Weekly Report

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1 Tasks

This week I was tasked with solving the steady state Boltzmann equation

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} f = \mathcal{Q}(f, f)$$

where

$$\mathcal{Q}(f, f)(\mathbf{v}) = \int_{\mathbb{R}^d} \int_{S^{d-1}} B(|\mathbf{v} - \mathbf{v}_*|, \cos\chi)[f(\mathbf{v}'_*)f(\mathbf{v}') - f(\mathbf{v}_*)f(\mathbf{v})] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_*.$$

and

$$\mathbf{v}' = \frac{\mathbf{v} + \mathbf{v}_*}{2} + \frac{|\mathbf{v} - \mathbf{v}_*|}{2}\sigma, \quad \mathbf{v}'_* = \frac{\mathbf{v} + \mathbf{v}_*}{2} - \frac{|\mathbf{v} - \mathbf{v}_*|}{2}\sigma$$

In particular, we want to replicate the results from from Section 5.2 in Hu et al. [2]. In this problem we are modeling 2D Maxwell molecules. We therefore have that $B(|\mathbf{v} - \mathbf{v}_*|, \cos\chi) = 1/2\pi$ for all velocities and angles.

2 Progress

2.1 Reading

I spent a good deal of time this week parsing these notes by Gyu Eun Lee. They provide the most intuitive explanation of the Boltzmann that I have seen. I understand the Boltzmann equation much better now.

2.2 Simplification of Boltzmann Equation for 2D Maxwell Molecules

Since the collision kernel is constant we can simplify the above integral. Note that both f and ρ (this will be defined later) are functions of $\mathbf{x}$ but we drop this dependence for convenience of notation.

$$\begin{aligned}
\mathcal{Q}(f, f)(\mathbf{v}) = \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{1}{2\pi} [f(\mathbf{v}'_*)f(\mathbf{v}') - f(\mathbf{v}_*)f(\mathbf{v})] \mathrm{d}\sigma \mathrm{d}\mathbf{v}_* = \\
\int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{1}{2\pi} f(\mathbf{v}'_*)f(\mathbf{v}') \mathrm{d}\sigma \mathrm{d}\mathbf{v}_* - \int_{\mathbb{R}^d} \int_{S^{d-1}} \frac{1}{2\pi} f(\mathbf{v}_*)f(\mathbf{v}) \mathrm{d}\sigma \mathrm{d}\mathbf{v}_* = \\
\mathcal{Q}^+(f, f)(\mathbf{v}) - \frac{1}{2\pi} f(\mathbf{v}) \int_{\mathbb{R}^d} \int_{S^{d-1}} f(\mathbf{v}_*) \mathrm{d}\sigma \mathrm{d}\mathbf{v}_* = \\
\mathcal{Q}^+(f, f)(\mathbf{v}) - \frac{1}{2\pi} f(\mathbf{v}) \int_{\mathbb{R}^d} 2\pi f(\mathbf{v}_*) \mathrm{d}\mathbf{v}_* = \\
\mathcal{Q}^+(f, f)(\mathbf{v}) - C\rho f(\mathbf{v})
\end{aligned}$$

The constant C comes from rescaling that is done in the derivation of the Boltzmann equation. This can be observed in this article. The ρ comes from the fact that we are integrating out velocity from the phase space probability function f which leaves us with only the spatial density, ρ.

2.3 Normal Shock Problem

We will tackle the normal shock problem as stated in section 5.2 of Hu et al. [2]. We take $R = 1$, $d = 2$, hence $\gamma = 2$, $M_L = u_L/\sqrt{2T_L}$. In the following, the spatial domain is chosen as $x_1 \in [-30,,30]$ with $N_{\mathbf{x}} = 1000$; and the velocity domain is $(v_1, v_2) \in [-L_{\mathbf{v}}, L_{\mathbf{v}}]^2$.

We choose the upstream and downstream conditions as

$$(\rho_L, \rho_R) = \left(1, \frac{3M_L^2}{M_L^2+2}\right), \quad (u_L, u_R) = \left(\sqrt{2}M_L, \frac{\rho_L u_L}{\rho_R}\right), \quad (T_L, T_R) = \left(1, \frac{4M_L^2-1}{\rho_R}\right)$$

and the downstream conditions as

$$\rho_0(x_1) = \frac{\tanh(\alpha x_1)+1}{2(\rho_R-\rho_L)} + \rho_L, \quad T_0(x_1) = \frac{\tanh(\alpha x_1)+1}{2(T_R-T_L)} + T_L, \quad \mathbf{u}_0(x_1) = \left(\frac{\tanh(\alpha x_1)+1}{2(u_R-u_L)}, 0\right),$$

with $\alpha = 0.5$.

When showing the numerical results, we are mainly interested in the macroscopic quantities: density $\rho(x_1)$, bulk velocity $u(x_1)$,and temperature $T(x_1)$. Their normalized calues will be plotted, which are defined by

$$\hat{\rho}(x_1) = \frac{\rho(x_1)-\rho_L}{\rho_R-\rho_L}, \quad \hat{u}(x_1) = \frac{u(x_1)-u_L}{u_R-u_L}, \quad \hat{T}(x_1) = \frac{T(x_1)-T_L}{T_R-T_L}.$$

In Hu et al. [2] there is an example of a strong as well as a weak shock. We will model both of these.

2.4 Numerical Scheme for the Normal Shock Problem

For our numerical scheme we will be applying the methods developed in Chen et al. [1] and applying them to the equation described above. For the left-to-right sweep the numerical discretization is

$$\frac{v+|v|}{2}\frac{f_i^{(l+1)}-f_{i-1}^{(l+1)}}{\Delta x} + \frac{v-|v|}{2}\frac{f_{i+1}^{(l)}-f_i^{(l+1)}}{\Delta x} = Q^+(f^{(l)}, f^{(l)}) - C\rho_i^{(l)} f_i^{(l+1)}$$

and for the right-to-left sweep the numerical discretization is

$$\frac{v+|v|}{2}\frac{f_i^{(l+1)}-f_{i-1}^{(l)}}{\Delta x} + \frac{v-|v|}{2}\frac{f_{i+1}^{(l+1)}-f_i^{(l+1)}}{\Delta x} = Q^+(f^{(l)}, f^{(l)}) - C\rho_i^{(l)} f_i^{(l+1)}.$$

The update rule is acquired by isolating $f_i^{(l+1)}$ on one side. For $Q^+(f^{(l)}, f^{(l)})$ we must integrate over the whole velocity domain as well as a sphere. For $\rho_i^{(l)}$ we must integrate only over the whole velocity domain. We can only use the data from the previous time step in these integral computations otherwise we do not know what we are integrating. The derivative discretization of this scheme is only first order so it would not make much sense to use a very high-order integration. I intend to use the trapezoidal method for the first attempt.

3 To Do

This coming week I will begin implementing the numerical methods to replicate the results in section 5.2 in Hu et al. [2] using the methods developed in Chen et al. [1].

References

[1] Weitao Chen, Ching-Shan Chou, and Chiu-Yen Kao. Lax–friedrichs fast sweeping methods for steady state problems for hyperbolic conservation laws. *Journal of Computational Physics*, 234:452–471, 2013.

[2] Jingwei Hu and Yubo Wang. An adaptive dynamical low rank method for the nonlinear boltzmann equation, 2021.