B-Spline Basis Functions

Problem Description

Given a non-decreasing sequence of knots $u_0, \ldots, u_N$ where $N > 2n$, the normalized B-spline basis functions of degree n are defined recursively by the Mansfield-de Boor-Cox formula:

$$N_i^0(x) = \begin{cases} 1 & \text{for } x \in [u_i, u_{i+1}), \\ 0 & \text{otherwise,} \end{cases}$$

$$N_i^n(x) = \frac{x - u_i}{u_{i+n} - u_i} N_i^{n-1}(x) + \frac{u_{i+n+1} - x}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(x) \quad \text{for } n > 0,$$

where $N_i^n(x) = 0$ if $u_i = \cdots = u_{i+n+1}$.

Algorithm for Computing B-Spline Basis Functions

/* x in $[u_k, u_{k+1})$ is a subset of $[u_n, u_{N-n})$ */

$b[k] = 1;$ /* $N_k^0 = 1$ */

for (j = 1; j <= n; j++) {

  $\beta = (u_{k+1} - x)/(u_{k+1} - u_{k-j+1});$ /* $\beta = \beta_{k-j+1}^{(j)}$ */

  $b[k-j] = \beta * b[k-j+1];$ /* $N_{k-j}^j = \beta N_{k-j+1}^{j-1}$ */

  for (i = k-j+1; i < k; i++) {

    $\alpha = 1 - \beta;$ /* $\alpha = \alpha_i^{(j)}$ */

    $\beta = (u_{i+j+1} - x)/(u_{i+j+1} - u_{i+1});$ /* $\beta = \beta_{i+1}^{(j)}$ */

    $b[i] = \alpha * b[i] + \beta * b[i+1];$ /* $N_i^j = \alpha N_i^{j-1} + \beta N_{i+1}^{j-1}$ */

  }

  $\alpha = 1 - \beta;$

  $b[k] *= \alpha;$ /* $N_k^j = \alpha N_k^{j-1}$ */

}

Algorithm for Computing B-Spline Function Values

Given the control points d_i and the B-spline basis functions $N_i^n(x)$, the B-spline function $s(x)$ is

$$s(x) = \sum_{i=0}^{N-n-1} d_i N_i^n(x) \quad \text{for } x \in [u_n, u_{N-n}).$$

/* $d_i^{(0)} = d_i$ for $i = k-n, \ldots, k, x$ in $[u_k, u_{k+1})$ is a subset of $[u_n, u_{N-n})$ */
for (j = 1; j <= n; j++)
  for (i = k-n+j; i <= k; i++) {
    $\alpha = (x - u_i)/(u_{i+n+1-j} - u_i);$ /* $\alpha = \alpha_i^{(n+1-j)}$ */
    $d_i^{(j)} = (1-\alpha) * d_{i-1}^{(j-1)} + \alpha * d_i^{(j-1)};$
  }
/* $d_k^{(n)} = s(x)$ */

Implementation

Write a program to find the B-spline function $s(x)$ that minimizes the sum of squared errors

$$E = \sum_{j=0}^{M} (s(x_j) - y_j)^2,$$

given data points (x_j, y_j). Compute the coefficients d_i by solving the linear least squares problem:

$$\sum_{i=0}^{N-4} a_{ji} d_i = y_j, \quad j = 0, \ldots, M,$$

where $a_{ji} = N_i^3(x_j)$. Use sparse matrix representation if necessary. Plot the resulting spline and the data points.