Iterative Inverse Kinematics with Manipulator Configuration Control

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***Abstract*—A new method, termed the *offset modification* method (OM method), for solving the manipulator inverse kinematics problem is presented. The OM method works by modifying the link offset values of a manipulator until it is possible to derive closed-form inverse kinematics equations for the resulting manipulator (termed the model manipulator). This procedure allows one to derive a set of three nonlinear equations in three unknowns that, when numerically solved, give an inverse kinematics solution for the original manipulator. The OM method can be applied to manipulators with any number of degrees of freedom, as long as the manipulator satisfies a given set of conditions (Theorem 1). The OM method is tested on a 6-degree-of-freedom manipulator that has no known closed-form inverse kinematics equations. It is shown that the OM method is applicable to real-time manipulator control, can be used to guarantee convergence to a desired endpoint position and orientation (if it exists), and allows one to directly choose which inverse kinematics solution the algorithm will converge to (as specified in the model manipulator closed-form inverse kinematics equations). Applications of the method to other 6-DOF manipulator geometries and to redundant manipulators (i.e. greater than 6 DOF geometries) are discussed.**

I. Introduction

A new theoretical approach for solving the manipulator inverse kinematics problem is presented in this paper. The inverse kinematics problem for a robot manipulator is one of obtaining the required manipulator joint values for a given desired endpoint position and orientation (see Fig. 1). Depending on the design of the manipulator, a given endpoint position and orientation may be obtainable using more than one set of joint values. For the purposes of this paper, each distinct set of joint values that gives the same endpoint position and orientation is termed a distinct manipulator configuration. Configuration control is here defined as the ability to choose one of a possible many manipulator configurations that give the same desired endpoint position and orientation.

Configuration control is an important part of inverse kinematics because not all sets of joint values that give the same endpoint position and orientation are applicable at all times for all tasks, and it is useful to have a method of choosing between them. For example, if one wishes to plan a path around an obstacle in the manipulator's workspace, only one of a possible many inverse kinematics solutions may allow a path to be executed that does not result in the arm colliding with the obstacle.

It is possible to derive closed-form inverse kinematics equations for many types of manipulators. For example, for 6-degree-of-freedom (DOF) manipulators, closed-form inverse kinematics equations can be derived for either the case when any three adjacent joints of the manipulator are intersecting [1], or for the case when any three adjacent joints are parallel [2]. The theory for redundant manipulators (i.e. 7 DOF or more) is not as well understood, however it is possible to derive closed-form equations for many manipulators that resemble the human arm [3].

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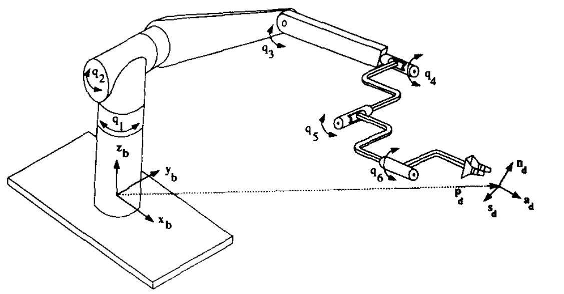

Fig. 1. The inverse kinematics problem. The inverse kinematics problem is one of calculating the required joint variable values $(q_1, q_2, \cdots, q_6)$ in order to achieve the desired endpoint position $(\boldsymbol{p}_d)$ and orientation (defined by the three-unit vectors $\boldsymbol{n}_d, \boldsymbol{s}_d, \boldsymbol{a}_d$).

Nonclosed-form methods for solving the inverse kinematics problem also exist. A brief review of some of these methods is given in the next section. Nonclosed-form methods involve numerical iteration until a desired endpoint position and orientation is reached to within a maximum allowable error. Such methods are usually intended to be used on manipulators that may not have closed-form inverse kinematics equations.

The iterative inverse kinematics method presented in this paper, termed the offset modification method (OM method), is applicable to manipulators of arbitrarily many degrees of freedom and is intended to be used on manipulators that have no known closed-form inverse kinematics equations. The main advantages of the OM method are summarized at the end of the next section. In this paper, the OM inverse kinematics method is tested on a 6 DOF manipulator that has no known closed-form inverse kinematics equations. It is shown that the method never fails and that it is applicable to real-time manipulator control.

A. Current Iterative Approaches to the Inverse Kinematics Problem

A summary of the characteristics of various inverse kinematics methods that are intended to be used on manipulators that have no known closed-form inverse kinematics equations is given Table I. A brief description of each of these methods is presented here.

Wang and Chen [4] developed a combined optimization method that can be used on any manipulator geometry. The method uses the cyclic coordinate descent method to find an initial guess at the desired solution and then uses the Broyden–Fletcher–Shanno variable metric method to find a solution that is within maximum tolerable error. The method is applicable to manipulators of arbitrary number of degrees of freedom, has guaranteed global convergence, and is not sensitive to manipulator singularity regions.

Tsai and Morgan [5] developed a method that is applicable to general 6- and 5-DOF manipulators and that can generate all possible solutions for a desired endpoint position and orientation. This was achieved by converting the problem into a system of eight second-order equations in eight unknowns and then solving this system, for all possible solutions, via continuation methods [6]. Wampler and Morgan [7] used a similar approach to generate "11 very simple polynomial equations" that was also able to generate all possible solutions, requiring approximately 10 s per solution.

TABLE I
SUMMARY OF CURRENT ITERATIVE INVERSE KINEMATICS METHODS

	References										
	[4]	[5], [7]	[13]	[14]	Inverse Jacobian Methods [16]–[20], [22], [37]	[28]	[30]	[31]	[12]	[8]–[11]	OM Method
Convergence conditions given.	Yes	No	No	Yes	No	No	No	No	No	No	Yes
Guaranteed convergence if solution exists.	Yes	No	No	No	No	No	No	No	No	No	Yes
Allows real time manipulator control.	Yes	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Allows direct control over configuration.	No	Yes	No	Yes	No	No	No	No	Yes	Yes	Yes
Works on manipulators of arbitrary # of DOF.	Yes	No	No	No	Yes	Yes	Yes	Yes	No	No	Yes

Lee and Liang [8] extended the work of Duffy [2] to obtain a general method for solving the inverse kinematics problem for 6 DOF manipulators with rotational joints. Their method involves deriving a 16th order polynomial equation in the tan-half-angle of the first joint (θ_1), and then (for each desired endpoint position and orientation) numerically solving for all of the zeroes of this polynomial equation. All the real roots obtained represent a viable joint angle value for θ_1 while the other five joint variable values are calculated using closed-form equations that are functions of θ_1. Raghavan and Roth obtained a similar type of solution for 6 DOF manipulators with rotational joints [9] and also extended it to other 6-DOF manipulators with various types of joint axes [10]. Manocha and Canny [11] modified these polynomial approaches to give fast and numerically stable convergence times for most desired solutions of a 6-DOF revolute manipulator. All of the above methods can be used to obtain all possible inverse kinematics solutions for a given desired endpoint position and orientation.

Takano [12] developed an inverse kinematics method for 6 DOF manipulators that requires closed-form inverse kinematics equations for the first three joints in terms of the endpoint position of link 3, and closed-form inverse kinematics equations for the last three links in terms of the orientation of the final link. Endpoint position and endpoint orientation are then solved for separately, one after the other in an iterative manner, until convergence is obtained to within required accuracy.

Manseur and Doty [13] developed a fast algorithm that can be used with 6 DOF manipulators that have revolute joints at the first and last links, and for which one is able to derive closed-form equations for joint variables 2 through 6 as a function of joint variable 1. Under these conditions, it is shown that a single nonlinear equation can be generated with the only unknown being joint variable 1. This equation is then solved using the Newton–Raphson method.

Tourassis and Ang [14] formulated a fast method that is applicable to general 6 DOF geometries. The method is based on creating a system of three nonlinear equations in three unknowns using the known forward kinematics and inverse kinematics of the first three joints and the last three joints separately. Fixed point theory [15] is used to solve for the three unknowns that represent the Cartesian position of the wrist. Conditions under which the method will convrge are given and the desired manipulator configuration can be easily specified. The algorithm is also applicable for real-time use.

Other methods that have been extensively used to solve the inverse kinematics problem are called Inverse Jacobian Methods and are based upon Newton algorithms for solving systems of nonlinear equations [16]–[22]. The nonlinear equations are usually derived from expressions in which the desired endpoint position and orientation are functions of the manipulator's joint variables (i.e., forward kinematics equations). These methods work by linearizing the system of equations using a Taylor's series expansion around the manipulator's initial position (i.e. calculating the Jacobian of the forward kinematics equations). This system of linear equations, when solved, provides an approximation to the joint values that give the desired endpoint position and orientation. Inverse Jacobian methods are generally applicable to all manipulator geometries, including redundant manipulators, and can be suitable for real-time control.

Some Newton-based algorithms use damped least-squares methods to compensate for problems that are associated with Jacobian singularities [17], [23]–[27]. Nakamura [20] made the damping factor variable by increasing it as the manipulator approached a Jacobian singularity region. Chan and Lawrence [21] made the damping factor a function of residual workspace error and showed that this gave good inverse kinematics control of manipulators both near and away from singularity regions. Hutchinson [22] showed that a least-squares approximated Jacobian used in a Newton–Raphson type algorithm has similar convergence properties to the damped least-squares method.

Novaković and Nemec [28] developed an algorithm based on sliding mode control and Lyapunov theory. The algorithm is tested on a 4-axis manipulator that has no known closed-form inverse kinematics equations. The method is shown to be computationally more efficient than Newton-based methods and is not prone to singularity problems.

Powell [29] developed a hybrid algorithm for solving systems of linear equations that uses the Newton–Raphson method, the gradient descent method, and a combination of the two. Powell's algorithm was applied to the inverse kinematics problem by Goldenberg [30]. This inverse kinematics formulation is applicable to general manipulator geometries, allows real-time control, and has better convergence properties than methods that are based solely on the Newton-Raphson approach.

Another approach to the inverse kinematics problem involves the integration of joint velocities that are obtained using inverse Jacobian techniques. Tsai and Orin [31] used a variation of this approach along with a special-purpose inverse kinematics processor to obtain a system that works in real-time. This method has good convergence properties in that usually only fails to converge at manipulator singularity regions.

From Table I, one can see that with the exception of the OM method described in this paper, none of the above methods simultaneously:

1) gives guaranteed convergence to the desired endpoint position and orientation if its exists,
2) is applicable to real-time manipulator control,
3) gives direct control over most manipulator configurations and,
4) is applicable to manipulators of arbitrarily many degrees of freedom (as long as it meets the conditions given in THEOREM I of *The Theoretical Framework* Section).

II. The Theoretical Development

A. Summary of the Approach

The OM inverse kinematics method consists of systematically modifying manipulator offset[1] values until it is possible to derive closed-form inverse kinematics equations for the resulting manipulator. These closed-form inverse kinematics equations are then used to obtain a system of three nonlinear equations in three unknowns that, when solved using standard numerical techniques [33], [29], [28], give an inverse kinematics solution for the original manipulator. Thus the OM inverse kinematics method can be applied to any manipulator that, via the modification of offsets values, can be converted to a manipulator that has closed-form inverse kinematics equations (see Theorem 1).

The manipulator for which the inverse kinematics solution is required is termed the *real* manipulator. The manipulator that is created via modification of the offset values (i.e. the values of the Denavit–Hartenberg parameters [34] a_i and d_i) of the real manipulator is termed the *model* manipulator. The process of selecting a model manipulator that has closed-form inverse kinematics equations is discussed later in this section.

B. Forward Kinematics Representation

The Denavit–Hartenberg representation [34] is here used to formulate the forward kinematics equations for a manipulator. This representation is based on the four geometric parameters α_i, d_i, a_i and θ_i. For a complete definition of these parameters see [32, pp. 36–44]. They are used in homogeneous transformation matrices to describe the geometric relationship between link $(i-1)$ and link i.

The forward kinematics equations for the manipulators presented in this paper are given in the form of a single 12-element column vector:

$$\boldsymbol{k}(\boldsymbol{q}) = [\boldsymbol{n}(\boldsymbol{q})\boldsymbol{s}(\boldsymbol{q})\boldsymbol{a}(\boldsymbol{q})\boldsymbol{p}(\boldsymbol{q})]^T. \tag{1}$$

The vectors $\boldsymbol{n} = [n_x n_y n_z], \boldsymbol{s} = [s_x s_y s_z]$ and $\boldsymbol{a} = [a_x a_y a_z]$ are orthonormal vectors that uniquely define the orientation of the manipulator's distal link (the *hand*) and are respectively termed the normal vector, the sliding vector, and the approach vector of the hand. The vector $\boldsymbol{p} = [p_x p_y p_z]$ defines the Cartesian endpoint position of the manipulator. The vector $\boldsymbol{q} = [q_1 q_2 \cdots q_N]^T$, where N is the number of degrees of freedom of the manipulator, contains the joint values of the manipulator. For a prismatic joint i the joint variable is d_i (i.e., $q_i = d_i$), and for a rotational joint i the joint variable is θ_i (i.e. $q_i = \theta_i$). The equations that give the vectors $\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{a}$, and $\boldsymbol{p}$ are derived using homogeneous transformation matrices $({}^{i-1}\boldsymbol{A}_i)$ [32]:

$$\begin{bmatrix} \boldsymbol{n}^T(\boldsymbol{q}) & \boldsymbol{s}^T(\boldsymbol{q}) & \boldsymbol{a}^T(\boldsymbol{q}) & \boldsymbol{p}^T(\boldsymbol{q}) \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0\boldsymbol{A}_1\, {}^1\boldsymbol{A}_2\, {}^2\boldsymbol{A}_3 \cdots {}^{N-1}\boldsymbol{A}_N. \tag{2}$$

The forward kinematics of the real manipulator, referred to by the subscript r, are given by the following equations:

$$\begin{aligned} \boldsymbol{x}_{r_d} &= [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_{r_d}]^T \\ &= \boldsymbol{k}_r(\boldsymbol{q}_r) = [\boldsymbol{n}_r(\boldsymbol{q}_r)\boldsymbol{s}_r(\boldsymbol{q}_r)\boldsymbol{a}_r(\boldsymbol{q}_r)\boldsymbol{p}_r(\boldsymbol{q}_r)]^T. \end{aligned} \tag{3}$$

[1] Offsets here refer to the manipulator Denavit–Hartenberg parameters d_i, a_i, which are defined in [32, pp. 36–34].

$\boldsymbol{x}_{r_d}$ is the column vector in (1) containing the 12 scalars that define the desired endpoint position and orientation of the real manipulator: $\boldsymbol{k}_r(\boldsymbol{q}_r)$ is a column vector containing the 12 forward kinematics functions of the real manipulator. The Denavit–Hartenberg parameters of the real manipulator are designated by $\alpha_{r_i}, a_{r_i}, d_{r_i}$, and θ_{r_i} (for $i = 1, 2, \cdots, N$).

Similarly, the forward kinematics equations of the model manipulator, referred to by the subscript m, are given by

$$\begin{aligned} \boldsymbol{x}_m &= [\boldsymbol{n}_m\boldsymbol{s}_m\boldsymbol{a}_m\boldsymbol{p}_m]^T \\ &= \boldsymbol{k}_m(\boldsymbol{q}_m) = [\boldsymbol{n}_m(\boldsymbol{q}_m)\boldsymbol{s}_m(\boldsymbol{q}_m)\boldsymbol{a}_m(\boldsymbol{q}_m)\boldsymbol{p}_m(\boldsymbol{q}_m)]^T \end{aligned} \tag{4}$$

where $\boldsymbol{x}_m$ is a column vector containing 12 scalars that define the endpoint position and orientation of the model manipulator; $\boldsymbol{k}_m(\boldsymbol{q}_m)$ is a column vector containing the 12 forward kinemtics functions of the model manipulator. The Denavit–Hartenberg parameters of the model manipulator are designated by $\alpha_{m_i}, a_{m_i}, d_{m_i}$, and θ_{m_i} (for $i = 1, 2, \cdots, N$).

C. The Theoretical Framework

Let the inverse kinematics of the model manipulator be symbolized by

$$\boldsymbol{q}_m = \boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c}). \tag{5}$$

The vector $\boldsymbol{c}$ contains the desired configuration (i.e. specifies one of all possible sets of values $\boldsymbol{q}_m$ that give the same $\boldsymbol{x}_m$) that both the real manipulator and model manipulator are required to have.

Let the joint values of the real manipulator be equal to the joint values of the model manipulator:

$$\boldsymbol{q} = \boldsymbol{q}_m = \boldsymbol{q}_r. \tag{6}$$

Delta kineamtics equations $(\Delta\boldsymbol{k}(\boldsymbol{q}))$, which give the difference between the forward kinematics equations of the model manipulator and the forward kinematics equations of the real manipulator, are derived as follows:

$$\begin{aligned} \Delta\boldsymbol{k}(\boldsymbol{q}) &= \boldsymbol{x}_m - \boldsymbol{x}_{r_d} = \boldsymbol{k}_m(\boldsymbol{q}) - \boldsymbol{k}_r(\boldsymbol{q}) \\ &= \begin{bmatrix} \boldsymbol{n}_m(\boldsymbol{q}) - \boldsymbol{n}_r(\boldsymbol{q}) \\ \boldsymbol{s}_m(\boldsymbol{q}) - \boldsymbol{s}_r(\boldsymbol{q}) \\ \boldsymbol{a}_m(\boldsymbol{q}) - \boldsymbol{a}_r(\boldsymbol{q}) \\ \boldsymbol{p}_m(\boldsymbol{q}) - \boldsymbol{p}_r(\boldsymbol{q}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta\boldsymbol{p}(\boldsymbol{q}) \end{bmatrix}. \end{aligned} \tag{7}$$

The first nine entries in the column vector $\Delta\boldsymbol{k}(\boldsymbol{q})$ are always zero because the real manipulator and the model manipulator always have identical endpoint orientations (see the proof of Theorem 1 in Appendix I). Therefore

$$[\boldsymbol{n}_m\boldsymbol{s}_m\boldsymbol{a}_m]^T = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}]^T. \tag{8}$$

By substituting (5) into the system (7), the following system of three equations in three unknowns is obtained:

$$\boldsymbol{p}_m = \boldsymbol{p}_{r_d} + \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}([\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m]^T, \boldsymbol{c})). \tag{9}$$

The only unknown in (9) is the three element row vector $\boldsymbol{p}_m = [p_{m_x} p_{m_y} p_{m_z}]$.

The unknown vector $(\boldsymbol{p}_m)$ represents the endpoint position of the model manipulator when the real manipulator and the model manipulator have identical joint variable values (i.e. $\boldsymbol{q}_m = \boldsymbol{q}_r$), and when the real manipulator is at the desired endpoint position $(\boldsymbol{p}_{r_d})$, endpoint orientation $(\boldsymbol{n}_{r_d}, \boldsymbol{s}_{r_d}$, and $\boldsymbol{a}_{r_d})$, and manipulator configuration $(\boldsymbol{c})$. Therefore, numerically solving the system (9) for $\boldsymbol{p}_m$ allows one to solve for the joint values of the model manipulator, and therefore for the real manipulator as well.

Because the system (9) implicitly contains information about manipulator configuration, the solution of system (9) gives direct control over all configurations of the real manipulator that are explicitly specified in the closed-form equations of the model manipulator (5). Thus the real manipulator will always have at least as many, and perhaps more (see the *Discussion* Section), inverse kinematics solutions at a given endpoint position and orientation, as the model manipulator.

One important characteristic about the system of equations (9) is that the unknown vector, $\boldsymbol{p}_m$, must lie within a sphere centered at $\boldsymbol{p}_{r_d}$, having a radius of $\Delta p_{\max}$, where

$$\Delta p_{\max} = \max_{\boldsymbol{q}}\{\|\Delta \boldsymbol{p}(\boldsymbol{q},\boldsymbol{c})\|\}. \tag{10}$$

The magnitude of $\Delta p_{\max}$ depends on the magnitude of the real manipulator Denavit–Hartenberg parameters a_{r_i} and d_{r_i} (for $i = 1, 2, \cdots, N$), which have been modified in order to produce a model manipulator that has closed-form inverse kinematics equations. Therefore, using the OM theoretical framework, the N dimensional manipulator inverse kinematics problem is converted into a bounded three-dimensional iterative problem.

The conditions under which the above formulation gives the solution to the inverse kinematics of the real manipulator, are given in Theorem 1. The proof of this theorem is contained in Appendix I. Note that $\|\boldsymbol{x}\|$ indicates the L2 norm of the vector $\boldsymbol{x}$.

Theorem 1: Given a real manipulator and a model manipulator, assume that the following conditions are true:

C1) The model manipulator and the real manipulator differ only in the values of their Denavit–Hartenberg parameters a_i and d_i.

C2) The model manipulator has closed-form inverse kinematics equations; specifically, for any fixed configuration $\boldsymbol{c}$, the function $\boldsymbol{q}_m = \boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c})$ is continuous with respect to $\boldsymbol{x}_m$, and, if $(\boldsymbol{x}_m, \boldsymbol{c}) \notin U_m$ (U_m defines the singularity region of the model manipulator) the function is also one to one.

C3) $\boldsymbol{q} = \boldsymbol{q}_m = \boldsymbol{q}_r$.

C4) For every endpoint position, endpoint orientation and manipulator configuration that lies in a singularity region of the model manipulator, there exists another endpoint position that is arbitrarily close to the previous endpoint position, but which is not in a singularity region; i.e. if all the previous conditions of the theorem are met, and if $([\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m^1], \boldsymbol{c}) \in U_m$ then there exists a $\boldsymbol{p}_m^2 \in \Re^3$ such that $\|\boldsymbol{p}_m^2 - \boldsymbol{p}_m^1\| < \varepsilon$, where ε is an arbitrarily small positive real number, and $([\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m^2], \boldsymbol{c}) \notin U_m$.

C5) The desired inverse kinematics solution exists.

Then, with the real manipulator at the desired endpoint position, endpoint orientation ($\boldsymbol{x}_{r_d} = \boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_{r_d}]^T$), and manipulator configuration ($\boldsymbol{c}$), for any $E_{\max} > 0 (E_{\max} \in \Re)$, there exists an approximation of the endpoint position of the model manipulator, $\boldsymbol{p}_{m_a} \in \Re^3$, such that (rewriting system (9)):

$$\|\boldsymbol{p}_{m_a} - \Delta \boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_{m_a}, \boldsymbol{c})) - \boldsymbol{p}_{r_d}\| < E_{\max} \tag{11}$$

where $\boldsymbol{x}_{m_a} = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_{m_a}]^T$.

D. Selecting a Model Manipulator

For most 6-DOF manipulators, the process of selecting a model manipulator is well-defined. Pieper [1] showed that if any three consecutive joint axes (which must give three independent degrees of freedom) of a 6-DOF manipulator are intersecting, then one can generate closed-form inverse kinematics equations for that manipulator. Thus, if one is able to modify the Denavit–Hartenberg parameters a_{r_i} and d_{r_i} of a 6-DOF manipulator, and thereby obtain a manipulator that satisfies Pieper's criteria, then one can formulate an inverse kinematics solution for that manipulator using the OM method.

The application of the OM method to a specific subclass of general purpose 6 DOF manipulators is given in Theorem 2. The conditions of Theorem 2 define this subclass and an algorithm for generating model manipulators is given in condition C4 of the theorem. The proof of Theorem 2 is given in Appendix II.

TABLE II
THE DENAVIT-HARTENBERG PARAMETERS OF TWO 7-DOF REVOLUTE JOINT MANIPULATORS WITH CLOSED-FORM INVERSE KINEMATICS EQUATIONS

	Manipulator No. 1			Manipulator No. 2		
Joint i	α_i	a_i	d_i	α_1	a_i	d_i
1	90°	0	d_1	90°	0	d_1
2	90°	0	0	0°	a_2	0
3	−90°	0	d_3	−90°	0	0
4	90°	0	0	90°	0	0
5	−90°	0	d_5	−90°	0	d_5
6	90°	0	0	90°	0	0
7	0°	0	0	0°	0	0

Theorem 2: Manipulators that satisfy all of the following conditions, satisfy the conditions of Theorem 1:

C1) The real manipulator has six serial degrees of freedom.

C2) Joint variables 1, 2, and 3 give three degrees of freedom for the Cartesian positioning of joint 4.

C3) Joints 4, 5, and 6 of the real manipulator are all

- rotational.
- nonparallel (i.e. $\alpha_{r_4} \neq 0°$ or $\pm 180°$ and $\alpha_{r_5} \neq 0°$ or $\pm 180°$).
- nonintersecting (i.e. not all of the a_{r_4}, a_{r_5} and d_{r_5} are zero).

C4) The model manipulator is generated using the following algorithm:

C5) For $i = 1, 2, 3, 6$ let

Step 1:
- $\alpha_{m_i} = \alpha_{r_i}$
- $a_{m_i} - a_{r_i}$
- $d_{m_i} = d_{r_i}$
- $\theta_{m_i} = \theta_{r_i}$

C6) For $i = 4, 5$ let

Step 2:
- $\alpha_{m_i} = \alpha_{r_i}$
- $a_{m_i} = 0$
- $d_{m_i} = \begin{cases} d_{r_i} & i = 4 \\ 0 & i = 5 \end{cases}$
- $\theta_{m_i} = \theta_{r_i}$

C7) $\boldsymbol{q}_m = \boldsymbol{q}_r$.

C8) The desired inverse kinematics solution exists.

Condition C2 ensures that the manipulator's endpoint will have three degrees of freedom in Cartesian space. Condition C3 ensures that the wrist joints give three degrees of freedom in hand orientation, and also excludes manipulators that have been shown by Pieper to have closed-form inverse kinematics equations [1].

The process of selecting a model manipulator for redundant manipulators (i.e. more than 6-DOF) is less well-defined. This stems from the fact that closed-form inverse kinematics equations for redundant manipulators have not as yet been as thoroughly analyzed as those for 6-DOF manipulators. There have, however, been closed-form equations published for some 7-DOF manipulators. Hollerbach [3], for example, analyzed two specific 7-DOF revolute mechanisms and gave closed-form inverse kinematics equations for such manipulators. The Denavit–Hartenberg parameters of these two manipulators are given in Table II. Therefore, if the Denavit–Hartenberg parameters of a 7-DOF manipulator (for which an inverse kinematics solution is required) can be modified to create a mechanism that is the same as one of the two given in Table II, and if the resulting real/model manipulator relationship meets the conditions of Theorem 1, then the OM inverse kinematics method can be used on that manipulator.

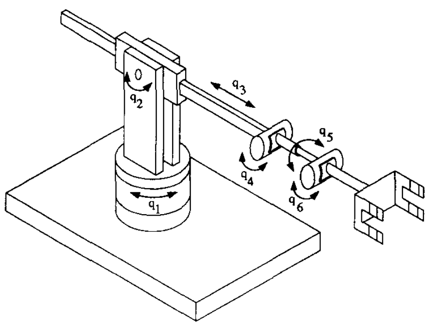

Fig. 2. The spherical manipulator.

TABLE III
THE DENAVIT–HARTENBERG PARAMETERS OF THE "REAL" SPHERICAL MANIPULATOR

Joint i	θ_{r_i}	α_{r_i}	a_{r_i}	d_{r_i}	Joint Range
1	q_1	90°	0	100.0 mm	$0° \leq q_1 \leq 360°$
2	q_2	−90°	15.0 mm	0	$0° \leq q_2 \leq 360°$
3	0°	0°	0	q_3	$-\infty < q_3 < \infty$
–	−90°	−90°	0	0	–
4	q_4	90°	0	0	$0° \leq q_4 \leq 360°$
5	q_5	90°	0	20.0 mm	$0° \leq q_5 \leq 360°$
6	q_6	0°	20.0 mm	0	$0° \leq q_6 \leq 360°$

The application of the OM method to other redundant manipulator geometries is briefly addressed in the discussion section.

E. The Nonlinear Equation Solver

The system of equations (9) from the OM formulation can be solved using any one of a number of standard numerical techniques [29], [28], [33]. No special numerical techniques were found to be necessary to give a fast theoretically guaranteed solution to these equations. In [35] a nonlinear equation solver is described that is a hybrid of the *fixed point algorithm* [33], *a modified Powell's algorithm* [29], and an exhaustive search strategy. The exhaustive search algorithm performs a 3-D search in a sphere, centered at the desired endpoint position, $\boldsymbol{p}_{r_d}$, and having a radius of $\Delta p_{\max}$, which must contain the desired solution (see (10)). The spacing of this exhaustive search corresponds to the maximum allowable endpoint positioning error, and thus, given Theorem 1, theoretically guarantees that if the desired solution exists, it will always be found. This type of exhaustive search is possible because 1) for most general purpose manipulators $\Delta p_{\max}$ is likely to be small, and 2) the search is bounded in three dimensions, rather than N dimensions for an N dimensional manipulator.

III. NUMERICAL EXAMPLE

A. The Spherical Manipulator

The OM inverse kinematics method has been tested on a 6 DOF manipulator that has no known closed-form inverse kinematics equations. This manipulator, called the *Spherical manipulator* (see Fig. 2), is taken from a paper by Tourassis and Ang [14].

The Denavit–Hartenberg parameters of the "real" Spherical manipulator [14] (see Fig. 2) are given in Table III. The Denavit–Hartenberg Parameters of the corresponding model manipulator (which has closed-form inverse kinematics equations) are given in Table IV. Note that the model manipulator was generated using the algorithm described in condition C4 of Theorem 2.

TABLE IV
THE DENAVIT-HARTENBERG PARAMETERS OF THE "MODEL" SPHERICAL MANIPULATOR

Joint i	θ_{m_i}	α_{m_i}	a_{m_i}	d_{m_i}	Joint Range
1	q_1	90°	0	100.0 mm	$0° \leq q_1 \leq 360°$
2	q_2	−90°	15.0 mm	0	$0° \leq q_2 \leq 360°$
3	0°	0°	0	q_3	$-\infty < q_3 < \infty$
–	−90°	−90°	0	0	–
4	q_4	90°	0	0	$0° \leq q_4 \leq 360°$
5	q_5	90°	0	0	$0° \leq q_5 \leq 360°$
6	q_6	0°	20.0 mm	0	$0° \leq q_6 \leq 360°$

The delta kinematics equations are

$$\Delta\boldsymbol{p}(\boldsymbol{q}) = \boldsymbol{p}_m(\boldsymbol{q}) - \boldsymbol{p}_r(\boldsymbol{q}) = \begin{bmatrix} d_{r_5}(C_1S_2S_4 - C_1C_2C_4) \\ d_{r_5}(S_1S_2S_4 - S_1C_2C_4) \\ d_{r_5}(-C_2S_4 - S_2C_4) \end{bmatrix}. \tag{12}$$

The notation $C_i = \cos q_i$ and $S_i = \sin q_i$ is used in this paper. The value of $\Delta p_{\max}$ for this manipulator is $|d_{r_5}|$ (see (10)).

The closed-form inverse kinematics equations of the model Spherical manipulator are given in Appendix III. These equations were obtained from [14]. The eight possible manipulator configurations are defined by ARM $= \pm 1$, ELBOW $= \pm 1$, and WRIST $= \pm 1$. The equation solver used to solve the system (9) is described in [35].

B. Simulation Results

The OM method was tested on over 100,000 randomly-generated desired endpoint positions, endpoint orientations and configurations of the real manipulator. The test points were generated using a uniform random distribution in joint space (joint variable 3 was limited to -500 mm $\leq q_3 \leq 500$ mm). The algorithm *always converged* to the desired solution. A manipulator configuration error never occurred and the orientation error, which was calculated as $\|\boldsymbol{n}_{r_d} - \boldsymbol{n}_{r_a}\| + \|\boldsymbol{s}_{r_d} - \boldsymbol{s}_{r_a}\| + \|\boldsymbol{a}_{r_d} - \boldsymbol{a}_{r_a}\|$, was always less than 10^{-8}. The real manipulator endpoint position error was always less than the maximum allowable error $E_{\max} = 0.1$ mm.

The algorithm was implemented in C code on a SUN SPARCstation 2. The average convergence time of the algorithm was 0.996 ms and the maximum convergence time was 14.074 s. Only 0.031% of the 100,000 points required convergence times of more than 54 ms.

As a comparison, Tourassis and Ang [14] applied their inverse kinematics algorithm to the Spherical manipulator and, when tested on 24 desired inverse kinematics solutions, it was not able to converge to 4 of these desired solutions. Their algorithm was written in C and was run on a SUN 3/60 workstation (Motorola 68020 with a 68881 floating point processor running at 20 MHz). For points at which Tourassis and Ang's algorithm found the desired solution, the average execution time in [14] was about 42 ms on a SUN 3/60 (the exact average time is not given in [14]) with a maximum of 65 ms.

The inverse kinematics method presented in this paper was also tested on the same desired 24 points. The maximum allowable error was chosen as $E_{\max} = 1.0 \times 10^{-8}$ mm. This error was chosen to be of the same order of magnitude as that used in [14]. The inverse kinematics method presented here converged to all of the 24 desired solutions. On a SUN 3/60 workstation (Motorola 68020 with a 68881 floating point processor running at 20 MHz), the average time of convergence (for all 24 points) was 14.17 ms with a maximum time of 20.0 ms.

Using the OM method, the average convergence time to points to which Tourassis and Ang's algorithm was not able to converge to was 15 ms, with a maximum time of 20 ms.

IV. Discussion

A. The Configuration Control Property of the OM Method

It is known that the maximum possible number of configurations that a 6 DOF manipulator can have is 16 [36]. However, all 16 configurations are not always real, and when they are real, may exist in only a small portion of the manipulator's workspace [8], [7]. It has in fact been difficult to find 6 DOF manipulators with 16 real configurations [8], [7]. The authors are not aware of any inverse kinematics method that allows *direct control* over all 16 configurations for a 6 DOF manipulator. Here *direct control* is defined as the ability to explicitly choose which of all possible solutions the inverse kinematics algorithm will converge to, and having it converge to *only* this solution. All known methods iterate to one out of a multiple number of solutions that satisfy a given set of equations. Other solutions that satisfy the same set of equations must be found by elimination of already discovered solutions; this we define as *indirect* configuration control.

In Theorem 1 it is shown that the OM method gives direct configuration control over all manipulator configurations that are explicitly specified in the closed-form equations of the model manipulator. Other configurations, if they exist, must be found indirectly. Thus, the OM method, when applied to the class of 6 DOF manipulators defined in Theorem 2, gives direct control over eight configurations[2] and, depending on manipulator geometry, can be used to indirectly find the remaining eight configurations, if they exist (i.e., are real).

B. Application to Redundant Manipulators

Inverse kinematics solutions for model manipulators with more than 6 degrees of freedom require additional specifications in order to define all possible solutions. For an N DOF redundant manipulator, the (N-6) redundancies can be specified using appropriately-defined (N-6) real variables (here termed *redundancy variables*[3]). Using redundancy variables, it is possible to write closed-form inverse kinematics equations for many types of redundant manipulators (see for example [3]). The desired redundancy variable values, which are specified as inputs to the inverse kinematics algorithm, are considered to be appended to the configuration vector $\boldsymbol{c}$.

By basing the choice of the values of each of the redundancy variables in the $\boldsymbol{c}$ vector on some appropriate optimization criteria, one can control redundant degrees of freedom in such a way as to optimize the control of the manipulator for various tasks. For example, the $\boldsymbol{c}$ vector can be used to control redundant degrees of freedom for the purpose of 1) avoiding obstalces by controlling the angle of the elbow plane of the manipulator [3], 2) maximizing the distance, in joint space, from manipulator joint limits at a given endpoint position and orientation (thus maximizing the dexterity of the manipulator at a given endpoint position and orientation), and/or 3) avoiding manipulator singularity regions. Thus, any advantage gained in using a redunant manipulator can be explicitly exploited by either a) manually specifying redundancy variable values within the $\boldsymbol{c}$ vector, or b) by using an appropriate optimization criterion to automatically generate desired redundancy variable values.

If one can reduce a *redundant real* manipulator to a *redundant model* manipulator that satisfies the conditions of Theorem 1, then one can formulate a solution to the inverse kinematics problem for the real manipulator using the OM method. The advantage gained in using the OM method on a redundant manipulator with no closed-form inverse kinematics equations, is that the complexity of the inverse kinematics problem never goes above that needed to solve a bounded system of three equations in three unknowns. This potentially gives real-time inverse kinematics control for redundant manipulators of any number of degrees of freedom. Other redundant manipulator inverse kinematics techniques, which utilize redundant degrees of freedom to maximize some optimization criterion [16]–[20], [37], [22], [30], require the manipulation and/or inversion of N by N matrices, which can be extremely computationally expensive when N is very large.

[2] This results from the observation that manipulators with three intersecting joints have a maximum of eight configurations [1].

[3] In [3] the redundancy variable is the angle that the plane of the arm makes with a vertical plane passing through the wrist and the shoulder.

V. Conclusion

A new theoretical framework (the OM method) that gives theoretically guaranteed convergence to a desired endpoint position and orientation, has been developed for solving the manipulator inverse kinematics problem. It has been shown that this theoretical framework converts the inverse kinematics problem for an N dimensional manipulator into a bounded 3-D problem (i.e. solving three nonlinear equations in three unknowns with the solution being contained within a known bounded three dimensional region). It has been proven that the OM method can be applied to any manipulator (of any number of degrees of freedom) that satisfies the conditions of Theorem 1, including manipulators that have no known closed-form inverse kinematics equations.

It has also been proven that the OM method allows one to directly control those manipulator configuration that are specified in the closed-form equations of the model manipulator (i.e. the offset-reduced manipulator created in the application of the OM method).

When the OM method was applied to the manipulator described in [14], rapid convergence was observed to all 24 points reported in [14], including the four points to which the algorithm of [14] was unable to converge to.

Appendix I
Proof of Theorem 1

Throughout the proof, condition C5 is assumed valid.

First, since the mapping $\boldsymbol{q}_m = \boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c})$ is continuous w.r.t. $\boldsymbol{x}_m$ (condition C2), and since the forward kinematics function $\Delta\boldsymbol{p}(\cdot)$ is also continuous (this is true because all forward kinematics functions are continuous), then the mapping $\boldsymbol{p}_m = \boldsymbol{p}_{r_d} + \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c}))$ must also be continuous.

The theorem will first be proven for the case where the desired solution is not in a singularity region of the model manipulator. A *singularity region*, U_m, is defined as the set of all $(\boldsymbol{x}_m, \boldsymbol{c})$, such that if for one set of joint variables, $\boldsymbol{q}^1$, one can write $(\boldsymbol{x}_m, \boldsymbol{c}) = \boldsymbol{k}_m(\boldsymbol{q}^1)$($\boldsymbol{k}_m(\cdot)$ is a function that gives both the forward kinematics and the configuration of the manipulator), then there exists a second set of joint variables, $\boldsymbol{q}^2$, where $\boldsymbol{q}^2 \neq \boldsymbol{q}^1$ and $(\boldsymbol{x}_m, \boldsymbol{c}) = \boldsymbol{k}_m(\boldsymbol{q}^2)$ [23]–[27]. Since the model manipulator is not in a singularity region, then, by condition C2, the inverse kinematics of the model manipulator is one to one; i.e. the mapping $\boldsymbol{q}_m = \boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c})$ is one to one. Since the model manipulator differs from the real manipulator by only the values of the Denavit–Hartenberg parameters a_i and d_i (condition C1), the hand (endpoint) orientation of the model manipulator is always identical to the hand orientation of the real manipulator; i.e. if condition C3 holds ($\boldsymbol{q} = \boldsymbol{q}_m = \boldsymbol{q}_r$) then $[\boldsymbol{n}_m \boldsymbol{s}_m \boldsymbol{a}_m] = [\boldsymbol{n}_{r_d} \boldsymbol{s}_{r_d} \boldsymbol{a}_{r_d}]$. This can be shown using homogeneous transformation matrices [32]. Thus the equation $\boldsymbol{p}_m = \boldsymbol{p}_{r_d} + \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c}))$ must completely define the relationship between the model manipulator and the real manipulator.

Since the mapping $\boldsymbol{p}_m = \boldsymbol{p}_{r_d} + \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_m, \boldsymbol{c}))$ is continuous and completely defines the relationship between the model manipulator and the real manipulator in the nonsingularity case, then by the definition of continuous functions, it is concluded that there exists a $\boldsymbol{p}_{m_a} \in \Re^3$ which is sufficiently close to $\boldsymbol{p}_m$, such that for $\|\boldsymbol{p}_{m_a} - \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_{m_a}, \boldsymbol{c})) - \boldsymbol{p}_{r_d}\| = E_{\max}$ (where $\boldsymbol{x}_{m_a} = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_{m_a}]^T$), and $E_{\max} \to 0, \boldsymbol{p}_{m_a} \to \boldsymbol{p}_m$. This proves the theorem for the case where the desired solution is not in a singularity region of the model manipulator.

By condition C4, if the model manipulator is in a singularity region then there exists a $\boldsymbol{p}_m^* \in \Re^3$ such that $\|\boldsymbol{p}_m^* - \boldsymbol{p}_m\| < \varepsilon$, where ε is an arbitrarily small positive real number, and $([\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m^*], \boldsymbol{c}) \notin U_m$. Therefore if we let $\boldsymbol{p}_{m_a} = \boldsymbol{p}_m^*$, then this $\boldsymbol{p}_{m_a}$, because it is not in a singularity region, must give a mapping $\boldsymbol{p}_{m_a} = \boldsymbol{p}_{r_d} + \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_{m_a}, \boldsymbol{c}))$ which is nonsingular. Therefore, as already proven above, there exists a $\boldsymbol{p}_{m_a} \in \Re^3$ which is sufficiently close to $\boldsymbol{p}_m$, such that for $\|\boldsymbol{p}_{m_a} - \Delta\boldsymbol{p}(\boldsymbol{k}_m^{-1}(\boldsymbol{x}_{m_a}, \boldsymbol{c})) - \boldsymbol{p}_{r_d}\| = E_{\max}$ (where $\boldsymbol{x}_{m_a} = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_{m_a}]^T$) as $E_{\max} \to 0, \boldsymbol{p}_{m_a} \to \boldsymbol{p}_m$. However, under these conditions, $E_{\max}$ can be arbitrarily small but never zero, and $\boldsymbol{p}_{m_a}$ can be arbitrarily close to $\boldsymbol{p}_m$, but never equal to it. QED.

Appendix II
Proof of Theorem 2

Condition C6 of Theorem 2 is identical to condition C5 of Theorem 1, and, Condition C5 of Theorem 2 is identical to condition C3 of Theorem 1. Thus it only remains to be proven that conditions C1, C2, C3, and C4 of Theorem 2 imply conditions C1, C2, and C4 of Theorem 1.

Satisfying Condition C1, of Theorem 1: This condition is explicitly satisfied by the algorithm given in condition C4 of Theorem 2; i.e. by condition C4 of Theorem 2, only the Denavit–Hartenberg parameters a_{r_4}, a_{r_5} and d_{r_5} are modified when creating the model manipulator.

Satisfying Condition C2, of Theorem 1: By condition C4 of Theorem 2, $a_{m_4} = 0, a_{m_5} = 0$ and $d_{m_5} = 0$. Therefore, the last three links of the model manipulator are intersecting and thus a complete solution to the inverse kinematics problem for the model manipulator must exist [1]. Thus condition C2 of Theorem 1 is satisfied.

Satisfying Condition C4, of Theorem 1: This condition will be shown to be true by proving it for 2 separate cases.

Case 1: Assume that the model manipulator is in a singularity region that is associated with the first three joint variables; i.e. for the desired manipulator configuration and hand (endpoint) position and orientation $(\boldsymbol{x}_m^1, \boldsymbol{c})$ (where $\boldsymbol{x}_m^1 = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m^1]$), there are more than one set of values for the joint variables (q_1^1, q_2^1, q_3^1), which give the required wrist position defined by $\boldsymbol{p}_w^1 = \boldsymbol{p}_m^1 - d_{m6}\boldsymbol{a}_{r_d} - a_{m6}\boldsymbol{n}_{r_d}$. Then, by condition C2 of Theorem 2, and because $\boldsymbol{p}_w^1 = \boldsymbol{p}_w(q_1^1, q_2^1, q_3^1)$ is a continuous forward kinematics function that gives the position of the wrist, one can find a set of joint variable values (q_1^2, q_2^2, q_3^2) such that $\|(q_1^2, q_2^2, q_3^2) - (q_1^1, q_2^1, q_3^1)\| < \varepsilon$ (where $\varepsilon \in \Re$ is an arbitrarily small positive number), and $\boldsymbol{p}_w^2 = \boldsymbol{p}_w(q_1^2, q_2^2, q_3^2) \neq \boldsymbol{p}_w^1$ (where $\boldsymbol{p}_m^2 = \boldsymbol{p}_w^2 + d_{m6}\boldsymbol{a}_{r_d} + a_{m6}\boldsymbol{n}_{r_d}, \boldsymbol{x}_m^2 = [\boldsymbol{n}_{r_d}\boldsymbol{s}_{r_d}\boldsymbol{a}_{r_d}\boldsymbol{p}_m^2]$), and $(\boldsymbol{x}_m^2, \boldsymbol{c})$ is no longer in a singularity region. Thus condition C4 of Theorem 1 is satisfied when the singularity is associated with the first three joint variables.

Case 2: Now assume that the singularity is a result of the desired hand orientation; i.e. for the desired hand configuration, orientation and position, $(\boldsymbol{x}_m^1, \boldsymbol{c})$, there is more than one set of valid joint variable values (q_4^1, q_5^1, q_6^1). Because the last three joints are all rotational, a hand orientation singularity can occur if and only if any two of the last three joints becomes parallel (and thus give the same motion in orientation space). By condition C3 of Theorem 2, the last three joints give three degrees of freedom to the orientation of the hand, and thus joint 4 is never parallel to joint 5, and joint 5 is never parallel to joint 6. Therefore, a hand orientation singularity will occur if and only if joint 4 becomes parallel to joint 6. As will be proven in the following lemma, this can only occur at specific discrete values of the joint variable q_5 (for example, a wrist singularity of the PUMA 560 manipulator occurs when $q_5 = 0°$).

LEMMA: By conditions C1, C2, C3 and C4 of Theorem 2, if $q_5^1 = \sigma$, where $\sigma \in \Re$ gives a hand orientation singularity, corresponds to the Cartesian wrist position $\boldsymbol{p}_w^1$, and if $0 < \|\boldsymbol{p}_w^2 - \boldsymbol{p}_w^1\| < \varepsilon$ (ε is an arbitrarily small positive real number) and $\boldsymbol{p}_w^2$ does not produce a singularity condition in the first three joints of the manipulator, then it can be concluded that $q_5^2 \neq \sigma$, and $\boldsymbol{p}_w^2$ does not correspond to a hand orientation singularity.

Proof of lemma (by contradiction): Assume that $\boldsymbol{p}_w^2$ is also in a hand orientation singularity region. Let ${}^0\boldsymbol{A}_3 = [\boldsymbol{n}_w^T\boldsymbol{s}_w^T\boldsymbol{a}_w^T\boldsymbol{p}_w^T]$ (see [32, pp. 40–44]) be the homogeneous transform from the base coordinate system to the coordinate system of joint 4. In this coordinate frame, the desired hand orientation is given by $[\boldsymbol{n}_{w_d}^T\boldsymbol{s}_{w_d}^T\boldsymbol{a}_{w_d}^T 0] = {}^0\boldsymbol{A}_3[\boldsymbol{n}_{r_d}^T\boldsymbol{s}_{r_d}^T\boldsymbol{a}_{r_d}^T 0]$. If we move form $\boldsymbol{p}_w^1$ to $\boldsymbol{p}_w^2$, then the joint variables (q_1^1, q_2^1, q_3^1) must change to (q_1^2, q_2^2, q_3^2) such that $\|(q_1^2, q_2^2, q_3^2) - (q_1^1, q_2^1, q_3^1)\| < \varepsilon$ (where $\varepsilon \in \Re$ is an arbitrarily small positive number). Therefore, $[\boldsymbol{n}_{w_d}^{1^T}\boldsymbol{s}_{w_d}^{1^T}\boldsymbol{a}_{w_d}^{1^T} 0] = {}^0\boldsymbol{A}_3^1[\boldsymbol{n}_{r_d}^T\boldsymbol{s}_{r_d}^T\boldsymbol{a}_{r_d}^T 0]$ must change to $[\boldsymbol{n}_{w_d}^{2^T}\boldsymbol{s}_{w_d}^{2^T}\boldsymbol{a}_{w_d}^{2^T} 0] = {}^0\boldsymbol{A}_3^2[\boldsymbol{n}_{r_d}^T\boldsymbol{s}_{r_d}^T\boldsymbol{a}_{r_d}^T 0]$. By conditions C2 and C3 of Theorem 2, and because $\boldsymbol{p}_w^2$ does not produce a singularity condition in the first three joints of the manipulator, it must be the case that $\boldsymbol{n}_{w_d}^{1^T} \neq \boldsymbol{n}_{w_d}^{2^T}, \boldsymbol{s}_{w_d}^{1^T} \neq \boldsymbol{s}_{w_d}^{2^T}$, and $\boldsymbol{a}_{w_d}^{1^T} \neq \boldsymbol{a}_{w_d}^{2^T}$. However, $\boldsymbol{p}_w^1$ is in a hand orientation singularity condition and therefore joint 4 is parallel to joint 6. Thus changes in q_4^1 and q_6^1 cause rotation about the $\boldsymbol{a}_{w_d}^{1^T}$ (this is true because, by the definition of the homogeneous transformation matrix [32, pp. 36–44], at $\boldsymbol{p}_w^1$, the z-axis of the wrist coodinate frame ${}^0\boldsymbol{A}_3^1$ is parallel to $\boldsymbol{a}_{w_d}^{1^T}$, and a change in q_4^1, which is parallel to q_6^1, causes rotation about the z-axis). Thus changes in q_4^1 and q_6^1 can only cause changes in $\boldsymbol{n}_{w_d}^{1^T}$ and $s_{w_d}^{1^T}$. Therefore, if $\boldsymbol{p}_w^2$ is in a hand orientation singularity condition, then $\boldsymbol{a}_{w_d}^{1^T} = \boldsymbol{a}_{w_d}^{2^T}$. This is a contradiction and therefore q_5 must change to a value such that $q_5^2 \neq \sigma = q_5^1$, and $\boldsymbol{p}_w^2$ cannot be in a hand orientation singularity condition. This completes the proof of the lemma.

Therefore, because the position of the coordinate system of joint 4 can be defined by $\boldsymbol{p}_w^1 = \boldsymbol{p}_m^1 - d_{m6}\boldsymbol{a}_{r_d} - a_{m6}\boldsymbol{n}_{r_d}$, by the above lemma, condition C4 of Theorem 1 is satisfied when the singularity is associated with the last three joint variables.

Thus condition C4 of Theorem 1 is satisfied. Q.E.D.

Appendix III
The Inverse Kinematics Equations of the Spherical Model Manipulator

$$q_1 = \operatorname{atan2}\left[\frac{\text{ARM} \cdot p_y}{\text{ARM} \cdot p_x}\right]$$

$$q_2 = \operatorname{atan2}\left[\frac{a_{m_2}b + \text{ELBOW} \cdot a\sqrt{a^2 + b^2 - a_{m_2}}}{a_{m_2}a - \text{ELBOW} \cdot b\sqrt{a^2 + b^2 - a_{m_2}}}\right]$$

$$q_3 = \begin{cases} \dfrac{b - a_{m_2}\sin q_2}{\cos q_2}, & \text{if } \cos q_2 \neq 0 \\ \dfrac{a - a_{m_2}\cos q_2}{-\sin q_2}, & \text{otherwise} \end{cases}$$

$$q_4 = \operatorname{atan2}\left[\frac{\text{WRIST} \cdot A_y}{\text{WRIST} \cdot A_x}\right]$$

$$q_5 = \operatorname{atan2}\left[\frac{\text{WRIST} \cdot \sqrt{1 - A_z^2}}{-A_z}\right]$$

$$q_6 = \operatorname{atan2}\left[\frac{-\text{WRIST} \cdot O_z}{\text{WRIST} \cdot N_z}\right]$$

where

$$p_x = p_{m_x} - a_{m_6} n_{m_x}$$

$$p_y = p_{m_y} - a_{m_6} n_{m_y}$$

$$p_z = p_{m_z} - a_{m_6} n_{m_z}$$

$$a = p_x \cos q_1 + p_y \sin q_1$$

$$b = p_z - d_{m_1}$$

$$A_x = (-\sin q_1 \cos q_2) a_{r_{d_x}} + (-\sin q_1 \sin q_2) a_{r_{d_y}} + (\cos q_2) a_{r_{d_z}}$$

$$A_y = (-\cos q_1 \cos q_2) a_{r_{d_x}} + (-\sin q_1 \cos q_2) a_{r_{d_y}} + (-\sin q_2) a_{r_{d_z}}$$

$$A_z = (\sin q_1) a_{r_{d_x}} + (-\cos q_1) a_{r_{d_y}}$$

$$O_z = (\sin q_1) s_{r_{d_x}} + (-\cos q_1) s_{r_{d_y}}$$

$$N_z = (\sin q_1) n_{r_{d_x}} + (-\cos q_1) n_{r_{d_x}}$$

The function atan2 is defined in [32, p55]. Note that the configuration vector is

$$c = [\text{ARM ELBOW WRIST}]. \tag{13}$$

The manipulator singularities are handled as follows.

1) If $p_x = p_y = 0$, then q_1 retains its previous value and the rest of the joint variables are calculated as defined previously.
2) If, in addition to the above singularity, $p_z = d_{m_1}$, then both q_1 and q_2 retain their previous values and the rest of the joint variables are calculated as defined previously.
3) If $A_x = A_y = 0$, then q_4 retains its previous value and

$$q_5 = 180^\circ, q_6 = -q_4 + \operatorname{atan2}\left(\frac{O_x}{-N_x}\right), \quad \text{if } A_z = 1 \tag{14}$$

or

$$q_5 = 0^\circ, q_6 = q_4 - \operatorname{atan2}\left(\frac{O_x}{N_x}\right) \quad \text{if } A_z = -1 \tag{15}$$

where

$$O_x = (-\sin q_1 \cos q_2) s_{r_{d_x}} + (-\sin q_1 \sin q_2) s_{r_{d_y}} + (\cos q_2) s_{r_{d_z}}$$

$$N_x = (-\sin q_1 \cos q_2) n_{r_{d_x}} + (-\sin q_1 \sin q_2) n_{r_{d_y}} + (\cos q_2) n_{r_{d_z}}$$

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