PRÁCTICA 3: *Souciones*

Pablo Verdes Dante Zanarini Pamela Viale Alejandro Hernandez Mauro Lucci

1.

- Caso base: $g_0^{(1)} = c^{(1)} \in FRP$.
- Caso inductivo: Supongamos $g_n^{(1)} \in FRP$ (H.I.). Luego:

$$g_{n+1}(x) = n + 1 = \Phi\left(s^{(1)}, \underbrace{g_n^{(1)}}_{H.I.}\right)(x) \in FRP$$

2.

a) $\Phi\left(s^{(1)}, \Phi\left(s^{(1)}, c^{(2)}\right)\right)(17,3) = s^{(1)}\left(\Phi\left(s^{(1)}, c^{(2)}\right)(17,3)\right) = s^{(1)}\left(s^{(1)}\left(c^{(2)}(17,3)\right)\right) = s^{(1)}\left(s^{(1)}(0)\right) = s^{(1)}(1) = 2.$

b) $Mas2^{(1)}(5) = \Phi\left(s^{(1)}, s^{(1)}\right)(5) = s^{(1)}\left(s^{(1)}(5)\right) = s^{(1)}(6) = 7.$

c) $R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(2,3) \underbrace{=}_{2>0} \Phi\left(s^{(1)}, p_3^{(3)}\right)\left(1, 3, R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(1,3)\right) =$

$\Phi\left(s^{(1)}, p_3^{(3)}\right)\left(1, 3, R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(1,3)\right) = s^{(1)}\left(p_3^{(3)}\left(1, 3, R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(1,3)\right)\right) =$

$s^{(1)}\left(R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(1,3)\right) \underbrace{=}_{1>0} s^{(1)}\left(\Phi\left(s^{(1)}, p_3^{(3)}\right)\left(0, 3, R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(0,3)\right)\right) =$

$s^{(1)}\left(s^{(1)}\left(p_3^{(3)}\left(\left(0, 3, R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(0,3)\right)\right)\right)\right) = s^{(1)}\left(s^{(1)}\left(R\left(p_1^{(1)}, \Phi\left(s^{(1)}, p_3^{(3)}\right)\right)(0,3)\right)\right)$

$\underbrace{=}_{0=0} s^{(1)}\left(s^{(1)}\left(p_1^{(1)}(3)\right)\right) = s^{(1)}\left(s^{(1)}(3)\right) = s^{(1)}(4) = 5.$

d) $R\left(c^{(0)}, p_1^{(2)}\right)(412) = p_1^{(2)}\left(411, R\left(c^{(0)}, p_1^{(2)}\right)(411)\right) = 411.$

3.

a)

- $\Pi^{(2)}(0,x) = 0x = 0 = c^{(1)}(x)$.
- $$\begin{aligned} \Pi^{(2)}(y,x) &= yx = (y-1)\,x + x = \Sigma^{(2)}\left[x, \Pi(y-1,x)\right] = \\ &= \Sigma^{(2)}\left\{p_2^{(3)}\left[y-1, x, \Pi^{(2)}(y-1,x)\right], p_3^{(3)}\left[y-1, x, \Pi^{(2)}(y-1,x)\right]\right\} = \\ &= \Phi\left(\Sigma^{(2)}, p_2^{(3)}, p_3^{(3)}\right)\left(y-1, x, \Pi^{(2)}(y-1,x)\right). \end{aligned}$$
- $\Pi^{(2)}(y,x) = R\left[c^{(1)}, \Phi\left(\Sigma^{(2)}, p_2^{(3)}, p_3^{(3)}\right)\right](y,x)$.

b)

- $Fac^{(1)}(0) = 0! = 1 = 0+1 = s^{(1)}\left[c^{(0)}()\right] = \Phi\left(s^{(1)}, c^{(0)}\right)()$.
- $$\begin{aligned} Fac^{(1)}(y) &= y! = y\,(y-1)! = [y-1]!\,[(y-1)+1] = \\ &= \Pi^{(2)}\left\{p_2^{(2)}\left(y-1, Fac^{(1)}(y-1)\right), \Phi\left(s^{(1)}, p_1^{(2)}\right)\left(y-1, Fac^{(1)}(y-1)\right)\right\} = \\ &= \Phi\left[\Pi^{(2)}, p_2^{(2)}, \Phi\left(s^{(1)}, p_1^{(2)}\right)\right]\left(y-1, Fac^{(1)}(y-1)\right). \end{aligned}$$
- $Fac^{(1)}(y) = R\left\{\Phi\left(s^{(1)}, c^{(0)}\right), \Phi\left[\Pi^{(2)}, p_2^{(2)}, \Phi\left(s^{(1)}, p_1^{(2)}\right)\right]\right\}(y)$.

c)

- $Exp^{(2)}(0,x) = x^0 = 1 = 0+1 = s^{(1)}\left[c^{(1)}(x)\right] = \Phi\left(s^{(1)}, c^{(1)}\right)(x)$.
- $$\begin{aligned} Exp^{(2)}(y,x) &= x^y = x^{y-1}x = \Pi^{(2)}\left[x, Exp^{(2)}(y-1,x)\right] = \\ &= \Pi^{(2)}\left[p_2^{(3)}\left(y-1, x, Exp^{(2)}(y-1,x)\right), p_3^{(3)}\left(y-1, x, Exp^{(2)}(y-1,x)\right)\right] = \\ &= \Phi\left(\Pi^{(2)}, p_2^{(3)}, p_3^{(3)}\right)\left(y-1, x, Exp^{(2)}(y-1,x)\right). \end{aligned}$$
- $Exp^{(2)}(y,x) = R\left[\Phi\left(s^{(1)}, c^{(1)}\right), \Phi\left(\Pi^{(2)}, p_2^{(3)}, p_3^{(3)}\right)\right](y,x)$.

d)

- $\widetilde{d}^{(2)}(0,x) = x \dot{-} 0 = x = p_1^{(1)}(x)$.
- $$\begin{aligned} \widetilde{d}^{(2)}(y,x) &= x \dot{-} y = \left[x \dot{-} (y-1)\right] - 1 = Pd^{(1)}\left[\widetilde{d}^{(2)}(y-1,x)\right] = \\ &= Pd^{(1)}\left[p_3^{(3)}\left(y-1, x, \widetilde{d}^{(2)}(y-1,x)\right)\right] = \\ &= \Phi\left(Pd^{(1)}, p_3^{(3)}\right)\left(y-1, x, \widetilde{d}^{(2)}(y-1,x)\right). \end{aligned}$$
- $\widetilde{d}^{(2)}(y,x) = R\left[p_1^{(1)}, \Phi\left(Pd^{(1)}, p_3^{(3)}\right)\right](y,x)$.

e)

- $D_0^{(1)}(0) = 1 = 0 + 1 = s^{(1)}\left[c^{(0)}()\right] = \Phi\left(s^{(1)}, c^{(0)}\right)()$.
- $D_0^{(1)}(y) = 0 = c^{(2)}\Big(y-1, D_0^{(1)}(y-1)\Big)$.
- $D_0^{(1)}(y) = R\left[\Phi\left(s^{(1)}, c^{(0)}\right), c^{(2)}\right](y)$.

f)

$$k^{(2)}(x,y) = \begin{cases} x \dot{-} y & x > y \\ y \dot{-} x & x \leq y \end{cases} = \left(x \dot{-} y\right) + \left(y \dot{-} x\right) = \widetilde{d}^{(2)}\left(p_2^{(2)}, p_1^{(2)}\right)(x,y) + \widetilde{d}^{(2)}\left(p_1^{(2)}, p_2^{(2)}\right)(x,y) =$$

$$= \Phi\left(\widetilde{d}^{(2)}, p_2^{(2)}, p_1^{(2)}\right)(x,y) + \Phi\left(\widetilde{d}^{(2)}, p_1^{(2)}, p_2^{(2)}\right)(x,y) =$$

$$= \Phi\left[\Sigma^{(2)}, \Phi\left(\widetilde{d}^{(2)}, p_2^{(2)}, p_1^{(2)}\right), \Phi\left(\widetilde{d}^{(2)}, p_1^{(2)}, p_2^{(2)}\right)\right](x,y).$$

g) $E^{(2)}(x,y) = D_0^{(1)}\left[k^{(2)}(x,y)\right] = \Phi\left(D_0^{(1)}, k^{(2)}\right)(x,y)$.

h) $\neg E^{(2)}(x,y) = \Phi\left(D_0^{(1)}, E^{(2)}\right)(x,y)$.

i) $sgn(y) = \Phi\left(D_0^{(1)}, D_0^{(1)}\right)(y)$.

4. $\hat{d}^{(2)}(x,y) = \Phi\left(\tilde{d}^{(2)}, p_2^{(2)}, p_1^{(2)}\right)(x,y)$.

5.

a)

- $F^{(2)}(0,x) = \sum_{z=0}^{0} f^{(2)}(z,x) = f^{(2)}(0,x) = \Phi\left(f^{(2)}, c^{(1)}, p_1^{(1)}\right)$.
- $$F^{(2)}(y,x) = \sum_{z=0}^{y} f^{(2)}(z,x) = \underbrace{\sum_{z=0}^{y-1} f^{(2)}(z,x)}_{p_3^{(3)}} + f^{(2)}(y,x) =$$
  $$= \Phi\left\{\Sigma^{(2)}, p_3^{(3)}, \Phi\left[f^{(2)}, \Phi\left(s^{(1)}, p_1^{(3)}\right), p_2^{(3)}\right]\right\}\left(y-1, x, F^{(2)}(y-1,x)\right)$$
- $F^{(2)} = R\left[\Phi\left(f^{(2)}, c^{(1)}, p_1^{(1)}\right), \Phi\left\{\Sigma^{(2)}, p_3^{(3)}, \Phi\left[f^{(2)}, \Phi\left(s^{(1)}, p_1^{(3)}\right), p_2^{(3)}\right]\right\}\right]$.
- $G^{(2)} = R\left[\Phi\left(f^{(2)}, c^{(1)}, p_1^{(1)}\right), \Phi\left\{\Pi^{(2)}, p_3^{(3)}, \Phi\left[f^{(2)}, \Phi\left(s^{(1)}, p_1^{(3)}\right), p_2^{(3)}\right]\right\}\right]$.

b) COMPLETAR.

6.

a)

- Caso base $y=0$: $\Sigma^{(2)}(0,x)=x=F^{(2)}(0,x)$.
- Caso inductivo $y=k$: Supongamos que $\Sigma^{(2)}(k,x)=F^{(2)}(k,x)$. Por definición de $F^{(2)}$ tenemos $F^{(2)}(k+1,x)=s^{(1)}\left(F^{(2)}(k,x)\right)\underbrace{=}_{H.I.}s^{(1)}\left(\Sigma^{(2)}(k,x)\right)\underbrace{=}_{def\Sigma}\Sigma^{(2)}(k+1,x)$.

b)

- $F^{(2)}(0,x)=x=p_1^{(1)}(x)$.
- $F^{(2)}(y,x)=f^{(1)}\left[F^{(2)}(y-1,x)\right]=\Phi\left(f^{(1)},p_3^{(3)}\right)\left(y-1,x,F^{(2)}(y-1,x)\right)$.
- $F^{(2)}=R\left[p_1^{(1)},\Phi\left(f^{(1)},p_3^{(3)}\right)\right]$.

c) $\hat{d}(x,y)=x\underbrace{-1-\ldots-1}_{y\text{ veces}}=Pd^y(x)$.

Conjuntos recursivos primitivos

7. Sea $A=\{n\}$, luego $\chi_A^{(1)}=\Phi\left(E^{(2)},f_n^{(1)},p_1^{(1)}\right)$ donde $f_n^{(1)}(x)=n$.

8.

- $\chi_{\neg A}^{(1)}=\Phi\left(D_0^{(1)},\chi_A^{(1)}\right)$.
- $\chi_{A\cap B}^{(1)}=\Phi\left(\Pi^{(2)},\chi_A^{(1)},\chi_B^{(1)}\right)$.
- $\chi_{A\cup B}^{(1)}=\Phi\left[Sgn^{(1)},\Phi\left(\Sigma^{(2)},\chi_A^{(1)},\chi_B^{(1)}\right)\right]$.

9. Lo demostraremos por inducción en el cardinal del conjunto:

- Caso base $c=0$: $\chi_A^{(1)}=c^{(1)}$.
- Caso inductivo $c=k$: Supongamos que los conjuntos de k elementos son CRP. Sea $A/\left|A\right|=k+1$. Para cualquier $x\in A$ resulta $\left|A-\{x\}\right|=k$ y por hipotesis inductiva este conjunto es CRP. Como $\{x\}$ es CRP y $A=(A-\{x\})\cup\{x\}$, por el ejercicio anterior sabemos que A es CRP.

10.

- Lo demostraremos por inducción en la aridad del conjunto:
  - Caso base $n = 1$: Trivial por ejercicio 7.
  - Caso inductivo $n = k$: Supongamos que todos los conjuntos unitarios de $\mathbb{N}^k$ son CRP. Veamos que pasa para los conjuntos unitarios de $\mathbb{N}^{k+1}$. Sean $A = \{(a_1, \ldots, a_k)\}$ y $B = \{(a_1, \ldots, a_k, b)\}$. Por hipotesis inductiva A es FRP, luego:

$$\chi_B^{(k+1)} = \Phi\left[\Pi^{(2)}, \Phi\left(\chi_A^{(k)}, p_1^{(k+1)}, \ldots, p_k^{(k+1)}\right), \Phi\left(E^{(2)}, p_{k+1}^{(k+1)}, f_b^{(k+1)}\right)\right]$$

- COMPLETAR.
- COMPLETAR.

11.

- $\chi_P^{(1)}(0) = 1 = s^{(1)}\left(c^{(0)}()\right) = \Phi\left(s^{(1)}, c^{(0)}\right)()$.
- $\chi_P^{(1)}(y) = D_0\left[\chi_P^{(1)}(y-1)\right] = \Phi\left(D_0^{(1)}, p_2^{(2)}\right)\left(y-1, \chi_P^{(1)}(y-1)\right)$.
- $\chi_P^{(1)} = R\left[\Phi\left(s^{(1)}, c^{(0)}\right), \Phi\left(D_0^{(1)}, p_2^{(2)}\right)\right]$.

12. Observemos que $r_3(x) = r_3(x-1) + 1$ salvo cuando $r_3(x-1) + 1 = 3$ en cuyo caso debe ser 0. Luego:

- $r_3^{(1)}(0) = 0 = c^{(0)}()$.
- $$\begin{aligned} r_3^{(1)}(x) =& \Pi^{(2)}\left\{\left[r_3^{(1)}(x-1)+1\right], \neg E^{(2)}\left[r_3^{(1)}(x-1)+1, 3\right]\right\} = \\ =& \Phi\left\{\Pi^{(2)}, \Phi\left[s^{(1)}, p_2^{(2)}\right], \Phi\left[\neg E^{(2)}, \Phi\left(s^{(1)}, p_2^{[2]}\right), f_3^{(2)}\right]\right\} \end{aligned}$$
- $\chi_3^{(1)} = \Phi\left(D_0^{(1)}, r_3^{(1)}\right)$.

Relaciones recursivas primitivas

13.

- $\chi_{=}^{(2)} = E^{(2)}$.
- $\chi_{\neq}^{(2)} = \Phi\left(D_0^{(1)}, E^{(2)}\right)$.
- $\chi_{\leq}^{(2)}(x,y) = \begin{cases} 1 & x \leq y \\ 0 & x > y \end{cases} = \Phi\left(D_0^{(1)}, \hat{d}^{(2)}\right)(x,y)$.
- $\chi_{>}^{(2)} = \Phi\left(D_0^{(1)}, \chi_{\leq}^{(2)}\right)$.

14.

- $X_T^{(2)} = \Phi\left(\Pi^{(2)}, \chi_R^{(2)}, \chi_S^{(2)}\right)$.
- $X_U^{(2)} = \Phi\left[Sgn^{(1)}, \Phi\left(\Sigma^{(2)}, \chi_R^{(2)}, \chi_S^{(2)}\right)\right]$.
- $X_{\neg R}^{(2)} = \Phi\left(D_0^{(1)}, \chi_R^{(2)}\right)$.

15. Puesto que $=$ y $\leq$ son RRP, tambien deben serlo $\neg =$ y $\neg \leq$; es decir: $\neq$ y $>$.

16.

- $\chi_{\wedge R}^{(2)}(x,y) = \chi_R^{(2)}(x,0)\,\chi_R^{(2)}(x,1)\cdots\chi_R^{(2)}(x,k) = \prod_{k=0}^{y}\chi_R(x,k)$.
- $\chi_{\vee R}^{(2)}(x,y) = Sgn^{(1)}\left[\sum_{k=0}^{y}\chi_R(x,k)\right]$.

Varios

17.

- $r(a,n) = r_n(a) = R\left(c^{(1)}, \Phi\left\{\Pi^{(2)}, \Phi\left[s^{(1)}, p_3^{(3)}\right], \Phi\left[\neg E^{(2)}, \Phi\left(s^{(1)}, p_3^{(3)}\right), p_2^{(3)}\right]\right\}\right)$.
- $\chi_{|}^{(2)} = \Phi\left[D_0^{(1)}, \Phi\left(r^{(2)}, p_2^{(2)}, p_1^{(2)}\right)\right]$.

18.

a) $f(x) = x^2 D_0 [r_3(x)] + (x-3) D_1 [r_3(x)] + Fac(x) D_2 [r_3(x)]$.

b) Puesto que:

- $x > y \Rightarrow \left(x \dot{-} y\right) + y = x - y + y = x$.
- $x < y \Rightarrow \left(x \dot{-} y\right) + y = 0 + y = y$.
- $x = y \Rightarrow \left(x \dot{-} y\right) + y = 0 + y = y = x$.

resulta:

$$max^{(2)}(y,x) = \left(x \dot{-} y\right) + y = \widetilde{d}^{(2)}(y,x) + y = \Phi\left(\Sigma^{(2)}, \widetilde{d}^{(2)}, p_1^{(2)}\right)(y,x)$$