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# Dynamics of a Cosmic Muon in a Magnetic Field

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June 2025  
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# Contents

<b>1</b>	<b>Introduction</b>	<b>4</b>
1.1	Initial Problem . . . . .	4
1.1.1	Assumptions . . . . .	4
1.2	Experimental Approach . . . . .	4
1.3	Theoretical Approach . . . . .	4
<b>2</b>	<b>Dynamics of the particle in classical physics</b>	<b>6</b>
2.1	Hamiltonian and Lorentz force . . . . .	6
2.2	Moment derivative . . . . .	6
2.2.1	In the $x$ -plane . . . . .	6
2.2.2	In the $z$ -plane . . . . .	7
2.2.3	In the $y$ -plane . . . . .	7
2.3	Position with respect to time . . . . .	8
2.3.1	Equations of Motion . . . . .	8
2.3.2	Position . . . . .	8
<b>3</b>	<b>Dynamics of the particle in quantum physics</b>	<b>11</b>
3.1	Hamiltonian . . . . .	11
3.2	Quantum Harmonic oscillator . . . . .	13
3.2.1	One Dimension . . . . .	13
3.2.2	Two Dimensions . . . . .	17
3.3	Kinetic orbital momentum in the $y$ direction . . . . .	19
3.3.1	Intermediate $\hat{L}_z$ . . . . .	19
3.3.2	Eigenfunctions of $\hat{L}_y$ . . . . .	20
<b>4</b>	<b>Final and global wave function</b>	<b>22</b>
4.1	Compact formulation . . . . .	22
4.2	Developped formulation . . . . .	22
<b>5</b>	<b>Numerical calculations and predictions</b>	<b>23</b>
<b>6</b>	<b>Conclusion</b>	<b>24</b>
<b>7</b>	<b>Acknowledgments</b>	<b>24</b>
<b>8</b>	<b>Appendix</b>	<b>24</b>
8.1	Detailed solution choice . . . . .	24
8.2	Harmonic oscillator . . . . .	25
8.2.1	Time-independent Schrödinger . . . . .	25
8.2.2	Schrödinger equation for the quantum harmonic oscillator . . . . .	25
8.3	Maclaurin series expansion . . . . .	26

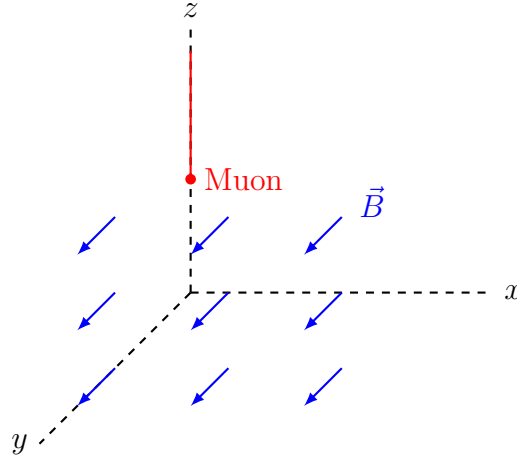
8.4	Hermite's polynomials . . . . .	27
8.5	Euler angles . . . . .	28

# 1 Introduction

## 1.1 Initial Problem

We consider a cosmic muon entering a uniform magnetic field. Therefore, the muon is subject to the Lorentz force, which deviates its trajectory.

**Our goal is to study the position of a cosmic muon in a uniform magnetic field, using both classical and quantum mechanics.**



### 1.1.1 Assumptions

In the following sections, we make the following assumptions:

- The magnetic field is uniform and given by  $\vec{B} = B\hat{y}$ .
- The muon has mass  $m$  and charge  $q$ .
- No electric field is present, and interactions with matter are neglected.

## 1.2 Experimental Approach

We plan to conduct an experiment in the context of the Physics Olympiad. We will publish the results once we complete the experiment.

## 1.3 Theoretical Approach

The next two sections will cover both quantum and classical mechanics, to understand the fundamental differences between them.

On the one hand, classical mechanics predicts the muon's trajectory merely using the Lorentz force. On the other hand, quantum mechanics introduces a probabilistic interpretation.

In this context, treating the muon as a quantum entity is more appropriate. However, we choose to explore both to evaluate their differences.

## 2 Dynamics of the particle in classical physics

### 2.1 Hamiltonian and Lorentz force

The Hamiltonian for a particle in a magnetic field is defined as follows:

$$H = \frac{1}{2m} \left[ \left( p_x - \frac{e}{c} A_x \right)^2 + \left( p_y - \frac{e}{c} A_y \right)^2 + \left( p_z - \frac{e}{c} A_z \right)^2 \right] \quad (1)$$

Using the right-hand rule, we see that the magnetic field in the  $y$  direction ( $B_y$ ) must be nonzero, as we have to deviate the muon on the  $x$  plane. The components of the magnetic field are given by

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \quad B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \quad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y},$$

Therefore, to obtain a nonzero  $B_y$ , either  $A_z$  depends on  $x$  or  $A_x$  depends on  $z$ , or both. With this established, the magnetic field exists only along the  $y$ -plane. Since  $A_y = 0$ , we have the following Hamiltonian:

$$H = \frac{1}{2m} \left[ \left( p_x - \frac{e}{c} A_x \right)^2 + \left( p_z - \frac{e}{c} A_z \right)^2 + p_y^2 \right] \quad (2)$$

This Hamiltonian is general and holds for any choice of the vector potentials, that we will define in the next parts.

### 2.2 Moment derivative

#### 2.2.1 In the $x$ -plane

For now, we can choose the vector potentials  $A_z = -bx$ ,  $A_x = 0$ , and  $A_y = 0$ . We may now rewrite the Hamiltonian by replacing the vector potentials by their values.

$$H = \frac{1}{2m} \left[ p_x^2 + \left( p_z + \frac{e}{c} bx \right)^2 + p_y^2 \right]$$

Thanks to Hamilton's equation we can easily calculate  $\dot{p}_x$ :

$$\begin{aligned} \dot{p}_x &= -\frac{\partial H}{\partial x} = -\frac{\partial}{\partial x} \left[ \frac{1}{2m} \left( p_x^2 + \left( p_z + \frac{e}{c} bx \right)^2 + p_y^2 \right) \right] \\ &= -\frac{1}{2m} \cdot 2 \left( p_z + \frac{e}{c} bx \right) \cdot \frac{e}{c} b \\ &= -\frac{eb}{cm} \left( p_z + \frac{e}{c} bx \right) = -\frac{eb}{cm} (mv_z - \frac{eb}{c} x + \frac{eb}{c} x) \\ &= -\frac{eb}{cm} mv_z = -\frac{eb}{c} v_z \end{aligned}$$

Since  $\dot{p}_x = ma_x + \frac{e}{c}\dot{A}_x$  and  $\dot{A}_x = 0$ , we correctly find the Lorentz force for  $v_z = v_0$ .

$$\dot{p}_x = ma_x = -\frac{eb}{c}v_z \quad (3)$$

### 2.2.2 In the $z$ -plane

We modify the vector potential components to:

$$A_z = 0, \quad A_x = bz, \quad A_y = 0.$$

This choice results in the following:

- The magnetic field along the  $y$ -plane becomes  $B_y = b$ .
- The Hamiltonien is now  $H = \frac{1}{2m} \left[ (p_x - \frac{e}{c}bz)^2 + p_z^2 + p_y^2 \right]$
- As you will see in the calculations,  $\dot{p}_z$  will be consistent with the Lorentz force.

Using the same method, we find the following.

$$\begin{aligned} \dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{\partial}{\partial z} \left[ \frac{1}{2m} \left( (p_x - \frac{e}{c}bz)^2 + p_z^2 + p_y^2 \right) \right] \\ &= -\frac{1}{2m} \cdot 2 \left( p_x - \frac{e}{c}bz \right) \cdot \left( -\frac{e}{c}b \right) \\ &= \frac{eb}{cm} \left( p_x - \frac{e}{c}bz \right) = \frac{eb}{cm} (mv_x - \frac{eb}{c}z + \frac{eb}{c}z) \\ &= \frac{eb}{cm} mv_x = \frac{eb}{c}v_x \end{aligned}$$

Since  $\dot{p}_z = ma_z + \frac{e}{c}\dot{A}_z$  and  $\dot{A}_z = 0$ , we correctly find the Lorentz force.

$$\dot{p}_z = ma_z = \frac{eb}{c}v_x \quad (4)$$

### 2.2.3 In the $y$ -plane

With the vector potentials chosen as

$$A_z = -bx, \quad A_x = bz, \quad A_y = 0,$$

the Hamiltonian reads  $H = \frac{1}{2m} \left[ (p_x - \frac{e}{c}bz)^2 + (p_z + \frac{e}{c}bx)^2 + p_y^2 \right]$ .

Since the Hamiltonian does not depend on the coordinate  $y$ ,  $\dot{p}_y$  is conserved  $\dot{p}_y = 0$ . The velocity component in the  $y$ -direction is given by

$$\dot{q}_y = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} = v_y \quad (5)$$

Therefore, the motion along the  $y$ -plane is free and unaffected by the magnetic field.

## 2.3 Position with respect to time

### 2.3.1 Equations of Motion

#### In the $x$ -plane

We defined  $\dot{p}_x = -\frac{eb}{c}v_z$  in equation (3), and we also have the general expression

$$\dot{p}_x = ma_x + \frac{e}{c} \frac{dA_x}{dt}$$

Equating both expressions with  $A_x = 0$ , we obtain:

$$ma_x = -\frac{eb}{c}v_z \quad \Longleftrightarrow \quad a_x = -\frac{eb}{mc}v_z$$

#### In the $z$ -plane

We defined  $\dot{p}_z = \frac{eb}{c}v_x$  in equation (4). Using the same reasoning as before, with  $A_z = 0$ , we obtain

$$ma_z = \frac{eb}{c}v_x \quad \Longleftrightarrow \quad a_z = \frac{eb}{mc}v_x$$

### 2.3.2 Position

The accelerations along the  $x$  and  $z$  directions are coupled through the velocity components:

$$a_x = -\frac{eb}{mc}v_z, \quad a_z = \frac{eb}{mc}v_x$$

Defining the cyclotron frequency as  $\omega = \frac{eb}{mc}$ , the system can be rewritten as

$$a_x = -\omega v_z, \quad a_z = \omega v_x$$

#### In the $x$ -plane

Differentiating the first equation with respect to time, and substituting  $a_z = \omega v_x$  yields:

$$\dot{a}_x = -\omega \dot{v}_z = -\omega a_z = -\omega^2 v_x$$

Since  $\dot{a}_x = \ddot{v}_x$ , we obtain the harmonic oscillator equation:

$$\ddot{v}_x + \omega^2 v_x = 0,$$

whose general solution is:

$$v_x(t) = A \cos \omega t + B \sin \omega t$$

For a detailed explanation of why this particular solution is chosen, we wrote one in the Appendix 8.1. We integrate with respect to time, in order to obtain the position of the muon in the  $x$ -plane:

Since the initial velocity along the  $x$ -axis is zero,  $v_x(0) = 0$ , the solution for  $v_x(t)$  must vanish at  $t = 0$ . Because  $\sin \omega t$  is zero at  $t = 0$  while  $\cos \omega t$  equals one, the term involving cosine must vanish as well ( $A = 0$ ). Thus, the position is expressed as

$$v_x(t) = B \sin \omega t$$

with  $B = v_{x,0}$

$$q_x(t) = -\frac{v_{x,0}}{\omega} \cos \omega t + x_0, \quad (6)$$

### In the $z$ -plane

Similarly, we have:

$$\dot{a}_z = \omega \dot{v}_x = \omega a_x = -\omega^2 v_z$$

and  $\ddot{v}_z + \omega^2 v_z = 0$ . Because the muon falls vertically into the magnetic field, its initial velocity along the  $z$ -axis is nonzero. Therefore, the cosine term must be present in the solution, and the sine term must vanish, so  $B=0$

$$v_z(t) = A \cos \omega t + B \sin \omega t = A \cos \omega t \quad (7)$$

with  $A = v_{z,0}$ , so

$$q_z(t) = \frac{v_{z,0}}{\omega} \sin \omega t + z_0 \quad (8)$$

### In the $y$ -plane

The motion along the  $y$ -axis differs from the other dimensions, as the magnetic field does not affect it. The velocity  $v_y$  remains constant. To make this explicit, we denote it as  $v_{y,0}$ . Integrating with respect to time then yields:

$$q_y = \int v_{y,0} dt = v_{y,0} t + C$$

Using the initial condition  $C = y_0$ , we find:

$$q_y(t) = v_{y,0} t + y_0 \quad (9)$$

## Conclusion

To conclude, we have first determined a global Hamiltonian, that holds for every vector potential. When computing  $\dot{p}_x$ ,  $\dot{p}_z$  and  $\dot{q}_y$ , we defined different vector potentials, corresponding to the direction in which we wanted to study the particle's motion. From there, we showed that the particle experiences a Lorentz force causing a circular motion in the  $(x, z)$ -plane. In order to address the Lorentz force, we introduced the cyclotron frequency  $\omega$ . However, the motion in the  $y$ -plane stays unaffected by the magnetic field.

The muon's trajectory is thus a **helix**, like a spring stretched along a line. In this case, the muon moves in circles in the  $(x, z)$ -plane due to the magnetic force, while also moving at a constant speed along the  $y$ -axis.

$$\vec{q}(t) = \begin{pmatrix} -\frac{v_{x,0}}{\omega} \cos \omega t + x_0 \\ v_{y,0}t + y_0 \\ \frac{v_{z,0}}{\omega} \sin \omega t + z_0 \end{pmatrix} \quad (10)$$

## 3 Dynamics of the particle in quantum physics

### 3.1 Hamiltonian

In classical mechanics, especially in the Hamiltonian, the dynamics of a system are described using canonical momenta. In quantum mechanics, however, the fundamental objects are operators acting on a Hilbert space. The canonical momentum is thus an operator  $\hat{p} = -i\hbar\nabla$ , and we obtain the equations of motion using the Schrödinger equation instead of Hamilton's equations. However, we still need the Hamiltonian, which will be a necessary operator to solve Schrödinger's equation. We will use the same Hamiltonian (1) as in classical mechanics, which is :

$$H = \frac{1}{2m} \left[ \left( p_x - \frac{e}{c} A_x \right)^2 + \left( p_y - \frac{e}{c} A_y \right)^2 + \left( p_z - \frac{e}{c} A_z \right)^2 \right]$$

but with different vector potentials. The term  $p_y$  disappears because  $p_y = mv_y + \frac{e}{c} A_y$  and both  $A_y$  and  $v_y$  are equal to zero. Using  $A_x = \frac{bz}{2}$ ,  $A_y = 0$  and  $A_z = -\frac{bx}{2}$  yields:

$$H = \frac{1}{2m} \left[ \left( p_x - \frac{e}{c} \frac{bz}{2} \right)^2 + \left( p_z + \frac{e}{c} \frac{bx}{2} \right)^2 \right]$$

Developing the two parts of the Hamiltonian gives:

$$\begin{aligned} \left( p_x - \frac{e}{c} \frac{bz}{2} \right)^2 &= p_x^2 + \frac{e^2}{c^2} \frac{b^2 z^2}{4} - 2p_x \frac{e}{c} \frac{bz}{2}, \\ \left( p_z + \frac{e}{c} \frac{bx}{2} \right)^2 &= p_z^2 + \frac{e^2}{c^2} \frac{b^2 x^2}{4} + 2p_z \frac{e}{c} \frac{bx}{2} \end{aligned}$$

The Hamiltonian becomes:

$$\begin{aligned} H &= \frac{1}{2m} \left( p_x^2 + \frac{e^2}{c^2} \frac{b^2 z^2}{4} - 2p_x \frac{e}{c} \frac{bz}{2} + p_z^2 + \frac{e^2}{c^2} \frac{b^2 x^2}{4} + 2p_z \frac{e}{c} \frac{bx}{2} \right) \\ H &= \frac{p_x^2 + p_z^2}{2m} + \frac{1}{8m} \frac{e^2 b^2}{c^2} (z^2 + x^2) - \frac{eb}{2cm} (xp_z - zp_x) \end{aligned}$$

Substituting the cyclotron frequency  $\omega = \frac{eb}{cm}$  defined in the previous section, we obtain the proper Hamiltonian:

$$H = \frac{p_x^2 + p_z^2}{2m} + \frac{1}{8} m \omega^2 (z^2 + x^2) + \frac{\omega}{2} (zp_x - xp_z) \quad (11)$$

We can notice two terms in this Hamiltonian: one that we will call  $H_o$ , as it corresponds to the Hamiltonian of a harmonic oscillator in two dimensions ( $x$  and  $z$ ), and one that corresponds to the orbital angular momentum operator along the  $y$ -axis, multiplied by  $\frac{\omega}{2}$ .

$$H_o = \frac{p_x^2 + p_z^2}{2m} + \frac{1}{8}m\omega^2(z^2 + x^2) \quad (12)$$

$$\frac{\omega}{2}\hat{L}_y = \frac{\omega}{2}(zp_x - xp_z) \quad (13)$$

The next two subsections will cover the resolution of both  $H_o$  and  $\frac{\omega}{2}\hat{L}_y$ .

## 3.2 Quantum Harmonic oscillator

The first part of the Hamiltonian, called  $H_o$ , corresponds to the harmonic oscillator in two dimensions. We will first solve it in the dimension  $x$ , and then add the second dimension  $z$ .

### 3.2.1 One Dimension

The Hamilton operator is defined as the sum of kinetic and potential energies.

$$\hat{H} = T + V = \frac{\hat{p}_x^2}{2m} + \frac{1}{8}m\omega^2 x^2$$

Substituting  $\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$ , we obtain:

$$\hat{H} = \frac{1}{2m} \left( i^2 \hbar^2 \frac{\partial^2}{\partial x^2} \right) + \frac{1}{8}m\omega^2 x^2 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{8}m\omega^2 x^2$$

The time-dependent Schrödinger equation reads:

$$\hat{H}\Phi(x, t) = i\hbar \frac{\partial}{\partial t} \Phi(x, t)$$

Assuming a stationary state solution of the form

$$\Phi(x, t) = e^{-\frac{iEt}{\hbar}} \Phi(x),$$

substitution into the time-dependent equation yields the time-independent Schrödinger equation:

$$\hat{H}\Phi(x) = E\Phi(x) \tag{14}$$

To simplify the equation, we multiply both sides by  $\frac{2}{\hbar\omega}$ , and define the dimensionless variables  $\gamma = \sqrt{\frac{m\omega}{4\hbar}}x$  and  $\tilde{E} = \frac{2E}{\hbar\omega}$ . This leads to:

$$\left[ -\frac{\partial^2}{\partial \gamma^2} + \gamma^2 \right] \Phi(\gamma) = \tilde{E}\Phi(\gamma)$$

We can subtract  $\tilde{E}\Phi(\gamma)$  from both sides, in order to get harmonic oscillator Schrödinger:

$$-\frac{\partial^2}{\partial \gamma^2} + (\gamma^2 - \tilde{E})\Phi(\gamma) = 0 \tag{15}$$

We may equivalently write the wave function in terms of  $\gamma$ :

$$\Phi(\gamma, t) = \Phi(\gamma)e^{-\frac{iEt}{\hbar}} \tag{16}$$

The detailed calculations are omitted here for clarity, but can be found in the Appendix 8.2.1 for the time-independent Schrödinger equation, and in the Appendix 8.2.2 for the harmonic oscillator Schrödinger.

## Resolution of the harmonic oscillator Schrödinger

Let us now examine the asymptotic behavior of this equation as  $\gamma \rightarrow +\infty$ :

$$\lim_{\gamma \rightarrow +\infty} \left[ -\frac{\partial^2}{\partial \gamma^2} + (\gamma^2 - \tilde{E}) \right] \Phi(\gamma) \approx \left[ -\frac{\partial^2}{\partial \gamma^2} + \gamma^2 \right] \Phi(\gamma)$$

We can propose a solution of the form:

$$\Phi(\gamma) = Ae^{-\frac{\gamma^2}{2}} + Be^{\frac{\gamma^2}{2}}$$

which is an *ansatz* (i.e. a trial solution based on the structure of the differential equation). However, the second term diverges as  $\gamma \rightarrow +\infty$ , which is unacceptable since the wavefunction must be square-integrable:

$$\int_{-\infty}^{+\infty} |\Phi(\gamma)|^2 d\gamma = 1$$

To prevent divergence, we must impose  $B = 0$ . Hence, the asymptotic solution is:

$$\Phi(\gamma) \sim Ae^{-\frac{\gamma^2}{2}} \quad \text{as } \gamma \rightarrow +\infty$$

We now seek solutions that oscillate around the origin. These oscillations can be encoded in a function  $h(\gamma)$ , so we write:

$$\Phi(\gamma) = h(\gamma)e^{-\frac{\gamma^2}{2}} \tag{17}$$

## Determination of $h(\gamma)$

We seek a differential equation for  $h(\gamma)$ . Differentiating  $\Phi(\gamma)$  two times yields:

$$\begin{aligned} \Phi'(\gamma) &= h'(\gamma)e^{-\frac{\gamma^2}{2}} - \gamma h(\gamma)e^{-\frac{\gamma^2}{2}} = [h'(\gamma) - \gamma h(\gamma)] e^{-\frac{\gamma^2}{2}} \\ \Phi''(\gamma) &= [h''(\gamma) - h(\gamma) - \gamma h'(\gamma)] e^{-\frac{\gamma^2}{2}} - \gamma [h'(\gamma) - \gamma h(\gamma)] e^{-\frac{\gamma^2}{2}} \\ &= [h''(\gamma) - h(\gamma) - \gamma h'(\gamma) - \gamma h'(\gamma) + \gamma^2 h(\gamma)] e^{-\frac{\gamma^2}{2}} \\ &= [h''(\gamma) - 2\gamma h'(\gamma) + (\gamma^2 - 1)h(\gamma)] e^{-\frac{\gamma^2}{2}}. \end{aligned}$$

Substituting  $\Phi(\gamma)$   $\Phi''(\gamma)$  in the harmonic oscillator Schrödinger defined in equation (15) yields:

$$\begin{aligned} -[h''(\gamma) - 2\gamma h'(\gamma) + (\gamma^2 - 1)h(\gamma)] e^{-\frac{\gamma^2}{2}} + (\gamma^2 - \tilde{E})h(\gamma)e^{-\frac{\gamma^2}{2}} &= 0 \\ -[h''(\gamma) - 2\gamma h'(\gamma) + (\gamma^2 - 1)h(\gamma)] + (\gamma^2 - \tilde{E})h(\gamma) &= 0 \\ -h''(\gamma) + 2\gamma h'(\gamma) - (\gamma^2 - 1)h(\gamma) + (\gamma^2 - \tilde{E})h(\gamma) &= 0 \\ -h''(\gamma) + 2\gamma h'(\gamma) + h(\gamma) - \tilde{E}h(\gamma) &= 0 \\ -h''(\gamma) + 2\gamma h'(\gamma) + (1 - \tilde{E})h(\gamma) &= 0 \end{aligned}$$

Writing the equation with a positive coefficient for  $h''(\gamma)$  is more convenient. Thus, we obtain:

$$h''(\gamma) - 2\gamma h'(\gamma) + (\tilde{E} - 1)h(\gamma) = 0 \quad (18)$$

We can find  $h(\gamma)$  by introducing Hermite polynomials, which are known to be the solutions of the quantum harmonic oscillator equation. Nevertheless, we can also use Maclaurin series, which are Taylor series centered at the origin. Although tedious and long, they provide an approach that allows us to obtain the quantization of energy levels, which is a fundamental theorem in quantum mechanics. We will first use the Maclaurin series, and then introduce Hermite polynomials.

### Time-independent wave function with Maclaurin series

In order to keep this part as clear as possible, we will write the Maclaurin series expansion in the Appendix 8.3. The harmonic oscillator Schrödinger equation defined in equation (15) is satisfied by  $\Phi(\gamma) = h(\gamma)e^{-\frac{\gamma^2}{2}}$  with  $h(\gamma) = \sum_{n=0}^{+\infty} a_n \gamma^n$ , provided that:

1. The coefficients  $a_n$  satisfy the recurrence relation:

$$a_{n+2} = \frac{2n + 1 - \tilde{E}}{(n + 1)(n + 2)} a_n$$

2. The series terminates, which imposes the quantization condition:

$$E_n = \hbar\omega \left( n + \frac{1}{2} \right)$$

The quantization condition is fundamental in quantum mechanics, as it shows that the energy levels are discrete and can only take specific values.

### Time-independent wave function with Hermite polynomials

Hermite's polynomials also satisfy the equation (18), they are defined as:

$$H_n(\gamma) = (-1)^n e^{\gamma^2} \frac{d^n}{d\gamma^n} e^{-\gamma^2}$$

For a proof of why Hermite's polynomials are a solution, see Appendix 8.4. Thus, we obtain a wave function in terms of  $\gamma$  and  $t$

$$\Phi_n(\gamma) = N_n H_n(\gamma) e^{-\frac{\gamma^2}{2}}$$

Note that we introduced  $N_n$  as a normalization factor, so that the wave function is square-integrable. It is defined as  $N_n = \frac{1}{2^{n/2} \sqrt{\frac{n!}{\sqrt{\pi}}}}$ .

The two last paragraphs lead to the same wave function, but the Hermite polynomials method is considered as the standard method for solving the quantum harmonic oscillator. Thus, we will continue our reasoning with Hermite polynomials.

### Time-dependent wave function

From the wave function in terms of  $\gamma$  defined in equation (16), we can write:

$$\Phi_n(\gamma, t) = \Phi_n(\gamma) e^{-\frac{iE_n t}{\hbar}} = N_n H_n(\gamma) e^{-\frac{\gamma^2}{2}} e^{-\frac{iE_n t}{\hbar}}$$

Since the dimensionless variable  $\gamma = \frac{m\omega}{4\hbar} x$  depends on the space variable  $x$ , we can write  $\Phi_n(x, t)$  instead:

$$\Phi_n(x, t) = N_n H_n(\gamma) e^{-\frac{\gamma^2}{2}} e^{-\frac{iE_n t}{\hbar}} \quad (19)$$

### Probability density

Unlike classical mechanics, in quantum mechanics, a particle is not located at a precise point. Instead, the particle's position is represented as the absolute square of its wave function  $\Phi(x, t)$ . This means that  $|\Psi(x, t)|^2 dx$  represents the probability of finding the particle between  $x$  and  $x + dx$ . Consequently, the particle's location is uncertain. The probability density gives the probability of finding the particle between two points,  $a$  and  $b$ , and is defined as:

$$P(a \leq x \leq b) = \int_a^b |\Psi_n(x, t)|^2 dx$$

When substituting the wave function of the 1D quantum harmonic oscillator defined in equation (19) in the probability density formula, we find:

$$\begin{aligned} P(a \leq x \leq b) &= \int_a^b |\Psi_n(x, t)|^2 dx = \int_a^b \left| \Psi_n(x) e^{-\frac{iE_n t}{\hbar}} \right|^2 dx \\ &= \int_a^b |\Psi_n(x)|^2 \left| e^{-\frac{iE_n t}{\hbar}} \right|^2 dx = \int_a^b |\Psi_n(x)|^2 e^{-\frac{iE_n t}{\hbar}} e^{\frac{iE_n t}{\hbar}} dx \\ &= \int_a^b |\Psi_n(x)|^2 dx \end{aligned}$$

This formulation will be particularly useful when extending the method to the two-dimensional harmonic oscillator, as discussed in the next section.

### 3.2.2 Two Dimensions

In the Hamiltonian, we have a harmonic oscillator in two dimensions. We just solved the harmonic oscillator in the dimension  $x$ , and we will use the results for the second dimension  $z$ . Our two dimensions harmonic oscillator takes the form:

$$H_o = \frac{p_x^2 + p_z^2}{2m} + \frac{1}{8}m\omega^2(z^2 + x^2)$$

#### Energy

In order to get the energy levels, we can separate the two dimensions, and use the quantization condition found with Maclaurin series:

$$E_{nx} = \frac{\hbar\omega}{2}(n_x + \frac{1}{2}), \quad E_{nz} = \frac{\hbar\omega}{2}(n_z + \frac{1}{2})$$

It is simply the one dimension energy in each situation, except that it is divided by two because it is just half of the global solution. The global solution is thus:

$$E_n = E_{nx} + E_{nz} = \frac{\hbar\omega}{2}(n_x + \frac{1}{2}) + \frac{\hbar\omega}{2}(n_z + \frac{1}{2}) = \hbar\omega(n_x + n_z + \frac{1}{2})$$

With  $n = n_x + n_z$ , the result is the same as the one in one dimension, but with  $n$  depending on two dimensions. We will see later that  $n$  is the principal quantum number in the solution of  $H$ .

$$E_n = \hbar\omega(n + \frac{1}{2}) \tag{20}$$

#### Wave function

The two wave functions of each dimensions can be combined in order to get a single wave function. In the previous subsection, we found these wave functions using Hermite polynomials:

$$\Phi_n(x, t) = N_{nx} H_{nx}(\gamma_x) e^{-\frac{\gamma_x^2}{2}} e^{-\frac{iE_{nx}t}{\hbar}}$$

$$\Phi_n(z, t) = N_{nz} H_{nz}(\gamma_z) e^{-\frac{\gamma_z^2}{2}} e^{-\frac{iE_{nz}t}{\hbar}}$$

with  $\gamma_x = \sqrt{\frac{m\omega}{4\hbar}}x$  and  $\gamma_z = \sqrt{\frac{m\omega}{4\hbar}}z$ .

Multiplying the wave functions yields:

$$\Phi_n(q, t) = \Phi_n(x) e^{-\frac{iE_{nx}t}{\hbar}} \Phi_n(z) e^{-\frac{iE_{nz}t}{\hbar}} \tag{21}$$

We obtain the wave function for the two dimensions harmonic oscillator, the first part of our Hamiltonian  $H_o$ .

### Probability density

When substituting the wave function of the 2D quantum harmonic oscillator defined in equation (21) in the probability density formula, we find:

$$\begin{aligned} P(x_a \leq x \leq x_b, z_a \leq z \leq z_b) &= \int_{x_a}^{x_b} \int_{z_a}^{z_b} \left| \Psi_{n_x}(x) e^{-\frac{iE_{n_x}t}{\hbar}} \Psi_{n_z}(z) e^{-\frac{iE_{n_z}t}{\hbar}} \right|^2 dz dx \\ &= \int_{x_a}^{x_b} \int_{z_a}^{z_b} |\Psi_{n_x}(x)|^2 |\Psi_{n_z}(z)|^2 dz dx. \end{aligned}$$

We can see that unlike the one dimension harmonic oscillator, the probability density (the absolute squared of the wave function) of the two dimension's one will depend of time, as the exponents will not vanish.

### 3.3 Kinetic orbital momentum in the $y$ direction

The second part of the Hamiltonian corresponds to the orbital angular momentum operator along the  $y$ -axis ( $\hat{L}_y$ ) multiplied by  $\frac{\omega}{2}$ . It reads:

$$\frac{\omega}{2}\hat{L}_y = \frac{\omega}{2}(zp_x - xp_z) \quad \text{with} \quad \hat{L}_y = zp_x - xp_z$$

*Eigenfunctions* (or *eigenstates*) of an operator are functions that, when the operator is applied on them, return a number called *eigenvalue* multiplied by the function itself. In our case, the wave function of the muon is one of these eigenfunctions.

To determine the eigenfunctions of  $\hat{L}_y$  more easily, it is convenient to express it in spherical coordinates instead of Cartesian coordinates. Substituting  $z = r \cos \theta$ ,  $x = r \sin \theta \cos \phi$ , and  $y = r \sin \theta \sin \phi$ , and recalling that each momentum component is represented by the quantum operator  $p_j = -i\hbar \frac{\partial}{\partial j}$ , the expression of  $\hat{L}_y$  becomes:

$$\hat{L}_y = -i\hbar(-\cos\phi \frac{\partial}{\partial \theta} + \cot\theta \sin\phi \frac{\partial}{\partial \phi}) \quad (22)$$

Finding the eigenfunctions of  $\hat{L}_y$  can be quite complicated. Therefore, we first switch to  $\hat{L}_z$  to find more convenient eigenfunctions. Afterwards, we apply the Wigner D-matrix to obtain eigenfunctions of  $\hat{L}_y$ .

#### 3.3.1 Intermediate $\hat{L}_z$

The intermediate  $\hat{L}_z$  offers two main advantages. First the kinetic orbital momentum along the  $z$ -direction is very simple:

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \phi}$$

Second, the eigenfunctions (i.e., the wave functions) of  $\hat{L}_z$  are the spherical harmonics, given by:

$$Y_l^m(\theta, \phi) = \frac{(-1)^l}{2^l l!} \sqrt{\frac{(2l+1)(l+m)!}{4\pi(l-m)!}} e^{im\phi} \sin^m \theta \frac{d^{l-m}}{d(\cos \theta)^{l-m}}$$

Here,  $l$  the azimuthal quantum number, which determines the shape of the orbital (the larger  $l$ , the more complex the orbital), and  $m$  the magnetic quantum number, which sets the orbital's orientation in space.

#### Eigenvalues

We can find the eigenvalues associated with the eigenfunctions  $Y_l^m$ . Applying  $\hat{L}_z$  to a spherical harmonic, with  $E$  the eigenvalue gives:

$$EY_l^m = \hat{L}_z Y_l^m = -i\hbar \frac{\partial}{\partial \phi} Y_l^m$$

Since the only term in  $Y_l^m$  depending on  $\phi$  is  $e^{im\phi}$ ,

$$EY_l^m = -i\hbar(imY_l^m) = -i^2 m\hbar Y_l^m$$

Therefore, the eigenvalue associated with  $Y_l^m$  is:

$$E_m = m\hbar$$

The result  $m\hbar$  is not the energy, but the eigenvalue of  $\hat{L}_z$ , which is the value we would measure for the  $z$ -component of the orbital angular momentum.

### 3.3.2 Eigenfunctions of $\hat{L}_y$

We may now determine the Wigner D-matrix to transform the eigenfunctions of  $\hat{L}_z$  into those of  $\hat{L}_y$ . The  $y$ -axis is rotated by an angle of  $\frac{\pi}{2}$  relative to the  $z$ -axis, and the Wigner D-matrix allows us to rotate using appropriate Euler angles. This gives the eigenfunctions of  $\hat{L}_y$  directly from the well-known eigenfunctions of  $\hat{L}_z$ .

The Wigner D-matrix  $D_{m'm}^n(\alpha, \beta, \lambda)$  describes the rotation of angular momentum eigenstates. It is given by:

$$D_{m'm}^n(\alpha, \beta, \lambda) = e^{-im'\alpha} d_{m'm}^n(\beta) e^{-im\lambda} \quad (23)$$

where  $n$  is the total angular momentum quantum number,  $m$  and  $m'$  are the magnetic quantum numbers before and after the rotation,  $\alpha, \beta, \lambda$  are Euler angles, and  $d_{m'm}^n(\beta)$  is the "small-d" matrix.

We may now find the appropriate Euler angles. The Wigner D-matrix has to be applied in the convention  $ZYZ$ , so we can not perform a direct rotation along the  $x$ -axis. To solve this, we establish the following rotation matrix equation, which allows us to find the Euler angles corresponding to a rotation of  $\frac{\pi}{2}$  around the  $x$  axis.

$$R_z(\alpha)R_y(\beta)R_z(\lambda) = R_x(\phi)$$

In our case,  $\phi = \frac{\pi}{2}$ :

$$R_z(\alpha)R_y(\beta)R_z(\lambda) = R_x\left(\frac{\pi}{2}\right) \quad (24)$$

Solving this equation gives the Euler angles  $\alpha = \frac{\pi}{2}$ ,  $\beta = \frac{\pi}{2}$  and  $\lambda = \frac{-\pi}{2}$ . Physically, it describes a rotation from the  $z$ -axis to the  $y$ -axis without rotating directly around the  $x$ -axis (convention  $ZYZ$ ):

- Rotation of  $90^\circ$  around the  $z$ -axis
- Rotation of  $90^\circ$  around the  $y$ -axis
- Rotation of  $-90^\circ$  around the  $z$ -axis

Consequently, the Wigner D-matrix must be taken as:

$$D_{m'm}^n \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2} \right)$$

For details of how we can solve the equation (24), see Appendix (8.5). This shows that the eigenfunction of  $\hat{L}_y$  can be obtained by rotating the eigenfunction of  $\hat{L}_z$  using the Wigner D-matrix with the specified Euler angles. The wave function associated with the orbital part of the Hamiltonian,  $\hat{L}_y$ , is therefore

$$\Phi(\theta, \phi) = D_{m'm}^n \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2} \right) Y_m^l(\theta, \phi) \quad (25)$$

The factor  $\frac{\omega}{2}$  does not appear in the eigenfunction itself, because in the second part of the Hamiltonian,  $\frac{\omega}{2}\hat{L}_y$ , the factor multiplies the operator, not the eigenfunction.

### Energy

The corresponding energy of this term is obtained by multiplying the eigenvalue of  $\hat{L}_y$  by  $\frac{\omega}{2}$ . The eigenvalue of  $\hat{L}_y$  depends on the magnetic quantum number  $m \in [-l, l]$ . Therefore, the contribution of energy of the orbital angular momentum term along the  $y$ -axis is:

$$E_m = \frac{\omega}{2} m \hbar \quad (26)$$

For a given orbital  $l$ , each possible value of  $m$  corresponds to a different eigenfunction  $\hat{L}_y$ , and thus a different energy contribution.

## 4 Final and global wave function

### 4.1 Compact formulation

We write the wave function in its minimal form: using the harmonic oscillator wave functions and the spherical harmonics rotated by the Wigner D-matrix.

$$\Phi_n(q, t) = \Phi_n(x) e^{\frac{-iE_{nx}t}{\hbar}} \Phi_n(z) e^{\frac{-iE_{nz}t}{\hbar}} D_{m'm}^n \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2} \right) Y_m^l(\theta, \phi) \cdot e^{\frac{-iE_m t}{\hbar}} \quad (27)$$

where:

- $\Phi_n(x) = N_{nx} H_{nx}(\gamma_x) e^{-\frac{\gamma^2}{2}}$ ,  $\Phi_n(z) = N_{nz} H_{nz}(\gamma_z) e^{-\frac{\gamma^2}{2}}$  — 1D harmonic oscillator wave functions
- $E_{nx}, E_{nz}$  — energy contribution of the harmonic oscillators along the  $x$  and  $z$ -axis
- $D_{m'm}^n(\alpha, \beta, \lambda)$  — Wigner D-matrix, used to obtain the eigenfunction of  $\hat{L}_y$
- $Y_m^l(\theta, \phi)$  — spherical harmonics, eigenfunctions of  $\hat{L}_z$
- $E_m = \frac{\omega}{2} m \hbar$  — energy contribution of  $\frac{\omega}{2} \hat{L}_y$

### 4.2 Developed formulation

By expressing  $x$  and  $z$  in spherical coordinates, the probability of finding the particle at a given radius  $r$  and angles  $(\theta, \phi)$  becomes easier to compute. We obtain:

$$\Phi(q, t) = N_{nx} H_{nx} e^{-\sqrt{\frac{m\omega}{4\hbar}} r \cos \phi / \sin \theta} e^{\frac{-iE_{nx}t}{\hbar}} N_{nz} H_{nz} e^{-\sqrt{\frac{m\omega}{4\hbar}} r \cos \theta} e^{\frac{-iE_{nz}t}{\hbar}} D_{m'm}^n \left( \frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2} \right) Y_m^l(\theta, \phi) e^{\frac{-iE_m t}{\hbar}}$$

## 5 Numerical calculations and predictions

## 6 Conclusion

## 7 Acknowledgments

## 8 Appendix

### 8.1 Detailed solution choice

We consider the harmonic oscillator equation:

$$\ddot{v}_x + \omega^2 v_x = 0$$

To solve it, we assume a trial solution of the form  $v_x(t) = e^{rt}$ , where  $r \in \mathbb{C}$ . This choice is motivated by the fact that:

- The derivatives of  $e^{rt}$  are proportional:  $\frac{d^k}{dt^k} e^{rt} = r^k e^{rt}$ .
- For equations with complex roots, like  $r = \pm i\omega$ , Euler's identity allows us to rewrite solutions with sines and cosines.

Substituting the trial solution into the differential equation yields:

$$r^{ert} + \omega^2 e^{rt} = 0 \quad \Longleftrightarrow \quad e^{rt}(r^2 + \omega^2) = 0$$

As  $e^{rt} \neq 0$  for all  $t \in \mathbb{R}$  (and even  $\mathbb{C}$ ), we know that  $r^2 + \omega^2 = 0$  and  $r^2 = -\omega^2$ . Solving in  $\mathbb{C}$  gives

$$r = \pm\sqrt{-\omega^2} = \pm\sqrt{-1}\sqrt{\omega^2} = \pm i\omega$$

Thus, we consider the two possible values of  $r$ :  $v_x(t) = C_1 e^{-i\omega t} + C_2 e^{i\omega t}$ . Applying Euler's identity  $e^{\pm i\omega} = \cos \omega \pm i \sin \omega$ :

$$\begin{aligned} v_x(t) &= C_1(\cos -\omega t + i \sin -\omega t) + C_2(\cos \omega t + i \sin \omega t) \\ &= C_1(\cos \omega t - i \sin \omega t) + C_2(\cos \omega t + i \sin \omega t) \\ &= C_1 \cos \omega t - C_1 i \sin \omega t + C_2 \cos \omega t + C_2 i \sin \omega t \\ &= (C_1 + C_2) \cos \omega t + (C_2 - C_1) i \sin \omega t \end{aligned}$$

Letting  $A = C_1 + C_2$ ,  $B = (C_2 - C_1)i$ , we obtain the general solution:

$$v_x(t) = A \cos \omega t + B \sin \omega t$$

## 8.2 Harmonic oscillator

### 8.2.1 Time-independent Schrödinger

Substituting  $\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$  and  $\Phi(x, t) = e^{-\frac{iEt}{\hbar}} \Phi(x)$  into the left-hand side of the time-dependent Schrödinger equation yields:

$$\begin{aligned} -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi(x, t)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \Phi(x, t) &= -\frac{\hbar^2}{2m} \left[ e^{-\frac{iEt}{\hbar}} \frac{\partial^2 \Phi(x)}{\partial x^2} \right] + \frac{1}{2}m\omega^2 x^2 e^{-\frac{iEt}{\hbar}} \Phi(x) \\ &= \left[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Phi(x)}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \Phi(x) \right] e^{-\frac{iEt}{\hbar}} \\ &= \hat{H}\Phi(x) e^{-\frac{iEt}{\hbar}} \end{aligned}$$

We can do the same in the right side of the Schrödinger equation:

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Phi(x, t) &= i\hbar \frac{\partial}{\partial t} \left[ \Phi(x) e^{-\frac{iEt}{\hbar}} \right] \\ &= i\hbar \left[ -\frac{iE}{\hbar} e^{-\frac{iEt}{\hbar}} \Phi(x) \right] \\ &= -i^2 E \Phi(x) e^{-\frac{iEt}{\hbar}} = E \Phi(x) e^{-\frac{iEt}{\hbar}} \end{aligned}$$

Equating both sides and dividing by  $e^{-\frac{iEt}{\hbar}} \neq 0$  gives:

$$\hat{H}\Phi(x) = E\Phi(x)$$

### 8.2.2 Schrödinger equation for the quantum harmonic oscillator

We multiply both sides of the time-independent Schrödinger equation

$$\hat{H}\Phi(x) = E\Phi(x)$$

by the constant factor  $\frac{2}{\hbar\omega}$ :

$$\frac{2\hat{H}\Phi(x)}{\hbar\omega} = \frac{2E}{\hbar\omega} \Phi(x)$$

We now introduce the following dimensionless quantities  $\tilde{E} = \frac{2E}{\hbar\omega}$  and  $\gamma = \sqrt{\frac{m\omega}{\hbar}} x$ . Substituting into the equation, we get:

$$\begin{aligned} \left[ -\frac{2}{\hbar\omega} \cdot \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{2}{\hbar\omega} \cdot \frac{1}{2}m\omega^2 x^2 \right] \Phi(x) &= \frac{2E}{\hbar\omega} \Phi(x) \\ \left[ -\frac{\hbar}{m\omega} \frac{\partial^2}{\partial x^2} + \frac{m\omega}{\hbar} x^2 \right] \Phi(x) &= \tilde{E} \Phi(x) \end{aligned}$$

Next, we change variables from  $x$  to the dimensionless parameter  $\gamma$ . Since

$$\gamma = \sqrt{\frac{m\omega}{\hbar}} x \quad \Longleftrightarrow \quad x = \sqrt{\frac{\hbar}{m\omega}} \gamma,$$

we apply the chain rule to express derivatives with respect to  $x$  in terms of  $\gamma$ :

$$\frac{\partial \Phi(x)}{\partial x} = \frac{\partial \gamma}{\partial x} \frac{\partial \Phi(\gamma)}{\partial \gamma} = \sqrt{\frac{m\omega}{\hbar}} \frac{\partial \Phi(\gamma)}{\partial \gamma} \implies \frac{\partial^2 \Phi(x)}{\partial x^2} = \left( \frac{m\omega}{\hbar} \right) \frac{\partial^2 \Phi(\gamma)}{\partial \gamma^2}$$

Substituting into the differential equation:

$$-\frac{\hbar}{m\omega} \frac{\partial^2 \Phi(x)}{\partial x^2} = -\frac{\hbar}{m\omega} \frac{m\omega}{\hbar} \frac{\partial^2}{\partial \gamma^2} \Phi(\gamma) = -\frac{\partial^2 \Phi(\gamma)}{\partial \gamma^2}$$

Thus, the Schrödinger equation becomes:

$$\left[ -\frac{\partial^2}{\partial \gamma^2} + \gamma^2 \right] \Phi(\gamma) = \tilde{E} \Phi(\gamma)$$

### 8.3 Maclaurin series expansion

The Maclaurin series is a special case of the Taylor series centered at the origin ( $x=0$ ). It offers a way to represent a function as an infinite sum of polynomial terms. We expand  $h(\gamma)$  in a Maclaurin series to solve the differential equation:

$$\begin{aligned} h(\gamma) &= \sum_{n=0}^{+\infty} a_n \gamma^n \quad \text{with} \quad a_n = \frac{f^{(n)}(0)}{n!} \\ h'(\gamma) &= \sum_{n=0}^{+\infty} n a_n \gamma^{n-1} \\ h''(\gamma) &= \sum_{n=0}^{+\infty} n(n-1) a_n \gamma^{n-2} \end{aligned}$$

Substituting  $h''(\gamma)$ ,  $h'(\gamma)$  and  $h(\gamma)$  in the differential equation (18) yields:

$$\sum_{n=0}^{+\infty} n(n-1) a_n \gamma^{n-2} - 2\gamma \sum_{n=0}^{+\infty} n a_n \gamma^{n-1} + (\tilde{E} - 1) \sum_{n=0}^{+\infty} a_n \gamma^n = 0$$

To combine these sums, all series must have the same power of  $\gamma$ .

#### Index shifting

For the first term  $h''(\gamma)$ , set  $k = n - 2 \iff n = k + 2$ :

$$h''(\gamma) = \sum_{n=0}^{+\infty} n(n-1) a_n \gamma^{n-2} = \sum_{k=0}^{+\infty} (k+2)(k+1) a_{k+2} \gamma^k = \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} \gamma^n$$

For the second term  $-2\gamma h'(\gamma)$ :

$$-2\gamma h'(\gamma) = -2\gamma \sum_{n=0}^{+\infty} n a_n \gamma^{n-1} = \sum_{n=0}^{+\infty} -2n a_n \gamma^n$$

Combining all terms:

$$\sum_{n=0}^{+\infty} \left[ (n+1)(n+2)a_{n+2} - 2na_n + (\tilde{E} - 1)a_n \right] \gamma^n = 0$$

This implies:

$$\begin{aligned} (n+1)(n+2)a_{n+2} - 2na_n + (\tilde{E} - 1)a_n &= 0 \\ (n+1)(n+2)a_{n+2} - (2n - \tilde{E} - 1)a_n &= 0 \\ a_{n+2} &= \frac{2n - \tilde{E} + 1}{(n+1)(n+2)} a_n \end{aligned}$$

As  $\Phi(\gamma) = h(\gamma) e^{-\frac{\gamma^2}{2}}$  and  $\Phi \in L^2(\mathbb{R})$ , we require that  $h(\gamma)$  does not grow exponentially as  $\gamma \rightarrow \infty$ . This condition is satisfied if the series terminates. Let  $n = n_{\max}$  such that  $a_{n_{\max}+2} = 0$ .

From  $n = n_{\max}$ ,  $a_{n_{\max}+2} = 0$ :

$$\begin{aligned} a_{n_{\max}+2} &= \frac{2n_{\max} - \tilde{E} + 1}{(n_{\max} + 1)(n_{\max} + 2)} a_{n_{\max}} = 0 \\ 2n_{\max} - \tilde{E} + 1 &= 0 \\ \tilde{E} &= 2n_{\max} + 1 \end{aligned}$$

Since  $\tilde{E} = \frac{2E}{\hbar\omega}$ , the quantized energy levels are:

$$E = \hbar\omega \left( n_{\max} + \frac{1}{2} \right) \quad \text{with} \quad n_{\max} \in \mathbb{N}$$

## 8.4 Hermite's polynomials

We will prove that Hermite's polynomials are a solution to the equation (18):

$$h''(\gamma) - 2\gamma h'(\gamma) + (\tilde{E} - 1)h(\gamma) = 0$$

Hermite's polynomials are defined as follow:

$$H_n(\gamma) = (-1)^n e^{\gamma^2} \frac{d^n}{d\gamma^n} e^{-\gamma^2}$$

Thus, we need to prove they are a solution for  $n \in \mathbb{N}$ . We will use a proof by recurrence.

## 8.5 Euler angles

We have to find the right Euler's angles to apply to the Wigner matrix in order to execute a rotation of  $\frac{\pi}{2}$  around the  $x$  axis. As the Wigner matrix is defined in the  $ZYZ$  convention (a first rotation around the  $Z$  axis, a second around the  $Y$  axis and a third around the  $Z$  axis), we have the following equation of the rotation matrix, as we can't execute a direct rotation around  $x$ .

$$\begin{aligned} R_z(\alpha)R_y(\beta)R_z(\alpha) &= R_x(\phi) \\ R_z(\alpha)R_y(\beta)R_z(\alpha) &= R_x\left(\frac{\pi}{2}\right) \end{aligned} \quad (28)$$

with

$$\begin{aligned} R_z(\alpha) &= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R_y(\beta) &= \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \\ R_z(\lambda) &= \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R_x(\phi = \frac{\pi}{2}) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned} \quad (29)$$

In order to simplify the triple multiplication between the different matrix, we will firstly multiply two of them to get a new matrix, that we will multiply by the last matrix.

$$R_z(\alpha)R_y(\beta) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

It gives us the new matrix:  $T = \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$  Now that we have this

new matrix, we can multiply it by the last matrix  $R_z(\lambda)$ , and then restrict the angles of the final matrix to obtain the same matrix as  $R_x(\frac{\pi}{2})$ .

$$R_z(\alpha)R_y(\beta)R_z(\lambda) = T \cdot R_z(\lambda) = R_x\left(\frac{\pi}{2}\right)$$

So

$$R_x\left(\frac{\pi}{2}\right) = \begin{pmatrix} \cos\alpha\cos\beta\cos\lambda - \sin\alpha\sin\lambda & -\sin\alpha\cos\alpha\cos\beta + \cos\alpha\cos\lambda & \cos\alpha\sin\beta \\ \sin\alpha\cos\alpha\cos\beta + \cos\alpha\sin\alpha & \sin\alpha\sin\lambda\cos\beta + \cos\lambda\cos\alpha & \sin\alpha\sin\beta \\ \cos\beta & \sin\beta\sin\alpha & \cos\beta \end{pmatrix} \quad (30)$$

and as  $R_x(\frac{\pi}{2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ , the only values that are possible for the different Euler's angles are  $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}, \lambda = \frac{-\pi}{2}$ . So, the Wigner matrix is  $D_{m'm}^n(\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2})$ , and we can now execute a rotation of the wave function of  $\hat{L}_z$  pf  $\frac{\pi}{2}$  around the  $x$  axis to get the wave function of  $\hat{L}_y$ .