Ramsey's theorem and cone avoidance

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Background

Definition

Let $X \subseteq \omega$ be an infinite set and $n, k \in \omega$.

1. $[X]^n := \{Y \subset X : |Y| = n\}$.
2. A *k-coloring on X of exponent n* is a function $f : [X]^n \to k = \{0, \ldots, k-1\}$.
3. A set $H \subseteq X$ is *homogeneous for f* if $f \upharpoonright [H]^n$ is constant.

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Ramsey's theorem

For every $n, k \geq 1$, every $f : [\omega]^n \to k$ admits an infinite homogeneous set.

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There exists a computable 2-coloring of $[\omega]^2$ *admitting no infinite computable homogeneous set.*

Theorem (Jockusch, 1972)

There exists a computable 2-coloring of $[\omega]^2$ *admitting no infinite homogeneous set computable in* $0'$.

Cone non-avoidance

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For every $n \geq 3$, there exists a computable 2-coloring of $[\omega]^n$ all of whose infinite homogeneous sets compute $0^{(n-2)}$.

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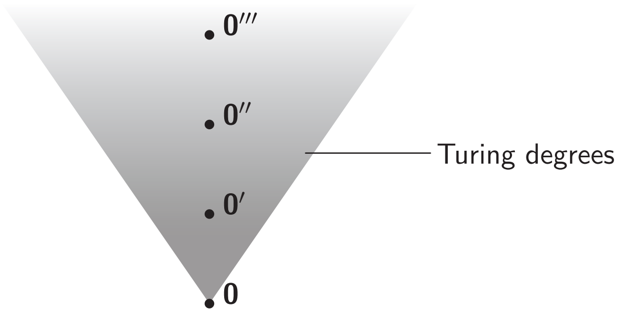

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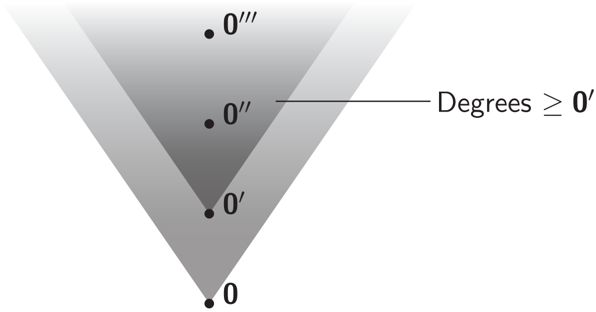

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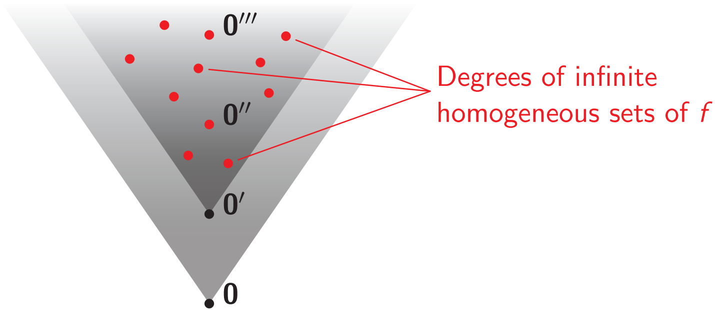
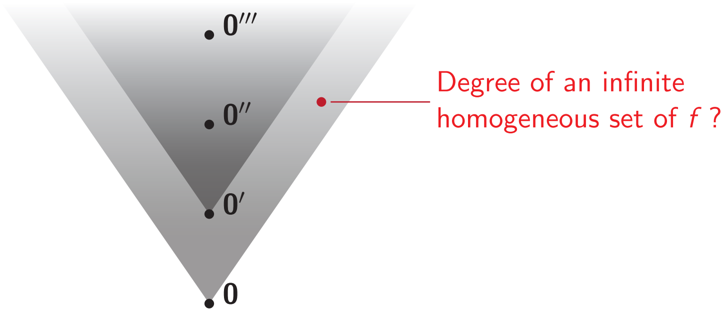


Cone avoidance

Question (Jockusch, 1972)

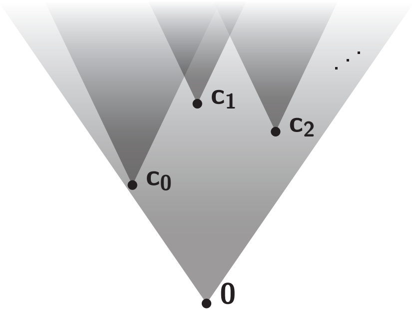
Does every computable 2-coloring of $[\omega]^2$ admit an infinite homogeneous set which does not compute $0'$?



Cone avoidance

Theorem (Seetapun, 1995)

Given $C_0, C_1, \ldots >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $C_i \not\leq_T H$ for all i.



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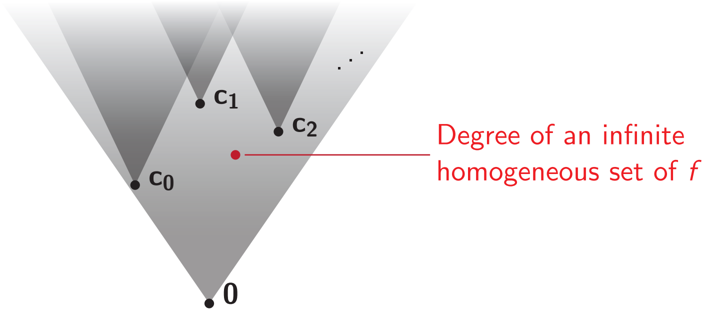

Lowness over $0^{(n)}$

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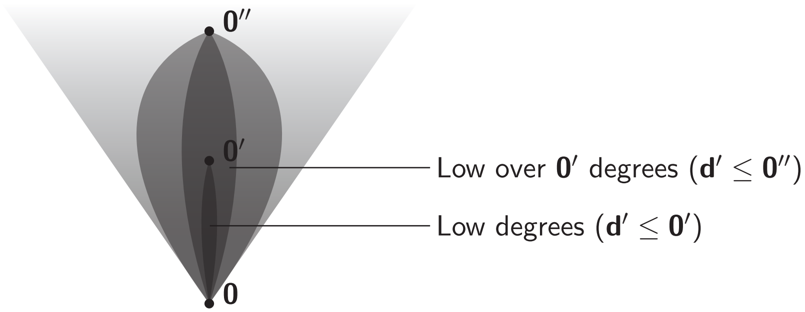
For every $n, k \geq 2$, every computable $f : [\omega]^n \to k$ has an infinite homogeneous set H with $H' \leq_T 0^{(n)}$.



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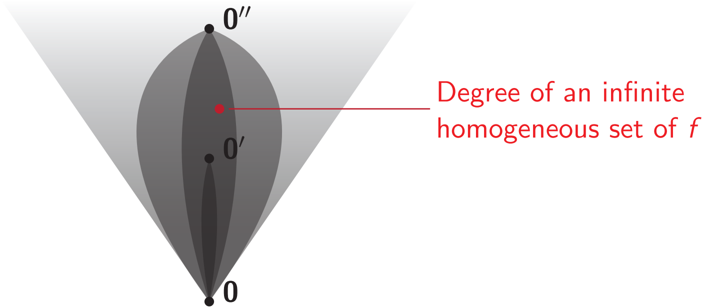
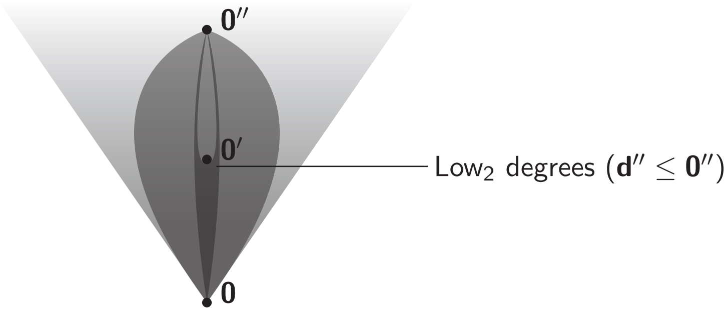


Low$_2$ness

Question (Jockusch, 1972)

Does every computable 2-coloring of $[\omega]^2$ admit an infinite homogeneous set H with $H'' \leq_T 0''$ (i.e., which is low$_2$)?



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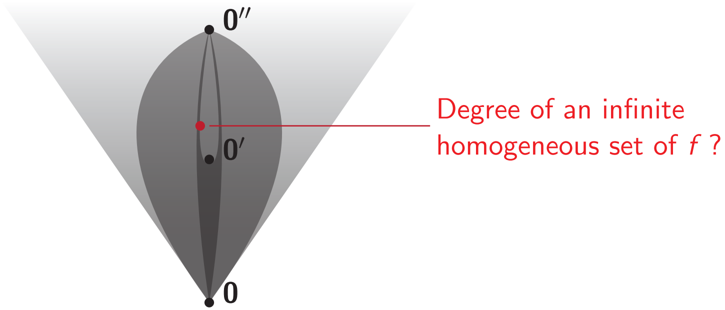

Low$_2$ness

Recall...

Definition

A degree $\mathbf{d}$ is *PA over* $\mathbf{0}'$, written $\mathbf{d} \gg \mathbf{0}'$, if every infinite $0'$-computable tree in $2^{<\omega}$ has an infinite path of degree $\leq \mathbf{d}$.

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- There exists an infinite $0'$-computable tree in $2^{<\omega}$ each of whose infinite paths has degree $\gg \mathbf{0}'$.
- By the Low Basis Theorem relativized to $\mathbf{0}'$, there exists a degree $\mathbf{d} \gg \mathbf{0}'$ which is low over $\mathbf{0}'$ (i.e., $\mathbf{d}' \leq \mathbf{0}''$).

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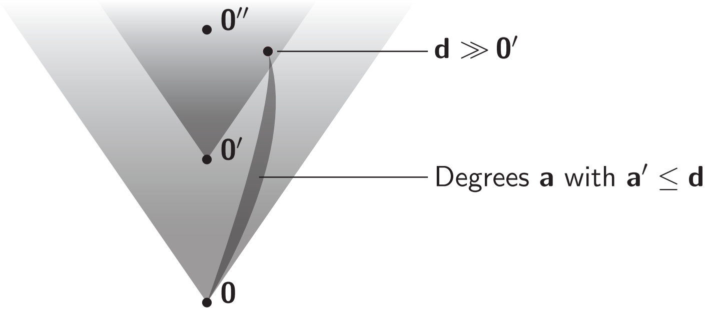
- By the Low Basis Theorem relativized to $\mathbf{0}'$, there exists a degree $\mathbf{d} \gg \mathbf{0}'$ which is low over $\mathbf{0}'$ (i.e., $\mathbf{d}' \leq \mathbf{0}''$).
- If $\mathbf{a}$ is low over $\mathbf{0}'$ and $\mathbf{b}' \leq \mathbf{a}$ then $\mathbf{b}'' \leq \mathbf{a}' \leq \mathbf{0}''$, so $\mathbf{b}$ is low$_2$.

Low$_2$ness

Theorem (Cholak, Jockusch, and Slaman, 2001)

Given $\mathbf{d} \gg \mathbf{0}'$, *every computable* $f : [\omega]^2 \to 2$ *admits an infinite homogeneous set* H *with* $\deg(H)' \leq \mathbf{d}$.



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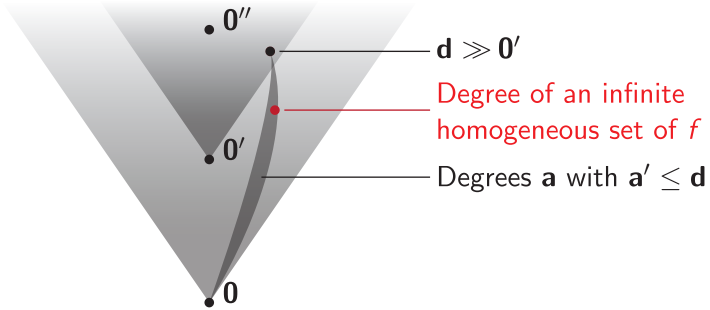
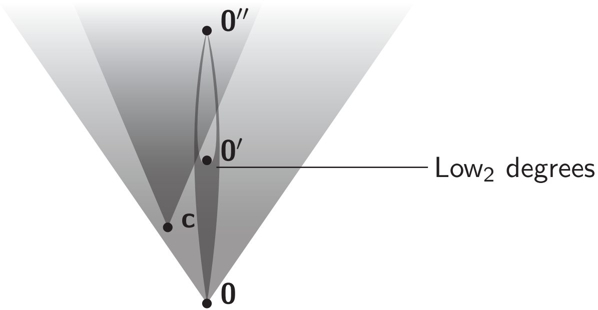


Cone avoidance and low$_2$ness

Question (Cholak, Jockusch, and Slaman, 2001)

Given $C >_T 0$, does every computable $f : [\omega]^2 \to 2$ admit an infinite low$_2$ homogeneous set H with $C \not\leq_T H$?



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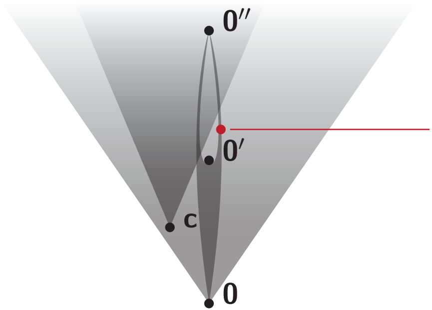
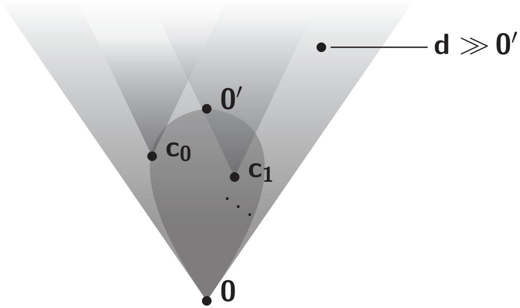


Cone avoidance and low$_2$ness

Theorem (Dzhafarov and Jockusch)

Given $C_0, C_1, \ldots >_T 0$ with $\oplus_i C_i \leq_T 0'$ and $\mathbf{d} \gg \mathbf{0}'$, every computable $f : [\omega]^2 \to 2$ admits an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ and $C_i \not\leq_T H$ for all i.



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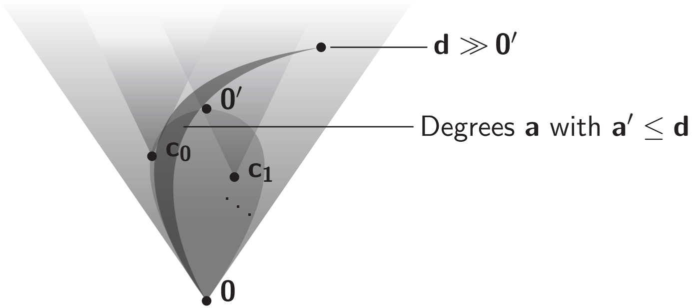

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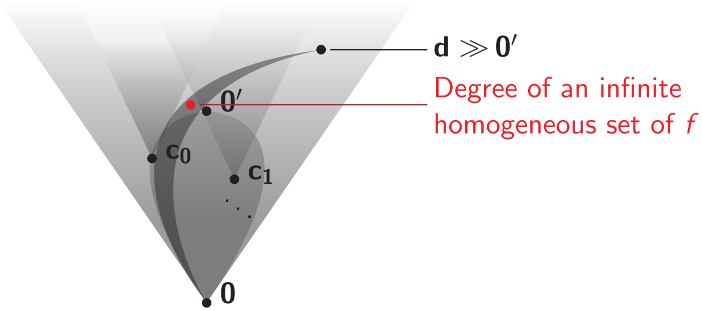

Cone avoidance and low$_2$ness

Corollary (Dzhafarov and Jockusch)

Given $0 <_T C \leq_T 0'$, every computable $f : [\omega]^2 \to 2$ admits an infinite low$_2$ homogeneous set H with $C \not\leq_T H$.

Cone avoidance and low_2ness

Corollary (Dzhafarov and Jockusch)

Given $0 <_T C \leq_T 0'$, every computable $f : [\omega]^2 \to 2$ admits an infinite low_2 homogeneous set H with $C \not\leq_T H$.

- The case $C \not\leq_T 0'$ is handled by analyzing the subcases $C \leq_T 0''$ and $C \not\leq_T 0''$.

Theorem (Dzhafarov and Jockusch)

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Cone avoidance and low_2ness

- A more careful case analysis yields the following extension:

Theorem (Dzhafarov and Jockusch)

Given $C_0, \ldots, C_n >_T 0$, every computable $f : [\omega]^2 \to 2$ admits an infinite low_2 homogeneous set H with $C_i \not\leq_T H$ for each $i \leq n$.

Low$_2$ness and minimal pairs

Recall...

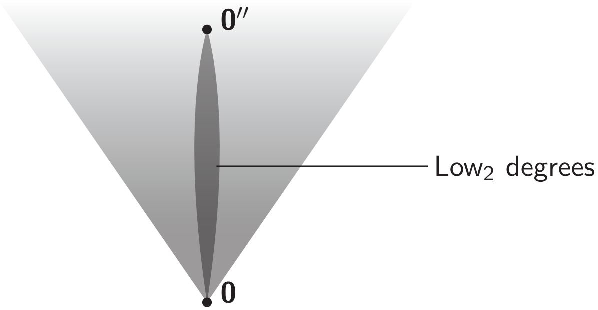
Definition

Degrees $\mathbf{a}$ and $\mathbf{b}$ form *a minimal pair* if $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.

Low$_2$ness and minimal pairs

Theorem (Dzhafarov and Jockusch)

Every computable $f : [\omega]^2 \to 2$ admits infinite low$_2$ homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair.



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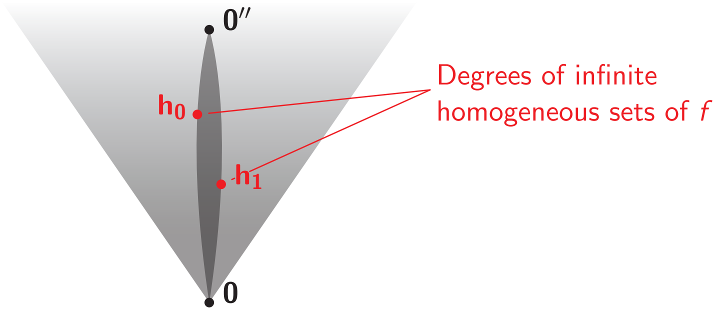

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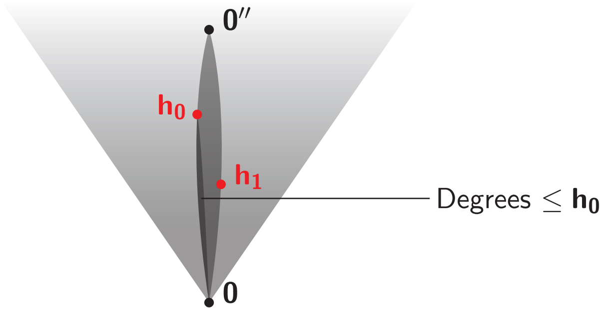

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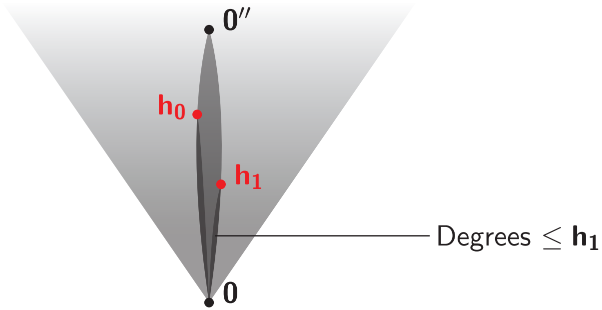

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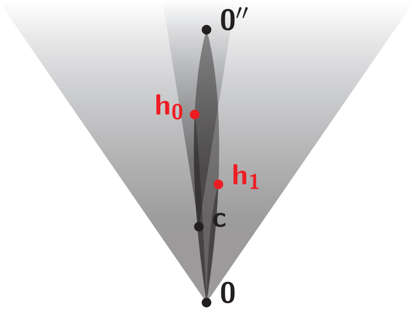

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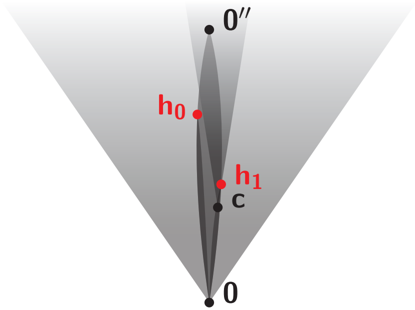

Questions

Open question

Given any noncomputable set C and any degree $\mathbf{d} \gg \mathbf{0}'$, does every computable $f : [\omega]^2 \to 2$ admit an infinite homogeneous set H with $\deg(H)' \leq \mathbf{d}$ *and $C \not\leq_T H$?*

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Open question (Simpson)

Does every computable $f : [\omega]^2 \to 2$ admit infinite homogeneous sets H_0, H_1 such that $\deg(H_0)$ and $\deg(H_1)$ form a minimal pair and $H_0 \oplus H_1$ is low$_2$?

Thank you for your attention.