1 Nondimensionalization of the Boltzmann equation

The Boltzmann equation for 3D hard sphere model reads

$$f_t + v\cdot\nabla_x f = r^2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|[f'_*f' - f_*f]\,d\sigma\,dv_*, \tag{1.1}$$

where

$$\begin{cases} v' = \dfrac{v+v_*}{2} + \dfrac{|v-v_*|}{2}\sigma, \\ v'_* = \dfrac{v+v_*}{2} - \dfrac{|v-v_*|}{2}\sigma. \end{cases} \tag{1.2}$$

Introduce $V = \frac{L}{T}$ macroscopic velocity, $c = \sqrt{\Theta}$ thermal speed (Θ reference temperature), and

$$\hat{x} = \frac{x}{L}, \quad \hat{t} = \frac{t}{T}, \quad \hat{v} = \frac{v}{c}, \quad \hat{f} = \frac{f}{N/(L^3c^3)}. \tag{1.3}$$

The equation becomes (neglect $\hat{}$ for all variables)

$$\frac{V}{c}f_t + v\cdot\nabla_x f = \frac{r^2N}{L^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|[f'_*f' - f_*f]\,d\sigma\,dv_*, \tag{1.4}$$

$\frac{V}{c}$ is the kinetic Strouhal number; $\frac{L^2}{r^2N}$ is the Knudsen number. Usually assume $V = c$, then the equation is

$$f_t + v\cdot\nabla_x f = \frac{r^2N}{L^2}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}|v-v_*|[f'_*f' - f_*f]\,d\sigma\,dv_*. \tag{1.5}$$

In general ($d = 2$ or 3), we consider

$$f_t + v\cdot\nabla_x f = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta)[f'_*f' - f_*f]\,d\sigma\,dv_*, \tag{1.6}$$

where v', v'_* are the same as above, $\cos\theta = \sigma\cdot\frac{v-v_*}{|v-v_*|}$. $B(|v-v_*|,\cos\theta) = |v-v_*|\Sigma(|v-v_*|,\cos\theta)$, Σ is the cross section (unit is length in 2d and area in 3d). Let

$$\hat{B}(|\hat{v}-\hat{v}_*|,\cos\theta) = \frac{B(|v-v_*|,\cos\theta)}{cr^{d-1}}, \tag{1.7}$$

then the nondimensionalized equation is

$$f_t + v\cdot\nabla_x f = \frac{r^{d-1}N}{L^{d-1}}\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta)[f'_*f' - f_*f]\,d\sigma\,dv_*. \tag{1.8}$$

2 About the collision kernel $B(|v-v_*|,\cos\theta)$ in 3D

In the usual 3D form

$$Q(f)(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(|v-v_*|,\cos\theta)[f'_*f'-f_*f]\,d\sigma\,dv_*, \tag{2.1}$$

the collision kernel $B(|v-v_*|,\cos\theta)=|v-v_*|\Sigma(|v-v_*|,\cos\theta)$, where Σ is the cross section. For hard sphere model: $\Sigma(|v-v_*|,\cos\theta)=r^2$, where r is the radius of the particle. For Coulomb interaction: $\Sigma(|v-v_*|,\cos\theta)=\frac{e^4}{64\pi^2\epsilon_0^2}\frac{1}{|v-v_*|^4\sin^4\frac{\theta}{2}}$.

In the important model case of inverse-power law potentials,

$$\phi(r)=\frac{1}{r^{s-1}},\quad 2<s\le\infty, \tag{2.2}$$

where r is the distance between particles, and the corresponding force is $O(\frac{1}{r^s})$, the collision kernel cannot be computed explicitly, but one can show that

$$B(|v-v_*|,\cos\theta)=b_\gamma(\cos\theta)|v-v_*|^\gamma,\quad \gamma=\frac{s-5}{s-1}, \tag{2.3}$$

so $-3<\gamma\le 1$. Particular cases are:

1. $s=2$, $\gamma=-3$: Coulomb interaction; $\phi(r)=\frac{1}{r}$, $B(|v-v_*|,\cos\theta)=b_{-3}(\cos\theta)|v-v_*|^{-3}$.
2. $s=3$, $\gamma=-1$: Manev interaction; $\phi(r)=\frac{1}{r^2}$, $B(|v-v_*|,\cos\theta)=b_{-1}(\cos\theta)|v-v_*|^{-1}$.
3. $s=5$, $\gamma=0$: ion-neutral interaction; $\phi(r)=\frac{1}{r^4}$, $B(|v-v_*|,\cos\theta)=b_0(\cos\theta)$ (Maxwellian molecule).
4. $s=7$, $\gamma=\frac{1}{3}$: Van der Waals interaction; $\phi(r)=\frac{1}{r^6}$, $B(|v-v_*|,\cos\theta)=b_{\frac{1}{3}}(\cos\theta)|v-v_*|^{\frac{1}{3}}$.
5. $s=\infty$, $\gamma=1$: hard sphere; $\phi(r)=\frac{1}{r^\infty}$, $B(|v-v_*|,\cos\theta)=b_1(\cos\theta)|v-v_*|$.

We call $\gamma<0$ soft potentials; $\gamma=0$ Maxwellian potentials, and $\gamma>0$ hard potentials. $\gamma>-1$ Boltzmann term dominants; $\gamma<-1$ mean-field term (Vlasov) dominants.

Function b_γ is only implicitly defined, locally smooth, and has a nonintegrable singularity for $\theta\to 0$:

$$\sin\theta b_\gamma(\cos\theta)\sim K\theta^{-1-\frac{2}{s-1}},\quad \sin\theta \text{ comes from the surface element.} \tag{2.4}$$

This happens as soon as forces are of infinite range, no matter how fast they decay at ∞. So consider cut-off collision kernel (without grazing collisions).

3 From usual (center of mass) form to reflected form

We use (center of mass parametrization)

$$Q(f)(v)=\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta)[f'_*f'-f_*f]\,d\sigma\,dv_*, \tag{3.1}$$

where $\cos\theta=\sigma\cdot\frac{v-v_*}{|v-v_*|}$, and

$$\begin{cases} v'=\dfrac{v+v_*}{2}+\dfrac{|v-v_*|}{2}\sigma, \\ v'_*=\dfrac{v+v_*}{2}-\dfrac{|v-v_*|}{2}\sigma. \end{cases} \tag{3.2}$$

Cercignani, Golse, Levermore use (reflection parametrization)

$$Q(f)(v)=\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} B_1(|v-v_*|,\cos\theta')[f'_*f'-f_*f]\,d\omega\,dv_*, \tag{3.3}$$

where $\cos\theta'=\omega\cdot\frac{v-v_*}{|v-v_*|}$, and

$$\begin{cases} v'=v-[(v-v_*)\cdot\omega]\omega, \\ v'_*=v_*+[(v-v_*)\cdot\omega]\omega. \end{cases} \tag{3.4}$$

In fact, $B_1(|v-v_*|,\cos\theta')=(2|\cos\theta'|)^{d-2}B(|v-v_*|,1-2\cos^2\theta')$. For 3D hard sphere model, $B=r^2|v-v_*|$, $B_1=2r^2|(v-v_*)\cdot\omega|$. Equivalence of these two forms can be seen as follows.

3.1 2D case

$$\begin{aligned}
\int_{\mathbb{S}^1} B(|v-v_*|,\cos\theta)[f'_*f'-f_*f]\,d\sigma &= \int_0^{2\pi} B(|v-v_*|,\cos\theta)[f'_*f'-f_*f]\,d\theta \\
\left(\theta'=\frac{\theta}{2}+\frac{\pi}{2}\right) &= 2\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} B(|v-v_*|,-\cos 2\theta')[f'_*f'-f_*f]\,d\theta' \\
\left(\text{or } \theta'=\frac{\theta}{2}+\frac{3\pi}{2}\right) &= 2\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} B(|v-v_*|,-\cos 2\theta')[f'_*f'-f_*f]\,d\theta' \\
(=2A=2B=A+B) &= \int_{\frac{\pi}{2}}^{\frac{5\pi}{2}} B(|v-v_*|,-\cos 2\theta')[f'_*f'-f_*f]\,d\theta' \\
(\text{integrand } \theta' \ 2\pi\text{-periodic}) &= \int_0^{2\pi} B(|v-v_*|,-\cos 2\theta')[f'_*f'-f_*f]\,d\theta' \\
&= \int_{\mathbb{S}^1} B(|v-v_*|,1-2\cos^2\theta')[f'_*f'-f_*f]\,d\omega.
\end{aligned} \tag{3.5}$$

3.2 3D case

$$
\begin{aligned}
\int_{\mathbb{S}^2} B(|v-v_*|,\cos\theta)[f_*'f'-f_*f]\,d\sigma &= \int_0^{2\pi}\int_0^{\pi} B(|v-v_*|,\cos\theta)[f_*'f'-f_*f]\sin\theta\,d\theta\,d\varphi \\
\left(\theta'=\frac{\theta}{2}+\frac{\pi}{2},\ \varphi'=\varphi\right) &= 2\int_0^{2\pi}\int_{\frac{\pi}{2}}^{\pi} B(|v-v_*|,-\cos 2\theta')[f_*'f'-f_*f](-\sin 2\theta')\,d\theta'\,d\varphi' \\
\left(\text{or } \theta'=\frac{\pi}{2}-\frac{\theta}{2},\ \varphi'=\varphi+\pi\right) &= 2\int_{\pi}^{3\pi}\int_0^{\frac{\pi}{2}} B(|v-v_*|,-\cos 2\theta')[f_*'f'-f_*f](\sin 2\theta')\,d\theta'\,d\varphi' \\
(\text{integrand } \theta' \ 2\pi\text{-periodic}) &= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} B(|v-v_*|,-\cos 2\theta')[f_*'f'-f_*f](\sin 2\theta')\,d\theta'\,d\varphi' \\
(=2A=2B=A+B) &= \int_0^{2\pi}\int_0^{\pi} B(|v-v_*|,-\cos 2\theta')[f_*'f'-f_*f]|\sin 2\theta'|\,d\theta'\,d\varphi' \\
&= 2\int_0^{2\pi}\int_0^{\pi} |\cos\theta'|B(|v-v_*|,1-2\cos^2\theta')[f_*'f'-f_*f]\sin\theta'\,d\theta'\,d\varphi' \\
&= 2\int_{\mathbb{S}^2} |\cos\theta'|B(|v-v_*|,1-2\cos^2\theta')[f_*'f'-f_*f]\,d\omega.
\end{aligned}
\tag{3.6}
$$

4 From Carleman form to usual form

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{B}(x, y)\, \delta(x \cdot y)[f(v+y)f(v+x) - f(v+x+y)f(v)]\, dx\, dy, \tag{4.1}$$

where

$$\tilde{B}(x, y) = \frac{2^{d-1}}{|x+y|^{d-2}} B\left(|x+y|, 1 - 2\left(\frac{x \cdot (x+y)}{|x||x+y|}\right)^2\right), \tag{4.2}$$

if we assume $x \cdot y = 0$,

$$\tilde{B}(x, y) = \frac{2^{d-1}}{(|x|^2 + |y|^2)^{\frac{d-2}{2}}} B\left(\sqrt{|x|^2 + |y|^2}, 1 - 2\frac{|x|^2}{|x|^2 + |y|^2}\right) = \tilde{B}(|x|, |y|). \tag{4.3}$$

1. Change y to $v_* = y + x + v$,

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{B}(x, v_* - v - x)\, \delta(x \cdot (v_* - v - x))[f(v_* - x)f(v+x) - f(v_*)f(v)]\, dx\, dv_*.$$

2. Change x to $x' = x + \frac{v - v_*}{2}$,

$$\begin{aligned} Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{B}\left(x' - \frac{v - v_*}{2}, \frac{v_* - v}{2} - x'\right) \delta\left(x'^2 - \frac{|v - v_*|^2}{4}\right) \\ \cdot \left[f\left(\frac{v + v_*}{2} - x'\right) f\left(\frac{v + v_*}{2} + x'\right) - f(v_*)f(v)\right] dx'\, dv_*. \end{aligned}$$

3. Let $x' = \rho\sigma$, $dx' = \rho^{d-1}\, d\rho\, d\sigma$,

$$\begin{aligned} Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \int_0^\infty \tilde{B}\left(\rho\sigma - \frac{v - v_*}{2}, \frac{v_* - v}{2} - \rho\sigma\right) \delta\left(\rho^2 - \frac{|v - v_*|^2}{4}\right) \\ \cdot \left[f\left(\frac{v + v_*}{2} - \rho\sigma\right) f\left(\frac{v + v_*}{2} + \rho\sigma\right) - f(v_*)f(v)\right] \rho^{d-1}\, d\rho\, d\sigma\, dv_*. \end{aligned}$$

4. Then $\rho = \frac{|v - v_*|}{2}$,

$$\begin{aligned} Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \tilde{B}\left(\rho\sigma - \frac{v - v_*}{2}, \frac{v_* - v}{2} - \rho\sigma\right) \frac{1}{2\rho} \\ \cdot \left[f\left(\frac{v + v_*}{2} - \rho\sigma\right) f\left(\frac{v + v_*}{2} + \rho\sigma\right) - f(v_*)f(v)\right] \rho^{d-1}\, d\sigma\, dv_*. \end{aligned}$$

Now define

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma \end{cases} \tag{4.4}$$

and

$$\tilde{B}\left(\rho\sigma - \frac{v - v_*}{2}, \frac{v_* - v}{2} - \rho\sigma\right) \frac{1}{2\rho}\rho^{d-1} = B(|v - v_*|, \cos\theta). \tag{4.5}$$

Finally

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \cos\theta)[f'_* f' - f_* f]\, d\sigma\, dv_*. \tag{4.6}$$

5 Three equivalent forms for 2D Maxwellian molecule and 3D hard sphere model

In the usual form

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \cos\theta)[f'_* f' - f_* f]\, d\sigma\, dv_*, \tag{5.1}$$

and the Carleman form

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{B}(x, y)\, \delta(x \cdot y)[f(v + y)f(v + x) - f(v + x + y)f(v)]\, dx\, dy, \tag{5.2}$$

there are two special cases: for 2D Maxwellian molecules $B = \frac{1}{2}$, and 3D hard sphere model $B = \frac{1}{4}|v - v_*|$, we have $\tilde{B} \equiv 1$. Another equivalent form for these two cases is

$$Q(f)(v) = \int_{\mathbb{R}^{3d}} \delta(v + v_* - v' - v'_*)\, \delta\left(\frac{v^2}{2} + \frac{v_*^2}{2} - \frac{v'^2}{2} - \frac{v_*'^2}{2}\right) [f'_* f' - f_* f]\, dv_*\, dv'\, dv'_*. \tag{5.3}$$

Indeed,

1. Make a change of variables $O = \frac{v'+v'_*}{2}$, $W = \frac{v'-v'_*}{2}$, Jacobian is 2^d,

$$Q(f)(v) = 2^d \int_{\mathbb{R}^{3d}} \delta(v + v_* - 2O)\, \delta\left(\frac{v^2}{2} + \frac{v_*^2}{2} - O^2 - W^2\right) [f'_* f' - f_* f]\, dv_*\, dO\, dW.$$

2. Let $W = \rho\sigma$, $dW = \rho^{d-1}\, d\rho\, d\sigma$,

$$Q(f)(v) = 2^d \int_{\mathbb{R}^{2d}} \int_{\mathbb{S}^{d-1}} \int_0^\infty \delta(v+v_*-2O)\, \delta\left(\frac{v^2}{2} + \frac{v_*^2}{2} - O^2 - \rho^2\right) [f'_* f' - f_* f] \rho^{d-1}\, d\rho\, d\sigma\, dv_*\, dO.$$

3. Then $\rho = \sqrt{\frac{v^2}{2} + \frac{v_*^2}{2} - O^2}$,

$$\begin{aligned} Q(f)(v) &= 2^d \int_{\mathbb{R}^{2d}} \int_{\mathbb{S}^{d-1}} \delta(v + v_* - 2O) \frac{1}{2\rho} [f'_* f' - f_* f] \rho^{d-1}\, d\sigma\, dv_*\, dO \\ &= \frac{1}{2} \int_{\mathbb{R}^{2d}} \int_{\mathbb{S}^{d-1}} \rho^{d-2} \delta\left(O - \frac{v + v_*}{2}\right) [f'_* f' - f_* f]\, d\sigma\, dv_*\, dO. \end{aligned}$$

4. Then $O = \frac{v+v_*}{2}$,

$$Q(f)(v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \rho^{d-2} [f'_* f' - f_* f]\, d\sigma\, dv_*.$$

Therefore, $\rho = \frac{|v-v_*|}{2}$, and

$$\begin{cases} v' = \dfrac{v + v_*}{2} + \dfrac{|v - v_*|}{2}\sigma; \\ v'_* = \dfrac{v + v_*}{2} - \dfrac{|v - v_*|}{2}\sigma. \end{cases} \tag{5.4}$$

Finally

$$Q(f)(v) = \frac{1}{2^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} |v - v_*|^{d-2} [f'_* f' - f_* f]\, d\sigma\, dv_*. \tag{5.5}$$

6 Truncation for usual form

Start from the usual form

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|v - v_*|, \cos\theta)[f'_* f' - f_* f] \, d\sigma \, dv_*, \tag{6.1}$$

we first change v_* to $g = v - v_*$,

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} B(|g|, \cos\theta)[f'_* f' - f_* f] \, d\sigma \, dg. \tag{6.2}$$

Then $v_* = v - g$, $v' = v - \frac{g}{2} + \frac{|g|}{2}\sigma$, and $v'_* = v - \frac{g}{2} - \frac{|g|}{2}\sigma$.

If $\mathrm{Supp}(f(v)) \subset \mathcal{B}_S$, then we have

1. $\mathrm{Supp}(Q(f)(v)) \subset \mathcal{B}_{\sqrt{2}S}$.

   This is because if $|v| > \sqrt{2}S$, then $f = 0$; also $v'^2 + v'^2_* \geq v^2 > 2S^2$, then $|v'| > S$ or $|v'_*| > S$, so $f' = 0$ or $f'_* = 0$; either way $Q(f)(v) = 0$.

2. It is enough to consider

$$Q(f)(v) = \int_{\mathcal{B}_R} \int_{\mathbb{S}^{d-1}} B(|g|, \cos\theta)[f'_* f' - f_* f] \, d\sigma \, dg, \tag{6.3}$$

   where $R = 2S$.

   This is because if $2S < |g| = |v - v_*| \leq |v| + |v_*|$, then $|v| > S$ or $|v_*| > S$, so $f = 0$ or $f_* = 0$; also $2S < |g| = |v - v_*| = |v' - v'_*| \leq |v'| + |v'_*|$, then $|v'| > S$ or $|v'_*| > S$, so $f' = 0$ or $f'_* = 0$; either way $Q(f)(v) = 0$.

3. Since $|v| \leq \sqrt{2}S$ and $|g| \leq 2S$, we have

   $|v_*| = |v - g| \leq |v| + |g| \leq (2 + \sqrt{2})S$;

   $|v'| = |v - \frac{g}{2} + \frac{|g|}{2}\sigma| \leq |v| + |g| \leq (2 + \sqrt{2})S$;

   $|v'_*| = |v - \frac{g}{2} - \frac{|g|}{2}\sigma| \leq |v| + |g| \leq (2 + \sqrt{2})S$.

4. To avoid aliasing, need

$$2T \geq (2 + \sqrt{2})S + S \quad \Rightarrow \quad T \geq \frac{3 + \sqrt{2}}{2} S. \tag{6.4}$$

7 Truncation for Carleman form

For the Carleman form

$$Q(f)(v) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tilde{B}(x, y)\, \delta(x \cdot y)[f'_* f' - f_* f]\, dx\, dy, \tag{7.1}$$

where $v'_* = v + y$, $v' = v + x$, and $v_* = v + x + y$.

If $\mathrm{Supp}(f(v)) \subset \mathcal{B}_S$, then we have

1. $\mathrm{Supp}(Q(f)(v)) \subset \mathcal{B}_{\sqrt{2}S}$.

   This is because if $|v| > \sqrt{2}S$, then $f = 0$; also $v'^2 + v'^2_* \geq v^2 > 2S^2$, then $|v'| > S$ or $|v'_*| > S$, so $f' = 0$ or $f'_* = 0$; either way $Q(f)(v) = 0$.

2. It is enough to consider

   $$Q(f)(v) = \int_{\mathcal{B}_R} \int_{\mathcal{B}_R} \tilde{B}(x, y)\, \delta(x \cdot y)[f'_* f' - f_* f]\, dx\, dy, \tag{7.2}$$

   where $R = 2S$.

   This is because if $|x| > 2S$ or $|y| > 2S$, we have $|x+y|^2 = |x|^2+|y|^2 > 4S^2$, i.e. $|x+y| > 2S$.

   Then $2S < |x + y| = |v - v_*| \leq |v| + |v_*|$, then $|v| > S$ or $|v_*| > S$, so $f = 0$ or $f_* = 0$; also $2S < |x + y| = |v - v_*| = |v' - v'_*| \leq |v'| + |v'_*|$, then $|v'| > S$ or $|v'_*| > S$, so $f' = 0$ or $f'_* = 0$; either way $Q(f)(v) = 0$.

3. Since $|x| \leq 2S$, $|y| \leq 2S$, we have $|x + y|^2 = |x|^2 + |y|^2 \leq 8S^2 \Rightarrow |x + y| \leq 2\sqrt{2}S$. Also $|v| \leq \sqrt{2}S$, then

   $|v'_*| = |v + y| \leq |v| + |y| \leq (2 + \sqrt{2})S$;

   $|v'| = |v + x| \leq |v| + |x| \leq (2 + \sqrt{2})S$;

   $|v_*| = |v + x + y| \leq |v| + |x + y| \leq 3\sqrt{2}S$;

4. To avoid aliasing, need

   $$2T \geq 3\sqrt{2}S + S \quad \Rightarrow \quad T \geq \frac{3\sqrt{2} + 1}{2} S. \tag{7.3}$$

8 Fourier expansion

Define inner product

$$\langle f, g \rangle = \frac{1}{(2T)^d} \int_{\mathcal{D}_T} f \bar{g} \, dv, \tag{8.1}$$

where $\mathcal{D}_T = [-T, T]^d$. Then

$$f(v) = \sum_{k=-\infty}^{\infty} \langle f(v), e^{i\frac{\pi}{T}k\cdot v} \rangle e^{i\frac{\pi}{T}k\cdot v} = \sum_{k=-\infty}^{\infty} \hat{f}_k e^{i\frac{\pi}{T}k\cdot v}, \tag{8.2}$$

where

$$\hat{f}_k = \frac{1}{(2T)^d} \int_{\mathcal{D}_T} f(v) e^{-i\frac{\pi}{T}k\cdot v} \, dv. \tag{8.3}$$

Also we know

$$e^{i\frac{\pi}{T}j\cdot v} = \sum_{k=-\infty}^{\infty} \langle e^{i\frac{\pi}{T}j\cdot v}, e^{i\frac{\pi}{T}k\cdot v} \rangle e^{i\frac{\pi}{T}k\cdot v}, \tag{8.4}$$

so

$$\langle e^{i\frac{\pi}{T}j\cdot v}, e^{i\frac{\pi}{T}k\cdot v} \rangle = \delta_{jk}. \tag{8.5}$$

9 Fourier expansion for quantum usual form

$$\begin{aligned}
Q(f)(v) &= \int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)[f'f'_*(1+f)(1+f_*) - ff_*(1+f')(1+f'_*)]\,d\sigma\,dg \\
&= \int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)[(f'f'_* - ff_*) + (f'f'_*f_* + f'f'_*f - f'f_*f - f_*f'_*f)]\,d\sigma\,dg,
\end{aligned} \tag{9.1}$$

where $v_* = v - g$, $v' = v - \frac{g}{2} + \frac{|g|}{2}\sigma$, and $v'_* = v - \frac{g}{2} - \frac{|g|}{2}\sigma$.

Insert Fourier expansion of f into $Q(f)$:

$$\begin{aligned}
Q(f) = &\int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)\Bigg[\sum_{l,m}\left(e^{i\frac{\pi}{T}l\cdot(v-\frac{g}{2}+\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}m\cdot(v-\frac{g}{2}-\frac{|g|}{2}\sigma)} - e^{i\frac{\pi}{T}l\cdot v}e^{i\frac{\pi}{T}m\cdot(v-g)}\right)\hat{f}_l\hat{f}_m \\
&+\sum_{l,m,n}\Big(e^{i\frac{\pi}{T}l\cdot(v-\frac{g}{2}+\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}m\cdot(v-\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}n\cdot(v-g)} + e^{i\frac{\pi}{T}l\cdot(v-\frac{g}{2}+\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}m\cdot(v-\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}n\cdot v} \\
&-e^{i\frac{\pi}{T}l\cdot(v-\frac{g}{2}+\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}m\cdot(v-g)}e^{i\frac{\pi}{T}n\cdot v} - e^{i\frac{\pi}{T}l\cdot(v-g)}e^{i\frac{\pi}{T}m\cdot(v-\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{i\frac{\pi}{T}n\cdot v}\Big)\hat{f}_l\hat{f}_m\hat{f}_n\Bigg]\,d\sigma\,dg \\
= &\int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)\Bigg[\sum_{l,m} e^{i\frac{\pi}{T}(l+m)\cdot v}\left(e^{-i\frac{\pi}{T}l\cdot(\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}m\cdot(\frac{g}{2}+\frac{|g|}{2}\sigma)} - e^{-i\frac{\pi}{T}m\cdot g}\right)\hat{f}_l\hat{f}_m \\
&+\sum_{l,m,n} e^{i\frac{\pi}{T}(l+m+n)\cdot v}\Big(e^{-i\frac{\pi}{T}l\cdot(\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}m\cdot(\frac{g}{2}+\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}n\cdot g} + e^{-i\frac{\pi}{T}l\cdot(\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}m\cdot(\frac{g}{2}+\frac{|g|}{2}\sigma)} \\
&-e^{-i\frac{\pi}{T}l\cdot(\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}m\cdot g} - e^{-i\frac{\pi}{T}l\cdot g}e^{-i\frac{\pi}{T}m\cdot(\frac{g}{2}+\frac{|g|}{2}\sigma)}\Big)\hat{f}_l\hat{f}_m\hat{f}_n\Bigg]\,d\sigma\,dg.
\end{aligned} \tag{9.2}$$

Define

$$\beta(l,m) = \int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)e^{-i\frac{\pi}{T}l\cdot(\frac{g}{2}-\frac{|g|}{2}\sigma)}e^{-i\frac{\pi}{T}m\cdot(\frac{g}{2}+\frac{|g|}{2}\sigma)}\,d\sigma\,dg, \tag{9.3}$$

then

$$\begin{aligned}
Q(f) = &\sum_{l,m} e^{i\frac{\pi}{T}(l+m)\cdot v}[\beta(l,m) - \beta(m,m)]\hat{f}_l\hat{f}_m \\
&+\sum_{l,m,n} e^{i\frac{\pi}{T}(l+m+n)\cdot v}[\beta(l+n,m+n) + \beta(l,m) - \beta(l+m,m) - \beta(l,l+m)]\hat{f}_l\hat{f}_m\hat{f}_n.
\end{aligned} \tag{9.4}$$

Therefore,

$$\begin{aligned}
\widehat{Q(f)}_k &= \langle Q(f), e^{i\frac{\pi}{T}k\cdot v}\rangle \\
&= \sum_{l+m=k}[\beta(l,m) - \beta(m,m)]\hat{f}_l\hat{f}_m \\
&+ \sum_{l+m+n=k}[\beta(l+n,m+n) + \beta(l,m) - \beta(l+m,m) - \beta(l,l+m)]\hat{f}_l\hat{f}_m\hat{f}_n.
\end{aligned} \tag{9.5}$$

β can be recast as

$$\beta(l,m) = \int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} B(|g|,\cos\theta)e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g + i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma\,dg. \tag{9.6}$$

9.1 Compute $\beta(l,m)$

In general, $B(|g|,\cos\theta) = b_\gamma(\cos\theta)|g|^\gamma$. Therefore,

$$\begin{aligned}\beta(l,m) &= \int_{\mathcal{B}_R}\int_{\mathbb{S}^{d-1}} b_\gamma\left(\frac{\sigma\cdot g}{|g|}\right)|g|^\gamma e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g + i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma\,dg \\ &= \int_{\mathcal{B}_R}|g|^\gamma e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g}\left(\int_{\mathbb{S}^{d-1}} b_\gamma\left(\frac{\sigma\cdot g}{|g|}\right)e^{i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma\right)dg. \qquad (9.7)\end{aligned}$$

Define

$$I(g,l-m) = \int_{\mathbb{S}^{d-1}} b_\gamma\left(\frac{\sigma\cdot g}{|g|}\right)e^{i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma. \qquad (9.8)$$

Then

$$\beta(l,m) = \int_{\mathcal{B}_R}|g|^\gamma e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g} I(g,l-m)\,dg. \qquad (9.9)$$

To simplify the problem, we now assume $b_\gamma(\cos\theta)\equiv 1$.

9.1.1 Special 2D case

$$\begin{aligned}I(g,l-m) &= \int_{\mathbb{S}^1} e^{i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma \\ (\sigma \text{ in } l-m \text{ coordinate}) &= \int_0^{2\pi} e^{i\frac{\pi}{T}|g|\frac{|l-m|}{2}\cos\theta}\,d\theta \\ (\theta' = \frac{\pi}{2}-\theta) &= \int_{-\frac{3}{2}\pi}^{\frac{\pi}{2}} e^{i\frac{\pi}{T}|g|\frac{|l-m|}{2}\sin\theta}\,d\theta \\ (\text{integrand } \theta\ 2\pi\text{-periodic}) &= \int_{-\pi}^{\pi} e^{i\frac{\pi}{T}|g|\frac{|l-m|}{2}\sin\theta}\,d\theta \\ &= 2\pi J_0\left(\frac{\pi}{T}|g|\frac{|l-m|}{2}\right). \qquad (9.10)\end{aligned}$$

Then

$$\begin{aligned}\beta(l,m) &= 2\pi\int_{\mathcal{B}_R}|g|^\gamma e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g}J_0\left(\frac{\pi}{T}|g|\frac{|l-m|}{2}\right)dg \\ (g=\rho\sigma \text{ in } l+m \text{ coordinate}) &= 2\pi\int_0^{2\pi}\int_0^R \rho^{\gamma+1}e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\theta}J_0\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)d\rho\,d\theta \\ &= 2\pi\int_0^R \rho^{\gamma+1}J_0\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)\left(\int_0^{2\pi}e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\theta}\,d\theta\right)d\rho \\ &= 2\pi\int_0^R \rho^{\gamma+1}J_0\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)I_1(\rho,l+m)\,d\rho. \qquad (9.11)\end{aligned}$$

$$
\begin{aligned}
I_1(\rho, l+m) &= \int_0^{2\pi} e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\theta}\,d\theta \\
(\theta' = \frac{3\pi}{2} - \theta) &= \int_{-\frac{\pi}{2}}^{\frac{3}{2}\pi} e^{i\frac{\pi}{T}\frac{|l+m|}{2}\rho\sin\theta}\,d\theta \\
(\text{integrand } \theta \ 2\pi\text{-periodic}) &= \int_{-\pi}^{\pi} e^{i\frac{\pi}{T}\frac{|l+m|}{2}\rho\sin\theta}\,d\theta \\
&= 2\pi J_0\left(\frac{\pi}{T}\rho\frac{|l+m|}{2}\right). \qquad (9.12)
\end{aligned}
$$

Finally,

$$\beta(l,m) = 4\pi^2 \int_0^R \rho^{\gamma+1} J_0\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right) J_0\left(\frac{\pi}{T}\rho\frac{|l+m|}{2}\right)\,d\rho. \qquad (9.13)$$

9.1.2 Special 3D case

$$
\begin{aligned}
I(g, l-m) &= \int_{\mathbb{S}^2} e^{i\frac{\pi}{T}|g|\frac{(l-m)}{2}\cdot\sigma}\,d\sigma \\
(\sigma \text{ in } l-m \text{ coordinate}) &= 2\pi\int_0^{\pi} e^{i\frac{\pi}{T}|g|\frac{|l-m|}{2}\cos\varphi}\sin\varphi\,d\varphi \\
&= 4\pi\,\mathrm{Sinc}\left(\frac{\pi}{T}|g|\frac{|l-m|}{2}\right). \qquad (9.14)
\end{aligned}
$$

Then

$$
\begin{aligned}
\beta(l,m) &= 4\pi\int_{\mathcal{B}_R} |g|^{\gamma} e^{-i\frac{\pi}{T}\frac{(l+m)}{2}\cdot g}\,\mathrm{Sinc}\left(\frac{\pi}{T}|g|\frac{|l-m|}{2}\right)\,dg \\
(g = \rho\sigma \text{ in } l+m \text{ coordinate}) &= 8\pi^2\int_0^{\pi}\int_0^R \rho^{\gamma+2} e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\varphi}\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)\sin\varphi\,d\rho\,d\varphi \\
&= 8\pi^2\int_0^R \rho^{\gamma+2}\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)\left(\int_0^{\pi} e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\varphi}\sin\varphi\,d\varphi\right)\,d\rho \\
&= 8\pi^2\int_0^R \rho^{\gamma+2}\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right) I_1(\rho, l+m)\,d\rho \qquad (9.15)
\end{aligned}
$$

$$I_1(\rho, l+m) = \int_0^{\pi} e^{-i\frac{\pi}{T}\frac{|l+m|}{2}\rho\cos\varphi}\sin\varphi\,d\varphi = 2\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l+m|}{2}\right). \qquad (9.16)$$

Finally,

$$\beta(l,m) = 16\pi^2\int_0^R \rho^{\gamma+2}\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l-m|}{2}\right)\,\mathrm{Sinc}\left(\frac{\pi}{T}\rho\frac{|l+m|}{2}\right)\,d\rho. \qquad (9.17)$$

10 Fourier expansion for quantum Carleman form

$$
\begin{aligned}
Q(f)(v) &= \int_{\mathcal{B}_R}\int_{\mathcal{B}_R} \tilde{B}(x,y)\,\delta(x\cdot y)[f'f'_*(1+f)(1+f_*) - ff_*(1+f')(1+f'_*)]\,dx\,dy \\
&= \int_{\mathcal{B}_R}\int_{\mathcal{B}_R} \tilde{B}(x,y)\,\delta(x\cdot y)[(f'f'_* - ff_*) + (f'f'_*f_* + f'f'_*f - f'f_*f - f_*f'_*f)]\,dx\,dy,
\end{aligned}
\tag{10.1}
$$

where $v'_* = v + y$, $v' = v + x$, and $v_* = v + x + y$.

Insert Fourier expansion of f into $Q(f)$:

$$
\begin{aligned}
Q(f) &= \int_{\mathcal{B}_R}\int_{\mathcal{B}_R} \tilde{B}(x,y)\,\delta(x\cdot y)\left[\sum_{l,m}\left(e^{i\frac{\pi}{T}l\cdot(v+x)}e^{i\frac{\pi}{T}m\cdot(v+y)} - e^{i\frac{\pi}{T}l\cdot v}e^{i\frac{\pi}{T}m\cdot(v+x+y)}\right)\hat{f}_l\hat{f}_m\right. \\
&\quad + \sum_{l,m,n}\left(e^{i\frac{\pi}{T}l\cdot(v+x)}e^{i\frac{\pi}{T}m\cdot(v+y)}e^{i\frac{\pi}{T}n\cdot(v+x+y)} + e^{i\frac{\pi}{T}l\cdot(v+x)}e^{i\frac{\pi}{T}m\cdot(v+y)}e^{i\frac{\pi}{T}n\cdot v}\right. \\
&\quad \left.\left. - e^{i\frac{\pi}{T}l\cdot(v+x)}e^{i\frac{\pi}{T}m\cdot(v+x+y)}e^{i\frac{\pi}{T}n\cdot v} - e^{i\frac{\pi}{T}l\cdot(v+x+y)}e^{i\frac{\pi}{T}m\cdot(v+y)}e^{i\frac{\pi}{T}n\cdot v}\right)\hat{f}_l\hat{f}_m\hat{f}_n\right]\,dx\,dy \\
&= \int_{\mathcal{B}_R}\int_{\mathcal{B}_R} \tilde{B}(x,y)\,\delta(x\cdot y)\left[\sum_{l,m} e^{i\frac{\pi}{T}(l+m)\cdot v}\left(e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot y} - e^{i\frac{\pi}{T}m\cdot(x+y)}\right)\hat{f}_l\hat{f}_m\right. \\
&\quad + \sum_{l,m,n} e^{i\frac{\pi}{T}(l+m+n)\cdot v}\left(e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot y}e^{i\frac{\pi}{T}n\cdot(x+y)} + e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot y}\right. \\
&\quad \left.\left. - e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot(x+y)} - e^{i\frac{\pi}{T}l\cdot(x+y)}e^{i\frac{\pi}{T}m\cdot y}\right)\hat{f}_l\hat{f}_m\hat{f}_n\right]\,dx\,dy.
\end{aligned}
\tag{10.2}
$$

Define

$$
\beta(l,m) = \int_{\mathcal{B}_R}\int_{\mathcal{B}_R} \tilde{B}(x,y)\,\delta(x\cdot y)e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot y}\,dx\,dy,
\tag{10.3}
$$

then

$$
\begin{aligned}
Q(f) &= \sum_{l,m} e^{i\frac{\pi}{T}(l+m)\cdot v}[\beta(l,m) - \beta(m,m)]\hat{f}_l\hat{f}_m \\
&\quad + \sum_{l,m,n} e^{i\frac{\pi}{T}(l+m+n)\cdot v}[\beta(l+n,m+n) + \beta(l,m) - \beta(l+m,m) - \beta(l,l+m)]\hat{f}_l\hat{f}_m\hat{f}_n.
\end{aligned}
\tag{10.4}
$$

Therefore,

$$
\begin{aligned}
\widehat{Q(f)}_k &= \langle Q(f), e^{i\frac{\pi}{T}k\cdot v}\rangle \\
&= \sum_{l+m=k}[\beta(l,m) - \beta(m,m)]\hat{f}_l\hat{f}_m \\
&\quad + \sum_{l+m+n=k}[\beta(l+n,m+n) + \beta(l,m) - \beta(l+m,m) - \beta(l,l+m)]\hat{f}_l\hat{f}_m\hat{f}_n.
\end{aligned}
\tag{10.5}
$$

10.1 Compute $\beta(l,m)$

WLOG, assume $\tilde{B}(x,y)=\tilde{B}(|x|,|y|)=a(|x|)b(|y|)$, otherwise $\tilde{B}(|x|,|y|)=\sum_t a_t(|x|)b_t(|y|)$. Therefore,

$$\beta(l,m)=\int_{\mathcal{B}_R}\int_{\mathcal{B}_R} a(|x|)b(|y|)\delta(x\cdot y)e^{i\frac{\pi}{T}l\cdot x}e^{i\frac{\pi}{T}m\cdot y}\,dx\,dy. \tag{10.6}$$

Let $x=\rho_1\sigma_1$, $dx=\rho_1^{d-1}\,d\rho_1\,d\sigma_1$, $y=\rho_2\sigma_2$, $dy=\rho_2^{d-1}\,d\rho_2\,d\sigma_2$,

$$\begin{aligned}
\beta(l,m) &= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^R\int_0^R a(\rho_1)b(\rho_2)\delta(\rho_1\rho_2\,\sigma_1\cdot\sigma_2)e^{i\frac{\pi}{T}\rho_1 l\cdot\sigma_1}e^{i\frac{\pi}{T}\rho_2 m\cdot\sigma_2}\rho_1^{d-1}\rho_2^{d-1}\,d\rho_1\,d\rho_2\,d\sigma_1\,d\sigma_2\\
&= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\int_0^R\int_0^R a(\rho_1)b(\rho_2)\delta(\sigma_1\cdot\sigma_2)e^{i\frac{\pi}{T}\rho_1 l\cdot\sigma_1}e^{i\frac{\pi}{T}\rho_2 m\cdot\sigma_2}\rho_1^{d-2}\rho_2^{d-2}\,d\rho_1\,d\rho_2\,d\sigma_1\,d\sigma_2\\
&= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_0^R a(\rho_1)\rho_1^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_0^R b(\rho_2)\rho_2^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
(\rho_1\to-\rho_1) &= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^0 a(-\rho_1)(-\rho_1)^{d-2}e^{-i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_0^R b(\rho_2)\rho_2^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
(\sigma_1\to-\sigma_1) &= \int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^0 a(-\rho_1)(-\rho_1)^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_0^R b(\rho_2)\rho_2^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
&= \frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^R a(|\rho_1|)|\rho_1|^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_0^R b(\rho_2)\rho_2^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
(\rho_2\to-\rho_2) &= \frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^R a(|\rho_1|)|\rho_1|^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_{-R}^0 b(-\rho_2)(-\rho_2)^{d-2}e^{-i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
(\sigma_2\to-\sigma_2) &= \frac{1}{2}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^R a(|\rho_1|)|\rho_1|^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_{-R}^0 b(-\rho_2)(-\rho_2)^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2\\
&= \frac{1}{4}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^R a(|\rho_1|)|\rho_1|^{d-2}e^{i\frac{\pi}{T}\rho_1\,l\cdot\sigma_1}\,d\rho_1\right]\left[\int_{-R}^R b(|\rho_2|)|\rho_2|^{d-2}e^{i\frac{\pi}{T}\rho_2\,m\cdot\sigma_2}\,d\rho_2\right]d\sigma_1\,d\sigma_2.
\end{aligned} \tag{10.7}$$

So

$$\beta(l,m)=\frac{1}{4}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\left[\int_{-R}^R a(|\rho|)|\rho|^{d-2}e^{i\frac{\pi}{T}\rho\,l\cdot\sigma_1}\,d\rho\right]\left[\int_{-R}^R b(|\rho|)|\rho|^{d-2}e^{i\frac{\pi}{T}\rho\,m\cdot\sigma_2}\,d\rho\right]d\sigma_1\,d\sigma_2. \tag{10.8}$$

Define

$$\phi_a(s)=\int_{-R}^R a(|\rho|)|\rho|^{d-2}e^{i\frac{\pi}{T}\rho s}\,d\rho,\quad \phi_b(s)=\int_{-R}^R b(|\rho|)|\rho|^{d-2}e^{i\frac{\pi}{T}\rho s}\,d\rho, \tag{10.9}$$

then

$$\beta(l,m)=\frac{1}{4}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{S}^{d-1}}\delta(\sigma_1\cdot\sigma_2)\phi_a(l\cdot\sigma_1)\phi_b(m\cdot\sigma_2)\,d\sigma_1\,d\sigma_2. \tag{10.10}$$

We have

$$\phi_{a,b}(s)=\phi_{a,b}(-s)=\phi_{a,b}(|s|). \tag{10.11}$$

For 2D Maxwellian molecules and 3D hard sphere model, $\tilde{B} \equiv 1$. Then

$$\phi(s) = \phi_{a,b}(s) = \int_{-R}^{R} |\rho|^{d-2} e^{i\frac{\pi}{T}\rho s}\, d\rho, \tag{10.12}$$

and when $d = 2$,

$$\phi(s) = 2R\,\mathrm{Sinc}\left(\frac{\pi}{T}Rs\right); \tag{10.13}$$

when $d = 3$,

$$\phi(s) = 2R^2\,\mathrm{Sinc}\left(\frac{\pi}{T}Rs\right) - R^2\,\mathrm{Sinc}^2\left(\frac{\pi}{2T}Rs\right). \tag{10.14}$$

In the following, $a \cdot b$ denotes the usual dot product; $a * b$ denotes the dot product computed in the same coordinate, i.e., $a * b = \sum_i a_i b_i$.

10.1.1 General 2D case

$$\begin{aligned}
\beta(l,m) &= \frac{1}{4}\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \delta(\sigma_1 \cdot \sigma_2)\phi_a(l \cdot \sigma_1)\phi_b(m \cdot \sigma_2)\, d\sigma_1\, d\sigma_2 \\
&= \frac{1}{4}\int_{\mathbb{S}^1} \phi_a(l \cdot \sigma_1)\left[\int_{\mathbb{S}^1} \delta(\sigma_1 \cdot \sigma_2)\phi_b(m \cdot \sigma_2)\, d\sigma_2\right] d\sigma_1 \\
&= \frac{1}{4}\int_{\mathbb{S}^1} \phi_a(l \cdot \sigma_1) I(m, \sigma_1)\, d\sigma_1.
\end{aligned} \tag{10.15}$$

$$\begin{aligned}
I(m,\sigma_1) &= \int_{\mathbb{S}^1} \delta(\sigma_1 \cdot \sigma_2)\phi_b(m \cdot \sigma_2)\, d\sigma_2 \\
(\sigma_2 \text{ in } \sigma_1 \text{ coordinate}) &= \int_0^{2\pi} \delta(\cos\theta)\phi_b(m(\sigma_1) * (\cos\theta, \sin\theta))\, d\theta \\
&= \phi_b(m_2(\sigma_1)) + \phi_b(-m_2(\sigma_1)) = 2\phi_b(|m_2(\sigma_1)|) \\
&= 2\phi_b(\sqrt{|m|^2 - (m \cdot \sigma_1)^2}).
\end{aligned} \tag{10.16}$$

Therefore,

$$\begin{aligned}
\beta(l,m) &= \frac{1}{2}\int_{\mathbb{S}^1} \phi_a(l \cdot \sigma_1)\phi_b(\sqrt{|m|^2 - (m \cdot \sigma_1)^2})\, d\sigma_1 \\
(\sigma_1 \text{ in original coordinate}) &= \frac{1}{2}\int_0^{2\pi} \phi_a(l * (\cos\theta, \sin\theta))\phi_b(\sqrt{|m|^2 - (m * (\cos\theta, \sin\theta))^2})\, d\theta \\
(\text{integrand } \theta\ \pi\text{-periodic}) &= \int_0^{\pi} \phi_a(l * (\cos\theta, \sin\theta))\phi_b(\sqrt{|m|^2 - (m * (\cos\theta, \sin\theta))^2})\, d\theta.
\end{aligned} \tag{10.17}$$

10.1.2 General 3D case

$$\begin{aligned}
\beta(l,m) &= \frac{1}{4}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \delta(\sigma_1 \cdot \sigma_2)\phi_a(l \cdot \sigma_1)\phi_b(m \cdot \sigma_2)\, d\sigma_1\, d\sigma_2 \\
&= \frac{1}{4}\int_{\mathbb{S}^2} \phi_a(l \cdot \sigma_1)\left[\int_{\mathbb{S}^2} \delta(\sigma_1 \cdot \sigma_2)\phi_b(m \cdot \sigma_2)\, d\sigma_2\right] d\sigma_1 \\
&= \frac{1}{4}\int_{\mathbb{S}^2} \phi_a(l \cdot \sigma_1) I(m, \sigma_1)\, d\sigma_1.
\end{aligned} \tag{10.18}$$

$$
\begin{aligned}
I(m,\sigma_1) &= \int_{\mathbb{S}^2} \delta(\sigma_1\cdot\sigma_2)\phi_b(m\cdot\sigma_2)\,d\sigma_2 \\
(\sigma_2 \text{ in } \sigma_1 \text{ coordinate}) &= \int_0^{2\pi}\int_0^{\pi} \delta(\cos\varphi)\phi_b(m(\sigma_1)*(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi))\sin\varphi\,d\varphi\,d\theta \\
&= \int_0^{2\pi} \phi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2}\cos\theta)\,d\theta \\
(\text{integrand } \theta\ \pi\text{-periodic}) &= 2\int_0^{\pi} \phi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2}\cos\theta)\,d\theta \\
&= 2\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2}),
\end{aligned} \tag{10.19}
$$

where

$$
\psi_b(s) = \int_0^{\pi} \phi_b(s\cos\theta)\,d\theta, \quad \psi_b(-s)=\psi_b(s)=\psi_b(|s|). \tag{10.20}
$$

Therefore,

$$
\begin{aligned}
\beta(l,m) &= \frac{1}{2}\int_{\mathbb{S}^2} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1 \\
&= \frac{1}{2}\left(\int_{\mathbb{S}^{2+}} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1 + \int_{\mathbb{S}^{2-}} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1\right) \\
(\text{second one } \sigma_1\to-\sigma_1) &= \frac{1}{2}\left(\int_{\mathbb{S}^{2+}} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1 + \int_{\mathbb{S}^{2+}} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1\right) \\
&= \int_{\mathbb{S}^{2+}} \phi_a(l\cdot\sigma_1)\psi_b(\sqrt{|m|^2-(m\cdot\sigma_1)^2})\,d\sigma_1 \\
(\sigma_1 \text{ in original coordinate}) &= \int_0^{\pi}\int_0^{\pi} \phi_a(l*(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi)) \\
&\quad\cdot\psi_b(\sqrt{|m|^2-(m*(\sin\varphi\cos\theta,\sin\varphi\sin\theta,\cos\varphi))^2})\sin\varphi\,d\varphi\,d\theta.
\end{aligned} \tag{10.21}
$$