Parametric Curves

Problem Description

A parametric curve is a set of vectors $S = \{s(t) : t \in [a, b]\}$ defined by a vector function s. If s has a continuous derivative almost everywhere, the length of the curve can be calculated as

$$\ell(S) = \int_a^b \|s'(t)\| \, dt.$$

A cubic B-spline curve is given by the parametrization

$$s(t) = \sum_{i=0}^{N-n-1} d_i N_i^n(t), \quad t \in [a, b),$$

using B-spline basis functions of degree n with non-decreasing knots $u_0, \ldots, u_N$ such that $u_n = a$ and $u_{N-n} = b$. The vector coefficients d_i are the control points of the curve.

Derivative of B-Spline Curve

The derivative of a cubic B-spline curve is a quadratic B-spline vector function. Therefore, the length calculation involves integrating the square root of a quartic spline function. This requires numerical integration using a suitable quadrature method.

$$s'(t) = \sum_{i=0}^{N-n-1} d_i \left(N_i^n(t)\right)', \quad t \in [a, b),$$

$$\frac{d}{dt} N_i^n(t) = \frac{n}{u_{i+n} - u_i} N_i^{n-1}(t) - \frac{n}{u_{i+n+1} - u_{i+1}} N_{i+1}^{n-1}(t).$$

Alternatively,

$$s'(t) = \sum_{i=0}^{N-n-2} \frac{n}{u_{i+n+1} - u_{i+1}} (d_{i+1} - d_i) N_{i+1}^{n-1}(t).$$

Numerical Integration

Implement a script to compute the length of the cubic B-spline curve based on given knots and control points. Assume knots are equally spaced $u_i = i$ for simplicity.

Use a composite quadrature method, such as Simpson's or Gauss-Legendre fourth order, to approximate the integral in subintervals where the integrand is smooth.

The script should optimize the number of quadrature nodes to achieve a given tolerance ϵ, comparing results from quadrature intervals of length h and $h/2$.