

The Knaster-Tarski theorem

문순원

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The statement

Theorem (Knaster-Tarski)

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f . Then (P, \sqsubseteq) forms a complete lattice.

Basic definitions

Definition (Bottom and top element)

A bottom (top) element of a poset is an element less (greater) than or equal to any elements in the poset. We write \perp and \top for a bottom element and a top element, respectively.

$$\forall x \in L, \perp \sqsubseteq x$$

$$\forall x \in L, x \sqsubseteq \top$$

Definition (Infimum and supremum)

Let (L, \sqsubseteq) be a poset and A be a subset of L . Then α is an infimum (meet) of A if α is a lower bound of A and $l \sqsubseteq \alpha$ for every lower bound l of A . Similarly, β is a supremum (join) of A if β is an upper bound of A and $\beta \sqsubseteq u$ for every upperbound u of A .

Bottom and top elements, as well as infimum and supremum, are unique if they exist.

Basic definitions

Definition (Complete lattice)

A poset (L, \sqsubseteq) is a complete lattice if every subset of L has an infimum and a supremum. We write $\sqcap A$ and $\sqcup A$ for the infimum and the supremum of A , respectively.

Example (Power set lattice)

Given a set A , $(\mathcal{P}(A), \sqsubseteq)$ forms a complete lattice.

Basic definitions

Lemma

Complete lattice has the bottom and the top element, and they can be represented as follows.

$$\perp = \bigcap L = \bigsqcup \emptyset$$

$$\top = \bigcap \emptyset = \bigsqcup L$$

Lemma

Let (L, \sqsubseteq) be a poset where every subset of L has a supremum. Then L is a complete lattice.

Proof.

Let A be a subset of L . Define D as the set of lower bounds of A , and $\alpha = \bigsqcup D$. Then α is the infimum of A . □

Basic definitions

Definition (Fixed point)

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$. Then $x \in L$ is said to be a fixed point, prefixed point, or postfix point of f when it satisfies $x = f(x)$, $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$, respectively.

Knaster-Tarski theorem

Lemma

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$ be an order-preserving function. Then f has the least fixpoint $\mu = \bigcap \{x \in L \mid f(x) \sqsubseteq x\}$ and the greatest fixpoint $\nu = \bigcup \{x \in L \mid x \sqsubseteq f(x)\}$.

Proof.

Let $D = \{x \in L \mid x \sqsubseteq f(x)\}$ and $\nu = \bigcup D$. Let $x \in D$. Then $x \sqsubseteq f(x)$ and $x \sqsubseteq \nu$, from which we know $x \sqsubseteq f(x) \sqsubseteq f(\nu)$ by monotonicity of f . Since x is arbitrary, $f(\nu)$ is an upper bound of D , and thus $\nu \sqsubseteq f(\nu)$. Again, by monotonicity of f , $f(\nu) \sqsubseteq f(f(\nu))$. But then $f(\nu) \in D$ and thus $f(\nu) \sqsubseteq \nu$. Then $\nu = f(\nu)$ by the antisymmetry of \sqsubseteq . The same argument can be used to show that μ is the least fixpoint. □

Knaster-Tarski theorem

Theorem (Knaster-Tarski)

Let (L, \sqsubseteq) be a complete lattice, and $f : L \rightarrow L$ be an order-preserving function. Let $P \subseteq L$ be the set of fixpoints of f . Then (P, \sqsubseteq) forms a complete lattice.

Proof.

Let W be a subset of P . Define D as the set of upper bounds of W . Then D is a induced subposet of L that is complete with the bottom element $\bigsqcup W$ and a supremum $\bigsqcup W \sqcup \bigsqcup A$ for all $A \subseteq D$.

Suppose $x \in D$, $w \in W$. Then $w \sqsubseteq x$ and $w = f(w) \sqsubseteq f(x)$, by monotonicity of f . Since w and x are arbitrary, $f(x) \in D$ for all $x \in D$, and thus $f(D) \subseteq D$.

Consider a restriction of f from D to D . Then f has the least fixpoint in D .

Call it α . Then it is evident that α is the supremum of W in P . □

Inductively defined set

$$\frac{}{\text{nil} \in \text{Tree}} \quad \frac{t_0 \in \text{Tree} \quad t_1 \in \text{Tree}}{\text{bin}(t_0, t_1) \in \text{Tree}}$$

Figure: An inductive definition of binary tree

Inductively defined set

$$\frac{}{\text{nil} \in \text{Tree}} \quad \frac{t_0 \in \text{Tree} \quad t_1 \in \text{Tree}}{\text{bin}(t_0, t_1) \in \text{Tree}}$$

Figure: An inductive definition of binary tree

Consider the power set lattice $(\mathcal{P}(T), \subseteq)$ where

$$\alpha \in \mathbf{2}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \dots\}$$

$$\hat{t} \in T = \mathbf{2}^*$$

Define $f : \mathcal{P}(T) \rightarrow \mathcal{P}(T)$ as follows.

$$\text{nil}(\alpha) = 0$$

$$\text{bin}(\hat{t}_0, \hat{t}_1)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \varepsilon \\ \hat{t}_0(\beta) & \text{if } \alpha = 0\beta \\ \hat{t}_1(\beta) & \text{if } \alpha = 1\beta \end{cases}$$

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

Inductively defined set

f is an order-preserving function.

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

By using the Knaster-Tarski theorem, we can define the set of binary trees with following properties as the least fixpoint of f .

$$\text{Tree} = \mu(f)$$

$$\text{Tree} = f(\text{Tree})$$

$$f(X) \subseteq X \rightarrow \text{Tree} \subseteq X$$

Inductively defined relation

$$\frac{}{\text{nil} \lesssim_{\text{Tree}} t} \quad \frac{t_0 \lesssim_{\text{Tree}} t'_0 \quad t_1 \lesssim_{\text{Tree}} t'_1}{\text{bin}(t_0, t_1) \lesssim_{\text{Tree}} \text{bin}(t'_0, t'_1)}$$

Figure: Inductive definition of tree relation

Consider $(\mathcal{P}(\text{Tree} \times \text{Tree}), \subseteq)$. We can define following partial order on Tree.

$$g_{\text{Tree}}(R) = \{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\text{Tree}}) = \mu(g_{\text{Tree}})$$

$$(\lesssim_{\text{Tree}}) = g_{\text{Tree}}(\lesssim_{\text{Tree}})$$

$$g_{\text{Tree}}(R) \subseteq R \rightarrow (\lesssim_{\text{Tree}}) \subseteq R$$

Inductive proof

The following proof is an example of inductive proof using Tarski's principle.

Example (Reflexivity of \lesssim_{Tree})

Let $X = \{t \mid t \lesssim_{\text{Tree}} t\}$. We have to show that $\text{Tree} \subseteq X$. By using Tarski's principle, it is enough to show $f(X) \subseteq X$. Let $t \in f(X)$. By unfolding the definition of f and X , we have to show $(t, t) \in (\lesssim_{\text{Tree}})$ whenever

$$t \in \{\text{nil}\} \cup \{\text{bin}(t_0, t_1) \mid t_0 \lesssim_{\text{Tree}} t_0, t_1 \lesssim_{\text{Tree}} t_1\}$$

(i) When $t = \text{nil}$,

$$(\text{nil}, \text{nil}) \in (\lesssim_{\text{Tree}}) = g_{\text{Tree}}(\lesssim_{\text{Tree}}) = \{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{\dots\}$$

(ii) When $t = \text{bin}(t_0, t_1)$ for some t_0, t_1 s.t. $t_0 \lesssim_{\text{Tree}} t_0$ and $t_1 \lesssim_{\text{Tree}} t_1$,

$$\begin{aligned} (\text{bin}(t_0, t_1), \text{bin}(t_0, t_1)) &\in (\lesssim_{\text{Tree}}) = g_{\text{Tree}}(\lesssim_{\text{Tree}}) = \\ &\{(\text{nil}, t) \mid t \in \text{Tree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid t_0 \lesssim_{\text{Tree}} t'_0, t_1 \lesssim_{\text{Tree}} t'_1\} \end{aligned}$$

Coinductively defined set and relation

The greatest fixpoint of f is a set of non-wellfounded trees.

$$f(X) = \{\text{nil}\} \cup \{\text{bin}(\hat{t}_0, \hat{t}_1) \mid \hat{t}_0, \hat{t}_1 \in X\}$$

$$\text{CoTree} = \nu(f)$$

$$\text{CoTree} = f(\text{Tree})$$

$$X \subseteq f(X) \rightarrow X \subseteq \text{CoTree}$$

Define a partial order of CoTree.

$$g_{\text{CoTree}}(R) = \{(\text{nil}, t) \mid t \in \text{CoTree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

$$(\lesssim_{\text{CoTree}}) = \nu(g_{\text{CoTree}})$$

$$(\lesssim_{\text{CoTree}}) = g_{\text{CoTree}}(\lesssim_{\text{CoTree}})$$

$$R \subseteq g_{\text{CoTree}}(R) \rightarrow R \subseteq (\lesssim_{\text{CoTree}})$$

Coinductive proof

Example (Transitivity of \lesssim_{CoTree})

Let $R = \{(t, t'') \mid \exists t', t \lesssim_{\text{CoTree}} t', t' \lesssim_{\text{CoTree}} t''\}$. We will show that $R \subseteq (\lesssim_{\text{CoTree}})$. By using Tarski's principle, it is enough to show $R \subseteq g_{\text{CoTree}}(R)$. Let $(t, t'') \in R$. Then there is t' s.t. $t \lesssim_{\text{CoTree}} t'$ and $t' \lesssim_{\text{CoTree}} t''$. By unfolding the definition of g_{CoTree} , we have to show

$$(t, t'') \in \{(\text{nil}, t) \mid t \in \text{CoTree}\} \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$

By using $(\lesssim_{\text{CoTree}}) = g_{\text{CoTree}}(\lesssim_{\text{CoTree}})$, we know that

$$(t, t'), (t', t'') \in \{(\text{nil}, t) \mid t \in \text{CoTree}\} \\ \cup \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid t_0 \lesssim_{\text{CoTree}} t'_0, t_1 \lesssim_{\text{CoTree}} t'_1\}$$

Then there are two possible cases.

(i) When $t = \text{nil}$,

$$(\text{nil}, t'') \in \{(\text{nil}, t) \mid t \in \text{CoTree}\}$$

(ii) When $t = \text{bin}(t_0, t_1)$, $t' = \text{bin}(t'_0, t'_1)$, and $t'' = \text{bin}(t''_0, t''_1)$ for some $t_0, t_1, t'_0, t'_1, t''_0, t''_1$ s.t. $t_0 \lesssim_{\text{CoTree}} t'_0$, $t_1 \lesssim_{\text{CoTree}} t'_1$, and $t'_0 \lesssim_{\text{CoTree}} t''_0$, $t'_1 \lesssim_{\text{CoTree}} t''_1$,

$$(\text{bin}(t_0, t_1), \text{bin}(t''_0, t''_1)) \in \{(\text{bin}(t_0, t_1), \text{bin}(t'_0, t'_1)) \mid (t_0, t'_0) \in R, (t_1, t'_1) \in R\}$$