Equation of an harmonic oscillator

December of 2024

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1 Introduction

The harmonic oscillator consists in a system, which is usually composed of a mass attached to a spring. The mass moves around an equilibrium position under the action of a restoring force. The harmonic oscillator has enabled scientists like Werner Heisenberg to test the quantum mechanic theory. We are trying to determine the evolution of the position x (the mass) in relation to time, whilst taking into account friction effects caused by air resistance.

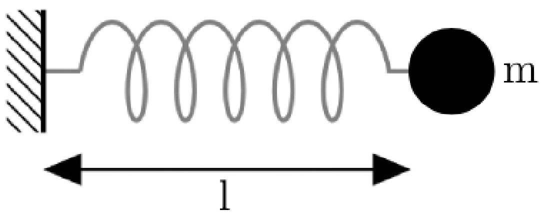


Figure 1: schema of an harmonic oscillator, composed of a mass m attached to a spring of length ℓ.

2 Equation of motion

2.1 Newton's second law

Newton's second law states that a resultant force applied on an object is always equal to the product of the object's mass and its acceleration. Moreover, the acceleration produced and the resultant force have the same orientation.

$$\Sigma\vec{F} = m\vec{a} \tag{1}$$

Yet, the second derivative of the position is the acceleration. We can thus rewrite the equation with a unit vector $\vec{u_{i,j}}$:

$$\Sigma\vec{F} = mx\overrightarrow{u_{i,j}}$$

2.2 Forces applied to an harmonic oscillator

As the mass moves to the right, the spring exerts a force in the opposite direction, to the left. This is described by Hooke's law.

$$\overrightarrow{F_{\text{ressort}}} = -kx\overrightarrow{u_{i,j}}$$

where k is the spring stiffness constant, which depends on its elasticity, and x is the position of the mass.

In particular, as we are taking into account air friction, which is proportionnal to velocity, we have to substract it to our previous expression:

$$\overrightarrow{F} = (-kx - \alpha\dot{x})\overrightarrow{u_{i,j}}$$

where α depends in particular on air density, but also on the shape of the mass.

2.3 Equation of motion

We have obtained two equivalent expressions of the forces exerted on an harmonic oscillator.

$$\overrightarrow{F} = m\ddot{x}\overrightarrow{u_{i,j}}$$
$$\overrightarrow{F} = (-kx - \alpha\dot{x})\overrightarrow{u_{i,j}}$$

We can deduce that:

$$m\ddot{x} = -kx - \alpha\dot{x} \tag{2}$$

since the unit vectors $u_{i,j}$ have the same direction, sense and norm.

$$\ddot{x} = \frac{-k}{m}x - \frac{\alpha}{m}\dot{x}$$

$$\ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

Let $\beta = \sqrt{\frac{\alpha}{m}}$ and $\omega = \sqrt{\frac{k}{m}}$

Which gives us the following equation:

$$\ddot{x} + \beta^2\dot{x} + \omega^2 x = 0 \tag{3}$$

3 Resolution of the x position in time

3.1 Rewriting the equation of motion

We just established the equation of motion, which is the following one:

$$\ddot{x} + \beta^2 \dot{x} + \omega^2 x = 0$$

We can factorize this equation by x:

$$\left(\frac{d^2}{dt^2} + \beta^2 \frac{d}{dt} + \omega^2\right) x = 0$$

Let the operator $X = \frac{d}{dt}$, we have the equation:

$$(X^2 + \beta^2 X + \omega^2) x = 0$$

Thus, we can solve $X^2 + \beta^2 X + \omega^2 = 0$, using the formal identity, as both part are commutating:

$$\left(X + \frac{\beta^2}{2} + \sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right)\left(X + \frac{\beta^2}{2} - \sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right) = 0 \tag{4}$$

Solving this equation under the condition $\beta^2 < 2\omega$, implies that $\beta^2 - 2\omega < 0$.

$$\beta^2 - 2\omega < 0 \Longleftrightarrow \frac{\beta^2}{2} - \omega < 0 \Longleftrightarrow \left(\frac{\beta^2}{2}\right)^2 - \omega^2 < 0$$

If $\left(\frac{\beta^2}{2}\right)^2 - \omega^2 < 0$, then $\sqrt{\left(\frac{\beta^2}{2}\right)^2 - \omega^2} \notin \mathbb{R}$.

This implies that under the condition $\beta^2 < 2\omega$, the first solution of the equation of motion is not a real number. We extend to complex numbers by introducing $i = \sqrt{-1}$, so that $i^2 = -1$. The equation of motion can now be written as:

$$\left(X + \frac{\beta^2}{2} + i\sqrt{\omega^2 - \left(\frac{\beta^2}{2}\right)^2}\right)\left(X + \frac{\beta^2}{2} - i\sqrt{\omega^2 - \left(\frac{\beta^2}{2}\right)^2}\right) x = 0 \tag{5}$$

3.2 Resolution of the first solution

Using the previous expression, we can solve two distinct equations:

$$\left(X+\frac{\beta^2}{2}+i\sqrt{(\frac{\beta^2}{2})^2-\omega^2}\right)x=0 \tag{6}$$

$$\left(X+\frac{\beta^2}{2}-i\sqrt{\omega^2-\left(\frac{\beta^2}{2}\right)^2}\right)x=0 \tag{7}$$

We will first resolve the first one.
We begin by isolating X by factoring out x from each term:

$$Xx+\frac{\beta^2}{2}x+\left(i\sqrt{(\frac{\beta^2}{2})^2-\omega^2}\right)x=0$$

Subtracting the remaining terms, we have:

$$Xx=-\left(\frac{\beta^2}{2}+i\sqrt{(\frac{\beta^2}{2})^2-\omega^2}\right)x$$

Since $X=\frac{d}{dt}$,

$$\frac{dx}{dt}=\left(-\frac{\beta^2}{2}-i\sqrt{(\frac{\beta^2}{2})^2-\omega^2}\right)x$$

By multiplying through by dt and dividing by x, we have:

$$\frac{dx}{x}=\left(-\frac{\beta^2}{2}-i\sqrt{\left(\frac{\beta^2}{2}\right)^2-\omega^2}\right)dt$$

Integrating both sides, we find:

$$\ln(x)=\left(-\frac{\beta^2}{2}-i\sqrt{\left(\frac{\beta^2}{2}\right)^2-\omega^2}\right)t+C$$

Exponentiating both sides, we obtain:

$$x(t)=e^{\left(-\frac{\beta^2}{2}-i\sqrt{\left(\frac{\beta^2}{2}\right)^2-\omega^2}\right)t+C}$$

Simplifying the constant e^C into a new constant A, we can rewrite the first solution as:

$$x(t)=Ae^{\left(-\frac{\beta^2}{2}-i\sqrt{\left(\frac{\beta^2}{2}\right)^2-\omega^2}\right)t} \tag{8}$$

3.3 Resolution of the second solution

We still have to find the second solution:

$$\left(X + \frac{\beta^2}{2} - i\sqrt{\omega^2 - \left(\frac{\beta^2}{2}\right)^2}\right)x = 0$$

The only difference compared to the first solution, is that we substract $i\sqrt{\omega^2 - \left(\frac{\beta^2}{2}\right)^2}$. Thus, the same reasoning applies, with the only difference of:

$$Xx = -\left(\frac{\beta^2}{2} - i\sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right)x$$

We obtain:

$$\frac{dx}{x} = \left(-\frac{\beta^2}{2} + i\sqrt{\left(\frac{\beta^2}{2}\right)^2 - \omega^2}\right)dt$$

$$\ln(x) = \left(-\frac{\beta^2}{2} + i\sqrt{\left(\frac{\beta^2}{2}\right)^2 - \omega^2}\right)t + C$$

$$x(t) = e^{\left(-\frac{\beta^2}{2} + i\sqrt{\left(\frac{\beta^2}{2}\right)^2 - \omega^2}\right)t + C}$$

Simplifying the constant e^C into a new constant B, we can rewrite the second solution as

$$x(t) = Be^{\left(-\frac{\beta^2}{2} + i\sqrt{\left(\frac{\beta^2}{2}\right)^2 - \omega^2}\right)t} \tag{9}$$

3.4 General solution

By summing the two solutions together, we find the general solution of the motion:

$$x(t) = Ae^{\left(-\frac{\beta^2}{2} - i\sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right)t} + Be^{\left(-\frac{\beta^2}{2} + i\sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right)t} \tag{10}$$

We can transform the expression $e^{\left(-\frac{\beta^2}{2} - i\sqrt{(\frac{\beta^2}{2})^2 - \omega^2}\right)}t$ into sin and cosin functions, thanks to the Euler formula:

$$e^{ix} = \cos(x) + i\sin(x) \tag{11}$$

Let $\lambda = i\sqrt{(\frac{\beta^2}{2})^2 - \omega^2} = \sqrt{-(\frac{\beta^2}{2})^2 + \omega^2}$ so that:

$$e^{-\lambda t} = C\cos(\lambda t) - B\sin(\lambda t) = \sqrt{C^2 + B^2}\left(\frac{C}{\sqrt{C^2 + B^2}}\cos(\lambda t) - \frac{B}{\sqrt{C^2 + B^2}}\sin(\lambda t)\right)$$

By restricting $\frac{C^2}{C^2+B^2} + \frac{B^2}{C^2+B^2} = 1$, we can associate them to sin and cosin functions, as $\cos^2(\phi) + \sin^2(\phi) = 1$. That gives us:

$$\sqrt{C^2 + B^2}\Big(\cos(\phi) \cdot \cos(\lambda t) - \sin(\phi) \cdot \sin(\lambda t)\Big)$$

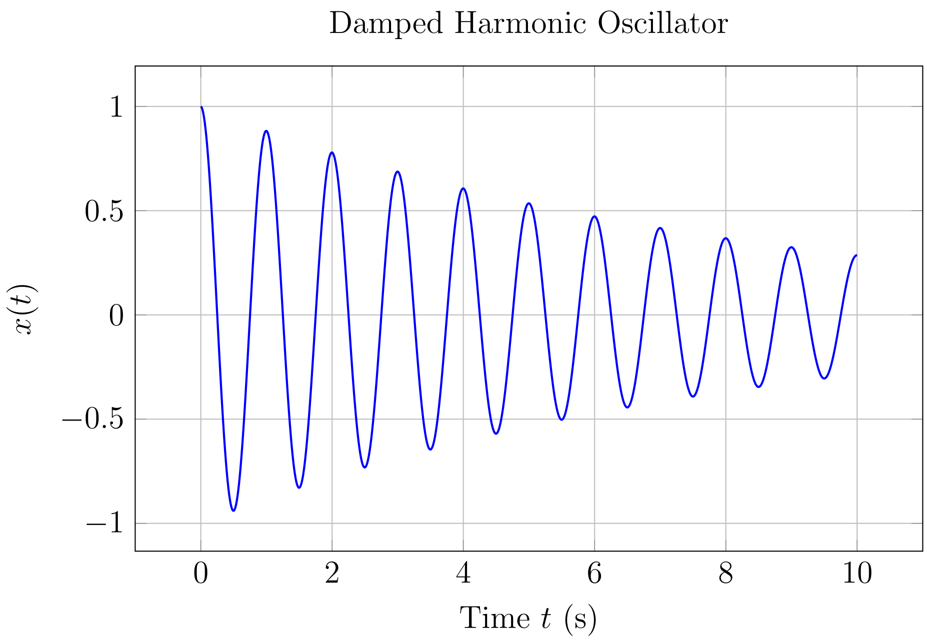
Using the trigonemetry addition formula $\cos(a) \times \cos(b) - \sin(a) \times \sin(b) = \cos(a + b)$, and defining $\sqrt{C^2 + B^2} = A$, we find that the previous term is equal to:

$$Acos(\lambda t + \phi)$$

By substituting this term into the equation of motion, we find the final result:

$$x(t) = Ae^{-\frac{\beta^2}{2}t} \cdot \cos(\lambda t + \phi)$$

$$\boxed{x(t) = Ae^{-\frac{\beta^2}{2}t} \cdot \cos\left(\sqrt{\frac{-\beta^4}{4} + \omega^2}\ t + \phi\right)} \tag{12}$$



On this plot, we can easily see how the displacement decreases over time. This means that the harmonic oscillator is slowing down, due to air resistance.

4 Interpretation

4.1 When β approaches zero

We have previously defined $x(t)$ as:

$$x(t) = Ae^{-\frac{\beta^2}{2}t} \cdot \cos\left(\sqrt{\frac{-\beta^4}{4} + \omega^2}\ t + \phi\right)$$

$$\lim_{\beta \to 0} e^{-\frac{\beta^2}{2}t} = e^0 = 1$$

This means the exponential energy loss is removed; the system is no longer damped. The square root term becomes:

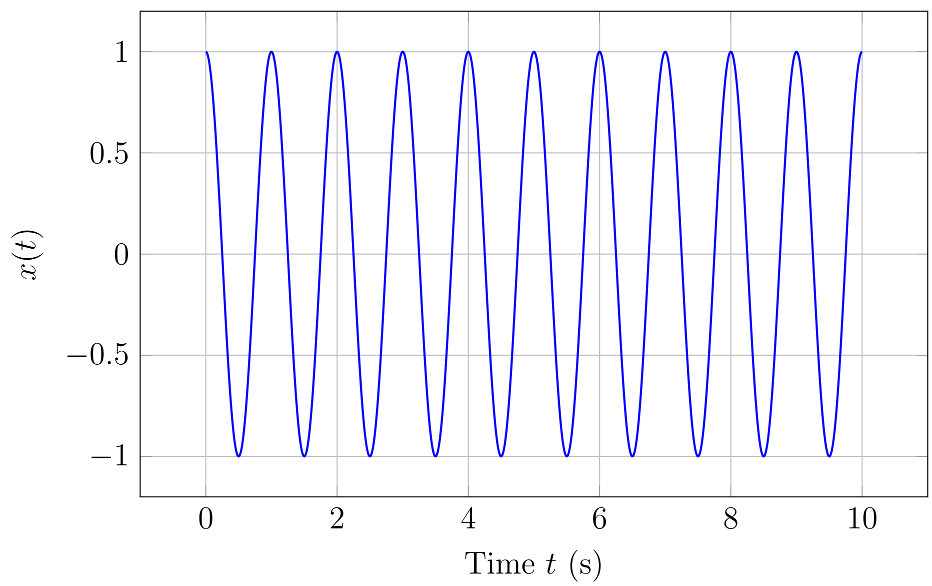
$$\lim_{\beta \to 0} \sqrt{\frac{-\beta^4}{4} + \omega^2} = \sqrt{0 + \omega^2} = \omega$$

Substituting these results into the original expression, we find:

$$x(t) = A\cos(\omega t + \phi)$$

In the limit as $\beta \to 0$, the oscillator behaves as a **simple harmonic oscillator** with:

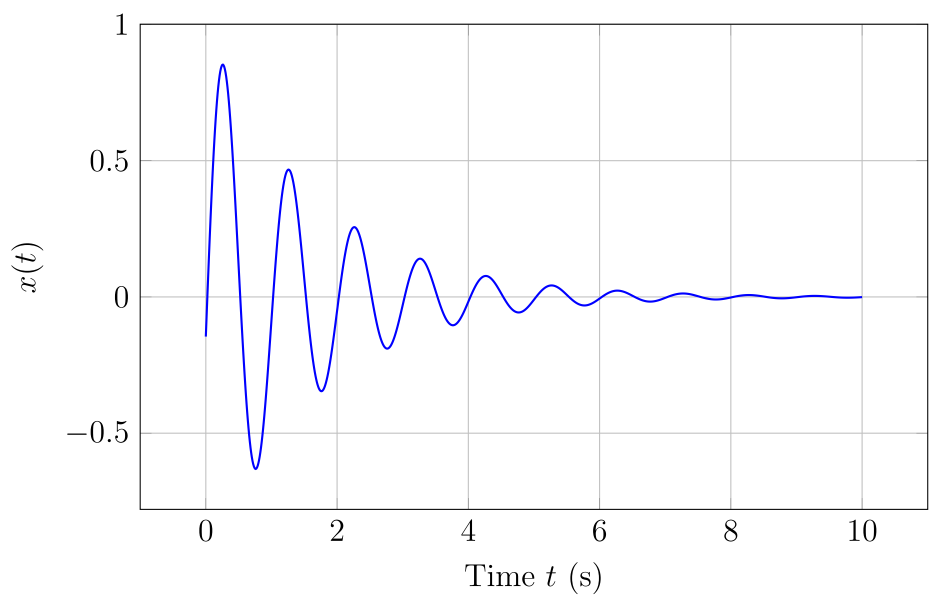
Undamped Harmonic Oscillator



This represents the displacement x(t) in relation to time, when = 0. We can observe that the system is no longer losing energy, as the mass travels the same distance.

On the contrary, if we increase β, it gives us the following plot:

Extremely damped Harmonic Oscillator



This shows how fast the Harmonic oscillator can lose energy if it's placed in an environment with a high friction resistance.