Vortices and sources on the sphere

August 18, 2020

Let's determine the fluid velocity of singularities on the sphere S^3.
Helmholz decomposition of the velocity:

$$\mathbf{V} = \nabla \times \mathbf{A} + \nabla \phi$$

We have

$$\boldsymbol{\Delta}\mathbf{A} = -\nabla \times \mathbf{V} = -\omega$$

and

$$\Delta \phi = \nabla \cdot \mathbf{V} = \sigma$$

with

$$\mathbf{w} = \sum_i \Gamma_i \mathbf{r_i} \delta\left(\mathbf{r} - \mathbf{r_i}\right)$$

$$\sigma = \sum_i m_i \delta\left(\mathbf{r} - \mathbf{r}_i\right)$$

The Green function G for the laplacian on the sphere satisfies:

$$\Delta G(\mathbf{r}) = \delta\left(\mathbf{r} - \mathbf{r}'\right)$$

and

$$G(\mathbf{r}) = \frac{\log\left(1 - \mathbf{r} \cdot \mathbf{r}'\right)}{4\pi}$$

With this,

$$\mathbf{A} = -\sum_i \frac{\Gamma_i \mathbf{r}_i}{4\pi} \ln(1 - \mathbf{r} \cdot \mathbf{r}_i)$$

and

$$\sigma = \sum_i \frac{m_i}{4\pi} \ln(1 - \mathbf{r} \cdot \mathbf{r}_i)$$

As a result

$$\mathbf{V} = \sum_i \frac{\Gamma_i}{4\pi} \frac{\mathbf{r_i} \times \mathbf{r}}{1 - \mathbf{r} \cdot \mathbf{r}_i} + \frac{m_i}{4\pi} \frac{\mathbf{r}_i - \left(\mathbf{r} \cdot \mathbf{r}_i\right) \mathbf{r}}{1 - \mathbf{r} \cdot \mathbf{r}_i}$$

The singularities positions evolve as $\frac{d\mathbf{r}_i}{dt} = \mathbf{V}(\mathbf{r}_i)$ where in the sum, the term i is taken our.