\}$$

4 Coinductive definition and proof

We defined the set of binary tree as the least fixpoint of some order preserving function f. On the other hand, the greatest fixpoint of the same function gives a set of non-wellfounded trees (tree of possibly infinite height).

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$
$$\text{CoTree} = \boldsymbol{\nu}(f)$$
$$\text{CoTree} = f(\text{Tree})$$
$$X \subseteq f(X) \rightarrow X \subseteq \text{CoTree}$$

Also, we can coinductively define a partial order of CoTree.

$$g_{\text{CoTree}}(R) = \{(\text{nil}, t) \mid t \in \text{CoTree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\text{CoTree}}) = \boldsymbol{\nu}(g_{\text{CoTree}})$$
$$(\lesssim_{\text{CoTree}}) = g_{\text{CoTree}}(\lesssim_{\text{CoTree}})$$
$$R \subseteq g_{\text{CoTree}}(R) \rightarrow R \subseteq (\lesssim_{\text{CoTree}})$$

The following is an example of coinductive proof.

Example 4.1 (Transitivity of $\lesssim_{\text{CoTree}}$)**.** *Let $R = \{(t, t'') \mid \exists t', t \lesssim_{CoTree} t', t' \lesssim_{CoTree} t''\}$. We will show that $R \subseteq (\lesssim_{CoTree})$. By using Tarski's principle, it is enough to show $R \subseteq g_{CoTree}(R)$. Let $(t, t'') \in R$. Then there is t' s.t. $t \lesssim_{CoTree} t'$ and $t' \lesssim_{CoTree} t''$. By unfolding the definition of g_{CoTree}, we have to show*

$$(t, t'') \in \{(nil, t) \mid t \in CoTree\} \cup \{(bin(t_0, t_1), bin(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

By using $(\lesssim_{CoTree}) = g_{CoTree}(\lesssim_{CoTree})$, we know that

$$\begin{aligned}(t, t'), (t', t'') \in \{&(nil, t) \mid t \in CoTree\} \\ &\cup \{(bin(t_0, t_1), bin(t'_0, t'_1)) \mid t_0 \lesssim_{CoTree} t'_0, t_1 \lesssim_{CoTree} t'_1\}\end{aligned}$$

Then there are two possible cases.

(i) When $t = nil$,

$$(nil, t'') \in \{(nil, t) \mid t \in CoTree\}$$

(ii) When $t = bin(t_0, t_1)$, $t' = bin(t'_0, t'_1)$, and $t'' = bin(t''_0, t''_1)$ for some $t_0, t_1, t'_0, t'_1, t''_0, t''_1$ s.t. $t_0 \lesssim_{CoTree} t'_0$, $t_1 \lesssim_{CoTree} t'_1$, and $t''_0 \lesssim_{CoTree} t''_1$,

$$(bin(t_0, t_1), bin(t''_0, t''_1)) \in \{(bin(t_0, t_1), bin(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$