Quadruple Tank Process

Building the Model

Physical models are based on conservation principles:

1. Mass
2. Energy
3. Momentum

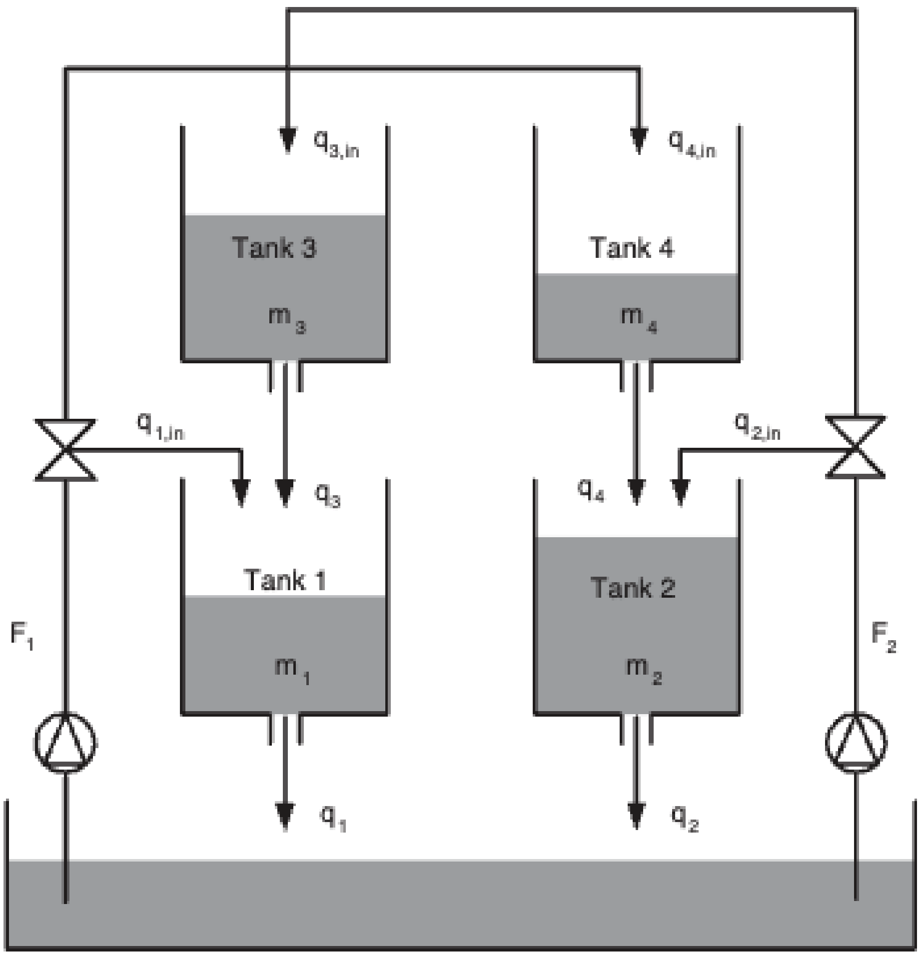
The general derivation of the system of equations have the form

$$\text{Accumulated} = \text{influx} \; - \; \text{Outflux} \; + (\underset{\text{(Generated)}}{\text{Produced}} \; - \; \text{Consumed})$$

For non-reactive systems, the generation term is absent

$$\text{Accumulated} = \text{Influx} - \text{Outflux}$$

The 4-Tank System



Tank 1

$$\text{Accumulated} = \text{Influx} - \text{Outflux}$$

with,

$$\begin{aligned}\text{Accumulated} &= m_1(t+\Delta t) - m_1(t)\\ \text{Influx} &= \rho q_{1,in}(t)\Delta t + \rho q_3(t)\Delta t\\ \text{Outflux} &= \rho q_1(t)\Delta t\end{aligned}$$

Therefore,

$$\begin{aligned} m_1(t+\Delta t) - m_1(t) &= \rho q_{1,in}(t)\Delta t + \rho q_3(t)\Delta t - \rho q_1(t)\Delta t\\ &= \rho \Delta t\left(q_{1,in}(t) + q_3(t) - q_1(t)\right)\\ \implies \frac{m_1(t+\Delta t) - m_1(t)}{\Delta t} &= \rho\left(q_{1,in}(t) + q_3(t) - q_1(t)\right)\end{aligned}$$

Let $\Delta t \to 0$

$$\frac{dm_1(t)}{dt} = \rho\left(q_{1,in}(t) + q_3(t) - q_1(t)\right)$$

Overview of process:

1. Conservation of mass.
2. Division by Δt
3. Taking the limit ($\Delta t \to 0$, i.e., differentiation from first principles).

Entire System

Mass Balances:

$$\begin{aligned}\frac{dm_1(t)}{dt} &= \rho\left(q_{1,in}(t) + q_3(t) - q_1(t)\right), \quad m_1(t_0) = m_{1,0}\\ \frac{dm_2(t)}{dt} &= \rho\left(q_{2,in}(t) + q_4(t) - \rho q_2(t)\right), \quad m_2(t_0) = m_{2,0}\\ \frac{dm_3(t)}{dt} &= \rho\left(q_{3,in}(t) - q_3(t)\right), \quad m_3(t_0) = m_{3,0}\\ \frac{dm_4(t)}{dt} &= \rho\left(q_{4,in}(t) - q_4(t)\right), \quad m_4(t_0) = m_{4,0}\end{aligned}$$

Inflows:

$$\begin{aligned}&q_{1,in}(t) = \gamma_1 F_1(t), \quad q_{2,in}(t) = \gamma_2 F_2(t)\\ &q_{1,in}(t) = (1-\gamma_2)F_2(t), \quad q_{2,in}(t) = (1-\gamma_1)F_1(t)\end{aligned}$$

Outflows:

$$q_i(t) = a_i\sqrt{2gh_i(t)}, \quad h_i(t) = \frac{m_i(t)}{\rho A_i}, \quad i \in \{1,2,3,4\}$$

Where the variables are:

- A_i : Cross-sectional area of tank i

- h_i : Water height of tank i
- a_i : Cross-sectional area of outlet of tank i

Converting to a System of ODEs

$$\dot{x}(t) = f(x(t), u(t), p), \quad x(t_0) = x_0$$

with vectors defined as,

$$x = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, \quad u = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$p = [a_1 \; a_2 \; a_3 \; a_4 \; A_1 \; A_2 \; A_4 \; \gamma_1 \; \gamma_2 \; g \; \rho]^\top$$

The Generic Input-Output Model

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t), u(t), p), \quad x(t_0) = x_0 \quad \text{(Process Model)} \\ y(t) &= f(x(t), p), \quad \text{(Sensor function)} \\ z(t) &= h(x(t), p), \quad \text{(Output function)} \end{aligned}$$

Sensor Function

These are sensors measuring the level/height of all the tanks.

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix} = \begin{bmatrix} \frac{m_1}{\rho A_1} \\ \frac{m_2}{\rho A_2} \\ \frac{m_3}{\rho A_3} \\ \frac{m_4}{\rho A_4} \end{bmatrix} = g\left(x, p\right)$$

Output Function

In this case, we will consider the output as the level in tank 1 and tank 2.

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{m_1}{\rho A_1} \\ \frac{m_2}{\rho A_2} \end{bmatrix} = h(x, p)$$