

Strong computable reducibility

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Problems.

A **problem** is a Π_2^1 statement of second-order arithmetic, thought of as

for every $X \in \text{Inst}(P)$, there is a $Y \in \text{Soln}(P, X)$,

where $\text{Inst}(P)$ and $\text{Soln}(P, X)$ are arithmetically-definable sets.

Examples.

RT_k^n . Every coloring $c : [\omega]^n \rightarrow k$ has an infinite homogeneous set.

COH. For every family $\vec{c} = \langle c_0, c_1, \dots \rangle$ of colorings $c_i : \omega \rightarrow 2$ there is an infinite set H that is almost homogeneous for each c_i , i.e., if for each i there is a finite set F such that $H - F$ is homogeneous for c_i .

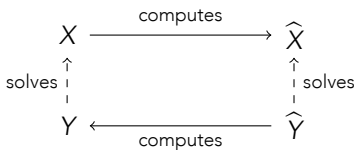
Reductions.

Let P and Q be problems.

P is **strongly computably reducible** to Q , written $P \leq_{sc} Q$, if

every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that

every $\hat{Y} \in \text{Soln}(Q, \hat{X})$ computes a $Y \in \text{Soln}(P, X)$.



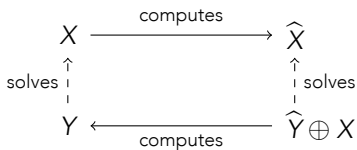
Reductions.

Let P and Q be problems.

P is **computably reducible** to Q , written $P \leq_c Q$, if

every $X \in \text{Inst}(P)$ computes an $\hat{X} \in \text{Inst}(Q)$, such that

every $\hat{Y} \in \text{Soln}(Q, \hat{X})$, **together with X** , computes a $Y \in \text{Soln}(P, X)$.



As a finer metric.

Most implications between problems are formalizations of (strong) computable or (strong) Weihrauch reductions.

Theorem (Cholak, Jockusch, and Slaman). $\text{RCA}_0 \vdash \text{RT}_2^2 \rightarrow \text{COH}$.

The proof is a formalization in RCA_0 that $\text{COH} \leq_{sW} \text{RT}_2^2$.

We can tease apart subtle differences that RCA_0 alone does not see.

For all j and k , we have $\text{RCA}_0 \vdash \text{RT}_j^n \leftrightarrow \text{RT}_k^n$.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer). If $j > k$, then $\text{RT}_j^n \not\leq_{sW} \text{RT}_k^n$.

Theorem (Hirschfeldt and Jockusch). If $j > k$, then $\text{RT}_j^n \not\leq_W \text{RT}_k^n$.

Theorem (Patey). If $j > k$, then $\text{RT}_j^n \not\leq_c \text{RT}_k^n$.

Two versions of Ramsey's theorem.

A coloring $c : [\omega]^2 \rightarrow 2$ is **stable** if $\lim_y c(x, y)$ exists for all x .

SRT₂². Every stable coloring has an infinite homogeneous set.

Theorem (Cholak, Jockusch, and Slaman). $\text{RT}_2^2 \equiv_{sW} \text{SRT}_2^2 \bullet \text{COH}$.

A set L is **limit-homogeneous** for a stable coloring c if there is an $i \in \{0, 1\}$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

D₂². Every stable coloring has an infinite limit-homogeneous set.

Observation. $\text{SRT}_2^2 \equiv_c D_2^2$.

Pf. Thin out a limit-homogeneous set to a homogeneous one.

Theorem (Chong, Lempp, and Yang). $\text{RCA}_0 \vdash \text{SRT}_2^2 \leftrightarrow D_2^2$.

Two versions of Ramsey's theorem.

Theorem (Hirschfeldt and Jockusch). $\text{SRT}_2^2 \leq_W D_2^2 \bullet D_2^2$.

Question (Hirschfeldt and Jockusch). Does $\text{SRT}_2^2 \leq_W D_2^2$? Does $\text{SRT}_2^2 \leq_{sc} D_2^2$?

If L is limit-homogeneous, but we do not know what color $i \in \{0, 1\}$ the elements in it limit to, then thinning it to a homogeneous set seems difficult.

Theorem (Dzhafarov). $\text{SRT}_2^2 \not\leq_W D_2^2$.

Theorem (Dzhafarov). There is a stable coloring c such that every other stable coloring d has an infinite limit-homogeneous set L that computes no infinite homogeneous set for c .

Corollary. $\text{SRT}_2^2 \not\leq_{sc} D_2^2$.

COH and D_2^2 .

Open question (Chong, Slaman, and Yang). Does SRT_2^2 (or D_2^2) imply COH in ω -models of RCA_0 ? Is $COH \leq_c SRT_2^2$? Equivalently, is $COH \leq_c D_2^2$?

Theorem (Dzhafarov, 2012). $COH \not\leq_{sc} D_2^2$.

The proof is a computable forcing argument. Any 3-generic yields a family $\langle X_0, X_1, \dots \rangle$ witnessing the theorem, so we can find one computable in $\emptyset^{(3)}$.

Theorem (Hirschfeldt and Jockusch; Patey). There is a family of sets $X = \langle X_0, X_1, \dots \rangle$ such that every stable coloring d has an infinite limit-homogeneous set L that computes no infinite X -cohesive set.

The X built by Hirschfeldt and Jockusch is non-hyperarithmetical. Patey's is Δ_2^0 .

Question. Given the differences between SRT_2^2 and D_2^2 under \leq_W and \leq_{sc} , what relationships hold between COH and SRT_2^2 ?

COH and SRT_2^2 .

It is possible to elaborate on the proof that $\text{COH} \not\leq_W \text{D}_2^2$ to obtain:

Theorem (Dzhafarov). $\text{COH} \not\leq_W \text{SRT}_2^2$ (via a computable instance).

Homogeneous sets, unlike limit-homogeneous ones, have internal structure.

E.g., suppose we are building a family of colorings \vec{c} and $\Phi^{\vec{c}}$ is to be stable.

To build a limit-homogeneous set L for $\Phi^{\vec{c}}$, we can build a finite portion F of L , and only later extend \vec{c} , say in a way to diagonalize some computation from F .

By Seetapun's argument, F can be chosen so that its elements' limits agree.

But to build a homogeneous set H for $\Phi^{\vec{c}}$, we cannot delay building \vec{c} in this way because homogeneity of any finite set directly depends on it.

Tree labeling method.

We define a certain subtree of $\omega^{<\omega}$ with labels on its nodes corresponding to diagonalization opportunities.

Paths give trivial wins (e.g., solutions that don't compute infinite sets).

If the tree is well-founded, we can use the labels to guide the construction of a homogeneous set.

Theorem (Dzhafarov). $\text{COH} \not\leq_{sc} \text{SRT}_2^2$.

The tree labeling method is quite powerful for separating principles under \leq_{sc} .

Theorem (Dzhafarov, Patey, Solomon, Westrick). If $j > k$ then $\text{RT}_j^1 \not\leq_{sc} \text{SRT}_k^2$.

Theorem (Nichols). $\text{SRT}_2^2 \not\leq_{sc} \text{SPT}_2^2$.

Hyperarithmetical instances.

The tree labeling method involves iteratively taking paths through subtrees of $\omega^{<\omega}$ so the instances it produces are non-hyperarithmetical.

Open question. Can the tree labeling method be made more effective?

Recall that a set X has a **self-modulus** if there is a function $f \equiv_T X$ such that $X \leq_T g$ from every function $g > f$. By a result of Solovay, X is hyperarithmetical.

Observation. If $\text{COH} \not\leq_{sc} \text{SRT}_2^2$ via an instance $\vec{c} = \langle c_0, c_1, \dots \rangle$ that has a self-modulus, then $\text{COH} \not\leq_c \text{SRT}_2^2$.

Theorem (Dzhafarov, Patey, Solomon, Westrick). $\text{COH} \not\leq_{sc} \text{SRT}_2^2$ via an instance \vec{c} computable in $\emptyset^{(\omega)}$ (and so at least hyperarithmetical).

Open question. Can the instance \vec{c} be chosen Δ_2^0 ?



Thank you.