Strong computable reducibility

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Problems.

A problem is a Π^1_2 statement of second-order arithmetic, thought of as

$$\text{for every } X \in \mathit{Inst}(P), \text{ there is a } Y \in \mathit{Soln}(P, X),$$

where $\mathit{Inst}(P)$ and $\mathit{Soln}(P, X)$ are arithmetically-definable sets.

Examples.

RT^n_k. Every coloring $c : [\omega]^n \to k$ has an infinite homogeneous set.

COH. For every family $\vec{c} = \langle c_0, c_1, \ldots \rangle$ of colorings $c_i : \omega \to 2$ there is an infinite set H that is almost homogeneous for each c_i, i.e., if for each i there is a finite set F such that $H - F$ is homogeneous for c_i.

Reductions.

Let P and Q be problems.

P is strongly computably reducible to Q, written $P \leq_{sc} Q$, if every $X \in Inst(P)$ computes an $\widehat{X} \in Inst(Q)$, such that every $\widehat{Y} \in Soln(Q, \widehat{X})$ computes a $Y \in Soln(P, X)$.

$$
\begin{tikzcd}
X \arrow[r, "\text{computes}"] & \widehat{X} \\
Y \arrow[u, dashed, "\text{solves}"] & \widehat{Y} \arrow[l, "\text{computes}"] \arrow[u, dashed, "\text{solves}"']
\end{tikzcd}
$$

Reductions.

Let P and Q be problems.

P is computably reducible to Q, written $P \leq_c Q$, if

every $X \in \mathit{Inst}(P)$ computes an $\widehat{X} \in \mathit{Inst}(Q)$, such that

every $\widehat{Y} \in \mathit{Soln}(Q, \widehat{X})$, together with X, computes a $Y \in \mathit{Soln}(P, X)$.

$$
\begin{tikzcd}
X \arrow[r, "\text{computes}"] & \widehat{X} \\
Y \arrow[u, dashed, "\text{solves}"] & \widehat{Y} \oplus X \arrow[u, dashed, "\text{solves}"'] \arrow[l, "\text{computes}"]
\end{tikzcd}
$$

As a finer metric.

Most implications between problems are formalizations of (strong) computable or (strong) Weihrauch reductions.

Theorem (Cholak, Jockusch, and Slaman). $\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \to \mathsf{COH}$.

The proof is a formalization in RCA_0 that $\mathsf{COH} \leq_{sW} \mathsf{RT}^2_2$.

We can tease apart subtle differences that RCA_0 alone does not see.

For all j and k, we have $\mathsf{RCA}_0 \vdash \mathsf{RT}^n_j \leftrightarrow \mathsf{RT}^n_k$.

Theorem (Dorais, Dzhafarov, Hirst, Mileti, Shafer). If $j > k$, then $\mathsf{RT}^n_j \not\leq_{sW} \mathsf{RT}^n_k$.

Theorem (Hirschfeldt and Jockusch). If $j > k$, then $\mathsf{RT}^n_j \not\leq_{W} \mathsf{RT}^n_k$.

Theorem (Patey). If $j > k$, then $\mathsf{RT}^n_j \not\leq_{c} \mathsf{RT}^n_k$.

Two versions of Ramsey's theorem.

A coloring $c : [\omega]^2 \to 2$ is stable if $\lim_y c(x, y)$ exists for all x.

SRT^2_2. Every stable coloring has an infinite homogeneous set.

Theorem (Cholak, Jockusch, and Slaman). $\mathsf{RT}^2_2 \equiv_{sW} \mathsf{SRT}^2_2 \bullet \mathsf{COH}$.

A set L is limit-homogeneous for a stable coloring c if there is an $i \in \{0, 1\}$ such that $\lim_y c(x, y) = i$ for all $x \in L$.

D^2_2. Every stable coloring has an infinite limit-homogeneous set.

Observation. $\mathsf{SRT}^2_2 \equiv_c D^2_2$.

Pf. Thin out a limit-homogeneous set to a homogeneous one.

Theorem (Chong, Lempp, and Yang). $\mathsf{RCA}_0 \vdash \mathsf{SRT}^2_2 \leftrightarrow \mathsf{D}^2_2$.

Two versions of Ramsey's theorem.

Theorem (Hirschfeldt and Jockusch). $\mathsf{SRT}^2_2 \leq_W \mathsf{D}^2_2 \bullet \mathsf{D}^2_2$.

Question (Hirschfeldt and Jockusch). Does $\mathsf{SRT}^2_2 \leq_W \mathsf{D}^2_2$? Does $\mathsf{SRT}^2_2 \leq_{sc} \mathsf{D}^2_2$?

If L is limit-homogeneous, but we do not know what color $i \in \{0, 1\}$ the elements in it limit to, then thinning it to a homogeneous set seems difficult.

Theorem (Dzhafarov). $\mathsf{SRT}^2_2 \not\leq_W \mathsf{D}^2_2$.

Theorem (Dzhafarov). There is a stable coloring c such that every other stable coloring d has an infinite limit-homogeneous set L that computes no infinite homogeneous set for c.

Corollary. $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_2$.

COH and D^2_2.

Open question (Chong, Slaman, and Yang). Does SRT^2_2 (or D^2_2) imply COH in ω-models of RCA_0? Is COH $\leq_c$ SRT^2_2? Equivalently, is COH $\leq_c$ D^2_2?

Theorem (Dzhafarov, 2012). COH $\not\leq_{sc}$ D^2_2.

The proof is a computable forcing argument. Any 3-generic yields a family $\langle X_0, X_1, \ldots \rangle$ witnessing the theorem, so we can find one computable in $\emptyset^{(3)}$.

Theorem (Hirschfeldt and Jockusch; Patey). There is a family of sets $X = \langle X_0, X_1, \ldots \rangle$ such that every stable coloring d has an infinite limit-homogeneous set L that computes no infinite X-cohesive set.

The X built by Hirschfeld and Jockusch is non-hyperarithmetical. Patey's is Δ^0_2.

Question. Given the differences between SRT^2_2 and D^2_2 under $\leq_W$ and $\leq_{sc}$, what relationships hold between COH and SRT^2_2?

COH and SRT^2_2.

It is possible to elaborate on the proof that COH $\nleq_W$ D^2_2 to obtain:

Theorem (Dzhafarov). COH $\nleq_W$ SRT^2_2 (via a computable instance).

Homogeneous sets, unlike limit-homogeneous ones, have internal structure.

E.g., suppose we are building a family of colorings $\vec{c}$ and $\Phi^{\vec{c}}$ is to be stable.

To build a limit-homogeneous set L for $\Phi^{\vec{c}}$, we can build a finite portion F of L, and only later extend $\vec{c}$, say in a way to diagonalize some computation from F.

By Seetapun's argument, F can be chosen so that its elements' limits agree.

But to build a homogeneous set H for $\Phi^{\vec{c}}$, we cannot delay building $\vec{c}$ in this way because homogeneity of any finite set directly depends on it.

Tree labeling method.

We define a certain subtree of $\omega^{<\omega}$ with labels on its nodes corresponding to diagonalization opportunities.

Paths give trivial wins (e.g., solutions that don't compute infinite sets).

If the tree is well-founded, we can use the labels to guide the construction of a homogeneous set.

Theorem (Dzhafarov). $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$.

The tree labeling method is quite powerful for separating principles under $\leq_{sc}$.

Theorem (Dzhafarov, Patey, Solomon, Westrick). If $j > k$ then $\mathsf{RT}^1_j \not\leq_{sc} \mathsf{SRT}^2_k$.

Theorem (Nichols). $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{SPT}^2_2$.

Hyperarithmetic instances.

The tree labeling method involves iteratively taking paths through subtrees of $\omega^{<\omega}$ so the instances it produces are non-hyperarithmetical.

Open question. Can the tree labeling method be made more effective?

Recall that a set X has a self-modulus if there is a function $f \equiv_T X$ such that $X \leq_T g$ from every function $g > f$. By a result of Solovay, X is hyperarithmetical.

Observation. If COH $\not\leq_{sc}$ SRT^2_2 via an instance $\vec{c} = \langle c_0, c_1, \ldots \rangle$ that has a self-modulus, then COH $\not\leq_c$ SRT^2_2.

Theorem (Dzhafarov, Patey, Solomon, Westrick). COH $\not\leq_{sc}$ SRT^2_2 via an instance $\vec{c}$ computable in $\emptyset^{(\omega)}$ (and so at least hyperarithmetic).

Open question. Can the instance $\vec{c}$ be chosen Δ^0_2?

Thank you.