

# fn discrete\_laplacian\_scale\_to\_accuracy

Michael Shoemate

April 25, 2023

This document contains materials associated with `discrete_laplacian_scale_to_accuracy`.

**Definition 0.1.** Let  $z$  be the true value of the statistic and  $X$  be the random variable the noisy release is drawn from. Define  $Y = |X - z|$ , the distribution of DP errors. Then for any statistical significance level `alpha`, denoted  $\alpha \in [0, 1]$ , and `accuracy`, denoted  $a \geq 0$ ,

$$\alpha = P[Y \geq a] \quad (1)$$

**Theorem 0.2.** For any `scale`  $\geq 0$  denoted  $s$ , when  $X \sim \mathcal{L}_{\mathbb{Z}}(z, s)$ ,

$$a = s \cdot \ln(2/(\alpha(e^{1/s} + 1))) + 1 \quad (2)$$

*Proof.* Consider that the distribution of  $(X - z) \sim \mathcal{L}_{\mathbb{Z}}(0, s)$ . Then the PMF of  $Y$  is:

$$\forall y \geq 0 \quad g(y) = (1 + 1[y \neq 0]) \frac{1 - e^{-1/s}}{1 + e^{-1/s}} e^{-y/s} \quad (3)$$

The purpose of the indicator function is to avoid double-counting zero.  
Now derive an expression for  $\alpha$ :

$$\begin{aligned} \alpha &= P[Y \geq a] \\ &= 1 - P[Y < a] \\ &= 1 - \sum_{y=0}^{a-1} g(y) && \text{where } g(y) \text{ is the distribution of } Y \\ &= 1 - \frac{1 - e^{-1/s}}{1 + e^{-1/s}} \left( 1 + 2 \sum_{y=0}^{a-1} e^{-y/s} \right) \\ &= 2 \frac{e^{(1-a)/s}}{e^{1/s} + 1} \end{aligned}$$

Invert to solve for  $a$ :

$$\begin{aligned} 2 \frac{e^{(1-a)/s}}{e^{1/s} + 1} &= \alpha \\ e^{(1-a)/s} &= \alpha(e^{1/s} + 1)/2 \\ a &= 1 - s \cdot \ln(\alpha(e^{1/s} + 1)/2) \\ a &= s \cdot \ln(2/(\alpha(e^{1/s} + 1))) + 1 \end{aligned}$$

□