

fn cdp_epsilon

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February 7, 2024

Proves soundness of `fn cdp_epsilon` in `cdp_epsilon.rs` at commit `0b8f4222` (outdated¹). This proof is an adaptation of [subsection 2.3](#) of [\[CKS20\]](#).

1 Bound Derivation

Definition 1.1. (Privacy Loss Random Variable). Let $M : \mathcal{X}^n \rightarrow \mathbb{Y}$ be a randomized algorithm. Let $x, x' \in \mathcal{X}^n$ be neighboring inputs. Define $f : \mathcal{Y} \rightarrow \mathbb{R}$ by $f(y) = \log \left(\frac{\mathbb{P}[M(x)=y]}{\mathbb{P}[M(x')=y]} \right)$. Let $Z = f(M(x))$, the privacy loss random variable, denoted $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$.

Lemma 1.2. [\[CKS20\]](#) Let $\epsilon, \delta \geq 0$. Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Then M satisfies (ϵ, δ) -differential privacy if and only if

$$\delta \geq \mathbb{E}_{Z \leftarrow \text{PrivLoss}(M(x)||M(x'))} [\max(0, 1 - e^{\epsilon - Z})] \quad (1)$$

(2)

for all $x, x' \in \mathcal{X}^n$ differing on a single element.

Proof. Fix neighboring inputs $x, x' \in \mathcal{X}^n$. Let $f : \mathcal{Y} \rightarrow \mathbb{R}$ be as in [1.1](#). For notational simplicity, let $Y = M(x)$, $Y' = M(x')$, $Z = f(Y)$ and $Z' = -f(Y')$. This is equivalent to $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$. Our first goal is to prove that

$$\sup_{E \subset \mathcal{Y}} \mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E}[\max\{0, 1 - e^{\epsilon - Z}\}]. \quad (3)$$

For any $E \subset \mathcal{Y}$, we have

$$\mathbb{P}[Y' \in E] = \mathbb{E}[\mathbb{I}[Y' \in E]] = \mathbb{E}[\mathbb{I}[Y \in E]e^{-f(Y)}]. \quad (4)$$

This is because $e^{-f(y)} = \frac{\mathbb{P}[Y'=y]}{\mathbb{P}[Y=y]}$.

Thus, for all $E \subset \mathcal{Y}$, we have

$$\mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E} \left[\mathbb{I}[Y \in E] (1 - e^{\epsilon - f(Y)}) \right] \quad (5)$$

Now it is easy to identify the worst event as $E = \{y \in \mathcal{Y} : 1 - e^{\epsilon - f(y)} > 0\}$. Thus

$$\sup_{E \subset \mathcal{Y}} \mathbb{P}[Y \in E] - e^\epsilon \mathbb{P}[Y' \in E] = \mathbb{E} \left[\mathbb{I}[1 - e^{\epsilon - f(Y)} > 0] (1 - e^{\epsilon - f(Y)}) \right] = \mathbb{E}[\max\{0, 1 - e^{\epsilon - Z}\}] \quad (6)$$

□

¹See new changes with `git diff 0b8f4222..2fc0be0 rust/src/combinators/measure_cast/zCDP_to_approxDP/cdp_epsilon.rs`

Theorem 1.3. [CKS20] Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Let $\alpha \in (1, \infty)$ and $\epsilon \geq 0$. Suppose $D_\alpha(M(x)||M(x')) \leq \tau$ for all $x, x' \in \mathcal{X}^n$ differing in a single entry.² Then M is (ϵ, δ) -differentially private for

$$\delta = \frac{e^{(\alpha-1)(\tau-\epsilon)}}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha \quad (7)$$

Proof. Fix neighboring $x, x' \in \mathcal{X}^n$ and let $Z \leftarrow \text{PrivLoss}(M(x)||M(x'))$. We have

$$\mathbb{E}[e^{(\alpha-1)Z}] = e^{(\alpha-1)D_\alpha(M(x)||M(x'))} \leq e^{(\alpha-1)\tau} \quad (8)$$

By 1.2, our goal is to prove that $\delta \geq \mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}]$. Our approach is to pick $c > 0$ such that $\max\{0, 1 - e^{\epsilon-Z}\} \leq ce^{(\alpha-1)z}$ for all $z \in \mathbb{R}$. Then

$$\mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}] \leq \mathbb{E}[ce^{(\alpha-1)z}] \leq ce^{(\alpha-1)\tau}. \quad (9)$$

We identify the smallest possible value of c :

$$c = \sup_{z \in \mathbb{R}} \frac{\max\{0, 1 - e^{\epsilon-z}\}}{e^{(\alpha-1)z}} = \sup_{z \in \mathbb{R}} e^{z-\alpha z} - e^{\epsilon-\alpha z} = \sup_{z \in \mathbb{R}} f(z) \quad (10)$$

where $f(z) = e^{z-\alpha z} - e^{\epsilon-\alpha z}$. We have

$$f'(z) = e^{z-\alpha z}(1-\alpha) - e^{\epsilon-\alpha z}(-\alpha) = e^{-\alpha z}(\alpha e^z - (\alpha-1)e^\epsilon) \quad (11)$$

Clearly $f'(z) = 0 \iff e^z = \frac{\alpha-1}{\alpha}e^\epsilon \iff z = \epsilon - \log(1 - 1/\alpha)$. Thus

$$c = f(\epsilon - \log(1 - 1/\alpha)) \quad (12)$$

$$= \left(\frac{\alpha}{\alpha-1}e^\epsilon\right)^{1-\alpha} - e^\epsilon \left(\frac{\alpha}{\alpha-1}e^\epsilon\right)^{-\alpha} \quad (13)$$

$$= \left(\frac{\alpha}{\alpha-1}e^\epsilon - e^\epsilon\right) \left(\frac{\alpha}{\alpha-1}e^{-\epsilon}\right)^\alpha \quad (14)$$

$$= \frac{e^\epsilon}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha e^{-\alpha\epsilon}. \quad (15)$$

Thus

$$\mathbb{E}[\max\{0, 1 - e^{\epsilon-Z}\}] \leq \frac{e^\epsilon}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha e^{-\alpha\epsilon} e^{(\alpha-1)\tau} = \frac{e^{(\alpha-1)(\tau-\epsilon)}}{\alpha-1} \left(1 - \frac{1}{\alpha}\right)^\alpha = \delta \quad (16)$$

□

Corollary 1. [CKS20] Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm. Let $\alpha \in (1, \infty)$ and $\epsilon \geq 0$. Suppose $D_\alpha(M(x)||M(x')) \leq \tau$ for all $x, x' \in \mathcal{X}^n$ differing in a single entry. Then M is (ϵ, δ) -differentially private for

$$\epsilon = \tau + \frac{\ln(1/\delta) + (\alpha-1)\ln(1-1/\alpha) - \ln(\alpha)}{\alpha-1} \quad (17)$$

Proof. This follows by rearranging 1.3. □

Corollary 2. Let $M : \mathcal{X}^n \rightarrow \mathcal{Y}$ be a randomized algorithm satisfying ρ -concentrated differential privacy. Then M is (ϵ, δ) -differentially private for any $0 < \delta \leq 1$ and

$$\epsilon = \inf_{\alpha \in (1, \infty)} \alpha\rho + \frac{\ln(1/\delta) + (\alpha-1)\ln(1-1/\alpha) - \ln(\alpha)}{\alpha-1} \quad (18)$$

Proof. This follows from 1 by taking the infimum over all divergence parameters α . □

²This is the definition of (α, τ) -Rényi differential privacy.

1.1 Efficient computation of ϵ

From 2, we have

$$\epsilon(\alpha) = \alpha\rho + \frac{\ln(1/\delta) + (\alpha - 1)\ln((\alpha - 1)/\alpha) - \ln(\alpha)}{\alpha - 1} \quad (19)$$

$$\epsilon'(\alpha) = \rho + \frac{\ln(\alpha\delta)}{(\alpha - 1)^2} \quad (20)$$

$$\epsilon''(\alpha) = \frac{2\alpha \ln(\alpha\delta) - \alpha + 1}{(\alpha - 1)^3\alpha} \quad (21)$$

Notice the curve is convex so long as

$$\delta < e^{1/2-1/(2\alpha)}/\alpha \quad (22)$$

Otherwise the curve is concave, with a non-negative derivative for any choice of α :

$$\epsilon'(\alpha) = \rho + \frac{\ln(\alpha\delta)}{(\alpha - 1)^2} \geq \rho + \frac{\alpha - 1}{2\alpha(\alpha + 1)^2} \geq 0 \quad (23)$$

We can find the minimizer α_* by conducting a binary search over the interval $(1, \alpha_{max})$, where α_{max} is discovered via exponential search for a positive derivative.

2 Pseudocode

Precondition

- Type Q must have trait Float.

Implementation

```
1 def cdp_epsilon(rho: Q, delta: Q) -> Q:
2     if rho.is_sign_negative():
3         raise "rho must be non-negative"
4
5     if not delta.is_sign_positive():
6         raise "delta must be positive"
7
8     if rho.is_zero():
9         return 0
10
11     # checks if derivative is positive
12     def deriv_pos(a):
13         return rho > -log(a * delta) / (a - 1)**2
14
15     # find bounds
16     a_min = 1.01
17     a_max = 2
18     while not deriv_pos(a_max):
19         a_max *= 2
20
21     # optimize alpha
22     while True:
23         diff = a_max - a_min
24
25         a_mid = a_min + diff / _2
26
27         if a_mid == a_max or a_mid == a_min:
28             break
29
```

```

30     if deriv_pos(a_mid):
31         a_max = a_mid
32     else:
33         a_min = a_mid
34
35     # back out epsilon
36     a_m1 = a_max.inf_sub(_1)
37
38     numer = (a_m1.inf_div(a_max).inf_ln().inf_mul(a_m1)) \
39             .inf_sub(a_max.inf_ln()) \
40             .inf_add(delta.recip().inf_ln())
41
42     denom = a_max.neg_inf_sub(_1)
43
44     epsilon = a_max.inf_mul(rho).inf_add(numer.inf_div(denom))
45
46     return max(epsilon, 0)

```

Postcondition

Either a valid epsilon is returned or an error is returned.

3 Proof

Theorem 3.1. For any possible setting of ρ and δ , `cdp_epsilon` either returns an error, or an ϵ such that any ρ -differentially private measurement is also (ϵ, δ) -differentially private.

Proof. The code always finds an $\alpha_* \approx \mathbf{a_max} \geq 1.01$. Since $\mathbf{a_max} \in (1, \infty)$, then by 2, any ρ -differentially private measurement is also $(\epsilon(\mathbf{a_max}), \delta)$ -differentially private. Define $\epsilon_{cons}(\alpha)$ as a “conservative” function for computing $\epsilon(\alpha)$, where floating-point arithmetic is computed with conservative rounding such that $\epsilon_{cons}(\alpha) \geq \epsilon(\alpha)$ for $\forall \alpha \in (1, \infty)$. Since $\mathbf{epsilon} = \epsilon_{cons}(\mathbf{a_max}) \geq \epsilon(\mathbf{a_max})$, then any $(\epsilon(\mathbf{a_max}), \delta)$ -differentially private measurement is also $(\mathbf{epsilon}, \delta)$ -differentially private. \square

References

[CKS20] Clément L. Canonne, Gautam Kamath, and Thomas Steinke. The discrete gaussian for differential privacy. *CoRR*, abs/2004.00010, 2020.