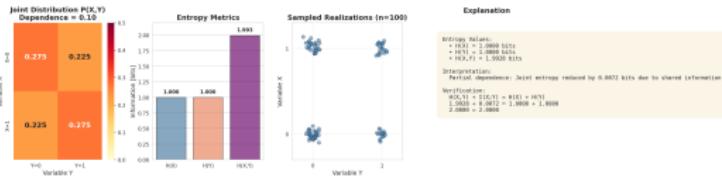


Information Theory

Instructor: Dr. GP Saggese - gsaggese@umd.edu

References:



[video](#)

- ***Information theory***
 - Entropy

- Information theory
 - *Entropy*

Entropy: Definition

- **Entropy** $H(X)$ of a discrete random variable X is defined as:

$$H(X) \triangleq - \sum_x p(x) \log_2 p(x)$$

- **Intuition**

- Entropy quantifies average *information*, *surprise*, or *uncertainty* in outcomes
 - High entropy = more unpredictability
 - Low entropy = more certainty
- Log in base 2 → entropy measured in bits

- **Examples**

- Fair coin
 - Two equally likely outcomes (heads or tails) → maximum uncertainty $H = 1$
- Biased coin
 - Coin lands heads 90% of the time, less uncertainty → less entropy $H < 1$
- “Broken” coin
 - Always lands on heads, no uncertainty → no entropy $H = 0$

Entropy as Expected Information Content

- Entropy measures how *surprising* a random variable is *on average*
- **Intuition:** information quantifies how much we *learn* when an event occurs
 - Information of outcome x_i is defined as $-\log p_i$
 - Rare events ($p_i \rightarrow 0$) yield more information: $-\log p_i \rightarrow \infty$
 - Common events ($p_i \rightarrow 1$) yield little information: $-\log p_i \rightarrow 0$
 - Entropy is the expected information content:

$$H(X) = \mathbb{E}[-\log p_i] \triangleq - \sum_i p_i \log p_i$$

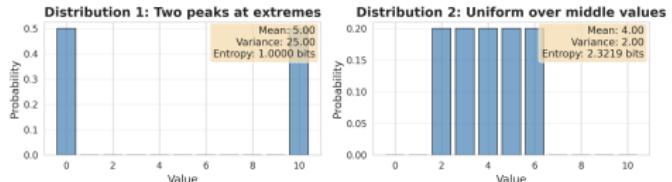
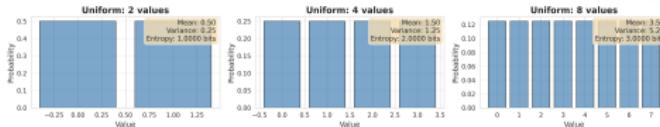
Axiomatic Derivation of Entropy

- Shannon's criteria for $H(p_1, \dots, p_n)$:
 1. Continuity: $H()$ is continuous in p_i
 - Small changes in probabilities should not cause large jumps in uncertainty
 2. Maximality: $H()$ is maximal when outcomes are equally likely
 3. Additivity: For composite outcomes, $H(X, Y) = H(X) + H(Y|X)$
 - Uncertainty about (X, Y) is uncertainty about H plus remaining uncertainty about Y
- The only functional form is:

$$H(X) \propto - \sum_i p_i \log p_i$$

Entropy and Probability Density Function (PDF)

- Entropy is related to variance but is not the same
 - Variance measures how far values are from the mean
 - Entropy measures how unpredictable a random draw is
- If a distribution has more spread, typically its entropy is larger
 - It is possible that variance increases, but entropy doesn't
 - E.g., a uniform distribution increasing its support
- Entropy is related to information and uncertainty
 - A flatter distribution has high entropy
 - A sharply peaked distribution has low



Joint Entropy

- **Joint entropy** $H(X, Y)$ of two variables X and Y is defined as:

$$H(X, Y) \triangleq - \sum_{x,y} p(x, y) \log p(x, y)$$

- Describes the information needed for the joint distribution of X and Y

- **Properties**

- Non-negative and zero if X and Y are perfectly determined
- For two independent binary variables X and Y , the joint entropy is the sum of the entropy

- **Applications**

- Identifies dependencies or correlations in datasets
- Aids in feature selection by finding informative variable combinations
- E.g., in sensor network data, joint entropy can highlight overlapping sensor information

Conditional Entropy

- **Conditional entropy** $H(Y|X)$ is defined as

$$H(Y|X) \triangleq - \sum_{x,y} p(x,y) \log p(y|x)$$

- Represents the average uncertainty in Y after observing X
- Measures the effectiveness of X in determining Y

- **Properties**

- Low $H(Y|X)$ implies stronger predictive power of X on Y
 - There is less uncertainty about Y after knowing X
 - I.e., Y has predictive power on X
- If $Y = X$, then $H(Y|X) = 0$
 - No uncertainty about Y once X is known
 - X completely determines Y
- If X and Y are independent, then $H(Y|X) = H(Y)$
 - Knowledge of X provides no new information about Y

- **Applications**

- In feature selection assess the predictive power of independent variables on dependent variables

Mutual Information

- The **mutual information** $I(X; Y)$ between X and Y is defined as:

$$I(X; Y) \triangleq H(X) - H(X|Y)$$

- Intuition**

- Measures how much knowing X reduces uncertainty about Y
- Gauges the shared information between two variables

- Properties**

- Non-negative: $I(X; Y) \geq 0$
- Symmetric: $I(X; Y) = I(Y; X)$
- If X and Y are independent, then $I(X; Y) = 0$
- Higher mutual information indicates greater relation between X and Y
- Related to the joint entropy but symmetric:

$$I(X; Y) = H(X) + H(Y) - H(X, Y)$$

- Applications**

- Selects features sharing high information with the target variable

- Used to reduce dimensionality



Kullback-Leibler (KL) Divergence

- The **KL divergence** $D_{\text{KL}}(P\|Q)$ between distributions P and Q is defined as:

$$D_{\text{KL}}(P\|Q) \triangleq \sum_x P(x) \log \frac{P(x)}{Q(x)}$$

- Intuition**

- Quantify how much one distribution deviates from another distribution

- Properties**

- Not symmetric: $D_{\text{KL}}(P\|Q) \neq D_{\text{KL}}(Q\|P)$
- $D_{\text{KL}}(P\|Q) = 0$ iff $P = Q$, i.e., $Q(x) = P(x)$ for all x
- It is a distance, but not a metric (not symmetric, no triangle inequality)

- Applications**

- In optimization of machine learning models by minimizing divergence
- E.g., variational autoencoders use KL divergence to ensure that the learned distribution is close to the true distribution

Cross-Entropy

- The **cross-entropy** $H(P, Q)$ between two distributions P and Q is defined as:

$$H(P, Q) \triangleq - \sum_x P(x) \log Q(x)$$

where:

- $P(x)$ is the true probability of the event x
- $Q(x)$ is the probability assigned by the model

- Intuition**

- Measures the average number of bits needed to encode data from P using a code optimized for Q
- Indicates inefficiency when the code for Q is used to represent P

- Properties**

- Cross-entropy is related to entropy and KL divergence:

$$H(P, Q) = H(P) + D_{\text{KL}}(P \| Q)$$

- Applications**

- Used to compare the similarity of the predicted outcomes probability distribution to the true distribution
 - E.g., as loss function in logistic regression
 - A perfect model has a cross-entropy of 0

Data Processing Inequality

- **Data processing inequality** states that:

Processing data cannot increase information, it can only lose information over time.

- Formally: if $X \rightarrow Y \rightarrow Z$, then $I(X; Z) \leq I(X; Y)$
 - After passing through an additional stage (from Y to Z), the mutual information with the initial stage (X) cannot increase
- **Examples**
 - If X is a raw image, Y is a compressed version
 - No additional processing Z will uncover more information about X than what Y already represents
 - If Y is a dataset derived from X (e.g., summary statistics)
 - Any analysis applied to Y alone cannot provide more insights into X than Y itself

• Applications

- Compression can only lead to information loss
- Identify “information bottlenecks” in a modeled process, ensuring model designs consider the constraints imposed by information processing



Chain Rule for Entropy and Mutual Information

- The **entropy chain rule**

$$H(X, Y) = H(X) + H(Y|X)$$

- In words, the joint entropy of two random variables can be decomposed into the entropy of one and the conditional entropy of the other
- Examples:
 - X represents the weather (sunny, rainy)
 - Y represents outdoor activity (park, cinema)
 - $H(Y|X)$ would provide information about outdoor activities given specific weather conditions
- The **mutual information chain rule**:

$$I(X, Y; Z) = I(X; Z) + I(Y; Z|X)$$

- **Applications**

- By using chain rules, one can decompose and understand the complexity of joint distributions with multiple interacting variables

• Useful in sequential models (e.g., speech recognition) and time-series analysis (e.g., stock market prediction)



Source Coding Theorem

- Aka “Shannon’s first theorem”

- **Statement**

- Compression cannot achieve an average code length less than the entropy of the source

- **Implications**

- It asserts a limit on lossless compression:
 - E.g., if a source has entropy $H(X) = 3$ bits, you cannot on average encode it with fewer than 3 bits per symbol

- **Examples**

- Lossless compression methods approach the entropy limit
 - E.g., Huffman coding encode data by creating variable-length codes

Noisy-Channel Coding Theorem

- Aka “Shannon’s second theorem”

- **Statement**

- For any discrete memoryless channel with capacity C , and for any desired level of reliability $\varepsilon > 0$, it is possible to transmit information at any rate $R < C$ with an arbitrarily small probability of error, using a sufficiently long encoding scheme

- **Implications**

- Accurate communication can be achieved even with noise
- Error correction techniques can be applied to achieve this
- It does not construct the code, it only proves existence
- Channel capacity defines the upper limit of information that can be transmitted reliably (e.g., 10 Mbps)
- Applications:
 - Fundamental principle for designing digital communication systems
 - E.g., mobile networks, satellite communications, and the internet

Redundancy and Compression

- **Redundancy** is the difference between actual and optimal code length
 - Measures excess information in the data
 - Redundancy implies room for compression, i.e., reduce data size without losing information
- **Compression techniques** remove redundancy while preserving information
 - Aim to make data smaller without losing meaning or important details
 - Useful for reducing storage or speeding up data transmission
- **Examples**
 - Run-length encoding
 - Compress by replacing consecutive identical elements with a single value and count
 - E.g., AAAABBCCDAA becomes 4A3B2C1D2A
 - Huffman coding
 - Uses variable-length codes for encoding
 - Frequently used symbols get shorter codes, reducing length

Typical Set

- Set of sequences with probability close to $2^{-nH(X)}$
 - E.g., if $H(X) = 2$, then for large n , sequences have a probability close to 2^{-2n}
- Central to proving coding theorems
 - The typical set is essential in demonstrating the efficiency of compression algorithms
- Almost all sequences in large samples lie in the typical set
 - E.g., for a sequence length n , the probability of falling outside the typical set decreases exponentially as n increases
- Enables asymptotic analysis of information theory
 - Used in deriving limits related to data compression and reliable communication

Rate-Distortion Theory

- Trade-off between compression rate R vs distortion D
 - **Compression rate** R : amount of data remaining after compression
 - **Distortion** D : difference between the original and compressed data
 - Balancing R and D is crucial for effective lossy compression
- **Goal:** reduce data size (lossy compression) while maintaining an acceptable level of quality
- Rate-distortion function $R(D)$ defines the minimal rate for a given distortion
 - Describes the lower bound of the data rate necessary to achieve a specified level of distortion
 - E.g., in image compression, $R(D)$ helps in determining the lowest bitrate for a desired image quality
- **Applications**
 - Widely used in image/audio/video compression, e.g., MP3, JPEG and MPEG formats
 - Important for streaming services and storage optimization

Fano's Inequality

- When X is guessed from Y , it holds:

$$H(X|Y) \leq h(P_e) + P_e \log(|X| - 1)$$

where

- X is a discrete random variable
- Y is the estimate variable of X based on some observations
- $H(X|Y)$ is the conditional entropy of X given Y
- $h(P_e)$ is the binary entropy function quantifying uncertainty of a binary random variable, where $h(p) = -p \log p - (1-p) \log(1-p)$

- Intuition**

- It relates the uncertainty remaining about X after observing Y to the probability of making an error in guessing X
- “You cannot simultaneously have low error and low entropy (uncertainty). If the entropy is high, your probability of error must also be high.”

Differential Entropy

- **Definition**

- The differential entropy $h(x)$ for continuous random variable X with density $p(x)$ is defined as:

$$h(X) \triangleq - \int p(x) \log p(x) dx$$

- **Concept**

- Extends Shannon entropy to continuous distributions
- Measures “spread” or uncertainty of X

- **Key Properties**

- Not invariant under variable change
 - E.g., scaling affects $h(X)$
- Can be negative, unlike non-negative discrete entropy
- Units depend on logarithm base

- **Example**

- For Gaussian $X \sim \mathcal{N}(\mu, \sigma^2)$:

$$h(X) = \frac{1}{2} \log(2\pi e \sigma^2)$$

- **Limitations**

SCIENCE • Cannot compare directly across variables with different units or scales
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Maximum entropy principle

- **Definition**
 - Use the prior with the largest entropy (i.e., the least informative) given the constraints of the problem
 - Can be solved as an optimization problem
- **Examples**
 - The distribution with largest entropy given a constraint is:
 - Without constraints: uniform
 - A positive mean: exponential
 - A given variance: normal distribution

Minimum Description Length (MDL)

- **Definition**

- The total description length of a dataset $MDL(H)$ is given by:

$$MDL(H) = L(H) + L(D | H)$$

where:

- $L(H)$ is the length (in bits) of the model or hypothesis
- $L(D | H)$ is the length of the data encoded using the model

- **Principle**

- MDL selects the hypothesis H that minimizes the total description length

$$MDL(H) = L(H) + L(D | H)$$

- **Intuition**

- Prefers the model that gives the shortest total description of data
- Based on Occam's Razor: simpler models are preferred
- Balances model complexity and data fit

- **Example**

- Given two decision trees for classifying email as spam:
 - Tree A is small and classifies 95% correctly
 - Tree B is large and classifies 96% correctly
 - MDL may prefer Tree A if the increased accuracy doesn't justify the extra complexity

Kolmogorov Complexity

- The Kolmogorov complexity $K(x)$ of a string x is the length of the shortest binary program that outputs x on a universal Turing machine
- **Examples**
 - A string of 1000 random bits: high Kolmogorov complexity, no compressible pattern
 - A string of 1000 repeated 0s: low Kolmogorov complexity, described by a short loop
- **Intuition**
 - Measures “algorithmic randomness” or compressibility of a string
 - A string is complex if it has no shorter description than itself
- **Formal Properties**
 - Incomputable: no algorithm computes $K(x)$ for all x
 - $K(x) \leq |x| + c$ for some constant c , showing the trivial upper bound (print x)
- **Relation to MDL**

MDL approximates Kolmogorov complexity by minimizing a practical description length



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Information Bottleneck

- Framework for extracting relevant information
- Trade-off: compression of X vs retention of info about Y
- Optimization: minimize $I(X; T)$ while preserving $I(T; Y)$
- Used in deep learning theory and representation learning

Multi-Information and Total Correlation

- Generalization of mutual information to multiple variables
 - Total correlation: $C(X_1, \dots, X_n) = \sum_i H(X_i) - H(X_1, \dots, X_n)$
 - Measures total dependency in a set of variables
 - Used in ICA, variational inference, and dependency modeling
2. Applications in Machine Learning
- Entropy: Feature selection, decision tree splitting criteria
 - Mutual Information: Dependency detection, feature relevance
 - Cross-Entropy: Loss function for classification
 - KL Divergence: Variational inference, model comparison
 - MDL: Model selection, preventing overfitting