Collateral volatility

Paul McCloud
Fixed Income Quantitative Research Nomura

May 13, 2013

Abstract

Collateral discounting recognises the value of funding for derivatives, which has gained prominence in recent years as basis spreads have widened in response to the financial crises. This article considers the impact of collateral volatility on discount factors and Libor and FX forwards, and re-examines the core assumptions of the approach. Convenient expressions are derived for convexity adjustments and collateral options, in a form that easily integrates into curve building and pricing. Analysis of the models with reasonable volatility assumptions suggests that these pricing adjustments are not negligible.

1 Introduction

One of the more persistent after-effects of the market dislocations of recent years has been the significant widening of basis spreads, which has in turn resulted in greater variability in the rates of return on collateral posted in different currencies. Combined with the recognition of the central role that collateral plays in the structure of a trade, this has led to the gradual adoption of a new paradigm for discounting. The increased complexity of discounting has not, however, been simultaneously matched by a growth in the liquid markets that could be used to hedge the novel risks this creates, and the vacuum in the pricing framework has been filled with unjustified assumptions, albeit pragmatic ones, whose implications have not been well understood.

Typical market quotes are provided for hedging instruments that use standard clearing house collateral, roughly summarised as local currency OIS for single-currency trades and USD OIS for cross-currency trades. From these quotes, a full population of single-currency (collateralised with local currency OIS) and cross-currency (collateralised with USD OIS) discount factors can be interpolated. The question remains: how can this data be extrapolated to generate a discount factor collateralised in a currency other than local or USD? The most commonplace, and arguably the most natural, assumption to make is that FX forwards do not depend on the collateral posted, so that discounting can be extended using the FX spot/forward no-arbitrage relationship. In this way, the discount factor rectangle of payment currency versus collateral index is completed.

But what is the theoretical basis for this assumption, and what, if any, are the economic implications? In reality, FX forwards are convexity adjusted when the collateral basis is correlated with the FX spot, implying an adjustment to the non-standard discount factors. Adjustments of a similar nature apply to Libor forwards, and this further complicates the stripping of market curves.

The complexity of collateral convexity, and its presence at the heart of pricing, combine to ensure this is a phenomenon often discussed but rarely addressed. This article considers two significant practical questions: how can collateral convexity be efficiently engineered into pricing, and how can it be quantified? The first question itself resolves into two parts: how is collateral convexity consistently incorporated into market curve stripping, and how does it affect the pricing of exotic payoffs? The second question then relates to the dynamic modelling of collateral convexity, analysing the risks that are generated and how they might be hedged.

2 Collateral convexity

Following the approach developed in [Johannes Sundaresan 2007] and later in [Piterbarg 2010] and [Fujii Takahashi 2011], the impact of the collateral cashflows is absorbed in the martingale condition for a collateralised price process. The result is a model for collateral discounting that looks in all aspects like the standard model of discounting, but with the risk-free rate replaced by the collateral rate. This brings collateral centre stage in the modelling of key financial variables, such as discount factors and FX forwards.

2.1 Collateral discounting

The traditional view of a derivative, supported by the language in a typical term sheet, is that of a discrete strip of future cashflows, whose price is their net present value. For cashflows paid in currency e, the standard model values the derivative using a combination of risk-neutral measure $\mathbb{E}^e$ and risk-free rate $\bar{r}^e_t$. The numeraire in this model is the money market account $\exp[\int_{\tau=0}^{t} \bar{r}^e_\tau \, d\tau]$, and the ratio of the price v^e_t over the numeraire is a martingale in the risk-neutral measure:

$$v_t^e = \mathbb{E}_t^e[\exp[-\int_{\tau=t}^T \bar{r}_\tau^e \, d\tau] v_T^e] \tag{2.1}$$

where $\mathbb{E}_t^e$ denotes the expectation conditional on the filtration at time t in the risk-neutral measure. For this expression to be valid, the payoff must reflect the actual cashflows received, incorporating in particular the consequences of default. This implies a divergence between the nominal payoff for the derivative, determined purely as a formula of market references, and the default-adjusted true payoff.

Collateralisation attempts to mitigate the effects of default, thereby re-establishing a clean link between the nominal payoff and the price of the derivative. When the derivative is collateralised, the net present value is given back as collateral which is rebased as the market moves. This provides both protection against default and a source of funding for the position. The derivative is then more accurately described as an entitlement to a continuous strip of rebalancing margin cashflows with zero net present value, where the price matches the level of collateral to be posted. In this regard, the collateralised derivative appears more akin to a futures contract.

The cross-currency model is augmented by the FX spot x_t^{ef} for each pair of currencies e and f. Collateralisation in the ideal sense discussed here requires for its construction the existence of liquid traded collateral with transparent market price. For the collateral labelled by c, take the unit price of collateral to be s_t^c in currency c, where for notational convenience the label for the collateral is also used to indicate its denomination currency. Consider the collateralised derivative with price $v_t^{c|e}$ in pay currency e. Exact matching of one unit of the derivative requires $v_t^{c|e}/x_t^{ec}s_t^c$ units of the collateral. If the derivative is held over the time interval from t to $t+dt$, the cashflows in currency e at time $t+dt$ are $v_{t+dt}^{c|e}$ from the derivative offset by $(x_{t+dt}^{ec}s_{t+dt}^c/x_t^{ec}s_t^c)v_t^{c|e}$ from the collateral, and the net present value of this package is zero. Applying the standard model to these cashflows generates the relationship:

$$0 = \mathbb{E}_t^e[\exp[-\bar{r}_t^e \, dt](v_{t+dt}^{c|e} - \frac{x_{t+dt}^{ec}s_{t+dt}^c}{x_t^{ec}s_t^c} v_t^{c|e})] \tag{2.2}$$

which is re-arranged to the drift condition:

$$\frac{\mathbb{E}_t^e[dv_t^{c|e}]}{v_t^{c|e}} = \frac{\mathbb{E}_t^e[d(x_t^{ec}s_t^c)]}{x_t^{ec}s_t^c} \tag{2.3}$$

This condition requires that the drift in the price of the collateralised derivative matches the drift in the price of the collateral denominated in the same currency. Define the collateral rate $r_t^{c|e}$ with collateral c and pay currency e by:

$$r_t^{c|e}\,dt = \frac{\mathbb{E}_t^e[d(x_t^{ec}s_t^c)]}{x_t^{ec}s_t^c} \tag{2.4}$$

Applying the drift condition to the case of a collateralised loan demonstrates that the collateral rate is the instantaneous repo rate for the collateral. The drift condition is then integrated to generate the pricing formula:

$$v_t^{c|e} = \mathbb{E}_t^e[\exp[-\int_{\tau=t}^{T} r_\tau^{c|e}\,d\tau]v_T^e] \tag{2.5}$$

for the payoff v_T^e in currency e at time T. The numeraire in this model is the collateral account $\exp[\int_{\tau=0}^{t} r_\tau^{c|e}\,d\tau]$, and the ratio of the collateralised price $v_t^{c|e}$ over the numeraire is a martingale in the risk-neutral measure.

Observe that the risk-free rate drops out of the pricing formula – the form of the standard pricing formula is retained, but with the risk-free rate usurped by the collateral rate. For this argument to be valid, the essential requirement of the money market numeraire is that it is predictable, so that the risk-free rate is eliminated from the pricing expression. The risk-neutral measure remains, and the collateral cashflows are absorbed by replacing the risk-free rate with the collateral rate. This strongly suggests that the risk-free rate is not a prerequisite of the formalism. Returning to first principles, [Piterbarg 2012] completely excises the risk-free rate, removing one of the less tangible precursors of the standard theory.

2.2 Collateral triangle relationships

The pricing model can be defined for different choices of collateral, and can also be expressed with respect to the risk-neutral measures of other pay currencies. It is essential that the models with different denominating currencies are economically equivalent: if the price $v_t^{c|e}$ satisfies the drift condition in the pay currency e, then the price $x_t^{fe}v_t^{c|e}$ should satisfy the equivalent drift condition in the alternative pay currency f. It is not immediately obvious that this can be achieved simultaneously for all collateral choices, as the only tool available to engineer the constraint is the Radon-Nikodym derivative $m_t^{ef} = d\mathbb{E}_t^f/d\mathbb{E}_t^e$ between the risk-neutral measures. As will be demonstrated below, consistency of pricing in different currencies imposes additional conditions that translate to constraints on the drifts of the FX spots and the bases between the collateral rates.

Consider the price $v_t^{c|e}$ that satisfies the drift condition with collateral c and pay currency e. Assuming only this drift condition, and that the measure change m_t^{ef} is a martingale in the measure $\mathbb{E}^e$, the drift of the price $x_t^{fe} v_t^{c|e}$ in pay currency f satisfies:

$$\frac{\mathbb{E}_t^f[d(x_t^{fe} v_t^{c|e})]}{x_t^{fe} v_t^{c|e}} = \frac{\mathbb{E}_t^e[d(x_t^{ec} s_t^c)]}{x_t^{ec} s_t^c} + \frac{\mathbb{E}_t^e[d(m_t^{ef} x_t^{fe})]}{m_t^{ef} x_t^{fe}} + \frac{\mathbb{E}_t^e[d(m_t^{ef} x_t^{fe})\, dv_t^{c|e}]}{m_t^{ef} x_t^{fe} v_t^{c|e}} \tag{2.6}$$

This expression is valid for any price that satisfies the drift condition, and so applies specifically to the collateral itself, $v_t^{c|e} = x_t^{ec} s_t^c$:

$$\frac{\mathbb{E}_t^f[d(x_t^{fc} s_t^c)]}{x_t^{fc} s_t^c} = \frac{\mathbb{E}_t^e[d(x_t^{ec} s_t^c)]}{x_t^{ec} s_t^c} + \frac{\mathbb{E}_t^e[d(m_t^{ef} x_t^{fe})]}{m_t^{ef} x_t^{fe}} + \frac{\mathbb{E}_t^e[d(m_t^{ef} x_t^{fe})\, d(x_t^{ec} s_t^c)]}{m_t^{ef} x_t^{fe} x_t^{ec} s_t^c} \tag{2.7}$$

Combining the general and the specific cases leads to the drift relationship:

$$\begin{aligned} \frac{\mathbb{E}_t^f[d(x_t^{fe} v_t^{c|e})]}{x_t^{fe} v_t^{c|e}} &= \frac{\mathbb{E}_t^f[d(x_t^{fc} s_t^c)]}{x_t^{fc} s_t^c} \\ &\quad + \mathbb{E}_t^e[\frac{d(m_t^{ef} x_t^{fe})}{m_t^{ef} x_t^{fe}}(\frac{dv_t^{c|e}}{v_t^{c|e}} - \frac{d(x_t^{ec} s_t^c)}{x_t^{ec} s_t^c})] \end{aligned} \tag{2.8}$$

The price $x_t^{fe} v_t^{c|e}$ satisfies the drift condition with collateral c and pay currency f only when m_t^{ef} is chosen so that the last term in this expression vanishes. Represented in this form, it is clear that the anomalous term vanishes when the product $m_t^{ef} x_t^{fe}$ is predictable:

$$\frac{d(m_t^{ef} x_t^{fe})}{m_t^{ef} x_t^{fe}} = -\mu_t^{ef}\, dt \tag{2.9}$$

for some drift μ_t^{ef}. This is integrated to generate the Radon-Nikodym derivative between the risk-neutral measures:

$$\frac{d\mathbb{E}_t^f}{d\mathbb{E}_t^e} = \exp[-\int_{\tau=0}^{t} \mu_\tau^{ef}\, d\tau] x_t^{ef} \tag{2.10}$$

There are many important consequences of these results. First, note that the martingale property for the measure change requires that the drift μ_t^{ef} is the drift of the FX spot x_t^{ef}:

$$\mu_t^{ef}\, dt = \frac{\mathbb{E}_t^e[dx_t^{ef}]}{x_t^{ef}} \tag{2.11}$$

so that the Radon-Nikodym derivative is the drift-adjusted FX spot. Feeding the measure change into the specific case of the drift condition above shows that the collateral rates are linked by the FX drifts. The relationships of the FX drifts and collateral bases with the collateral rates are then summarised by:

FX drift The FX drift μ_t^{ef} is the difference between the collateral rates $r_t^{c|e}$ and $r_t^{c|f}$:

$$\mu_t^{ef} = r_t^{c|e} - r_t^{c|f} \tag{2.12}$$

for the pay currencies e and f. This relationship is independent of the choice of collateral c. Consequently, the FX drifts satisfy the triangle relationship:

$$\mu_t^{ef} + \mu_t^{fg} = \mu_t^{eg} \tag{2.13}$$

for the pay currencies e, f and g.

Collateral basis The collateral basis β_t^{cb} is the difference between the collateral rates $r_t^{c|e}$ and $r_t^{b|e}$:

$$\beta_t^{cb} = r_t^{c|e} - r_t^{b|e} \tag{2.14}$$

for the collaterals c and b. This relationship is independent of the choice of pay currency e. Consequently, the collateral bases satisfy the triangle relationship:

$$\beta_t^{cb} + \beta_t^{ba} = \beta_t^{ca} \tag{2.15}$$

for the collaterals c, b and a.

The triangle relationships constrain the FX drifts and collateral bases, and these cannot be arbitrarily specified in the model. A convenient way to represent the complete set of consistent FX drifts and collateral bases is to define them relative to a nominated domestic collateral d. The independent variables are the FX drift μ_t^{de} against the domestic currency for each pay currency e, the collateral basis β_t^{cd} against the domestic collateral for each collateral c, and the domestic collateral rate $r_t^{d|d}$. The FX drift μ_t^{ef}, collateral basis β_t^{cb} and collateral rate $r_t^{c|e}$ are then uniquely expanded as:

$$\begin{aligned}
\mu_t^{ef} &= \mu_t^{df} - \mu_t^{de} \\
\beta_t^{cb} &= \beta_t^{cd} - \beta_t^{bd} \\
r_t^{c|e} &= \beta_t^{cd} + r_t^{d|d} - \mu_t^{de}
\end{aligned} \tag{2.16}$$

The domestic collateral rate plays a pivotal role in the decomposition of the collateral rate. The collateral basis adjusts to the alternative collateral c, and the FX drift adjusts to the alternative pay currency e.

Given the predominance of USD OIS collateral in the market for cross-currency hedging instruments, it makes sense to take the domestic currency to be USD and the corresponding collateral rate to be USD OIS. This benchmark is mapped to other collaterals and pay currencies using the collateral bases and FX drifts.

2.3 Collateral convexity of discount factors and FX forwards

The model is now used to generate expressions for collateralised discount factors and FX forwards. The discount factor $p^{c|e}_{tT}$, with collateral c, currency e and maturity T, is defined by the payoff $v^e_T = 1$ in currency e at time T. The pricing formula becomes:

$$p^{c|e}_{tT} = \mathbb{E}^e_t[\exp[-\int_{\tau=t}^{T} r^{c|e}_\tau \, d\tau]] \tag{2.17}$$

Similarly, the FX forward $x^{c|ef}_{tT}$, with collateral c, currencies e and f and maturity T, is defined by the payoff $v^e_T = x^{ef}_T - x^{c|ef}_{tT}$ in currency e at time T. Since this payoff has zero net present value by design, the pricing formula is re-arranged to:

$$x^{c|ef}_{tT} = x^{ef}_t \frac{\mathbb{E}^f_t[\exp[-\int_{\tau=t}^{T} r^{c|f}_\tau \, d\tau]]}{\mathbb{E}^e_t[\exp[-\int_{\tau=t}^{T} r^{c|e}_\tau \, d\tau]]} \tag{2.18}$$

This is equivalent to the FX spot/forward no-arbitrage relationship for collateralised discount factors and FX forwards.

At short maturities, the discount factor and FX forward become:

$$\begin{aligned} \frac{d}{dT} \log[p^{c|e}_{tT}]\Big|_{T=t} &= -r^{c|e}_t \\ \frac{d}{dT} \log[x^{c|ef}_{tT}]\Big|_{T=t} &= \mu^{ef}_t \end{aligned} \tag{2.19}$$

The short maturity discount factor decays at the collateral rate in the pay currency. The short maturity FX forward grows at the FX drift, irrespective of the collateral. This offers a market mechanism for determining the FX drifts.

As with the standard model, the pricing formula is simplified when expressed in terms of the terminal measure associated with the discount factor numeraire. The T-terminal measure $\mathbb{E}^{c|e}_T$ with collateral c and pay currency e is defined by the Radon-Nikodym derivative:

$$\frac{d\mathbb{E}^{c|e}_{tT}}{d\mathbb{E}^{e}_{t}} = \exp[-\int_{\tau=0}^{t} r^{c|e}_{\tau}\, d\tau] p^{c|e}_{tT} \tag{2.20}$$

where $\mathbb{E}^{c|e}_{tT}$ denotes the expectation conditional on the filtration at time t in the terminal measure. Price is then discounted expectation of payoff:

$$v^{c|e}_{t} = p^{c|e}_{tT}\mathbb{E}^{c|e}_{tT}[v^{e}_{T}] \tag{2.21}$$

The relationships between the various risk-neutral and terminal measures are encapsulated in the Radon-Nikodym derivatives:

$$\begin{aligned}
\frac{d\mathbb{E}^{f}_{t}}{d\mathbb{E}^{e}_{t}} &= \exp[-\int_{\tau=0}^{t} \mu^{ef}_{\tau}\, d\tau] x^{ef}_{t} \\
\frac{d\mathbb{E}^{c|f}_{tT}}{d\mathbb{E}^{e}_{t}} &= \exp[-\int_{\tau=0}^{t} r^{c|e}_{\tau}\, d\tau] x^{ef}_{t} p^{c|f}_{tT} \\
\frac{d\mathbb{E}^{c|f}_{tT}}{d\mathbb{E}^{b|e}_{tS}} &= \exp[-\int_{\tau=0}^{t} \beta^{cb}_{\tau}\, d\tau] x^{ef}_{t} \frac{p^{c|f}_{tT}}{p^{b|e}_{tS}}
\end{aligned} \tag{2.22}$$

Whether paying a rate in an unnatural currency or at an unnatural time, the convexity adjustment to the corresponding forward is generated by the correlation between the rate and the measure change that translates from natural to unnatural pricing measure. In the case of collateral convexity, the natural measure is taken to be the T-terminal measure $\mathbb{E}^{d|e}_{T}$ for some nominated domestic collateral d, and the unnatural measure is taken to be the T-terminal measure $\mathbb{E}^{c|e}_{T}$ with alternative collateral c. Applying the relevant Radon-Nikodym derivative to the payoffs for the discount factor and FX forward leads to:

$$\begin{aligned}
p^{c|e}_{tT} &= s^{cd}_{tT} q^{cd|e}_{tT} p^{d|e}_{tT} \\
x^{c|ef}_{tT} &= \frac{q^{cd|f}_{tT}}{q^{cd|e}_{tT}} x^{d|ef}_{tT}
\end{aligned} \tag{2.23}$$

where:

$$\begin{aligned}
s^{cd}_{tT} &= \mathbb{E}^{d|d}_{tT}[\exp[-\int_{\tau=t}^{T} \beta^{cd}_{\tau}\, d\tau]] \\
q^{cd|e}_{tT} &= \frac{\mathbb{E}^{d|d}_{tT}[\exp[-\int_{\tau=t}^{T} \beta^{cd}_{\tau}\, d\tau] x^{de}_{T}]}{\mathbb{E}^{d|d}_{tT}[\exp[-\int_{\tau=t}^{T} \beta^{cd}_{\tau}\, d\tau]]\mathbb{E}^{d|d}_{tT}[x^{de}_{T}]}
\end{aligned} \tag{2.24}$$

The adjustments to the discount factors and FX forwards factorise into two components: the spread adjustment s^{cd}_{tT} driven by the collateral basis, and the convexity adjustment $q^{cd|e}_{tT}$ generated by the correlation between the collateral basis and the FX spot. While collateral convexity complicates the generation of discount factors and FX forwards, this decomposition of the adjustments simplifies their representation within market curve stripping algorithms, and modelling is facilitated by the expressions for the adjustments in terms of the collateral basis and FX spot.

Market quotes that are used for stripping discount factors and FX forwards are commonly based on standard clearing house rules, with USD OIS collateral for cross-currency trades and local currency OIS for single-currency trades. Disregarding for now convexity issues that arise with the stripping of FX forwards from cross-currency basis swaps, assume that for each local currency c the available market instruments are the discount factors $p^{c|c}_{tT}$ collateralised in the local currency OIS, and the FX forwards $x^{d|dc}_{tT}$ against the domestic currency collateralised in the domestic currency OIS.

The FX no-arbitrage constraint generates the cross-currency discount factor $p^{d|e}_{tT}$:

$$p^{d|e}_{tT} = \frac{x^{d|de}_{tT}}{x^{de}_{t}} p^{d|d}_{tT} \tag{2.25}$$

so the discount factors collateralised in the domestic currency are unambiguously defined. This is not the case for other collateral choices. The convexity relationships generate the discount factor $p^{c|e}_{tT}$ in terms of the market variables:

$$p^{c|e}_{tT} = \frac{q^{cd|e}_{tT}}{q^{cd|c}_{tT}} \frac{p^{c|c}_{tT}}{p^{d|c}_{tT}} p^{d|e}_{tT} \tag{2.26}$$

The model for discount factors and FX forwards is completed by the rectangle of discounting convexity curves $q^{cd|e}_{tT}$ for each collateral c and pay currency e. This represents a substantial increase in the market data required to generate the discount factors and FX forwards.

For the positive payoff v^{e}_{T} in currency e at time T, a natural generalisation of the discounting convexity curve $q^{cd|e}_{tT}$ is the collateral convexity curve $Q^{cd|e}_{tT}[v^{e}_{T}]$ defined by:

$$Q^{cd|e}_{tT}[v^{e}_{T}] = \frac{\mathbb{E}^{d|d}_{tT}[\exp[-\int_{\tau=t}^{T} \beta^{cd}_{\tau}\, d\tau] x^{de}_{T} v^{e}_{T}]}{\mathbb{E}^{d|d}_{tT}[\exp[-\int_{\tau=t}^{T} \beta^{cd}_{\tau}\, d\tau]] \mathbb{E}^{d|d}_{tT}[x^{de}_{T} v^{e}_{T}]} \tag{2.27}$$

The discounting convexity curve then corresponds to the unit payoff, $Q^{cd|e}_{tT}[1] = q^{cd|e}_{tT}$.

The effects of collateral convexity are not restricted to discount factors and FX forwards, and further adjustments apply when the payoff is correlated with the collateral basis. This has implications both for market curve stripping and for the pricing of payoffs with non-standard collateral. These issues are explored in the next two sections.

2.4 Collateral convexity of Libor forwards

The Libor rate is the market benchmark for unsecured lending, and typically occurs in single-currency Libor-fixed swaps and cross-currency Libor-Libor basis swaps. There are convexity issues with Libor forwards due to the mis-matched collateralisation of these swaps, similar to the collateral convexity of discount factors and FX forwards. A further complication arises for basis swaps, as the market convention in many currencies is to rebase the notionals of the domestic leg to match the notionals of the foreign leg, using the FX spot at the start of each coupon period. This leads to an additional mark-to-market convexity for the Libor forwards.

Consider the Libor rate l^e_{ST} in currency e, that sets at time S and pays at time T with daycount δ. The Libor forward $l^{c|e}_{tST}$ with collateral c is defined by the payoff $v^e_T = (1 + l^e_{ST}\delta) - (1 + l^{c|e}_{tST}\delta)$ in currency e at time T. Since this payoff has zero net present value by design, the pricing formula is re-arranged to:

$$1 + l^{c|e}_{tST}\delta = \frac{\mathbb{E}^e_t[\exp[-\int_{\tau=t}^T r^{c|e}_\tau \, d\tau](1 + l^e_{ST}\delta)]}{\mathbb{E}^e_t[\exp[-\int_{\tau=t}^T r^{c|e}_\tau \, d\tau]]} \tag{2.28}$$

Equivalently, the Libor forward payoff $(1 + l^{c|e}_{tST}\delta)$ is a martingale in the T-terminal measure $\mathbb{E}^{c|e}_T$ with collateral c and pay currency e.

The Libor forwards with different collateral are related by the convexity adjustment:

$$1 + l^{c|e}_{tST}\delta = Q^{ce|e}_{tT}[1 + l^e_{ST}\delta](1 + l^{e|e}_{tST}\delta) \tag{2.29}$$

The convexity adjustment from the Libor forward $l^{e|e}_{tST}$ with local collateral e to the Libor forward $l^{c|e}_{tST}$ with collateral c resolves into the collateral convexity curve $Q^{ce|e}_{tT}[1 + l^e_{ST}\delta]$, driven by the correlation between the collateral basis and the Libor rate.

In the mark-to-market case, the Libor rate is paid with notional matching the spot price of the unit notional in the rebasing currency at set time, with the notional exchanged at set and pay times. The mark-to-market Libor forward is then defined to be the par rate paid with notional matching the forward price of the unit notional in the rebasing currency at set time, with the notional exchanged at set and pay times. The mark-to-market Libor forward $l^{c|ef}_{tST}$ with collateral c and rebasing currency f is defined by the payoff $v^e_T = (1 + l^e_{ST}\delta)x^{ef}_S - (1 + l^{c|ef}_{tST}\delta)x^{c|ef}_{tS}$ in currency e at time T. Since this payoff has zero net present value by design, the pricing formula is re-arranged to:

$$(1 + l^{c|ef}_{tST}\delta)x^{c|ef}_{tS} = \frac{\mathbb{E}^e_t[\exp[-\int_{\tau=t}^T r^{c|e}_\tau \, d\tau](1 + l^e_{ST}\delta)x^{ef}_S]}{\mathbb{E}^e_t[\exp[-\int_{\tau=t}^T r^{c|e}_\tau \, d\tau]]} \tag{2.30}$$

Equivalently, the mark-to-market Libor forward payoff $(1 + l^{c|ef}_{tST}\delta)x^{c|ef}_{tS}$ is a martingale in the T-terminal measure $\mathbb{E}^{c|e}_T$ with collateral c and pay currency e.

The mark-to-market Libor forwards with different collateral are related by the convexity adjustment:

$$(1 + l^{c|ef}_{tST}\delta)x^{c|ef}_{tS} = Q^{ce|e}_{tT}[(1 + l^e_{ST}\delta)x^{ef}_S](1 + l^{e|ef}_{tST}\delta)x^{e|ef}_{tS} \tag{2.31}$$

Beyond collateral convexity, there is now an additional mark-to-market convexity to account for the notional rebasing, required to relate the mark-to-market Libor forward to its standard counterpart:

$$1 + l^{e|ef}_{tST}\delta = \hat{Q}^{ef}_{tST}(1 + l^{e|e}_{tST}\delta) \tag{2.32}$$

This expression introduces the mark-to-market convexity:

$$\hat{Q}^{ef}_{tST} = \frac{\mathbb{E}^{e|e}_{tS}[p^{e|e}_{ST}(1 + l^e_{ST}\delta)x^{ef}_S]}{\mathbb{E}^{e|e}_{tS}[p^{e|e}_{ST}(1 + l^e_{ST}\delta)]\mathbb{E}^{e|e}_{tS}[x^{ef}_S]} \tag{2.33}$$

The Libor basis $p^{e|e}_{ST}(1+l^e_{ST}\delta)$ that occurs in this definition measures the spread between unsecured and collateralised lending. The convexity adjustment from the Libor forward $l^{e|e}_{tST}$ with local collateral e to the mark-to-market Libor forward $l^{c|ef}_{tST}$ with collateral c and rebasing currency f resolves into the collateral convexity curve $Q^{ce|e}_{tT}[(1 + l^e_{ST}\delta)x^{ef}_S]$, driven by the correlation between the collateral basis and the rebased Libor rate, and the mark-to-market convexity curve $\hat{Q}^{ef}_{tST}$, driven by the correlation between the Libor basis and the rebasing FX spot.

In a typical market configuration, for each currency e the available market data comprises the single-currency swaps with both legs in currency e and the cross-currency basis swaps with legs in currency e and domestic currency d. The swap is collateralised in currency e, so that this market instrument references the Libor forward $l^{e|e}_{tST}$. The basis swap is collateralised in domestic currency d, and moreover the domestic leg is rebased to the currency e, so that this market instrument references the Libor forward $l^{d|e}_{tST}$ and the domestic mark-to-market Libor forward $l^{d|de}_{tST}$. Consistent stripping of the market data thus requires the inclusion of the collateral convexity curve $Q^{de|e}_{tT}[1 + l^e_{ST}\delta]$ and the mark-to-market convexity curve $\hat{Q}^{de}_{tST}$ for each currency e.

The significance of this extends beyond the consistent determination of Libor forwards, as the discount factors that are used for pricing cross-currency instruments are inferred from the combination of single-currency swaps and cross-currency basis swaps. Additional Libor convexity curves are then also required to price Libor swaps with other collaterals and rebasing currencies.

2.5 Collateral convexity of general payoffs

Sadly, the complexities of collateral convexity are not limited to the determination of discount factors and Libor and FX forwards. For the positive payoff v^e_T in currency e at time T, the convexity is:

$$v^{c|e}_t = Q^{ce|e}_{tT}[v^e_T]\frac{p^{c|e}_{tT}}{p^{e|e}_{tT}}v^{e|e}_t \tag{2.34}$$

Applying only the adjustments to the discount factors and Libor forwards fails to capture the idiosyncratic convexity arising from the payoff, which can be similar in magnitude.

As an example, consider the Libor caplet with payoff $v^e_T = ((1 + l^e_{ST}\delta) - (1 + k\delta))^+$ where k is the strike. The price of the caplet is:

$$v^{c|e}_t = p^{c|e}_{tT}((1 + l^{c|e}_{tST}\delta)\Delta^c_l - (1 + k\delta)\Delta^c_k) \tag{2.35}$$

where the deltas are the probabilities of exercise in the corresponding measures:

$$\begin{aligned} \Delta^c_l &= \mathbb{E}^{c|e}_{tT}[(l^e_{ST} > k)\frac{1 + l^e_{ST}\delta}{\mathbb{E}^{c|e}_{tT}[1 + l^e_{ST}\delta]}] \\ \Delta^c_k &= \mathbb{E}^{c|e}_{tT}[(l^e_{ST} < k)] \end{aligned} \tag{2.36}$$

The convexity of the caplet is comparable with the discounting and Libor convexities in rough proportion to their respective deltas, and has a complex moneyness dependence that is hard to capture using the basic convexity curves. If the deltas were independent of the collateral, the convexity of the caplet would be generated purely from the discounting and Libor convexities. This is not the case, however, as additional adjustments are required for the deltas:

$$\begin{aligned} \Delta^c_l &= \Delta^e_l Q^{ce|e}_{tT}[(l^e_{ST} > k)(1 + l^e_{ST}\delta)] \\ \Delta^c_k &= \Delta^e_k Q^{ce|e}_{tT}[(l^e_{ST} < k)] \end{aligned} \tag{2.37}$$

This implies a convexity adjustment to the caplet volatility smile, over and above the adjustments for the discount factors and Libor forwards, generated by the correlation between the collateral basis and the Libor digital payoff.

Collateral convexity is increasingly intractable as the payoff becomes more exotic, and adding new convexity curves for every possible payoff is simply not feasible. The value of a scheme that captures only the discounting and Libor convexities is questionable in these cases, and reserving the convexity may be the preferred strategy.

3 Quantifying collateral convexity

Collateral convexity has been factorised into a collection of convexity curves, that must be incorporated into the market curves in order to identify discount factors and FX forwards with non-standard collateral. This section considers how these convexity curves can be modelled in terms of a small number of easily interpreted volatility parameters.

3.1 Mean reverting normal model for collateral

The expression for collateral convexity simplifies considerably when the variables are lognormal, and this is a useful model for understanding the factors that drive the convexity. To this end, recall that the mean reverting normal model for the rate r_t defined in terms of the vector w_t of uncorrelated Brownian factors by the equation:

$$dr_t = \theta_t(\kappa_t - r_t)\,dt + \phi_t \cdot dw_t \tag{3.1}$$

is integrated to:

$$\begin{aligned} r_T &= \gamma_{tT}[\theta]r_t + \int_{\tau=t}^{T} \gamma_{\tau T}[\theta](\theta_\tau\kappa_\tau\,d\tau + \phi_\tau \cdot dw_\tau) \\ \int_{\tau=t}^{T} r_\tau\,d\tau &= \Gamma_{tT}[\theta]r_t + \int_{\tau=t}^{T} \Gamma_{\tau T}[\theta](\theta_\tau\kappa_\tau\,d\tau + \phi_\tau \cdot dw_\tau) \end{aligned} \tag{3.2}$$

where:

$$
\begin{aligned}
\gamma_{tT}[\theta] &= \exp[-\int_{\tau=t}^{T} \theta_\tau \, d\tau] \\
\Gamma_{tT}[\theta] &= \int_{\tau=t}^{T} \gamma_{t\tau}[\theta] \, d\tau
\end{aligned}
\tag{3.3}
$$

Collateral convexity is generated from the correlation between the collateral basis and the FX spot. For the domestic collateral d, take the uncorrelated Brownian factors w_t to be driftless in the domestic risk-neutral measure $\mathbb{E}^d$. The model considered here and in the following is specified as:

Domestic collateral rate

$$
dr_t^{d|d} = \theta_t(\kappa_t - r_t^{d|d}) \, dt + \phi_t \cdot dw_t \tag{3.4}
$$

The domestic collateral rate $r_t^{d|d}$ is mean reverting normal with mean reversion θ_t and normal volatility ϕ_t.

Collateral basis

$$
d\beta_t^{cd} = \theta_t^c(\kappa_t^c - \beta_t^{cd}) \, dt + \phi_t^c \cdot dw_t \tag{3.5}
$$

For each collateral c, the collateral basis β_t^{cd} is mean reverting normal with mean reversion θ_t^c and normal volatility ϕ_t^c.

FX drift

$$
d\mu_t^{de} = \vartheta_t^e(\varkappa_t^e - \mu_t^{de}) \, dt + \varphi_t^e \cdot dw_t \tag{3.6}
$$

For each pay currency e, the FX drift μ_t^{de} is mean reverting normal with mean reversion ϑ_t^e and normal volatility φ_t^e.

FX spot

$$
\frac{dx_t^{de}}{x_t^{de}} = \mu_t^{de} \, dt + \sigma_t^e \cdot dw_t \tag{3.7}
$$

For each pay currency e, the FX spot x_t^{de} is lognormal with volatility σ_t^e.

The volatility of the domestic collateral rate then cancels in the convexity, leaving only the contribution from the correlation between the collateral basis and the FX spot:

$$
q_{tT}^{cd|e} = \exp[-\int_{\tau=t}^{T} \Gamma_{\tau T}[\theta^c]\phi_\tau^c \cdot (\Gamma_{\tau T}[\vartheta^e]\varphi_\tau^e + \sigma_\tau^e) \, d\tau] \tag{3.8}
$$

Simplify the model further, by assuming that the mean reversions and volatilities are constant. With this simplified model, it is possible to get a representation of the convexity with a minimum of opaque parameters:

$$q_{tT}^{cd|e} = \exp[-(\varrho^{ce}\phi^c\varphi^e)\int_{\tau=t}^{T}\Gamma_{\tau T}[\theta^c]\Gamma_{\tau T}[\vartheta^e]\,d\tau - (\rho^{ce}\phi^c\sigma^e)\int_{\tau=t}^{T}\Gamma_{\tau T}[\theta^c]\,d\tau] \tag{3.9}$$

where:

$$\begin{aligned}\Gamma_{tT}[\theta] &= \frac{1}{\theta}(1-\exp[-\theta(T-t)])\\ \int_{\tau=t}^{T}\Gamma_{\tau T}[\theta]\,d\tau &= \frac{1}{\theta}(\Gamma_{tT}[0]-\Gamma_{tT}[\theta])\\ \int_{\tau=t}^{T}\Gamma_{\tau T}[\theta]\Gamma_{\tau T}[\vartheta]\,d\tau &= \frac{1}{\theta\vartheta}(\Gamma_{tT}[0]-\Gamma_{tT}[\theta]-\Gamma_{tT}[\vartheta]+\Gamma_{tT}[\theta+\vartheta])\end{aligned} \tag{3.10}$$

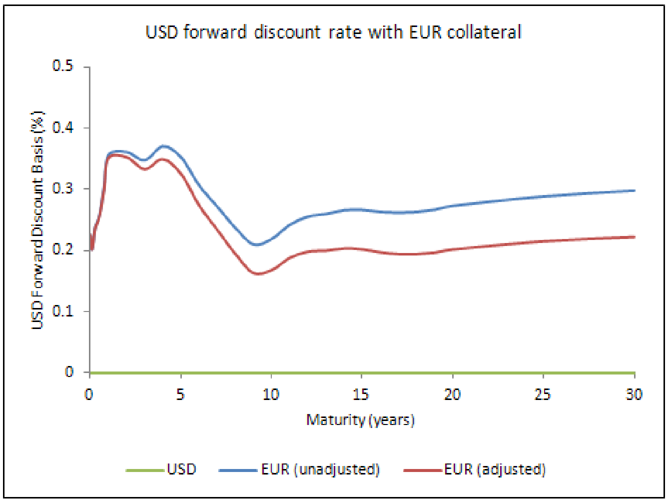
For the collateral c and pay currency e, the model as presented captures the drivers of the discounting convexity curve $q_{tT}^{cd|e}$ in eight parameters: the time-to-maturity $T-t$, the mean reversion θ^c and normal volatility ϕ^c of the collateral basis, the mean reversion ϑ^e and normal volatility φ^e of the FX drift, the lognormal volatility σ^e of the FX spot, the correlation ϱ^{ce} between the collateral basis and the FX drift, and the correlation ρ^{ce} between the collateral basis and the FX spot.

3.2 Case study: USD discounting convexity with EUR collateral

This case study looks at the collateral convexity of USD OIS discount factors with EUR EONIA collateral, based on market data from 21 March 2013. The volatility parameters for the EUR-USD collateral basis and the EURUSD FX spot and drift are taken to be:

$\theta^{€}$	15%
$\phi^{€}$	0.50%
$\vartheta^{€}$	15%
$\varphi^{€}$	0.40%
$\sigma^{€}$	10%
$\varrho^{€€}$	50%
$\rho^{€€}$	10%

The impact of the collateral convexity can be seen by comparing the underlying adjustment to the USD forward discount rates, due to the basis between USD and EUR forward collateral rates, with the convexity adjustment, due to the volatility of the collateral basis. This is shown in the graph below. The graphs on the following page then show how this adjustment is influenced by the volatility parameters.

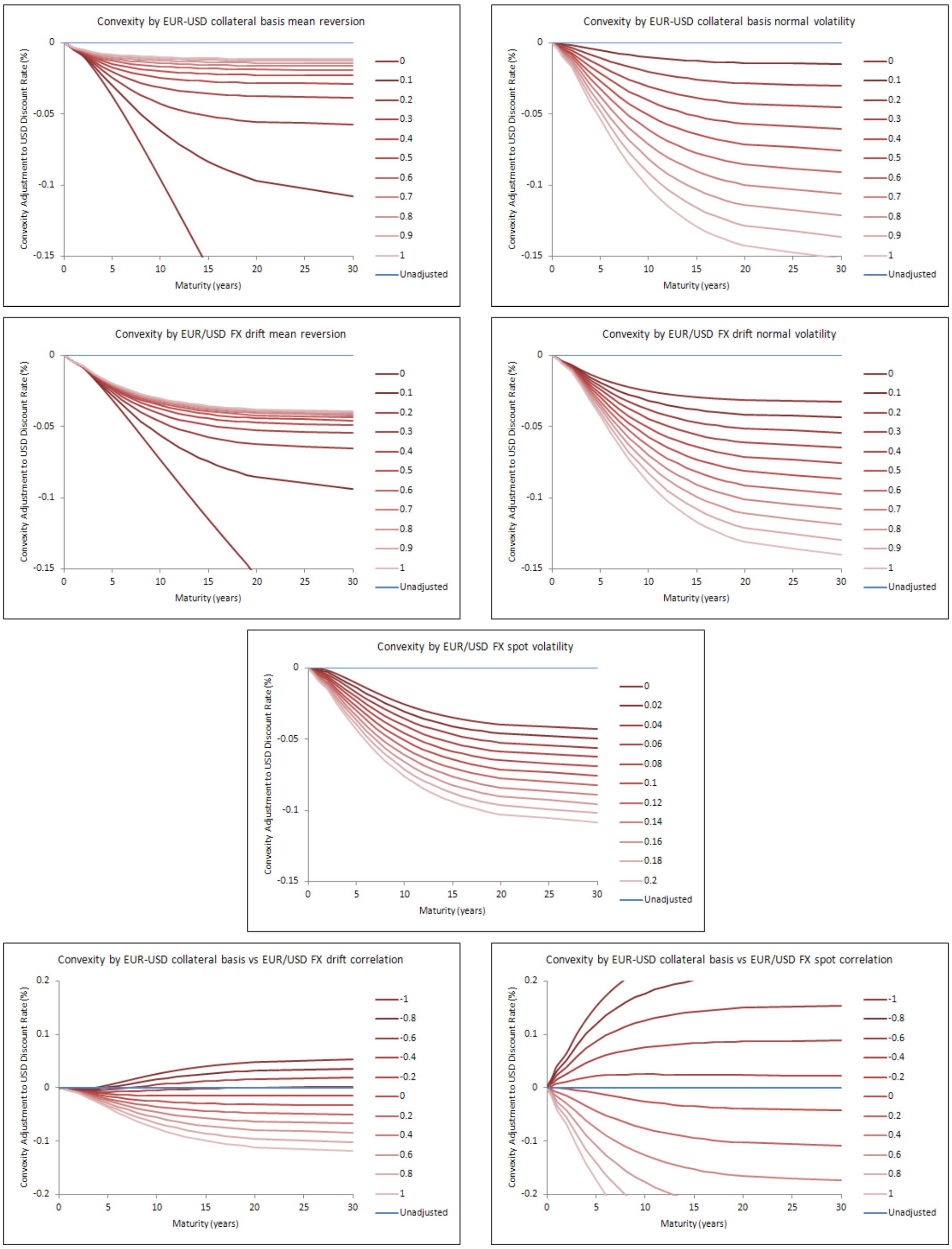


USD discounting convexity with EUR collateral

The basis adjustment to the USD discount factor is familiar, and recognises the difference between funding the discount bond with USD and EUR. The additional convexity correction prices in the hedging bias due to the correlation between the EUR-USD collateral basis and the EURUSD FX. A wide range of convexity adjustments can be achieved in the model within a reasonable range of volatility parameters. This is unfortunate, considering the difficulties in estimating the parameters. A preferable approach is to mark the model to market, though this requires the existence of liquid swap curves in all combinations of collateral and pay currency, a significant increase in market data requirements. In practice, the model serves more as a means to interpolate and extrapolate from a sparse data set for swaps with non-standard collateral.

4 Special topics

There are a number of features of real markets that are not captured by the idealised framework described above, and the model defined in the previous section for the collateral basis and FX spot can also be applied to these situations. In this section, the model is used to evaluate collateral options, where more than one collateral can be posted, and OIS forwards, which can be geometric or arithmetic compounded.



Impact of volatility parameters on the discounting convexity

4.1 Collateral options of discount factors

Many collateral arrangements admit a choice for the collateral that can be posted, and optimality dictates that the choice should be the cheapest-to-deliver. Notwithstanding the legal and practical obstructions to exercising the collateral option, in its purest form the option permits the switching of the entire collateral balance at any point in time, leading to the pricing formula:

$$v_t^{\vee c|e} = \mathbb{E}_t^e[\exp[-\int_{\tau=t}^T \max_i[r_\tau^{c_i|e}]\,d\tau]v_T^e] \tag{4.1}$$

for the payoff v_T^e in currency e at time T, where collateral may be posted in any of the collaterals c_i. In this expression, the optimal collateral rate is the maximum of the available collateral rates, compared from the perspective of the pay currency.

A bound for the pricing formula is obtained by interchanging the integral and maximum operators. When the payoff v_T^e is positive:

$$\begin{aligned} v_t^{\vee c|e} &\leq \mathbb{E}_t^e[\min_i[\exp[-\int_{\tau=t}^T r_\tau^{c_i|e}\,d\tau]]v_T^e] \\ &= \sum_i p_{tT}^{c_i|e}\mathbb{E}_{tT}^{c_i|e}[v_T^e 1_i] \end{aligned} \tag{4.2}$$

where 1_i indicates that the ith collateral account $\exp[\int_{\tau=t}^T r_\tau^{c_i|e}\,d\tau]$ is the largest. By changing to the T-terminal measures $\mathbb{E}_T^{c_i|e}$ with collaterals c_i and pay currency e, the second expression decomposes the bound in terms of the probabilities of the indicators and the corresponding conditional expectations of the payoff in each terminal measure.

The discount factor $p_{tT}^{\vee c|e}$, with collaterals c_i, currency e and maturity T, corresponds to the payoff $v_T^e = 1$ in currency e at time T. The pricing bound becomes:

$$p_{tT}^{\vee c|e} \leq \sum_i p_{tT}^{c_i|e}\mathbb{E}_{tT}^{c_i|e}[1_i] \tag{4.3}$$

so that the discount factor is bounded by a linear combination of the underlying discount factors, weighted by their probabilities in the respective terminal measures. Casting to the risk-neutral measure in domestic currency d, the bound is converted into a form that can be evaluated using the model of the previous section:

$$p_{tT}^{\vee c|e} \leq \mathbb{E}_t^d[\min_i[p_{tT}^{c_i|e}X_i]] \tag{4.4}$$

where the normalised variables X_i are given by:

$$X_i = \frac{\exp[-\int_{\tau=t}^{T}(r_\tau^{d|d} + \beta_\tau^{c_i d})\,d\tau]x_T^{de}}{\mathbb{E}_t^d[\exp[-\int_{\tau=t}^{T}(r_\tau^{d|d} + \beta_\tau^{c_i d})\,d\tau]x_T^{de}]} \tag{4.5}$$

The bound is the switch option between a collection of variables whose means are the underlying discount factors. For the case of two collaterals c_1 and c_2 the bound can be arranged as a spread option between the variables:

$$p_{tT}^{c_1 \vee c_2|e} \leq p_{tT}^{c_1|e} - \mathbb{E}_t^d[(p_{tT}^{c_1|e} X_1 - p_{tT}^{c_2|e} X_2)^+] \tag{4.6}$$

The normalised variables are lognormal in the mean reverting normal model, and their volatilities and correlations are computed using the relationships of the previous section. When there are only two collateral choices, the pricing bound is given by:

$$p_{tT}^{c_1 \vee c_2|e} \leq p_{tT}^{c_1|e} - \mathrm{BS}[p_{tT}^{c_1|e}, p_{tT}^{c_2|e}, \varepsilon_{tT}^{c_1 c_2}, T - t] \tag{4.7}$$

where $\mathrm{BS}[f, k, \varepsilon, \tau]$ is the Black-Scholes formula as a function of the forward f, strike k, lognormal volatility ε and time-to-maturity τ. The spread volatility for the collateral option is:

$$\varepsilon_{tT}^{c_1 c_2} = \sqrt{\frac{1}{T-t}\int_{\tau=t}^{T}(\Gamma_{\tau T}[\theta^{c_1}]\phi_\tau^{c_1} - \Gamma_{\tau T}[\theta^{c_2}]\phi_\tau^{c_2})^2\,d\tau} \tag{4.8}$$

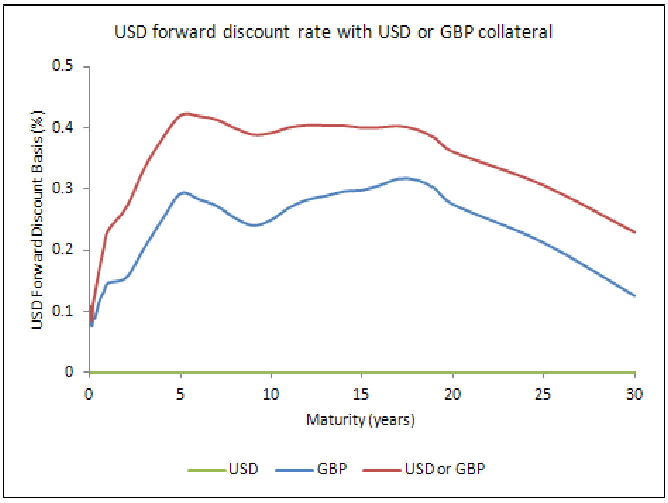
The volatility of the domestic collateral rate and the FX spot do not contribute to the spread volatility, and only the collateral basis volatility remains. Note, however, that the other volatilities still contribute to the convexity adjustments of the underlying discount factors. While the model only provides a bound for the discount factor, it has the benefit of generating a smooth transition between the underlying discount factors that satisfies appropriate boundary conditions and is efficiently computed.

4.2 Case study: USD discounting option with USD or GBP collateral

This case study looks at the collateral option of USD OIS discount factors with USD OIS or GBP SONIA collateral, based on market data from 21 March 2013. The volatility parameters for the GBP-USD collateral basis are taken to be:

$\theta^{£}$	15%
$\phi^{£}$	0.50%

The impact of the collateral option can be seen by comparing the adjustment to the USD forward discount rates, due to the basis between USD and GBP collateral rates, with the option adjustment, due to the choice of USD or GBP collateral. This is shown in the graph below. The graphs on the following page then show how this adjustment is influenced by the volatility parameters.



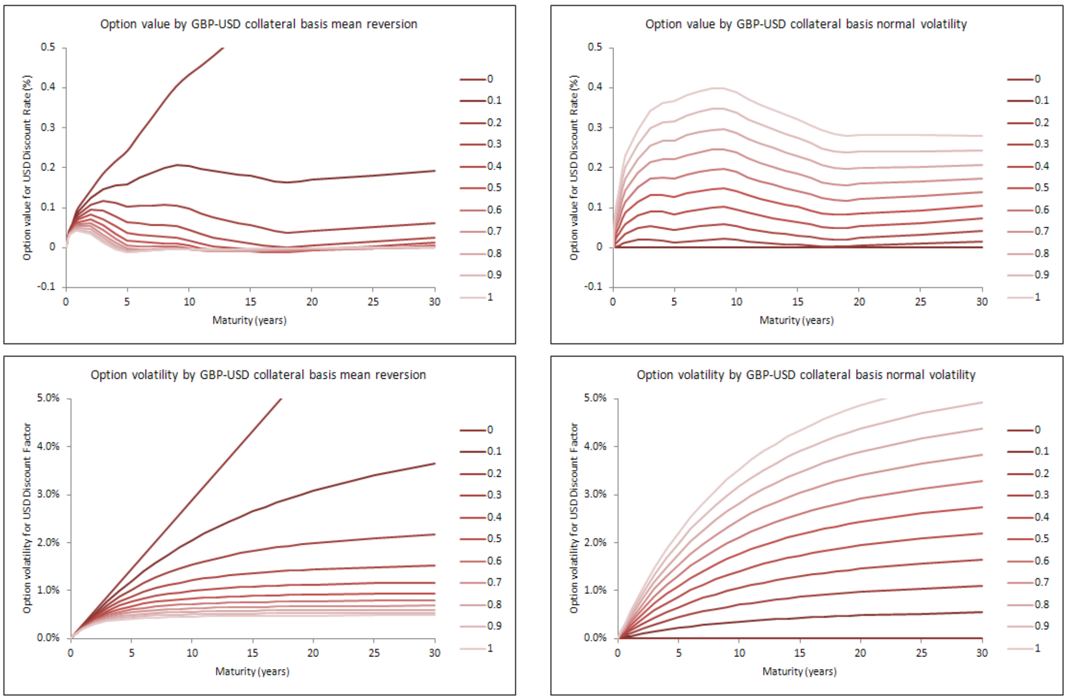
USD discounting option with USD or GBP collateral

The USD discount rate depends on whether the collateral is USD or GBP. When either of these collaterals can be chosen, the discount rate must be at least the maximum of the two underlying discount rates. The model then supplies the additional adjustment for the option value. This can be expressed as a margin on the discount rate, or as the implied lognormal volatility of the ratio of underlying discount factors.

4.3 Compounding convexity of OIS forwards

The convention for Libor-OIS basis swaps in USD, that are commonly used in combination with Libor-fixed swaps to determine OIS forwards, is to take the arithmetic average of the collateral rates over the coupon period. This is in contrast to the geometric compounding that applies to USD collateral earning the collateral rate. As observed in [Takada 2011], the OIS forwards need to be adjusted before they can be used to determine USD collateralised discount factors.

Consider the OIS rate o^d_{ST} in domestic currency d, that sets at time S and pays at time T with daycount δ. The OIS forward $o^{d|d}_{tST}$ with collateral d is defined by the payoff $v^d_T = (1 + o^d_{ST}\delta) - (1 + o^{d|d}_{tST}\delta)$ in currency d at time T. Since this payoff has zero net



Impact of volatility parameters on the discounting option present value by design, the pricing formula is re-arranged to:

$$1 + o^{d|d}_{tST}\delta = \frac{\mathbb{E}^d_t[\exp[-\int_{\tau=t}^{T} r^{d|d}_\tau \, d\tau](1 + o^d_{ST}\delta)]}{\mathbb{E}^d_t[\exp[-\int_{\tau=t}^{T} r^{d|d}_\tau \, d\tau]]} \tag{4.9}$$

The result then depends on whether the collateral rate is compounded geometrically or arithmetically:

Geometric compounding The geometric OIS rate is defined by:

$$o^d_{ST} = \frac{1}{\delta}(\exp[\int_{\tau=S}^{T} r^{d|d}_\tau \, d\tau] - 1) \tag{4.10}$$

The geometric OIS forward is then:

$$1 + o^{d|d}_{tST}\delta = \frac{p^{d|d}_{tS}}{p^{d|d}_{tT}} \tag{4.11}$$

Arithmetic compounding The arithmetic OIS rate is defined by:

$$\tilde{o}^d_{ST} = \frac{1}{\delta}\int_{\tau=S}^{T} r^{d|d}_\tau \, d\tau \tag{4.12}$$

The arithmetic OIS forward is then:

$$1 + \tilde{o}^{d|d}_{tST}\delta = \frac{p^{d|d}_{tS}}{p^{d|d}_{tT}} \mathbb{E}^{d|d}_{tS}[\exp[-\tilde{o}^d_{ST}\delta](1 + \tilde{o}^d_{ST}\delta)] \tag{4.13}$$

The integrand in the expression for the arithmetic OIS forward is a negatively convex function of $\exp[-\tilde{o}^d_{ST}\delta]$, and the mean of this variable is:

$$\mathbb{E}^{d|d}_{tS}[\exp[-\tilde{o}^d_{ST}\delta]] = \frac{1}{1 + o^{d|d}_{tST}\delta} \tag{4.14}$$

Jensen's inequality then dictates that:

$$\tilde{o}^{d|d}_{tST} \leq \frac{1}{\delta}\log[1 + o^{d|d}_{tST}\delta] \tag{4.15}$$

The remaining adjustment from geometric to arithmetic compounding depends on the distribution of the arithmetic OIS rate $\tilde{o}^d_{ST}$ in the S-terminal measure $\mathbb{E}^{d|d}_S$. Suppose that the arithmetic OIS rate is normal in this measure, with mean μ and variance σ^2.

In this case, the expectations become:

$$\begin{aligned} \mathbb{E}^{d|d}_{tS}[\exp[-\tilde{o}^d_{ST}\delta]] &= \exp[-\mu\delta + \frac{1}{2}\sigma^2\delta^2] \\ \mathbb{E}^{d|d}_{tS}[\exp[-\tilde{o}^d_{ST}\delta](1+\tilde{o}^d_{ST}\delta)] &= \exp[-\mu\delta + \frac{1}{2}\sigma^2\delta^2](1+\mu\delta-\sigma^2\delta^2) \end{aligned} \tag{4.16}$$

Combining these expressions, the relationship between the forward rates with geometric and arithmetic compounding is:

$$\begin{aligned} \tilde{o}^{d|d}_{tST} &= \bar{o}^{d|d}_{tST} + \hat{o}^{d|d}_{tST} \\ \bar{o}^{d|d}_{tST} &= \frac{1}{\delta}\log[1+o^{d|d}_{tST}\delta] \\ \hat{o}^{d|d}_{tST} &= -\frac{1}{2}\sigma^2\delta \end{aligned} \tag{4.17}$$

The adjustment for arithmetic compounding decomposes into two terms, representing the underlying adjustment and the convexity adjustment. Both of these adjustments are negative. The underlying adjustment is determined by the geometric OIS forward. The convexity adjustment is independent of the geometric OIS forward, and depends on the normal volatility of the arithmetic OIS rate.

The arithmetic OIS rate is normal when the collateral rate is modelled as mean reverting normal:

$$\begin{aligned} \tilde{o}^d_{ST}\delta &= \\ &\cdots + \int_{\tau=t}^{T} \Gamma_{\tau T}[\theta]\phi_\tau \cdot dw_\tau && (S \leq t) \\ &\cdots + \int_{\tau=t}^{S} \gamma_{\tau S}[\theta]\Gamma_{ST}[\theta]\phi_\tau \cdot dw_\tau + \int_{\tau=S}^{T} \Gamma_{\tau T}[\theta]\phi_\tau \cdot dw_\tau && (S > t) \end{aligned} \tag{4.18}$$

where the deterministic terms have been omitted. The first version covers the case of an OIS rate that has partially fixed, and the second version is for an OIS rate that is entirely in the future. Note that the measure change from $\mathbb{E}^d$ to $\mathbb{E}^{d|d}_S$ does not introduce any additional volatile terms, as the Brownian drift is deterministic:

$$\mathbb{E}^{d|d}_{tS}[dw_t] = -\Gamma_{tS}[\theta]\phi_t\,dt \tag{4.19}$$

Reading the variance from these expressions, the compounding convexity in the mean reverting normal model is:

$$\hat{o}^{d|d}_{tST} = \begin{cases} -\frac{1}{2\delta}\int_{\tau=t}^{T} \Gamma_{\tau T}[\theta]^2 \phi_\tau^2 \, d\tau & (S \leq t) \\ -\frac{1}{2\delta}(\int_{\tau=t}^{S} \gamma_{\tau S}[\theta]^2 \Gamma_{ST}[\theta]^2 \phi_\tau^2 \, d\tau + \int_{\tau=S}^{T} \Gamma_{\tau T}[\theta]^2 \phi_\tau^2 \, d\tau) & (S > t) \end{cases} \tag{4.20}$$

There are two terms in the second version of the convexity adjustment, arising from the volatility up to the start of the coupon period and the volatility during the coupon period. Only the second term appears when the OIS rate has already begun fixing. When the mean reversion and volatility are constant, the model simplifies to:

$$\hat{o}^{d|d}_{tST} = \begin{cases} -\frac{\phi^2}{2\delta}\int_{\tau=t}^{T} \Gamma_{\tau T}[\theta]^2 \, d\tau & (S \leq t) \\ -\frac{\phi^2}{2\delta}(\Gamma_{tS}[2\theta]\Gamma_{ST}[\theta]^2 + \int_{\tau=S}^{T} \Gamma_{\tau T}[\theta]^2 \, d\tau) & (S > t) \end{cases} \tag{4.21}$$

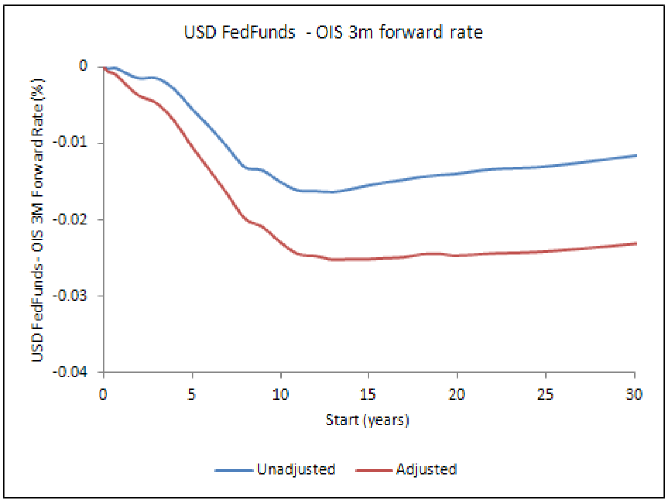
Exploiting the simplifications that arise when the collateral rate is normal, this model represents the compounding convexity in terms of the normal volatility and mean reversion of the collateral rate. As an alternative approach, [Takada 2011] demonstrates that the convexity can be replicated using OIS caplets and floorlets, which captures smile effects in a model-independent manner.

4.4 Case study: Compounding convexity of USD FedFunds forwards

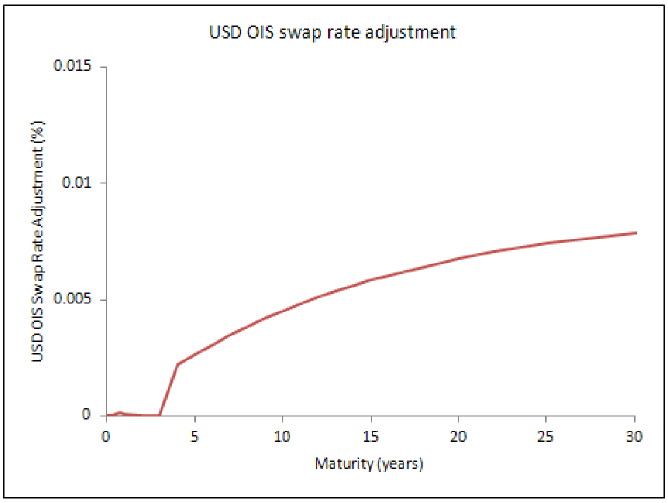
This case study looks at the compounding convexity of USD FedFunds forwards, based on market data from 21 March 2013. The volatility parameters for the USD collateral rate are taken to be:

θ	5%
ϕ	1.00%

The impact of the compounding convexity can be seen by comparing the unadjusted and adjusted FedFunds forwards with the corresponding OIS forwards, where the unadjusted version accounts for the arithmetic averaging and the adjusted version also incorporates the convexity. This is shown in the first graph below. In the case that the USD curve is calibrated to FedFunds forwards, the convexity is instead observed as an adjustment to OIS swap rates. This is shown in the second graph below. The graphs on the following page then show how the adjustment is influenced by the volatility parameters.

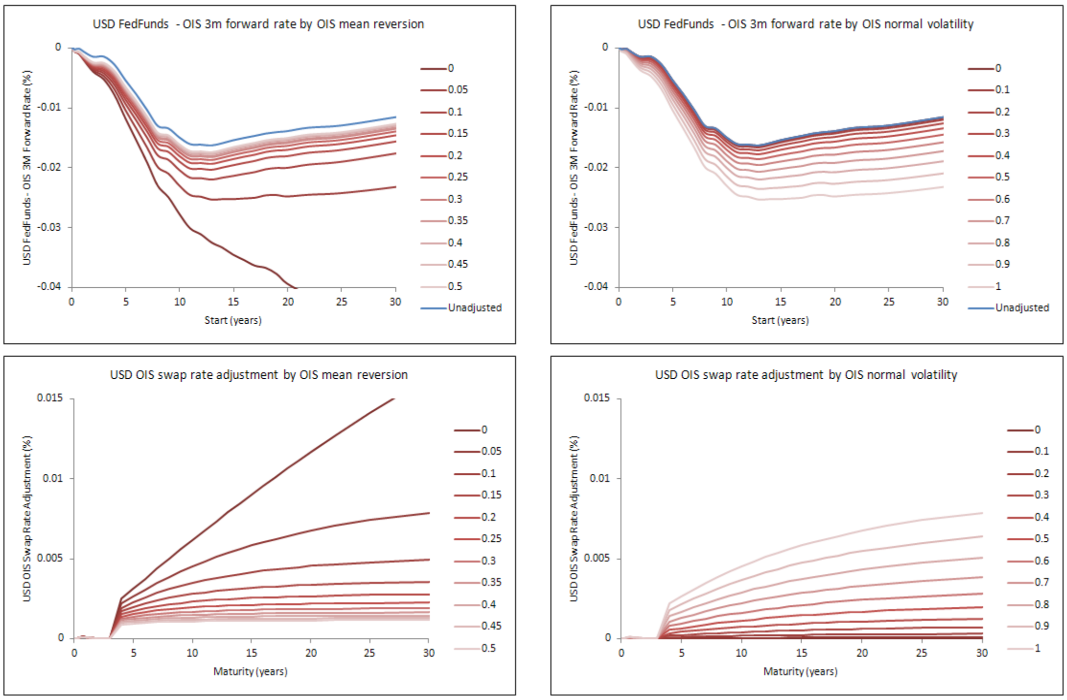


Compounding convexity of USD FedFunds forwards



Compounding convexity of USD OIS swap rates

Convexity of the relationship between arithmetic and geometric compounding ensures that both the underlying and volatility adjustments are directional – downwards in the case of the FedFunds forward adjustment relative to given OIS rates, and upwards in



Impact of volatility parameters on the compounding convexity the case of the OIS swap rate adjustment relative to given FedFunds rates.

5 Conclusion

Collateral convexity is driven by the dynamic relationship of the collateral basis, measured as the spread between the collateral rates compared in the same currency, with the FX spot in the pay currency. The convexity can be significant for long-dated FX forwards, and this has implications for the discounting of cashflows with non-standard collateralisation. Discounting is further complicated when FX forwards are stripped from cross-currency basis swaps, as the convexity adjustments of Libor forwards then contribute. While the various convexities can be easily expressed in terms of a collection of intuitive volatilities and correlations, these parameters are not so easily estimated from market series, and it is not obvious that there is a suitable hedge for the convexity, even less a constructible arbitrage that might exploit the adjustments. In these circumstances, it is debatable how, or even if, the convexity should be marked in pricing.

Limited evidence from the market suggests that collateral convexity is not currently regarded as a contributing factor in pricing, though this may change in the future. The model presented here aims to highlight the potential scale of the issue, and demonstrate the volatility factors that drive the convexity. As markets evolve, the model provides a tool for extrapolating the convexity from the prices of liquid traded swaps with non-standard collateralisation. In any case, the model is useful as a tool for risk management, as it provides convenient and natural risk measures for the convexity that are nettable across maturities and asset classes, and can be used to determine sensible reserve levels for the unhedged risks arising from the convexity.

Efforts to contain the collateral convexity centre on reducing the collateral basis volatility to manageable levels, either by standardising the acceptable forms of collateral, or by ensuring the drivers of the basis – liquidity and default risk – are mitigated. Standardisation of collateral rates has been assisted by developments in the OIS markets, which serve as imperfect proxies for the risk-free ideal. With liquid OIS curves available it becomes less clear what role Libor should play in the inter-bank market, at least insofar as discounting is concerned, as to intermediate the OIS hedge through Libor swaps simply confuses the risk picture, and the volatile OIS/Libor basis places an unwanted burden on the modelling of more exotic structures. Perhaps the real significance of the recognition of collateral convexity is that it further incentivises the push to OIS.

6 References

This work would not have been possible without support and contributions from Zhengyun Hu at Nomura. Thanks also for interesting discussions with Kenjiro Oya, Frederic Gabern and Rob Smith at Nomura, Manlio Trovato and Diana Ribeiro at Lloyds and Nick Lynn at CIBC.

- Johannes, Michael and Sundaresan, Suresh
  'The Impact of Collateralization on Swap Rates'
  The Journal of Finance, February 2007.

- Fujii, Masaaki, Shimada, Yasufumi and Takahashi, Akihiko
  'A Note on Construction of Multiple Swap Curves with and without Collateral'
  CARF Working Paper Series No. CARF-F-154, January 2010. Available at SSRN: http://ssrn.com/abstract=1440633.

- Piterbarg, Vladimir
  'Funding beyond discounting: collateral agreements and derivatives pricing'
  Risk, February 2010.

- Takada, Katsumi
  'Valuation of Arithmetic Average of Fed Funds Rates and Construction of the US Dollar Swap Yield Curve'
  September 2011. Available at SSRN: http://ssrn.com/abstract=1981668.

- Fujii, Masaaki and Takahashi, Akihiko
  'Derivative Pricing under Asymmetric and Imperfect Collateralization and CVA'
  December 2011. Available at SSRN: http://ssrn.com/abstract=1731763.

- Piterbarg, Vladimir
  'Cooking with Collateral'
  Risk, August 2012.

- McCloud, Paul
  'Collateral convexity complexity'
  Risk, May 2013.