Knaster-Tarski theorem

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Basic definitions

Definition (Bottom and top element)

A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write $\bot$ and $\top$ for a bottom element and a top element, respectively.

$$\forall x \in L, \bot \sqsubseteq x$$

$$\forall x \in L, x \sqsubseteq \top$$

Definition (Infimum and supremum)

Let $(L, \sqsubseteq)$ be a poset and A be a subset of L. Then α is an infimum (meet) of A if α is a lower bound of A and $l \sqsubseteq \alpha$ for every lower bound l of A. Similary, β is a supremum (join) of A if β is an upper bound of A and $\beta \sqsubseteq u$ for every upperbound u of A.

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

Basic definitions

Definition (Complete lattice)

A poset $(L, \sqsubseteq)$ is a complete lattice if every subset of L has an infimum and a supremum. We write $\bigsqcap A$ and $\bigsqcup A$ for the infimum and the supremum of A, respectively.

Example (Power set lattice)

Given a set A, $(\mathcal{P}(A), \subseteq)$ forms a complete lattice.

Lemma

Complete lattice has the bottom and the top element, and they can be represented as follows.

$$\bot = \bigsqcap L = \bigsqcup \varnothing$$
$$\top = \bigsqcap \varnothing = \bigsqcup L$$

Remark

Existence of supremum is enough for a poset to be a complete lattice. Let $(L, \sqsubseteq)$ be a poset where every subset of L has a supremum, and let A be a subset of L. Define $D = \{l \mid l \text{ is a lower bound of } A\}$ and $\alpha = \bigsqcup D$. Then α is the infimum of A.

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Definition (Fixed point)

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfixed point when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.

Lemma

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Then f has the least fixpoint $\mu = \bigsqcap\{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigsqcup\{x \in L \mid x \sqsubseteq f(x)\}$.

Proof.

Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigsqcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of f. Since x is arbitrary, $f(\nu)$ is an upper bound of D, and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of f, $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of $\sqsubseteq$. The same argument can be used to show that μ is the least fixpoint. $\square$

Knaster-Tarski theorem

Theorem (Knaster-Tarski)

Let $(L, \sqsubseteq)$ be a complete lattice, and $f : L \to L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f. Then $(P, \sqsubseteq)$ forms a complete lattice.

Proof.

Let W be a subset of P. Define D as the set of upper bounds of W, and suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of f. Since w is arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$. Consider a restriction of f from D to D. Then f has the least fixpoint in D. Call it α. Then it is evident that α is the supremum of W in P. □

Inductively defined set

$$\frac{}{\mathsf{nil} \in \mathsf{Tree}} \qquad \frac{t_0 \in \mathsf{Tree} \quad t_1 \in \mathsf{Tree}}{\mathsf{bin}(t_0, t_1) \in \mathsf{Tree}}$$

Figure: An inductive definition of binary tree

Inductively defined set

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Figure: An inductive definition of binary tree

Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$$

$$\hat{t} \in T = \mathbf{2}^{\mathbf{2}^*}$$

Define $f : \mathcal{P}(T) \to \mathcal{P}(T)$ as follows.

$$\mathsf{nil}(\alpha) = 0$$

$$\mathsf{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

Inductively defined set

f is an order-preserving function.

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

By using Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of f.

$$\mathsf{Tree} = \boldsymbol{\mu}(f)$$

$$\mathsf{Tree} = f(\mathsf{Tree})$$

$$f(X) \subseteq X \Rightarrow \mathsf{Tree} \subseteq X$$

Inductively defined relation

$$\frac{}{\mathsf{nil} \lesssim_{\mathsf{Tree}} t} \qquad \frac{t_0 \lesssim_{\mathsf{Tree}} t_0' \quad t_1 \lesssim_{\mathsf{Tree}} t_1'}{\mathsf{bin}(t_0, t_1) \lesssim_{\mathsf{Tree}} \mathsf{bin}(t_0', t_1')}$$

Figure: Inductive definition of tree relation

Consider $(\mathcal{P}(\mathsf{Tree} \times \mathsf{Tree}), \subseteq)$. We can define following partial order on Tree.

$$g_{\mathsf{Tree}}(R) = \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid (t_0, t_0') \in R, (t_1, t_1') \in R\}$$

$$(\lesssim_{\mathsf{Tree}}) = \boldsymbol{\mu}(g_{\mathsf{Tree}})$$

$$(\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}})$$

$$g_{\mathsf{Tree}}(R) \subseteq R \Rightarrow (\lesssim_{\mathsf{Tree}}) \subseteq R$$

Inductive proof

The following proof is an example of inductive proof using Tarski's principle.

Example (Reflexivity of $\lesssim_{\mathsf{Tree}}$)

Let $X = \{t \mid t \lesssim_{\mathsf{Tree}} t\}$. We have to show that Tree $\subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of f and X, we have to show $(t,t) \in (\lesssim_{\mathsf{Tree}})$ whenever

$$t \in \{\mathsf{nil}\} \cup \{\mathsf{bin}(t_0, t_1) \mid t_0 \lesssim_{\mathsf{Tree}} t_0, t_1 \lesssim_{\mathsf{Tree}} t_1\}$$

(i) When $t = \mathsf{nil}$,

$$(\mathsf{nil}, \mathsf{nil}) \in (\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}}) = \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{\dots\}$$

(ii) When $t = \mathsf{bin}(t_0, t_1)$ for some t_0, t_1 s.t. $t_0 \lesssim_{\mathsf{Tree}} t_0$ and $t_1 \lesssim_{\mathsf{Tree}} t_1$,

$$\begin{aligned}(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0, t_1)) \in (\lesssim_{\mathsf{Tree}}) = g_{\mathsf{Tree}}(\lesssim_{\mathsf{Tree}}) = \\ \{(\mathsf{nil}, t) \mid t \in \mathsf{Tree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t_0', t_1')) \mid t_0 \lesssim_{\mathsf{Tree}} t_0', t_1 \lesssim_{\mathsf{Tree}} t_1'\}\end{aligned}$$

Coinductively defined set and relation

The greatest fixpoint of f is a set of non-wellfounded trees.

$$f(X) = \{\mathsf{nil}\} \cup \{\mathsf{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

$$\mathsf{CoTree} = \boldsymbol{\nu}(f)$$

$$\mathsf{CoTree} = f(\mathsf{Tree})$$

$$X \subseteq f(X) \Rightarrow X \subseteq \mathsf{CoTree}$$

Define a partial order of CoTree.

$$g_{\mathsf{CoTree}}(R) = \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\mathsf{CoTree}}) = \boldsymbol{\nu}(g_{\mathsf{CoTree}})$$

$$(\lesssim_{\mathsf{CoTree}}) = g_{\mathsf{CoTree}}(\lesssim_{\mathsf{CoTree}})$$

$$R \subseteq g_{\mathsf{CoTree}}(R) \Rightarrow R \subseteq (\lesssim_{\mathsf{CoTree}})$$

Coinductive proof

Example (Transitivity of $\lesssim_{\mathsf{CoTree}}$)

Let $R = \{(t, t'') \mid \exists t', t \lesssim_{\mathsf{CoTree}} t', t' \lesssim_{\mathsf{CoTree}} t''\}$. We will show that $R \subseteq (\lesssim_{\mathsf{CoTree}})$. By using Tarski's principle, it is enough to show $R \subseteq g_{\mathsf{CoTree}}(R)$. Let $(t, t'') \in R$. Then there is t' s.t. $t \lesssim_{\mathsf{CoTree}} t'$ and $t' \lesssim_{\mathsf{CoTree}} t''$. By unfolding the definition of g_{CoTree}, we have to show

$$(t, t'') \in \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

By using $(\lesssim_{\mathsf{CoTree}}) = g_{\mathsf{CoTree}}(\lesssim_{\mathsf{CoTree}})$, we know that

$$\begin{aligned}(t, t'), (t', t'') \in\ & \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\} \\ & \cup \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t'_0, t'_1)) \mid t_0 \lesssim_{\mathsf{CoTree}} t'_0, t_1 \lesssim_{\mathsf{CoTree}} t'_1\}\end{aligned}$$

Then there are two possible cases.
(i) When $t = \mathsf{nil}$,

$$(\mathsf{nil}, t'') \in \{(\mathsf{nil}, t) \mid t \in \mathsf{CoTree}\}$$

(ii) When $t = \mathsf{bin}(t_0, t_1)$, $t' = \mathsf{bin}(t'_0, t'_1)$, and $t'' = \mathsf{bin}(t''_0, t''_1)$ for some $t_0, t_1, t'_0, t'_1, t''_0, t''_1$ s.t. $t_0 \lesssim_{\mathsf{CoTree}} t'_0$, $t_1 \lesssim_{\mathsf{CoTree}} t'_1$, and $t''_0 \lesssim_{\mathsf{CoTree}} t''_1$,

$$(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t''_0, t''_1)) \in \{(\mathsf{bin}(t_0, t_1), \mathsf{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$