Vortices and sources on the sphere

Let's determine the fluid velocity of singularities on the sphere S^3

Helmholz decomposition of the velocity:

$$\vec{V} = \vec{\nabla} \times \vec{A} + \vec{\nabla}\phi$$

We have $\vec{\Delta}\vec{A} = -\vec{\nabla}\times\vec{V} = -\vec{\omega}$
and $\Delta\phi = \vec{\nabla}\cdot\vec{V} = \sigma$

with $\vec{\omega} = \sum_i \Gamma_i \vec{n_i}\, \delta(\vec{n} - \vec{n_i})$

$$\sigma = \sum_i m_i\, \delta(\vec{n} - \vec{n_i}) \qquad \vec{n}, \vec{n_i} \in S^3$$

The Green function G for the laplacian on the sphere satisfies

$$\Delta G(\vec{n}) = \delta(\vec{n} - \vec{n}')$$

and $$G(\vec{n}) = \frac{\ln(1 - \vec{n}\cdot\vec{n}')}{4\pi}$$

With this, $\vec{A} = -\sum_i \frac{\Gamma_i \vec{n_i}}{4\pi} \ln(1 - \vec{n}\,\vec{n_i})$

and $\sigma = \sum_i \frac{m_i}{4\pi} \ln(1 - \vec{n}\,\vec{n_i})$

As a result $$\vec{V} = \sum_i \frac{\Gamma_i}{4\pi}\frac{\vec{n_i}\times\vec{n}}{1 - \vec{n}\cdot\vec{n_i}} + \frac{m_i}{4\pi}\frac{\vec{n_i} - (\vec{n}\,\vec{n_i})\vec{n}}{1 - \vec{n}\,\vec{n_i}}$$

The singularities positions evolve as

$$\frac{d\vec{r}_i}{dt} = \vec{V}(\vec{r}_i)$$, where in the sum, the term i is taken out.