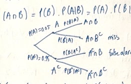
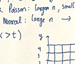
Probability MITx 6431x — Đàm Minh Tiến

+ Consequences of the axioms: $P(A\cup B\cup C) = P(A) + P(A^c\cap B) + P(A^c\cap B^c\cap C)$
+ The multiplication rule: $P(A|B) = \frac{P(A\cap B)}{P(B)}$; $P(A\cap B) = P(B).P(A|B) = P(A).P(B|A)$
+ Total prob theorem: $P(B) = \sum P(A_i).P(B|A_i)$
+ $P(A_i|B) = \frac{P(A_i\cap B)}{P(B)} = \frac{P(A_i).P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$



+ Independence: $P(A\cap B) = P(A).P(B)$; $P(X\le x, Y\le y) = P(X\le x).P(Y\le y)$.
  $f_{X,Y}(x,y) = f_X(x).f_Y(y)$ $\Rightarrow E(XY) = E[X].E(Y)$; $V(X+Y) = V(X)+V(Y)$
  $P(A|B) = P(A)$ if A and B are independent; $Cov(X,Y) = 0 \to$ uncorrelated, not necessarily dependence
+ Conditional Independence: $P(A\cap B|C) = P(A|C).P(B|C) \to$ A, B are conditionally indep given C
+ Pairwise Independence does not imply mutual independence.
+ Disjoint events with non-zero prob are not independence; zero covariance does not imply independence
+ Discrete / Continuous r.v. $E[X] = \sum_x x p_X(x)$; $E[g(X)] = \sum g(x).p_X(x)$; $V(X) = E[(X-\mu)^2] = E[X^2] - E[X]^2$
  $= \int_{-\infty}^{\infty} x.f_X(x)dx$; $V(aX+b) = a^2 V(X)$; $= \int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$
+ Moments: n-th moment: $E[X^n] = \begin{cases}\sum_x x^n p_X(x) & \text{(discrete)}\\ \int_{-\infty}^{\infty} x^n.f_X(x)dx & \text{(continuous)}\end{cases}$; central moment: $\mu_n' = E[(X-E(X))^n]$
  moment generating function: $M_X(t) = E[e^{tX}]$; $E[X^n] = M_X^{(n)}(0)$
+ Common discrete r.v: → binomial distribution → approx. Poisson: large n, small p → $\lambda = np$
  ↳ approx Normal: large n → $\mathcal{N}(np, npq)$
+ Memorylessness property: $P(X > s+t | X > s) = P(X > t)$
+ Joint PMF: $p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$



  Conditional: $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
  Independence: X, Y are independence if $p_{X,Y}(x,y) = p_X(x).p_Y(y)$
+ Prob mass function PMF: $p_X(x) = P(X = x)$; $p_X(x) \ge 0$; $\sum_x p_X(x) = 1$ } discrete r.v
  $P(a\le X\le b) = \sum_{a\le x\le b} p_X(x)$
+ Prob density function PDF: $P(a\le X\le b) = \int_a^b f_X(x)dx$; $f_X(x)\ge 0$; $\int_{-\infty}^{\infty} f_X(x)dx = 1$ } continuous r.v.
  $P(a\le X\le a+\delta) \approx f_X(a).\delta$; $P(X=a) = 0$
+ Cumulative distribution function CDF: $F_X(x) = P(X\le x) = \int_{-\infty}^{x} f_X(t)dt$ ⟨ non-decreasing; $\lim_{x\to-\infty}F_X(x) = 0$; $\lim_{x\to+\infty}F_X(x) = 1$; $P(a < X\le b) = F_X(b) - F_X(a)$
+ linear functions of normal r.v. Let $Y = aX+b$, $X\sim\mathcal{N}(\mu,\sigma^2) \to Y\sim\mathcal{N}(a\mu+b, a^2\sigma^2)$
+ Standardizing a r.v. $X\sim\mathcal{N}(\mu,\sigma^2)$; let $Y = \frac{X-\mu}{\sigma} \to Y\sim\mathcal{N}(0,1)$; $X\sim\mathcal{N}(6,4)$; $P(2\le X\le 8) = P(\frac{2-6}{2}\le\frac{X-6}{2}\le\frac{8-6}{2}) = P(-2\le Y\le 1)$
+ Conditional PDF of X given that $X\in A$: $f_{X|X\in A}(x) = \begin{cases} 0 & \text{, if } x\notin A\\ \frac{f_X(x)}{P(A)} & \text{, if } x\in A\end{cases}$

+ Conditional expectation of X, given that $x\in A$
  $E[X|A] = \sum_x x\, p_{X|A}(x)$; $E[X|A] = \int x.f_{X|A}(x)dx$
+ Memorylessness of exponential PDF: $P(T > x) = e^{-\lambda x}$; $P(X > x | T > t) = e^{-\lambda x}$
+ Joint continuous r.v and joint PDFs: $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\partial y}F_{X,Y}(x,y) \ge 0$
  $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$; $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$; $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx\,dy = 1$
+ Given distribution of X; $Y = aX+b \to p_Y = p_X(\frac{y-b}{a})$; $f_Y(y) = \frac{1}{|a|}f_X(\frac{y-b}{a})$
+ The distribution of X+Y: $p_Z(z) = \sum_x p_X(x)p_Y(z-x)$; $f_Z(x) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$
+ $cov(X,Y) = E[(X-E[X]).(Y-E[Y])] = E[XY] - E[X]E[Y]$
+ $Var(X_1+X_2) = var(X_1) + var(X_2) + 2cov(X_1,X_2)$
+ correlation coefficient: $\rho(X,Y) = E\left[\frac{(X-E[X])}{\sigma_X}\frac{(Y-E[Y])}{\sigma_Y}\right] = \frac{cov(X,Y)}{\sigma_X\sigma_Y}$
+ Law of iterated expectations: $E[E[X|Y]] = E[X]$
+ Chebyshev inequality: $P(|X-\mu|\ge c) \le \frac{\sigma^2}{c^2}$; + Markov inequality: $X\ge 0, a\ge 0$; $P(X\ge a)\le\frac{E[X]}{a}$
+ CLT: let $X_1, X_2, \dots X_n$ be a sequence of i.i.d r.v. with: $E[X_i] = \mu$, $Var[X_i] = \sigma^2$
  $S_n = X_1 + X_2 + \dots + X_n$; $\bar{X}_n = \frac{1}{n}.S_n \to Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma}$
  Then, as $n\to\infty$, $Z_n \to \mathcal{N}(0,1)$
+ Confident interval (CI):
  CI for mean (σ known): $CI = \bar{x}\pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$
  CI for mean (σ unknown): $CI = \bar{x}\pm t_{\alpha/2,n-1}\left(\frac{s}{\sqrt{n}}\right)$; t-values depend on degrees of freedom
  CI for proportion: $CI = \hat{p}\pm z_{\alpha/2}\left(\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right)$

Moments of Gaussian

Order	Non-central	Central moment
1	μ	0
2	$\mu^2+\sigma^2$	σ^2
3	$\mu^3+3\mu\sigma^2$	0
4	$\mu^4+6\mu^2\sigma^2+3\sigma^4$	$3\sigma^4$
5	$\mu^5+10\mu^3\sigma^2+15\mu\sigma^4$	0
6	$\mu^6+15\mu^4\sigma^2+45\mu^2\sigma^4+15\sigma^6$	$15\sigma^6$
7	$\mu^7+21\mu^5\sigma^2+105\mu^3\sigma^4+105\mu\sigma^6$	0
8	$\mu^8+28\mu^6\sigma^2+210\mu^4\sigma^4+420\mu^2\sigma^6+105\sigma^8$	$105\sigma^8$

- Pivotal quantity — A function of observations and unobservable parameters, that the function's probility distribution does not dependent on unknown parameter.
- Uniform distribution MLE — Maximum likelihood estimator for parameter θ^* of $X\sim Unif[0,\theta^*]$: $\max_{i=1,\dots,n} X_i$
- Empirical measure — A random measure arising from a particular realization of a (usually finite) sequence of r.v.
- Valid confidence interval should not depend on parameter that is unknown (to be estimated)
- $\equiv$ — congruence

- $f_X(x)$, $F_X(x)$ — pdf and cdf of X (continuous)
- $p_X(x)$ — pmf of X (discrete)
- $\triangleq$ — equal by definition
- ⇝ — leads to
- argmax f(x) argmin f(x) — arguments of maxima/minima (values at which f(x) is max/min)
- affine function — both convex and concave
- A priori — knowledge independent from experience
- A posteriori — knowledge that depends on empirical evidence
- Well defined — (unambiguous) an expression which assign it a unique interpretation (value)
- Improper integral — the limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number, ∞, -∞, or in some instances as both endpoints approach limits.
- w.r.t. — with regards to
- Right continuity of CDF —

  Example: $X = \begin{cases}1 & 50\%\\ 2 & 50\%\end{cases}$
  $P(X\le 1) = P(X\le 1+\delta) = 0.5$ $\delta\to 0$
  $\ne P(X\le 1-\delta) = 0$
- Consistent v.s. Unbiased Estimators — Consistent: Estimator T_n of parameter θ is consistent if it converges in Probability to true value of parameter: $T_n \xrightarrow{P} \theta$
  Unbiased: Estimator's expected value and true value is 0. $E[T_n] = \theta$
- Intergration by parts — $\int uv' dx = u\int v' dx - \int u'(\int v' dx)dx$
- $\perp\!\!\!\perp$ — Independent symbol
- uncorrelated — meaning covariance is 0, does not gaurentee independence. A and B uncorrelated only imply independence if jointly gaussian

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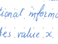
- Asymptotic — Refers to how an estimator behave as sample size grow larger (i.e tends to infinity)
- Well specified model (Not to confuse with well defined) — The class of distribution C you are assuming for your modeling contains the unknown probability distribution p from where the sample is drawn.
- Iff — If and only if
- s.t. — such that
- CMT — Continuous Mapping Theorem. Continuous functions preserve limits even if their arguments are sequences of random variables. Continuous function (Heine's definition) is such a function that maps convergent sequences into convergent sequences: if $x_n\to x$ then $g(x_n)\to g(x)$
  CMT: this will also be true if we replace deterministic sequence $\{x_n\}$ with a sequence of r.v. $\{X_n\}$ and replace "→" with one of the types of convergence of r.v.
- R.V. — Random variable
- := — Defined as
- $q_{\alpha/2}$ — $P(X\le q_{\alpha/2}) = 1-\alpha/2$



- Φ — cdf of standard gaussian
- Asymptotic normal — $\sqrt{n}(\hat\theta - \theta_0) \xrightarrow{d} \mathcal{N}(0, \sigma^2_\theta)$
- Sufficient statistics — No other statistic that can be calculated from the same sample provides any additional information as to the value of parameter
- $f: x\mapsto x^2$ — A function that takes value x to value y
- $\sup_{k\ge 0} T^{(k)}(N)$ — Supremum, the largest value $T^{(k)}(N)$ could get to as k varies
- supremum (sup) infimum (inf) — Sup of subset S of partially ordered set I is the least element greater or equal to all element of S. Inf of subset S of partially ordered set I is the greatest element in I that is less than or equal to all elements of S

Moment Generating Functions (MGFs)

- Moment:
  $E(X)$: 1st moment
  $E(X^2)$: 2nd moment
  $E(X^k)$: kth moment
- central moment: $E[(X-E(X))^k]$
- Definition: MGF for a random variable is:
  $M_X(t) = E(e^{tX}) = \begin{cases}\sum_{\text{all } k} e^{tk}p_X(k) & \text{(discrete)}\\ \int_{-\infty}^{\infty} e^{tx}f_X(x)dx & \text{(continuous)}\end{cases}$
  *Provided the expectation exist for some t in a neighborhood of 0.
  $E(X^n) = \frac{d^n}{dt^n}M_X(t)\big|_{t=0}$
- Proof:
  $\begin{cases} e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\\ e^{tx} = 1 + tx + \frac{t^2}{2!}x^2 + \frac{t^3}{3!}x^3 + \dots\end{cases}$
  Therefore: $E(e^{tX}) = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \dots$
  Therefore: $\frac{d}{dt}E(e^{tX}) = 0 + E(X) + tE(X^2) + \dots$
  $\frac{d}{dt}E(e^{tX})\big|_{t=0} = 0 + E(X) + 0 + \dots$
  and $\frac{d^2}{dt^2}E(e^{tX})\big|_{t=0} = 0 + 0 + E(X^2) + tE(X^3) + \dots = 0 + 0 + E(X^2) + 0 \dots$
- Application:
  - Bernoulli distribution:
    ex $X\sim Ber(p)$ $\begin{cases}p_X(1) = p\\ p_X(0) = 1-p\end{cases}$ $Var(X) = ?$
    $M_X(t) = E(e^{tX}) = \sum_{\text{all } x} e^{tx}p_X(x) = e^{t\cdot 0}p_X(0) + e^t p_X(1) = 1-p+pe^t$
    $Var(X) = E(X^2) - [E(X)]^2$ $\begin{cases}E(X) = \frac{d}{dt}M_X(t)\big|_{t=0} = pe^t\big|_{t=0} = p\\ E(X^2) = \frac{d^2}{dt^2}M_X(t)\big|_{t=0} = pe^t\big|_{t=0} = p\end{cases}$ $\Rightarrow Var(X) = p - p^2$

- Binomial distribution:
  ex. let $X_1,\dots,X_n \overset{i.i.d.}{\sim} Ber(p)$
  $\Rightarrow \sum_{i=1}^n X_i \sim Bin(n,p)$. Find $E(\sum_{i=1}^n X_i)$, $Var(\sum_{i=1}^n X_i)$
  $M_X(t) = E(e^{tX}) = E(e^{t\sum_{i=1}^n X_i}) = E[e^{tX_1}\cdot e^{tX_2}\cdots e^{tX_n}]$
  ⇓ i.i.d.
  $= [1-p+pe^t][1-p+pe^t]\cdots[1-p+pe^t]$
  $= [1-p+pe^t]^n$
  $E(\sum_{i=1}^n X_i) = \frac{d}{dt}(1-p+pe^t)^n\big|_{t=0} = n(1-p+pe^t)^{n-1}.pe^t\big|_{t=0} = np$
  $E[(\sum_{i=1}^n X_i)^2] = \frac{d^2}{dt^2}(1-p+pe^t)^n\big|_{t=0} = n(n-1)(1-p+pe^t)^{n-2}pe^t.pe^t + n(1-p+pe^t)^{n-1}.pe^t\big|_{t=0}$
  $= n(n-1)p^2 + np = n^2p^2 - np^2 + np$
  ⇓
  $Var(\sum_{i=1}^n X_i) = np - np^2 = np(1-p)$
  ☆ If $X_1,\dots,X_n$ are i.i.d. and X_i has MGF $M_{X_i}(t)$, then $\sum_{i=1}^n X_i$ has MGF: $M_{\sum X_i}(t) = [M_{X_i}(t)]^n$

Law of Large Numbers (LLN) & Central Limit Theorem (CLT)

Assumptions: ① $E[|X_i|] < \infty$ for all i ② $X_1,\dots X_n$ independent and identically distributed

Law(s) (weak and strong) of large numbers:
$\bar{X}_n := \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow[n\to\infty]{P,\ a.s.} \mu$ → convergence in probability ⇒ weak; → convergence almost surely ⇒ strong

Central limit theorem (CLT)
Assumptions: ① $E[|X_i|] < \infty$ for all i ② $Var(X_i) < \infty$ for all i ③ $X_1,\dots X_n$ iid
$\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$ → convergence in distribution

Hoeffding's inequality

Given n (n > 0) i.i.d. random variables $X_1, X_2 \dots X_n \overset{i.i.d.}{\sim} X$ that are almost surely bounded ($P(X\notin[a,b]) = 0$)
$P(|\bar{X}_n - E[X]| \ge \varepsilon) \le 2\exp\left(-\frac{2n\varepsilon^2}{(b-a)^2}\right)$ for all $\varepsilon > 0$
* n does not need to be large

Markov inequality

For a random variable $X\ge 0$ with mean $\mu > 0$, and any number $t > 0$
$P(X\ge t) \le \frac{\mu}{t}$
* Restricted to non-negative r.v.

Chebyshev inequality

For a random variable X with (finite) mean μ and variance σ^2, and any number $t > 0$,
$P(|X-\mu|\ge t) \le \frac{\sigma^2}{t^2}$

Slutsky's Theorem

- Let X_n, Y_n be sequences of scalar/vector/matrix random elements (Generalization of r.v. to more complicated spaces)
- If $\begin{cases}X_n \text{ converges in distribution to a random element } X\\ Y_n \text{ converges in probability to a constant } c\end{cases}$
- $X_n + Y_n \xrightarrow{d} X + c$
  $X_n Y_n \xrightarrow{d} Xc$
  $X_n / Y_n \xrightarrow{d} X/c$ (provided that c is invertible)

Continuous Mapping Theorem (also called Mann-Wald theorem)

- Continuous functions perserve limits even if their arguments are sequences of random variables.
  → Heine's definition: is a function that maps convergent sequences to convergent sequences: if $x_n\to x$ then $g(x_n)\to g(x)$
- Let $\{X_n\}$, X be random element defined on metric space S. Suppose a function $g: S\to S'$ (S' is another metric space) has the set of discontinuity points D_g such that $Pr[X\in D_g] = 0$. Then:
  $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$
  $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$
  $X_n \xrightarrow{a.s.} X \Rightarrow g(X_n) \xrightarrow{a.s.} g(X)$

Unit 2 Foundation of Inference

Lecture 3 Parametric Statistical Models
Lecture 4 Parametric Estimation & confidence intervals

Trinity of statistical Inference

① Estimation
② Confidence Intervals
③ Hypothesis testing

Formal Definition of Statistical Models

Let observed outcome of a statistical experiment be a sample $X_1,\dots X_n$ of n i.i.d. random variables in some measurable space E (usually $E\subseteq\mathbb{R}$) and denote by $\mathbb{P}$ their common distribution. A statistical model associated to that statistical experiment is a pair
$(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ → E: sample space; $(\mathbb{P}_\theta)$: family of probability measures on E; Θ: any set, called parameter set

* Well specified model: $\exists\theta$ such that $\mathbb{P} = \mathbb{P}_\theta$
* This particular θ is called true unknown parameter. The aim of statistical experiment is to θ, or check it's properties when they have special meaning.

Parametric / Non-parametric

- Parametric: $\Theta\subseteq\mathbb{R}^d$ ($d\ge 1$)
- Nonparametric: Θ is of infinite dimensional
- Semiparametric: $\Theta = \Theta_1\times\Theta_2$
  Θ_1: finite ⇑ estimate; Θ_2: infinite ⇑ nuisance parameter

Identifiability of Models

The parameter θ is identifiable iff the map $\theta\in\Theta\mapsto\mathbb{P}_\theta$ is injective, i.e.,
$\theta\ne\theta' \Rightarrow \mathbb{P}_\theta\ne\mathbb{P}_{\theta'}$
or equivalently:
$\mathbb{P}_\theta = \mathbb{P}_{\theta'} \Rightarrow \theta = \theta'$

Parameter Estimation

Definitions:
- Statistic: Any measurable (→ can compute with given data) function of the sample
  e.g. $\bar{X}_n$, $\max_i X_i$, $X_1 + \log(1+|X_n|)\dots$
- Estimator: Any statistic whose expression does not depend on θ.
  → Consistent estimator: $\hat\theta_n \xrightarrow[n\to\infty]{\mathbb{P}/a.s.} \theta$ → weakly consistent / strongly consistent
  → Asymtotically normal: $\sqrt{n}(\hat\theta_n - \theta) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\sigma^2)$ → asymptotic variance of $\hat\theta_n$

Bias of estimator

$bias(\hat\theta_n) = \mathbb{E}[\hat\theta_n] - \theta$
* Unbiased: $bias(\hat\theta_n) = 0$

Quadric risk

$R(\hat\theta_n) = \mathbb{E}[|\hat\theta_n - \theta|^2]$
= variance + bias²

Variance of estimator

An estimator is a random variable so we can compute its variance
$var(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = E[X^2] - (E[X])^2$

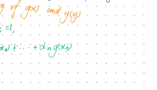
Confidence Interval

Let $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ be a statistical model based on observations $X_1,\dots X_n$ and assume $\Theta\subseteq\mathbb{R}$. Let $\alpha\in(0,1)$.
- Confidence Interval (C.I.) of level $1-\alpha$ for θ:
  Any random (depending on $X_1\dots X_n$) interval I whose boundaries do not depend on θ and such that
  $\mathbb{P}_\theta[\mathcal{I}\ni\theta] \ge 1-\alpha$, $\forall\theta\in\Theta$
- Confidence Interval of asymptotic level $1-\alpha$ for θ:
  Any random interval I whose bounderies do not depend on θ and such that:
  $\lim_{n\to\infty}\mathbb{P}[\mathcal{I}\ni\theta] \ge 1-\alpha$, $\forall\theta\in\Theta$

* Jensen's Inequality

If X is a random variable and φ is a convex function then
$\varphi(E[X]) \le E[\varphi(X)]$
* $E(\varphi[X]) - \varphi(E[X])$ = Jensen gap

Proof of Jensen's Inequality
* $\begin{cases}\text{Convex function: } g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha)g(y)\\ \text{Concave function: } g(\alpha x + (1-\alpha)y) \ge \alpha g(x) + (1-\alpha)g(y)\end{cases}$
(weighted avg of x & y; weighted avg of g(x) and g(y))



⇓
Therefore, for a convex function $g: I\to\mathbb{R}$ and $x_1, x_2\dots x_n$ in I and nonnegative α_i s.t. $\sum_{i=1}^n\alpha_i = 1$,
we have $g(\alpha_1x_1 + \alpha_2x_2 + \dots + \alpha_nx_n) \le \alpha_1g(x_1) + \alpha_2g(x_2) + \dots + \alpha_ng(x_n)$
⇓ $\alpha_i = P(X = x_i) = p_X(x_i)$
$g(E[X]) \le E[g(X)]$

Confidence Interval Example

- $R_1,\dots R_n \overset{iid}{\sim} Ber(p)$ for some unknown $p\in(0,1)$
- Statistical model: $(\{0,1\}, (Ber(p))_{p\in(0,1)})$
- Estimator for p: $\hat{p} = \bar{R}_n$
- CLT: $\sqrt{n}\frac{\bar{R}_n - p}{\sqrt{p(1-p)}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$
  ↳ cdf of $\sqrt{n}\frac{\bar{R}_n-p}{\sqrt{p(1-p)}} := \Phi_n(x)$
  ↳ when n becomes large, $\Phi_n(x)\approx\Phi(x)$ → standard gaussian cdf
  Therefore:
  $\mathbb{P}[|\bar{R}_n - p| \ge x] \simeq 2(1 - \Phi\frac{x\sqrt{n}}{\sqrt{p(1-p)}})$
- For a fixed $\alpha\in(0,1)$, if $q_{\alpha/2}$ is $(1-\alpha/2)$-quantile of $\mathcal{N}(0,1)$, then with probability $\simeq 1-\alpha$ (if n is large enough)
  $\bar{R}_n \in \left[p - \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}}, p + \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}}\right]$
  ⇓
  $\lim_{n\to\infty}\mathbb{P}\left(\left[\bar{R}_n - \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}}, \bar{R}_n + \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}}\right]\ni p\right) = 1-\alpha$
  * This is not a confidence interval, because it depends on p! → true parameter
- Solution 1: Conservative bound
  ↳ $p(1-p) \le \frac{1}{4}$
  ↳ $\bar{R}_n \in \left[p - \frac{q_{\alpha/2}\sqrt{\frac{1}{4}}}{\sqrt{n}}, p + \frac{q_{\alpha/2}\sqrt{\frac{1}{4}}}{\sqrt{n}}\right]$
  ↳ $\mathcal{I}_{conserv} = \left[\bar{R}_n - \frac{q_{\alpha/2}}{2\sqrt{n}}, \bar{R}_n + \frac{q_{\alpha/2}}{2\sqrt{n}}\right]$
- Solution 2: Solving the (quadratic) equation of p
  ↳ $\bar{R}_n - \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}} \le p \le \bar{R}_n + \frac{q_{\alpha/2}\sqrt{p(1-p)}}{\sqrt{n}} \Rightarrow (p - \bar{R}_n)^2 \le \frac{q^2_{\alpha/2}(p\cdot(1-p))}{n}$
  ↳ Find roots $p_1 < p_2$ of
  $(1 + \frac{q^2_{\alpha/2}}{n})p^2 - (2\bar{R}_n + \frac{q^2_{\alpha/2}}{n})p + \bar{R}_n^2 = 0$
  ↳ $\mathcal{I}_{solve} = [p_1, p_2]$

- Solution 3: plug-in
  ↳ LLN: $\hat{p} = \bar{R}_n \xrightarrow[n\to\infty]{P,\ a.s.} p$
  ↳ Slutsky:
  $\sqrt{n}\frac{\bar{R}_n - p}{\sqrt{\hat{p}(1-\hat{p})}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,1)$
  ↳ $\mathcal{I}_{plug\text{-}in} = \left[\bar{R}_n - \frac{q_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}, \bar{R}_n + \frac{q_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})}}{\sqrt{n}}\right]$

Delta Method

- Let $(Z_n)_{n\ge 1}$ sequence of r.v. that satisfies
  $\sqrt{n}(Z_n - \theta) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2)$
  for some $\theta\in\mathbb{R}$ and $\sigma^2 > 0$ ($(Z_n)_{n\ge 1}$ asymptotically normal around θ)
- Let $g: \mathbb{R}\to\mathbb{R}$ be continuously differentiable at point θ.
  Then,
  ⌊ $(g(Z_n))_{n\ge 1}$ is also asymptotically normal
  ⌊ $\sqrt{n}(g(Z_n) - g(\theta)) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, (g'(\theta))^2\sigma^2)$ — asymptotical variance

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Proof of Delta Method

- Let g be continuous differentiable everywhere in R
  Let μ be arbitrary
  Mean value theorem (0^{th} order statement of Taylor's theorem)
  for $z > \mu$: $g(z) = g(\mu) + g'(c_z)(z-\mu)$ $c_z\in(\mu, z)$ → is a function of z
  ⇓ works also for $z < \mu$
  $g(z) = g(\mu) + g'(c_z)(z-\mu)$ $|c_z - \mu| < |z-\mu|$
  ⇓ implies
  for r.v. Z $g(Z) - g(\mu) = g'(c_Z)(Z-\mu)$ $|c_Z - \mu| < |Z-\mu|$
  ⇓ given an arbitrary sequence $(Z_n)_{n\ge 1}$ and for any μ
  $g(Z_n) - g(\mu) = g'(c_{Z_n})(Z_n - \mu)$
- Let $X_1,\dots X_n \overset{i.i.d.}{\sim} X$, $Z_n = \bar{X}_n$, $\mu = \mathbb{E}[X]$
  $\sqrt{n}(g(\bar{X}_n) - g(\mu)) = g'(c_{\bar{X}_n})(\sqrt{n}(\bar{X}_n - \mu))$ $|c_{\bar{X}_n} - \mu| < |\bar{X}_n - \mu|$
  ⇓ CLT: $(\sqrt{n}(\bar{X}_n - \mu)) \xrightarrow[n\to\infty]{d} \mathcal{N}(0,\sigma^2)$
  ⇓ $P(|c_{\bar{X}_n} - \mu| > \varepsilon) \le P(|\bar{X}_n - \mu| > \varepsilon)$
  ⇓ $\bar{X}_n \xrightarrow[n\to\infty]{P} \mu$
  $c_{\bar{X}_n} \xrightarrow[n\to\infty]{P} \mu$
  ⇓ g' is continuous — continuous mapping theorem
  $g'(c_{\bar{X}_n}) \xrightarrow[n\to\infty]{P} g'(\mu)$
  Slutsky Theorem ⇒
  $\sqrt{n}(g(\bar{X}_n) - g(\mu)) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, (g'(\mu))^2\sigma^2)$

Hypothesis testing

Statistical formulation

- Consider a sample $X_1,\dots,X_n$ of i.i.d. random variables and a statistical model $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$.
- Let Θ_0 and Θ_1 be disjoint subset of Θ
- Consider the two hypothesis: $\begin{cases}H_0: \theta\in\Theta_0 & \text{(null hypothesis)}\\ H_1: \theta\in\Theta_1 & \text{(alternative hypothesis)}\end{cases}$

If we believe θ is either in Θ_0 or Θ_1, we may want to test H_0 against H_1
We want to decide whether to reject H_0 (look for evidence against H_0 in the data)

Asymmetry in the hypothesis

- H_0 and H_1 do not play a symmetric role: the data is only used to disprove H_0
- In particular, lack of evidence, does not mean H_0 is true (innocent until proven guilty)
- A test is a statistic $\psi\in\{0,1\}$ such that:
  $\begin{cases}\text{If } \psi = 0, H_0 \text{ is not rejected}\\ \text{If } \psi = 1, H_0 \text{ is rejected}\end{cases}$

Errors

- Rejection region of test ψ:
  $R_\psi = \{x\in E^n : \psi(x) = 1\}$ → sample space $(X_1,\dots,X_n)$; $\psi(x) = \mathbb{1}(x\in R_\psi)$
- Type 1 error of test ψ (rejecting H_0 when it is actually true):
  $\alpha_\psi: \Theta_0 \to \mathbb{R}$ → or [0,1]
  $\theta \mapsto \mathbb{P}_\theta[\psi = 1]$
- Type 2 error of test ψ (not rejecting H_0 although H_1 is actually true)
  $\beta_\psi: \Theta_1 \to \mathbb{R}$
  $\theta \mapsto \mathbb{P}_\theta[\psi = 0]$
- Power of test ψ:
  $\pi_\psi = \inf_{\theta\in\Theta_1}(1 - \beta_\psi(\theta))$ (1 - largest possible $\beta_\psi(\theta)$)

Level, test statistic and rejection region

- A test ψ has level α if:
  $\alpha_\psi(\theta) \le \alpha, \ \forall \theta \in \Theta_0$
- A test has asymptotic level α if:
  $\lim_{n\to\infty} \alpha_{\psi_n}(\theta) \le \alpha, \ \forall \theta \in \Theta_0$
- In general, a test has the form
  $\psi = \mathbb{1}\{T_n > c\}$ → test statistic
  for some statistic T_n and threshold $c \in \mathbb{R}$
  * Rejection region is $R_\psi = \{T_n > c\}$

One-sided v.s. two sided tests

When $\theta \in \Theta \subset \mathbb{R}$ and H_0 is of the form
$H_0: \theta = \theta_0 \iff \Theta_0 = \{\theta_0\}$

- If $H_1: \theta \ne \theta_0$: two-sided test
- If $H_1: \theta > \theta_0$ or $H_1: \theta < \theta_0$: one sided test

p-value

Definition: The (asymptotic) p-value of a test ψ_α is the smallest (asymptotic) level α at which ψ_α rejects H_0.
It is random. It depends on the sample.

Golden rule: p-value $\le \alpha \iff H_0$ is rejected by ψ_α, at the (asymptotic) level α.

* The smaller the p-value, the more confident one can reject H_0

Unit 3: Methods of Estimation

Lecture 8: Distance measures between distributions

Total variation distance

- Motivation:
  - Let $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ be a statistical model associated with a sample of i.i.d. r.v. $X_1, \dots, X_n$. Assume that there exist $\theta^* \in \Theta$ that $X_1 \sim \mathbb{P}_{\theta^*}$: θ^* is the true parameter
  - Statistician's goal: given $X_1, \dots X_n$, find an estimator $\hat\theta = \hat\theta(X_1, \dots, X_n)$ s.t. $\mathbb{P}_{\hat\theta}$ is close to $\mathbb{P}_{\theta^*}$ for true parameter θ^*
  - This means: $|\mathbb{P}_{\hat\theta}(A) - \mathbb{P}_{\theta^*}(A)|$ is small for all $A \subset E$
- Definition:
  Total variation distance between two probability measures $\mathbb{P}_\theta$ and $\mathbb{P}_{\theta'}$:
  $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = \max_{A\subset E} |\mathbb{P}_\theta(A) - \mathbb{P}_{\theta'}(A)|$
- Discrete sample space:
  $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = \frac{1}{2}\sum_{x\in E}|p_{\theta'}(x) - p_\theta(x)|$
- Continuous sample space:
  $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = \frac{1}{2}\int |f_\theta(x) - f_{\theta'}(x)|dx$
- Properties of total variation
  - Symmetric: $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = TV(\mathbb{P}_{\theta'}, \mathbb{P}_\theta)$
  - Positive: $0 \le TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) \le 1$
  - Definite: If $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = 0$, then $\mathbb{P}_\theta = \mathbb{P}_{\theta'}$
  - Triangle inequality: $TV(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) \le TV(\mathbb{P}_\theta, \mathbb{P}_{\theta''}) + TV(\mathbb{P}_{\theta''}, \mathbb{P}_{\theta'})$

  These imply that TV is a distance between probability distributions

Kullback-Leibler (KL) divergence

- Definition:
  The KL divergence between two probability measures $\mathbb{P}_\theta$ and $\mathbb{P}_{\theta'}$ is defined by
  $$KL(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = \begin{cases} \sum_{x\in E} p_\theta(x)\log\left(\frac{p_\theta(x)}{p_{\theta'}(x)}\right) & \text{if } E \text{ is discrete} \\ \int_E f_\theta(x)\log\left(\frac{f_\theta(x)}{f_{\theta'}(x)}\right)dx & \text{if } E \text{ is continuous} \end{cases}$$
- Properties of KL divergence:
  - ~~Symmetric~~: $KL(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) \ne KL(\mathbb{P}_{\theta'}, \mathbb{P}_\theta)$ in general
  - Positive: $KL(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) \ge 0$
  - Definite: If $KL(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) = 0$, then $\mathbb{P}_\theta = \mathbb{P}_{\theta'}$
  - ~~Triangle inequality~~: $KL(\mathbb{P}_\theta, \mathbb{P}_{\theta'}) \not\le KL(\mathbb{P}_\theta, \mathbb{P}_{\theta''}) + KL(\mathbb{P}_{\theta''}, \mathbb{P}_{\theta'})$ in general

  Not a distance

Lecture 9: Introduction to MLE

Maximum Likelihood Estimation

- Derivation:
  $KL(\mathbb{P}_{\theta^*}, \mathbb{P}_\theta) = \mathbb{E}_{\theta^*}\left[\log\left(\frac{p_{\theta^*}(X)}{p_\theta(X)}\right)\right] = \mathbb{E}_{\theta^*}[\log p_{\theta^*}(X)] - \mathbb{E}_{\theta^*}[\log p_\theta(X)]$
  (first term: constant; $h(x) = \log p_\theta(x)$, by LLN: $\mathbb{E}_{\theta^*}[h(X)] \rightsquigarrow \frac{1}{n}\sum_{i=1}^n h(X_i)$)

  Estimator of KL:
  $\widehat{KL}(\mathbb{P}_{\theta^*}, \mathbb{P}_\theta) = \text{"constant"} - \frac{1}{n}\sum_{i=1}^n \log p_\theta(X_i)$

  $\min_{\theta\in\Theta} \widehat{KL}(\mathbb{P}_{\theta^*}, \mathbb{P}_\theta) \iff \min_{\theta\in\Theta} -\frac{1}{n}\sum_{i=1}^n \log p_\theta(X_i)$
  $\iff \max_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^n \log p_\theta(X_i)$
  $\iff \max_{\theta\in\Theta} \log\left[\prod_{i=1}^n p_\theta(X_i)\right]$
  $\iff \max_{\theta\in\Theta} \prod_{i=1}^n p_\theta(X_i)$

  Maximum Likelihood Principle

Likelihood: Discrete Case

- Let $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ be a statistical model associated with a sample of i.i.d. r.v. $X_1, \dots X_n$. Assume that E is discrete (i.e. finite or countable)
- Definition:
  The likelihood of the model is the map L_n (or just L) defined as:
  $L_n: E^n \times \Theta \to \mathbb{R}$
  $(x_1, \dots x_n, \theta) \mapsto \mathbb{P}_\theta[X_1 = x_1, \dots, X_n = x_n]$
  $\prod_{i=1}^n \mathbb{P}_\theta[X_i = x_i]$

Likelihood: Continuous Case

- Let $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ be a statistical model associated with a sample of i.i.d. r.v. $X_1, \dots X_n$. Assume that all the $\mathbb{P}_\theta$ have density f_θ
- Definition:
  The likelihood of the model is the map L defined as:
  $L: E^n \times \Theta \to \mathbb{R}$
  $(x_1, \dots, x_n, \theta) \mapsto \prod_{i=1}^n f_\theta(x_i)$

Maximum likelihood estimator

- Let $X_1, \dots X_n$ be an i.i.d. sample associated with a statistical model $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ and let L be the corresponding likelihood.
- Definition:
  The maximum likelihood estimator of θ is defined as:
  $\hat\theta_n^{MLE} = \underset{\theta\in\Theta}{\text{argmax}}\ L(X_1, \dots, X_n, \theta)$, provided it exists

* In practice, we use the fact that $\hat\theta_n^{MLE} = \underset{\theta\in\Theta}{\text{argmax}} \log L(X_1, \dots, X_n, \theta)$ (log-likelihood estimator)

Concave and Convex

- Definition:
  A twice differentiable function $h: \Theta \subset \mathbb{R} \to \mathbb{R}$ is said to be concave if its second derivative satisfies:
  $h''(\theta) \le 0; \ \forall \theta \in \Theta$
  It is said to be strictly concave if the inequality is strict: $h''(\theta) < 0$
  Moreover, h is said to be (strictly) convex if $-h$ is (strictly) concave.
- Multivariate concave function
  For a multivariate function: $h: \Theta \subset \mathbb{R}^d \to \mathbb{R}$, $d \ge 2$, define the
  Gradient vector:
  $$\nabla h(\theta) = \begin{pmatrix} \frac{\partial h}{\partial\theta_1}(\theta) \\ \frac{\partial h}{\partial\theta_2}(\theta) \\ \vdots \\ \frac{\partial h}{\partial\theta_d}(\theta) \end{pmatrix} \in \mathbb{R}^d$$
  Hessian matrix:
  $$\mathbf{H}h(\theta) = \begin{pmatrix} \frac{\partial^2 h}{\partial\theta_1\partial\theta_1}(\theta) & \cdots & \frac{\partial^2 h}{\partial\theta_1\partial\theta_d} \\ \vdots & & \vdots \\ \frac{\partial^2 h}{\partial\theta_d\partial\theta_1}(\theta) & \cdots & \frac{\partial^2 h}{\partial\theta_d\partial\theta_d} \end{pmatrix} \in \mathbb{R}^{d\times d}$$
  h is concave $\iff x^T \mathbf{H}h(\theta) x \le 0 \quad \forall x \in \mathbb{R}^d, \theta \in \Theta$
  h is strictly concave $\iff x^T \mathbf{H}h(\theta) x < 0 \quad \forall x \in \mathbb{R}^d, x \ne 0, \theta \in \Theta$

Optimality Conditions

Strictly concave functions are easy to maximize: if they have a maximum then it is unique. It is the unique solution to
$h'(\theta) = 0$ (single-variate) / $\nabla h(\theta) = 0 \in \mathbb{R}^d$ (multi-variate)
Convex optimization: close form solution

Lecture 10: Consistency of MLE, Covariance Matrices, and Multivariate Statistics

* Examples of maximum likelihood estimators
  Bernoulli trials: $\hat p_n^{MLE} = \bar X_n$
  Poisson model: $\hat\lambda_n^{MLE} = \bar X_n$
  Gaussian model: $(\hat\mu_n, \hat\sigma_n^2) = (\bar X_n, \hat S_n)$, $\hat S_n = \frac{1}{n}\sum_{i=1}^n (X_i - \bar X_n)^2$

Consistency of MLE

- Under mild regularity conditions, we have:
  $\hat\theta_n^{MLE} \xrightarrow[n\to\infty]{\mathbb{P}} \theta^*$
  This is because for all $\theta \in \Theta$
  $\frac{1}{n}\log L(X_1, \dots, X_n, \theta) \xrightarrow[n\to\infty]{\mathbb{P}} \text{"constant"} - KL(\mathbb{P}_{\theta^*}, \mathbb{P}_\theta)$ (minimizer is θ^* if parameter identifiable)
  Technical conditions allow to transfer this convergence to the minimizer / maximizer

Covariance

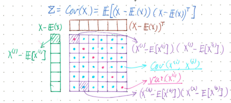
- In general, when $\theta \subset \mathbb{R}^d$, $d \ge 2$, its coordinates are not necessarily independent
- Definition:
  $Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))]$
  $= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ → if one of X, Y is 0 mean, $Cov(X,Y) = \mathbb{E}[XY]$.
  $= \mathbb{E}[X\cdot(Y - \mathbb{E}[Y])] = \mathbb{E}[Y\cdot(X - \mathbb{E}[X])]$
- Properties:
  - $Cov(X, X) = Var(X)$
  - $Cov(X, Y) = Cov(Y, X)$
  - If X and Y are independent $\Rightarrow Cov(X, Y) = 0$

* In general, $Cov(X,Y) = 0$ does not guarantee X, Y independent.
  Except if $(X, Y)^T$ is a Gaussian vector
  ↓ $\alpha X + \beta Y$ is gaussian for all $(\alpha, \beta) \in \mathbb{R}^2$ except for $\{0, 0\}$

* X is gaussian & Y is gaussian does not guarantee $(X, Y)^T$ is gaussian vector
  e.g. $X \sim \mathcal{N}(0,1)$, $B \sim Ber(1/2)$, $R = 2B - 1 \sim Rad(1/2)$
  $Y = R\cdot X \sim \mathcal{N}(0,1)$
  → $\alpha = \beta = 1$, we get:
  $X + Y = \begin{cases} 2X & \text{with prob } 1/2 \\ 0 & \text{with prob } 1/2 \end{cases}$
  $Cov(X, Y) = 0$, but X, Y not independent, and $(X, Y)^T$ not gaussian

Covariance matrix

- The covariance matrix of a random vector $X = (X^{(1)}, \dots X^{(d)})^T \in \mathbb{R}^d$ is given by
  $\Sigma = Cov(X) = \mathbb{E}[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T]$



- Linear property:
  $Cov(AX + B) = Cov(AX) = A\,Cov(X)A^T = A\Sigma A^T$

Diagonalization of the covariance matrix

Let Σ be a covariance matrix of size $d\times d$, Σ has properties:
(All entries of Σ would be non-negative)

- Σ is symmetric, that is $\Sigma = \Sigma^T$
- Σ is diagonalizable to a diagonal matrix D via transformation $D = U\Sigma U^T$ (orthogonal matrix: $AA^T = A^TA = I$, identity matrix)
  This implies: $\Sigma = U^T D U$
- Σ is positive semidefinite → diagonal matrix D has diagonal entries that are all non-negative
- Σ has a unique positive semidefinite square root matrix.
  That is, there exist a positive semidefinite matrix $\Sigma^{\frac{1}{2}}$ that is unique, such that $\Sigma^{\frac{1}{2}}\cdot\Sigma^{\frac{1}{2}} = \Sigma$
- Σ has d orthonormal eigenvectors (even if there are repeated eigen values).
  Furthermore, if U is a matrix with rows corresponding to the orthonormal eigenvectors, the the diagnol matrix $D = U\Sigma U^T$ contains the eigen values of Σ along its diagonal. Therefore, diagonalization of a symmetric matrix involves finding its eigenvalues and orthonormal eigenvectors.
- If Σ is positive definite $\iff$ the diagonal matrix $D = U\Sigma U^T$ has diagonal entries that are strictly positive, the it is invertible and the inverse Σ^{-1} satisfies: $\Sigma^{-\frac{1}{2}}\cdot\Sigma^{-\frac{1}{2}} = \Sigma^{-1}$, where $\Sigma^{-\frac{1}{2}}$ is the inverse of $\Sigma^{\frac{1}{2}}$.

Multivariate Gaussian distribution

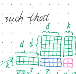
A Gaussian vector $X \in \mathbb{R}^d$ is completely determined by it expected value $\mathbb{E}[X] = \mu \in \mathbb{R}^d$ and covariance matrix Σ
$X \sim \mathcal{N}_d(\mu, \Sigma)$
The pdf is given by:
$$f_X(x) = f(x^{(1)}, \dots, x^{(d)}) = \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)} \quad (x \in \mathbb{R}^d)$$

Multivariate CLT

The CLT may be generalized to averages or random vectors (also vectors of averages)
Let $X_1, \dots X_n \in \mathbb{R}^d$ be independent copies of a random vector X such that $\mathbb{E}[X] = \mu$, $Cov(X) = \Sigma$,
$\sqrt{n}(\bar X_n - \mu) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, \Sigma)$
$\Updownarrow$
$\sqrt{n}\,\Sigma^{-\frac{1}{2}}(\bar X_n - \mu) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, I_d)$

Multivariate Delta Method

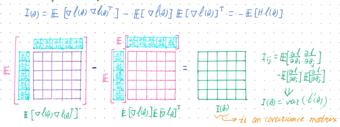
Let $(T_n)_{n\ge1}$ be sequence of random vectors in $\mathbb{R}^d$ such that
$\sqrt{n}(T_n - \theta) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, \Sigma)$
for some $\theta \in \mathbb{R}^d$ and some covariance $\Sigma \in \mathbb{R}^{d\times d}$



Let $g: \mathbb{R}^d \to \mathbb{R}^k$ $(k \ge 1)$ be continuously differentiable at θ. Then:
$[g_1, g_2, \dots g_k]$
$\sqrt{n}(g(T_n) - g(\theta)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k(0, \nabla g(\theta)^T\Sigma\nabla g(\theta))$
$$\frac{\partial g}{\partial\theta}(\theta) = \left(\frac{\partial g_j}{\partial\theta_i}\right)_{1\le i\le d,\ 1\le j\le k} \in \mathbb{R}^{d\times k}$$
$$\begin{bmatrix} \frac{\partial g_1}{\partial\theta_1} & \frac{\partial g_2}{\partial\theta_1} & \cdots & \frac{\partial g_k}{\partial\theta_1} \\ \frac{\partial g_1}{\partial\theta_2} & \frac{\partial g_2}{\partial\theta_2} & \cdots & \frac{\partial g_k}{\partial\theta_2} \\ \vdots & & & \vdots \\ \frac{\partial g_1}{\partial\theta_d} & \frac{\partial g_2}{\partial\theta_d} & \cdots & \frac{\partial g_k}{\partial\theta_d} \end{bmatrix}$$

Fisher Information

- Definition:
  Define the log-likelihood for one observation as:
  $\ell(\theta) = \log L_1(X, \theta)$, $\theta \in \Theta \subset \mathbb{R}^d$
  Assume that ℓ is a.s. twice differentiable. Under some regularity conditions, the Fisher information of the statistical model is:
  $I(\theta) = \mathbb{E}[\nabla\ell(\theta)\nabla\ell(\theta)^T] - \mathbb{E}[\nabla\ell(\theta)]\mathbb{E}[\nabla\ell(\theta)]^T = -\mathbb{E}[\mathbf{H}\ell(\theta)]$



If $\Theta \subset \mathbb{R}$, we get:
$I(\theta) = var(\ell'(\theta)) = -\mathbb{E}[\ell''(\theta)]$

* If $\Theta \subset \mathbb{R}$, $I(\theta) = var[\ell'(\theta)] = -\mathbb{E}[\ell''(\theta)]$

Lecture 11: Fisher information, Asymptotic Normality of MLE, and the Method of Moments

- Proof of Fisher Information Equivalent Formulas for 1 Dimension

Let $(\mathbb{R}, \{P_\theta\}_{\theta\in\mathbb{R}})$ denote a continuous statistical model. Let $f_\theta(x)$ denote the pdf of the continuous distribution P_θ. Assume that $f_\theta(x)$ is twice-differentiable as a function of the parameter θ.

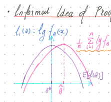
Proof: $I(\theta) = Var(\ell'(\theta)) = -\mathbb{E}[\ell''(\theta)]$ $\quad (\ell(\theta) = \ln f_\theta(x))$

① $\int f_\theta(x)dx = 1 \Rightarrow \begin{cases} \int \frac{\partial}{\partial\theta}L_1(x,\theta)dx = 0 & (1) \\ \int \frac{\partial^2}{\partial\theta^2}L_1(x,\theta)dx = 0 & (2) \end{cases}$, $f_\theta(x) = L_1(x,\theta)$

② $\ell'(\theta) = \frac{\partial}{\partial\theta}\log L_1(x,\theta) = \frac{\partial}{\partial\theta}L_1(x;\theta)/L_1(x,\theta)$
$\mathbb{E}[\ell'(\theta)] = \int \frac{\frac{\partial}{\partial\theta}L_1(x,\theta)}{L_1(x,\theta)}L_1(x,\theta)dx = \int\frac{\partial}{\partial\theta}L_1(x,\theta)dx \overset{(1)}{=} 0$
$var(\ell'(\theta)) = E[(\ell'(\theta) - E[\ell'(\theta)])^2] = E[(\ell'(\theta))^2]$
$= \int\left(\frac{\frac{\partial}{\partial\theta}L_1(x,\theta)}{L_1(x,\theta)}\right)^2 L_1(x,\theta)dx$
$= \int\frac{(\frac{\partial}{\partial\theta}L_1(x,\theta))^2}{L_1(x,\theta)}dx$

③ $\ell''(\theta) = \frac{\frac{\partial^2}{\partial\theta^2}L_1(x,\theta)\cdot L_1(x,\theta) - (\frac{\partial}{\partial\theta}L_1(x,\theta))^2}{(L_1(x,\theta))^2}$
$-E[\ell''(\theta)] = -\int\frac{\frac{\partial^2}{\partial\theta^2}L_1(x,\theta)\cdot L_1(x,\theta) - (\frac{\partial}{\partial\theta}L_1(x,\theta))^2}{(L_1(x,\theta))^2}L_1(x,\theta)dx$
$= -\int\frac{\partial^2}{\partial\theta^2}L_1(x,\theta)dx + \int\frac{(\frac{\partial}{\partial\theta}L_1(x;\theta))^2}{L_1(x,\theta)}dx$ (first term $\overset{(2)}{=} 0$)
$= \int\frac{(\frac{\partial}{\partial\theta}L_1(x;\theta))^2}{L_1(x;\theta)}dx = var(\ell'(\theta))$

- Informal Idea of Proof of Asymptotic Normality of MLE



$\frac{\partial}{\partial\theta}\sum_{i=1}^n \ell_i(\hat\theta) = \sum_{i=1}^n \ell_i'(\hat\theta) = 0$
$E[\ell'(\theta^*)] = 0$
$0 = \sum_{i=1}^n \ell_i'(\hat\theta) \cong \sum_{i=1}^n[\ell_i'(\theta^*) + (\hat\theta - \theta^*)\ell''(\theta^*)]$ (first order Taylor Expansion)
$= \sum_{i=1}^n[(\ell_i'(\theta^*) - E[\ell_i'(\theta^*)]) + (\hat\theta - \theta^*)\ell''(\theta^*)]$
↓ CLT
$\frac{1}{\sqrt n}\sum_{i=1}^n[\ell_i'(\theta^*) - E[\ell_i'(\theta^*)]] \xrightarrow[n\to\infty]{d} \mathcal{N}(0, var(\ell_i'(\theta^*)))$, $var(\ell_i'(\theta^*)) = I(\theta^*)$
↓
$0 \cong \mathcal{N}(0, I(\theta^*)) + \sqrt n(\hat\theta - \theta^*)\frac{1}{n}\sum_{i=1}^n \ell''(\theta^*)$
↓ LLN: $\frac{1}{n}\sum_{i=1}^n \ell''(\theta^*) \xrightarrow[n\to\infty]{P} -I(\theta)$
↓
$\sqrt n(\hat\theta - \theta^*) \sim \mathcal{N}\left(0, \frac{I(\theta^*)}{(-I(\theta^*))^2}\right) = \mathcal{N}(0, I(\theta^*)^{-1})$

- Asymptotic normality of the MLE
  - Theorem:
    Let $\theta^* \in \Theta$ (the true parameter). Assume the following:
    1. The parameter is identifiable;
    2. For all $\theta \in \Theta$, the support of $\mathbb{P}_\theta$ does not depend on θ;
    3. θ^* is not on the boundary of Θ;
    4. $I(\theta)$ is invertible in a neighborhood of θ^*;
    5. A few more technical conditions.
  - Then, $\hat\theta_n^{MLE}$ satisfies:
    $\hat\theta_n^{MLE} \xrightarrow[n\to\infty]{\mathbb{P}} \theta^*$ w.r.t. $\mathbb{P}_{\theta^*}$
    $\sqrt n(\hat\theta_n^{MLE} - \theta^*) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, I(\theta^*)^{-1})$ w.r.t. $\mathbb{P}_{\theta^*}$

Method of Moments

- Moments:
  Let $X_1, \dots, X_n$ be an iid sample associated with a statistical model $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$
  Assume that $E \subseteq \mathbb{R}$ and $\Theta \subseteq \mathbb{R}^d$, for some $d \ge 1$
  Population moments: Let $m_k(\theta) = \mathbb{E}_\theta[X_1^k]$, $1 \le k \le d$
  Empirical moments: Let $\hat m_k = \overline{X_n^k} = \frac{1}{n}\sum_{i=1}^n X_i^k$, $1 \le k \le d$
  From LLN,
  $\hat m_k \xrightarrow[n\to\infty]{P/a.s.} m_k(\theta)$
  more compactly, we say that the whole vector converges:
  $(\hat m_1, \dots, \hat m_d) \xrightarrow[n\to\infty]{P/a.s.} (m_1(\theta), \dots, m_d(\theta))$

- Moments estimator:
  Let
  $M: \Theta \to \mathbb{R}^d$
  $\theta \mapsto M(\theta) = (m_1(\theta), \dots, m_d(\theta))$
  Assume M is one to one:
  $\theta = M^{-1}(m_1(\theta), \dots, m_d(\theta))$

Definition:

Moments estimator of θ:
$\hat\theta_n^{MM} = M^{-1}(\hat m_1, \dots, \hat m_d)$,
provided it exists

- Generalized Method of Moments
  * Applying the multivariate CLT and Delta method yields:
  - Theorem:
    $\sqrt n(\hat\theta_n^{MM} - \theta) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, \Gamma(\theta))$ (w.r.t. $\mathbb{P}_\theta$)
    where $\Gamma(\theta) = \left[\frac{\partial M^{-1}}{\partial\theta}(M(\theta))\right]^T\Sigma(\theta)\left[\frac{\partial M^{-1}}{\partial\theta}(M(\theta))\right]$
    $\sqrt n(M^{-1}(\hat m) - M^{-1}(M(\theta))) \longrightarrow \mathcal{N}(0, (\nabla M^{-1})^T\Sigma(\nabla M^{-1}))$

MLE v.s. Moment estimator

- Quadratic risk: In general, the MLE is more accurate
- MLE still gives good results if model is misspecified
- Computational issue: Sometimes, MLE is intractable but MM is easier (polynomial equations)

Lecture 12: M-Estimation

M-estimators:

Idea:
- Let $X_1, \dots X_n$ be iid with some unknown distribution $\mathbb{P}$ in some sample space E ($E \subseteq \mathbb{R}^d$ for some $d \ge 1$)
- No statistical model needs to be assumed (similar to ML)
- Goal: estimate some parameter μ^* associated with $\mathbb{P}$, e.g. its mean, variance, median, other quantiles, the true parameter in some statistical model.
- Find a function $f: E \times M \to \mathbb{R}$, where M is the set of all possible values for the unknown μ^*, such that:
  $Q(\mu) := \mathbb{E}[f(X_1, \mu)]$
  achieves its minimum at $\mu = \mu^*$

* Common examples
  - Mean: $f(x, \theta) = \frac{(x-\theta)^2}{2}$
  - Median: $f(x, \theta) = |x - \theta|$

MLE is an M-estimator

Assume that $(E, \{\mathbb{P}_\theta\}_{\theta\in\Theta})$ is a statistical model associated with the data.
Theorem
Let $M = \Theta$ and $f(x, \theta) = -\log L_1(x, \theta)$, provided the likelihood is positive everywhere. Then,
$\mu^* = \theta^*$
where $\mathbb{P} = \mathbb{P}_{\theta^*}$ (i.e. θ^* is the true value of the parameter)

Asymptotic normality

Let $\mu^* \in M$ (the true parameter). Assume the following:
1. μ^* is the only minimizer of the function Q
2. $J(\mu)$ is invertible for all $\mu \in M$
3. A few more technical conditions

$J(\mu) = \frac{\partial^2 Q}{\partial\mu\partial\mu^T}(\mu) \doteq \mathbb{E}\left[\frac{\partial^2 f}{\partial\mu\partial\mu^T}(X_1, \mu)\right]$ (under some regularity conditions)

Then $\hat\mu_n$ satisfies:
$\hat\mu_n \xrightarrow[n\to\infty]{\mathbb{P}} \mu^*$
$\sqrt n(\hat\mu_n - \mu^*) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, J(\mu^*)^{-1}K(\mu^*)J(\mu^*)^{-1})$, $K(\mu) = Cov\left[\frac{\partial f}{\partial\mu}(X_1, \mu)\right]$

Unit 4: Hypothesis testing

Lecture 13: Chi Squared Distribution T-test

Motivation:

- Clinical trials – test a drug that is supposed to lower LDL ("bad cholesterol").
  Let $\Delta_d > 0$ denote expected decrease of LDL level for a patient that used the drug
  Let $\Delta_c \ge 0$ denote expected decrease of LDL level for a patient that used placebo
  We want to know if $\Delta_d > \Delta_c$
  We observe two independent samples: $\begin{cases} X_1, \dots X_n \overset{iid}{\sim} \mathcal{N}(\Delta_d, \sigma_d^2) & (\text{test}) \\ Y_1, \dots Y_m \overset{iid}{\sim} \mathcal{N}(\Delta_c, \sigma_c^2) & (\text{control}) \end{cases}$

- Hypothesis testing:
  $H_0: \Delta_c = \Delta_d$ v.s. $H_1: \Delta_d > \Delta_c$
  * Since data is Gaussian, we don't need CLT
  $\bar X_n \sim \mathcal{N}(\Delta_d, \frac{\sigma_d^2}{n})$ $\quad \bar Y_m \sim \mathcal{N}(\Delta_c, \frac{\sigma_c^2}{m})$
  ↓ See probability 6.419x Lec 12, sum of independent normal r.v.'s
  $$\frac{\bar X_n - \bar Y_m - (\Delta_d - \Delta_c)}{\sqrt{\frac{\sigma_d^2}{n} + \frac{\sigma_c^2}{m}}} \sim \mathcal{N}(0, 1)$$

- Asymptotic test:
  ① Assume that $m = cn$ and $n \to \infty$
  ② Using Slutsky's lemma, we have:
  $$\frac{\bar X_n - \bar Y_m - (\Delta_d - \Delta_c)}{\sqrt{\frac{\hat\sigma_d^2}{n} + \frac{\hat\sigma_c^2}{m}}} \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, 1)$$
  → using estimator of variance because true variance unknown
  $\hat\sigma_d^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar X_n)^2$ $\quad \hat\sigma_c^2 = \frac{1}{m-1}\sum_{i=1}^m (Y_i - \bar Y_m)^2$ (unbiased estimator)
  ③ We get following test at asymptotic level α: * one-sided, two sample test
  $$R_\psi = \left\{\frac{\bar X_n - \bar Y_m}{\sqrt{\frac{\hat\sigma_d^2}{n} + \frac{\hat\sigma_c^2}{m}}} > q_\alpha\right\}$$ → $(1-\alpha)$ quantile of $\mathcal{N}(0,1)$

· Small sample size:

- Can not realistically apply Slutsky's lemma
- We need to find the (asymptotic) distribution of quantities of the form

$$\frac{\bar{X}_n - \mu}{\sqrt{\sigma^2}}$$

when $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$

- It turns out that this distribution does not depend on μ or σ, so we can compute its quantiles.

The χ^2 distribution

Definition:

For a positive integer d, the χ^2 (pronounced "Kai-squared") distribution with d degree of freedom is the law of the random variable

$$Z_1^2 + Z_2^2 + \dots + Z_d^2 \quad \text{where } Z_1, \dots, Z_d \overset{iid}{\sim} \mathcal{N}(0,1)$$

Examples:

- If $Z \sim \mathcal{N}_d(0, I_d)$, then $\|Z\|_2^2 \sim \chi^2_d$ (I_d → d×d identity matrix)
- $\chi^2_2 = Exp(1/2)$

Properties for $V \sim \chi^2_k$:

$$E[V] = E[Z_1^2] + \dots + E[Z_d^2] = d$$
$$var[V] = Var(Z_1^2) + \dots + Var(Z_d^2) = 2d$$

Important example: the sample variance

- Sample variance is given by:

$$S_n = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n}\sum_{i=1}^n X_i^2 - (\bar{X}_n)^2$$

- Cochran's theorem: for $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$, if S_n is the sample variance:

$\bar{X}_n \perp\!\!\!\perp S_n$ (for all n)

$$\frac{nS_n}{\sigma^2} \sim \chi^2_{n-1}$$

* We often prefer the unbiased estimator of σ^2: $\tilde{S}_n = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$

Student's T distribution

· Definition:

For a positive integer d, the Student's T distribution with d degrees of freedom (denoted by t_d) is the law of the random variable $\frac{Z}{\sqrt{V/d}}$ where $Z \sim \mathcal{N}(0,1)$, $V \sim \chi^2_d$ and $Z \perp\!\!\!\perp V$ (Z is independent of V)

· Student's T test (one sample, two-sided)

① Let $X_1, \dots, X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ where both μ and σ^2 are unknown

② We want to test:

$H_0: \mu = 0$ v.s. $H_1: \mu \neq 0$

③ Test statistic:

$$T_n = \frac{\bar{X}_n}{\sqrt{\tilde{S}_n/n}} = \frac{\sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma}}{\sqrt{\tilde{S}_n/\sigma^2}} \quad \text{numerator} \sim \mathcal{N}(0,1), \ \tilde{S}_n/\sigma^2 \sim \frac{\chi^2_{n-1}}{n-1}$$

④ Since $\sqrt{n}\bar{X}_n/\sigma \sim \mathcal{N}(0,1)$ and $\tilde{S}_n/\sigma^2 \sim \frac{\chi^2_{n-1}}{n-1}$ are independent (Cochran's theorem)

$T_n \sim t_{n-1}$ (Student's T distribution with $n-1$ degrees of freedom)

⑤ Student's test with (non asymptotic) level $\alpha \in (0,1)$:

$$\psi_\alpha = \mathbb{1}\{|T_n| > q_{\alpha/2}\}$$

$q_{\alpha/2}$ → $(1-\alpha/2)$-quantile of t_{n-1}

· Student's T test (one sample, one-sided)

① ... as above

② We want to test:

$H_0: \mu \le \mu_0$ v.s. $H_1: \mu > \mu_0$

③ Test statistic:

$$T_n = \frac{\bar{X}_n - \mu_0}{\sqrt{\tilde{S}_n/n}} \sim t_{n-1} \quad (\text{under } H_0)$$

④ Student's test with (non asymptotic) level $\alpha \in (0,1)$:

$$\psi_\alpha = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of t_{n-1}

· Two-sample T-test

① Back to clinical trial example (motivation)

② We want to know distribution of:

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_d - \Delta_c)}{\sqrt{\frac{\sigma_d^2}{n} + \frac{\sigma_c^2}{m}}}$$

③ We approximate:

$$\frac{\bar{X}_n - \bar{Y}_m - (\Delta_d - \Delta_c)}{\sqrt{\frac{\hat{\sigma}_d^2}{n} + \frac{\hat{\sigma}_c^2}{m}}} \sim t_N$$

where
$$N = \frac{(\hat{\sigma}_d^2/n + \hat{\sigma}_c^2/m)^2}{\frac{\hat{\sigma}_d^4}{n^2(n-1)} + \frac{\hat{\sigma}_c^4}{m^2(m-1)}} \ge \min(n, m), \quad \le m+n$$

* Welch-Satterthwaite formula

Advantage & drawback of Student's test

· Advantage: Non-asymptotic / Can be run on small samples + can always use if large sample size

· Drawback: assumption that sample is Gaussian

Lecture 14: Wald's Test, Likelihood Ratio Test and Implicit Hypothesis Test

Wald's Test: A test based on the MLE

① Consider an iid sample $X_1, \dots, X_n$ with statistical model $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$, where $\Theta \subseteq \mathbb{R}^d$ ($d \ge 1$) and let $\theta_0 \in \Theta$ be fixed and given.

② Hypothesis: $\begin{cases} H_0: \theta^* = \theta_0 \\ H_1: \theta^* \neq \theta_0 \end{cases}$

③ Let $\hat{\theta}^{MLE}$ be the MLE estimate. Assume the MLE technical conditions are satisfied.

④ If H_0 is true, then:

Asymptotic normality of MLE:

$$\sqrt{n}\, I(\theta_0)^{1/2} \cdot (\hat{\theta}_n^{MLE} - \theta_0) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, I_d)$$

$\sqrt{n}\, I(\theta^*)^{1/2}$ (since $\theta^* = \theta_0$)

$\sqrt{n}\, I(\hat{\theta}^{MLE})^{1/2}$ (slutsky)

If $Z \sim \mathcal{N}_d(0, I_d)$, then $\|Z\|_2^2 \sim \chi^2_d$

⑤ Hence,

$$\underbrace{n(\hat{\theta}_n^{MLE} - \theta_0)^T I(\hat{\theta}^{MLE})(\hat{\theta}_n^{MLE} - \theta_0)}_{T_n} \xrightarrow[n\to\infty]{d} \chi^2_d$$

⑥ Wald's test with asymptotic level $\alpha \in (0,1)$:

$$\psi = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of χ^2_d

* Wald's test is also valid if H_1 has the form "$\theta > \theta_0$" or "$\theta < \theta_0$" or "$\theta = \theta_1$"... (But less powerful)

Likelihood ratio test: A test based on the log-likelihood (MLE)

① Consider an iid sample $X_1, \dots, X_n$ with statistical model $(E, (\mathbb{P}_\theta)_{\theta \in \Theta})$, where $\Theta \subseteq \mathbb{R}^d$ ($d \ge 1$)

② Suppose the null hypothesis has the form

$$H_0: (\theta_{r+1}, \dots, \theta_d) = (\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d)$$

($d-r$ parameters) for some fixed and given numbers $\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d$

③ Let

$$\hat{\theta}_n = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \ell_n(\theta) \quad (MLE)$$

and

$$\hat{\theta}^c_n = \underset{\theta \in \Theta^c}{\operatorname{argmax}}\ \ell_n(\theta) \quad (\text{"constrained MLE"})$$

$\Theta^c = \{\theta \in \Theta : (\theta_{r+1}, \dots, \theta_d) = (\theta^{(0)}_{r+1}, \dots, \theta^{(0)}_d)\}$

④ Test statistic:

$$T_n = 2(\ell_n(\hat{\theta}_n) - \ell_n(\hat{\theta}^c_n))$$

⑤ Wilks' Theorem:

Assume H_0 is true and MLE technical conditions are satisfied, then,

$$T_n \xrightarrow[n\to\infty]{(d)} \chi^2_{d-r}$$

⑥ Likelihood ratio test with asymptotic level $\alpha \in (0,1)$:

$$\psi = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of χ^2_{d-r}

Wald's test for implicit hypothesis

① Let $X_1, \dots, X_n$ be iid random variables and let $\theta \in \mathbb{R}^d$ be a parameter associated with the distribution of X_1 (e.g. a moment, the parameter of a statistical model, etc...)

② Let $g: \mathbb{R}^d \to \mathbb{R}^k$ be continuously differentiable (with $k < d$)

③ Consider the following hypothesis

$H_0: g(\theta) = 0$ v.s. $H_1: g(\theta) \neq 0$

e.g. $g(\theta) = (\theta_1, \theta_2)$ ($k=2$), or $g(\theta) = \theta_1 - \theta_2$ ($k=1$), ...

④ Suppose an asymptotically normal estimator $\hat{\theta}_n$ is available:

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_d(0, \Sigma(\theta))$$

↓ Delta method

$$\sqrt{n}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k(0, \Gamma(\theta))$$

$$\Gamma(\theta) = \nabla g(\theta)^T \Sigma(\theta) \nabla g(\theta) \in \mathbb{R}^{k \times k}$$

⑤ Assume $\Sigma(\theta)$ is invertible and $\nabla g(\theta)$ has rank k. So, $\Gamma(\theta)$ is invertible and:

$$\sqrt{n}\,\Gamma(\theta)^{-1/2}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k(0, I_k)$$

↓ Slutsky, if $\Gamma(\theta)$ is continuous in θ,

$$\sqrt{n}\,\Gamma(\hat{\theta})^{-1/2}(g(\hat{\theta}_n) - g(\theta)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}_k(0, I_k)$$

⑥ Hence, if θ_0 is true, i.e., $g(\theta) = 0$,

$$\underbrace{n\, g(\hat{\theta}_n)^T \Gamma^{-1}(\hat{\theta}_n)\, g(\hat{\theta}_n)}_{T_n} \xrightarrow[n\to\infty]{(d)} \chi^2_k$$

⑦ Test with asymptotic level α:

$$\psi = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of χ^2_k

Lecture 15: Goodness of Fit Test for Discrete Distributions

Goodness of fit tests - motivation

e.g. Let X be a r.v.. Given iid copies of X we want to answer:

- Does X have distribution $\mathcal{N}(0,1)$?
- Does X have distribution $\mathcal{U}[0,1]$?
- Does X have PMF $p_1 = 0.3$, $p_2 = 0.5$, $p_3 = 0.2$

These are all goodness of fit (GoF) tests: we want to know if the hypothesized distribution is a good fit for the data

* Key characteristic: no parametric

→ null hypothesis would be ~parametric; potential alternative could be more complicated

Discrete distribution - Probability Simplex

Let $E = \{a_1, \dots, a_K\}$ be a finite space, and $(\mathbb{P}_p)_{p \in \Delta_K}$ be the family of all probability distributions on E:

Δ_K: Probability Simplex in $\mathbb{R}^K$, is the set of all vectors $p = [p_1, \dots, p_K]^T$ such that:

$$p^T \mathbf{1} = \mathbf{1}^T p = 1, \quad p_i \ge 0 \text{ for all } i = 1, \dots, K$$

$\mathbf{1} = (1, 1, \dots, 1)^T$

Equivalently:

$$\Delta_K = \{p = (p_1, \dots, p_K) \in [0,1]^K : \sum_{i=1}^K p_i = 1\}$$

For $p \in \Delta_K$ and $X \sim \mathbb{P}_p$,

$$\mathbb{P}_p[X = a_j] = p_j, \quad j = 1, \dots, K$$

Multinomial Distribution

· Definition:

Multinomial distribution with K modalities (K possible outcomes) is a generalization of binomial distribution. It models probabilities of counts of K possible outcomes of experiment in n' iid trials of experiment.

· Parameters: n', $p_1, \dots, p_K$

n' → number of iid trials; p_i → probability of observing outcome i in any trial, $p_i \ge 0$, $\sum_{i=1}^K p_i = 1$. Let $p \triangleq [p_1\ p_2 \dots p_K]^T$, $p \in \Delta_K$

· Representation:

Multinomial distribution can be represented by random vector $N \in \mathbb{Z}^K$

$N^{(i)}$: number of instance of outcome i; $\sum_{i=1}^K N^{(i)} = n'$

Multinomial pmf:
$$p_N(N^{(1)} = n^{(1)}, \dots, N^{(K)} = n^{(K)}) = \frac{n'!}{n^{(1)}!\, n^{(2)}! \cdots n^{(K)}!} \prod_{i=1}^K p_i^{n^{(i)}}$$

Categorical (Generalized Bernoulli) Distribution and likelihood

· Multinomial distribution with:

- $n' = 1$: categorical distribution
  - $K = 2$, outcomes $\{0, 1\}$: Bernoulli distribution
  - $K > 2$: Generalized Bernoulli distribution

· Categorical random variable sample space:

$$E = \{a_1, \dots, a_K\}$$

· Categorical random variable pmf:

$$P(X = a_j) = \prod_{i=1}^K p_i^{\mathbb{1}(a_i = a_j)} = p_j, \quad j = 1, \dots, K$$

· Categorical statistical model:

$$(\{a_1, \dots, a_K\}, \{P_p\}_{p \in \Delta_K})$$

· Categorical likelihood: $X_1, \dots, X_n \overset{iid}{\sim} X$, outcomes K, number of occurrences N_i, $i = 1, \dots, K$

$L_n(X_1, \dots, X_n, p_1, \dots, p_K) = p_1^{N_1} p_2^{N_2} \cdots p_K^{N_K}$ ← for n iid sequence of outcomes

$L(X, p_1, \dots, p_K) = \prod_{i=1}^K p_i^{\mathbb{1}(X = a_i)}$ ← for r.v. X

Goodness of fit test (discrete, χ^2)

① Let $X_1, \dots, X_n \overset{iid}{\sim} \mathbb{P}_p$, for some unknown $p \in \Delta_K$, and $p^0 \in \Delta_K$ is fixed

Test: $H_0: p = p^0$ v.s. $H_1: p \neq p^0$ with asymptotic level $\alpha \in (0,1)$

② Likelihood of model:

$$L_n(X_1, \dots, X_n, p) = p_1^{N_1} p_2^{N_2} \cdots p_K^{N_K}$$

where $N_j = \#\{i = 1, \dots, n : X_i = a_j\}$

Let $\hat{p}$ be the MLE:

$$\hat{p}_j = \frac{N_j}{n}, \quad j = 1, \dots, K$$

→ $\hat{p}$ maximizes $\log L_n(X_1, \dots, X_n, p)$ under the constraint

③ χ^2 test:

· If H_0 is true, then: $\sqrt{n}(\hat{p} - p^0)$ is asymptotically normal, and the following holds

Theorem (under H_0)

$$\underbrace{n \sum_{j=1}^K \frac{(\hat{p}_j - p_j^0)^2}{p_j^0}}_{T_n} \xrightarrow[n\to\infty]{(d)} \chi^2_{K-1}$$

· χ^2 test with asymptotic level α:

$$\psi_\alpha = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of χ^2_{K-1}

· Asymptotic p-value of this test:

$$\text{p-value} = \mathbb{P}[Z > T_n | T_n]$$

$Z \sim \chi^2_{K-1}$, $Z \perp\!\!\!\perp T_n$

* Pointwise convergence:

$$\lim_{n\to\infty} g_n(x) = g(x)$$

* Uniform convergence:

$$\lim_{n\to\infty} \sup_{x \in \mathbb{R}} |g_n(x) - g(x)| = 0$$

For every $M > 0$, there exists an n_M s.t. $\sup_x |g_n(x) - g(x)| < M$ for all $n > n_M$

χ^2 test for a family of distribution:

① $X_1, \dots, X_n \overset{iid}{\sim} P$ denote iid discrete r.v. supported on $\{0, \dots, K\}$

Test: $H_0: P \in \{Bin(K, \theta)\}_{\theta \in (0,1)}$

$H_1: P \notin \{Bin(K, \theta)\}_{\theta \in (0,1)}$

② Let f_θ denote pmf of $Bin(K, \theta)$

$\hat{\theta}$ denote MLE of parameter θ

N_j denote number of times j ($j \in \{0, 1, \dots, K\}$) appears in data set $X_1, \dots, X_n$

③ χ^2 test

$$T_n := n \sum_{j=0}^K \frac{\left(\frac{N_j}{n} - f_{\hat{\theta}}(j)\right)^2}{f_{\hat{\theta}}(j)} \xrightarrow[n\to\infty]{(d)} \chi^2_{(K+1)-d-1}$$

$(K+1)$ → support size; d → dimension of θ; use $f_{\hat{\theta}}(j)$ to estimate $P(X=j)$

Lecture 16: Goodness of fit tests continued
Kolmogorov-Smirnov, Kolmogorov-Lilliefors Tests, QQ plots

CDF and empirical CDF

Let $X_1, \dots, X_n$ be iid real random variables.

· CDF: $F(t) = \mathbb{P}[X_1 \le t]$, $\forall t \in \mathbb{R}$ (completely characterizes distribution of X_1)

· Empirical cdf: $F_n(t) = \frac{1}{n}\sum_{i=1}^n \mathbb{1}\{X_i \le t\} = \frac{\#\{i = 1, \dots, n : X_i \le t\}}{n}$, $\forall t \in \mathbb{R}$

- Consistency:

(by LLN): for all $t \in \mathbb{R}$, $F_n(t) \xrightarrow[n\to\infty]{a.s.} F(t)$

Glivenko-Cantelli Theorem (Fundamental theorem of statistics)

$$\sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n\to\infty]{a.s.} 0$$

- Asymptotic normality:

(by CLT): for all $t \in \mathbb{R}$, $\sqrt{n}(F_n(t) - F(t)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, F(t)(1 - F(t)))$

Donsker's Theorem: If F is continuous, then

$$\sqrt{n} \sup_{t \in \mathbb{R}} |F_n(t) - F(t)| \xrightarrow[n\to\infty]{(d)} \sup_{0 \le x \le 1} |\mathbb{B}(x)|$$

(sup on $\mathbb{R}$ → $[0,1]$, range of $F(t)$); $\mathbb{B}$ → Brownian bridge on $[0,1]$, pivotal

Goodness of fit test (continuous, Kolmogorov-Smirnov)

① Let $X_1, \dots, X_n$ be iid real random variable with unknown cdf F

Let F^0 be a continuous cdf

Test: $H_0: F = F^0$ versus $H_1: F \neq F^0$

② Let F_n be empirical cdf of samples $X_1, \dots, X_n$

* If $F = F^0$, then $F_n(t) \approx F^0(t)$, for all $t \in \mathbb{R}$

③ Kolmogorov-Smirnov test

$$T_n = \sup_{t \in \mathbb{R}} \sqrt{n}\, |F_n(t) - F^0(t)|$$

By Donsker's theorem, if H_0 is true, then:

$T_n \xrightarrow[n\to\infty]{(d)} Z$ → supremum of a Brownian bridge

④ KS test with asymptotic level α:

$$\delta_\alpha^{KS} = \mathbb{1}\{T_n > q_\alpha\}$$

q_α → $(1-\alpha)$-quantile of Z (obtained in tables)

⑤ p-value of KS test:

$$\mathbb{P}[Z > T_n | T_n]$$

* Computational issues

Let $X_{(1)} \le X_{(2)} \le \dots \le X_{(n)}$ be reordered sample.

$$T_n = \sqrt{n} \max_{i=1,\dots,n} \left\{ \max\left(\left|\frac{i-1}{n} - F^0(X_{(i)})\right|, \left|\frac{i}{n} - F^0(X_{(i)})\right| \right) \right\}$$

* Pivotal distribution

T_n is called a pivotal statistic: If H_0 is true, the distribution of T_n does not depend on distribution of X_i's and it is easy to reproduce it in simulations

Let $U_i = F^0(X_i)$, $i = 1, \dots, n$. Let G_n be empirical cdf of $U_1, \dots, U_n$.

If H_0 is true, then: $U_1, \dots, U_n \overset{iid}{\sim} Unif[0,1]$

$$T_n = \sup_{0 \le x \le 1} \sqrt{n}\, |G_n(x) - x|$$

* Quantiles and p-values

For some large integer M:

Simulate M iid copies of $T_n^1, \dots, T_n^M$ of T_n;

Estimate $(1-\alpha)$-quantile $q_\alpha^{(n)}$ of T_n by taking the sample $(1-\alpha)$-quantile $\hat{q}_\alpha^{(n,M)}$ of $T_n^1, \dots, T_n^M$.

Test with approximate level α:

$$\delta_\alpha = \mathbb{1}\{T_n > \hat{q}_\alpha^{(n,M)}\}$$

Approximate p-value of this test:

$$\text{p-value} \approx \frac{\#\{j = 1, \dots, M : T_n^j > T_n\}}{M}$$

* Other goodness of fit tests

We want to know distance of two functions: $F_n(t)$ and $F(t)$

- Kolmogorov-Smirnov: $d(F_n, F) = \sup_{t \in \mathbb{R}} |F_n(t) - F(t)|$ (L_∞)
- Cramér-Von Mises: $d^2(F_n, F) = \int_{\mathbb{R}} [F_n(t) - F(t)]^2 dF(t)$ (L_2)
- Anderson-Darling: $d^2(F_n, F) = \int_{\mathbb{R}} \frac{[F_n(t) - F(t)]^2}{F(t)(1 - F(t))} dF(t)$

* Motivation for Composite Goodness of Fit Tests

What if I want to test: "Does X have Gaussian distribution?" but I don't have parameters?

Plug in: $\sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\hat{\mu}, \hat{\sigma}^2}(t)|$

$\hat{\mu} = \bar{X}_n$, $\hat{\sigma}^2 = S_n^2$, $\Phi_{\hat{\mu}, \hat{\sigma}^2}(t)$: cdf of $\mathcal{N}(\hat{\mu}, \hat{\sigma}^2)$

⇓

Donsker's Theorem is no longer valid

Goodness of fit test (continuous, Kolmogorov-Lilliefors)

Test statistics: $\sup_{t \in \mathbb{R}} |F_n(t) - \Phi_{\hat{\mu}, \hat{\sigma}^2}(t)|$ → pivot, use table

Quantile-Quantile (QQ) plots

· A visual way of performing GoF tests

· Not a formal test, but quick and easy way to check if a distribution is plausible

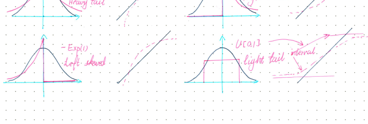
· Main idea: check visually if the plot F_n is close to that of F or equivalently if the plot of F_n^{-1} is close that of F^{-1}

· Define: $F_n^{-1}(i/n) = X_{(i)}$ → i-th largest observation

· Check if points:

$$\left(F^{-1}(\tfrac{1}{n}), F_n^{-1}(\tfrac{1}{n})\right), \left(F^{-1}(\tfrac{2}{n}), F_n^{-1}(\tfrac{2}{n})\right), \dots, \left(F^{-1}(\tfrac{n-1}{n}), F_n^{-1}(\tfrac{n-1}{n})\right)$$

are near $y = x$



Unit 5: Bayesian Statistics
Lecture 17: Introduction to Bayesian Statistics

Frequentist Approach

· Assume Statistical Model $(E, \{P_\theta\}_{\theta \in \Theta})$

· Assume that data $X_1, \dots, X_n$ was drawn iid from $\mathbb{P}_{\theta^*}$ for some unknown fixed θ^*

· When we use MLE for example, we looked at all possible $\theta \in \Theta$

Before seeing the data, we did not prefer a choice of $\theta \in \Theta$ over another

Bayesian Approach

· In many practical contexts, we have prior belief of θ^*

· Using the data, we want to update that belief and transform it into posterior belief.

Prior and Posterior

① Prior distribution: a probability distribution on a parameter space Θ, with some pdf $\pi(\cdot)$

② Let $X_1, \dots, X_n$ be a sample of n random variables

③ Denote by $L_n(\cdot|\theta)$ the joint pdf of $X_1, \dots, X_n$ conditionally on θ, where $\theta \sim \pi$

$L_n(X_1, \dots, X_n|\theta)$: likelihood used in frequentist approach

④ Posterior distribution: conditional distribution of θ given by $X_1, \dots, X_n$

Denote by $\pi(\cdot|X_1, \dots, X_n)$ its pdf

Baye's formula

$$\pi(\theta|X_1, \dots, X_n) = \frac{\pi(\theta) L_n(X_1, \dots, X_n|\theta)}{\int_\Theta \pi(\theta) L_n(X_1, \dots, X_n|\theta)} \quad \forall \theta \in \Theta$$

(denominator → constant, does not depend on θ)

⇓

$$\pi(\theta|X_1, \dots, X_n) \propto \pi(\theta) L_n(X_1, \dots, X_n|\theta)$$

e.g.

Bernoulli experiment with Beta prior

① $p \sim Beta(a, a)$: $\pi(p) \propto p^{a-1}(1-p)^{a-1}$, $p \in (0,1)$

② Given p, $X_1, \dots, X_n \overset{iid}{\sim} Ber(p)$, so: $L_n(X_1, \dots, X_n|p) = p^{\sum_{i=1}^n X_i}(1-p)^{n - \sum_{i=1}^n X_i}$

③ Hence, $\pi(p|X_1, \dots, X_n) \propto p^{a-1+\sum_{i=1}^n X_i}(1-p)^{a-1+n-\sum_{i=1}^n X_i}$

→ $Beta(a + \sum_{i=1}^n X_i,\ a + n - \sum_{i=1}^n X_i)$

* Conjugate prior (posterior is same family with prior)

Non-informative priors

* We can still use a Bayesian approach if we have no prior information

How to pick π?

· Good candidate: $\pi(\theta) \propto 1$ (constant pdf on Θ)

- If Θ bounded: uniform
- If Θ unbounded: does not define a proper pdf on Θ

Improper prior: a measurable, nonnegative function $\pi(\cdot)$ defined on Θ that is not integrable

In general, one can still define a posterior distribution using an improper prior, using Baye's formula

Jefferys prior

· Jefferys prior: $\pi_J(\theta) \propto \sqrt{\det I(\theta)}$

$I(\theta)$ → Fisher information matrix of statistical model associated with $X_1, \dots, X_n$ in the frequentist approach (provided it exists)

more weight → ① when $I(\theta)$ is high; ② values of θ where MLE has less uncertainty; ③ data giving more info; ④ potential outcomes more sensitive to changes

· Examples:

- Bernoulli: $\pi_J(p) \propto \frac{1}{\sqrt{p(1-p)}}$, $p \in (0,1)$, is a $Beta(\frac{1}{2}, \frac{1}{2})$
- Gaussian ($\mathcal{N}(\theta, 1)$): $\pi_J(\theta) \propto 1$, $\theta \in \mathbb{R}$, is an improper prior

· Jefferys prior satisfies a reparametrization invariance principle:

- If η is reparametrization of θ (i.e., $\eta = \phi(\theta)$ for some one-to-one map ϕ), then the pdf $\tilde{\pi}(\cdot)$ of η satisfies:

$$\tilde{\pi}(\eta) \propto \sqrt{\det \tilde{I}(\eta)}$$

$\tilde{I}(\eta)$ → fisher information of statistical model parametrized by η instead of θ

- If η is reparametrization of θ, $\phi(\theta) = \eta$ and ϕ is invertible:

$$\tilde{\pi}_J(\eta) = \pi_J(\phi(\theta))\frac{d\theta}{d\eta} = \pi_J(\phi(\theta))\frac{1}{\phi'(\theta)}$$

Bayesian confidence regions

· Definition: Bayesian confidence region with level α ($\alpha \in (0,1)$) is a random subset $\mathcal{R}$ of parameter space Θ, which depends on sample $X_1, \dots, X_n$, such that:

$$\mathbb{P}(\theta \in \mathcal{R}|X_1, \dots, X_n) = 1 - \alpha$$

($\mathcal{R}$ ← depend on prior $\pi(\cdot)$)

* "Bayesian confidence region" and "confidence interval" are two distinct notions

Bayesian estimation:

* The Bayesian framework can also be used to estimate the true underlying parameter (hence, in frequentist approach).

In this case, prior distribution does not reflect a prior belief: It is just an artificial tool used in order to define a new class of estimators.

· Bayesian estimator:

$$\hat{\theta}^{(\pi)} = \int_\Theta \theta \cdot \pi(\theta|X_1, \dots, X_n)\, d\theta \quad (\text{posterior mean})$$

→ depends on choice of prior distribution π (hence superscript π)

Another choice: $\hat{\theta}^{MAP} = \underset{\theta \in \Theta}{\operatorname{argmax}}\ \pi(\theta|X_1, \dots, X_n)$ (MAP → maximum a posteriori, provided it exists...)

$$= \underset{\theta \in \Theta}{\operatorname{argmax}}\ \pi(\theta) \cdot L_n(X_1, \dots, X_n|\theta)$$

* In general, the asymptotic properties of Bayes estimator do not depend on choice of prior

Unit 6: Linear Regression

Lecture 19: Linear Regression 1

Modeling Assumption

- (X_i, Y_i), $i=1,\dots,n$ are iid for some unknown joint distribution $\mathbb{P}$
- $\mathbb{P}$ can be described entirely by (assuming all exist)
  - Either a joint PDF $h(x,y)$
  - The marginal density of X $h(x)=\int h(x,y)\,dy$ and the conditional density $h(y|x)=\frac{h(x,y)}{h(x)}$

Partial modeling

We can also describe the distribution only partially, e.g. using
- The expectation of Y: $\mathbb{E}[Y]$
- The conditional expectation of Y given $X=x$: $\mathbb{E}[Y|X=x]$
  The function
  $$x \mapsto f(x) := \mathbb{E}[Y|X=x] = \int y\, h(y|x)\,dy$$
  is called regression function
- Other possibilities:
  - conditional median: $m(x)$ such that $\int_{-\infty}^{m(x)} h(y|x)\,dy = \frac{1}{2}$
  - conditional quantile
  - conditional variance (no information about location)

Linear regression

Focus on modeling the regression function $f(x)=\mathbb{E}[Y|X=x]$

* Too many possible regression functions f (nonparametric)

⇓ Useful to restrict to simple functions that are described by a few parameters

⇓ Simplest: $f(x)=a+bx$ linear / affine function

⇓ Under this assumption, we discuss linear regression.

Probabilistic analysis

① Let X and Y be two real r.v. (not necessarily independent) with two moments and such that $var(X)>0$

② The theoretical linear regression of Y on X is the line $x \mapsto a^* + b^* x$ where
$$(a^*, b^*) = \underset{(a,b)\in\mathbb{R}^2}{\operatorname{argmin}}\ \mathbb{E}[(Y-a-bX)^2]$$

③ Setting partial derivatives to 0 gives:
$$b^* = \frac{cov(X,Y)}{var(X)}$$
$$a^* = \mathbb{E}[Y] - b^*\mathbb{E}[X] = \mathbb{E}[Y] - \frac{cov(X,Y)}{var(X)}\mathbb{E}[X]$$

④ Noise:
Clearly the points are not exactly on the line $x\mapsto a^*+b^*x$ if $Var(Y|X=x)>0$
The random variable $\varepsilon = Y-(a^*+b^*X)$ is called noise and satisfies:
$$Y = a^* + b^*X + \varepsilon$$
with: $\begin{cases} E[\varepsilon]=0 \\ cov(X,\varepsilon)=0 \end{cases}$

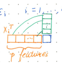
Least squares

- Definition: The least squares estimator (LSE) of (a,b) is the minimizer of sum of squared errors:
$$\sum_{i=1}^n (Y_i - a - bX_i)^2$$
- $\hat a, \hat b$ is given by $\begin{cases} \hat b = \dfrac{\overline{XY} - \bar X \bar Y}{\overline{X^2} - \bar X^2} \\ \hat a = \bar Y - \hat b \bar X \end{cases}$
- Residuals: $\hat\varepsilon_i = |y_i - (\hat a + \hat b x_i)|$

Lecture 20: Linear Regression 2

Multivariate regression

$$Y_i = X_i^T \beta^* + \varepsilon_i, \quad i=1,\dots,n$$
$$X_i = \begin{pmatrix} X_i^{(1)} \\ X_i^{(2)} \\ \vdots \\ X_i^{(p)} \end{pmatrix}$$



① Vector of explanatory variables / covariates: $X_i \in \mathbb{R}^p$ → wlog, assume its first coordinate is 1
Response / dependent variable: Y_i

② $\beta^* = (a^*, b^{*T})^T$; $\beta_1^* (=a^*)$ is called intercept
$\{\varepsilon_i\}_{i=1,\dots,n}$: noise terms satisfying $cov(X_i, \varepsilon_i)=0$

*Definition:
The least squares estimator (LSE) of β^* is the minimizer of squared error
$$\hat\beta = \underset{\beta\in\mathbb{R}^p}{\operatorname{argmin}} \sum_{i=1}^n (Y_i - X_i^T\beta)^2$$

LSE in matrix form:

① Let $Y = (Y_1, \dots, Y_n)^T \in \mathbb{R}^n$
$\mathbb{X}$ be $n\times p$ matrix whose rows are $X_1^T, \dots, X_n^T$ (design matrix) $\begin{bmatrix} - X_1 - \\ - X_2 - \\ \vdots \\ - X_n - \end{bmatrix}$
$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)^T \in \mathbb{R}^n$ (unobserved noise)

② $Y = \mathbb{X}\beta^* + \varepsilon$ (β^* unknown)

③ The LSE $\hat\beta$ satisfies:
$$\hat\beta = \underset{\beta\in\mathbb{R}^p}{\operatorname{argmin}} \|Y - \mathbb{X}\beta\|_2^2$$

④ Closed form solution:
Assume $rank(\mathbb{X})=p$, analytical computation of the LSE
$$\hat\beta = (\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T Y$$
* Geometric interpretation of the LSE: $\mathbb{X}\hat\beta$ is the orthogonal projection of Y onto the subspace spanned by the columns of $\mathbb{X}$:
$$\mathbb{X}\hat\beta = PY$$
→ $P = \mathbb{X}(\mathbb{X}^T\mathbb{X})^{-1}\mathbb{X}^T$

Statistical inference:

$$Y = \mathbb{X}\beta^* + \varepsilon$$

*To make inference (confidence regions, tests) we need more assumptions
Assumptions:
① The design matrix $\mathbb{X}$ is deterministic and $rank(\mathbb{X})=p$
② The model is homoscedastic: $\varepsilon_1, \dots, \varepsilon_n$ are iid
③ The noise vector is Gaussian: $\varepsilon \sim \mathcal{N}_n(0, \sigma^2 I_n)$ for some known/unknown $\sigma^2>0$
⇓ $Y \sim \mathcal{N}_n(\mathbb{X}\beta^*, \sigma^2 I_n)$

Properties of LSE

① LSE = MLE
② Distribution of $\hat\beta$: $\hat\beta \sim \mathcal{N}_p(\beta^*, \sigma^2(\mathbb{X}^T\mathbb{X})^{-1})$
③ Quadratic risk of $\hat\beta$: $\mathbb{E}[\|\hat\beta - \beta\|_2^2] = \sigma^2 tr((\mathbb{X}^T\mathbb{X})^{-1})$
④ Predicted error: $\mathbb{E}[\|Y - \mathbb{X}\hat\beta\|_2^2] = \sigma^2(n-p)$
⑤ Unbiased estimator of σ^2: $\hat\sigma^2 = \frac{\|Y - \mathbb{X}\hat\beta\|_2^2}{n-p} = \frac{1}{n-p}\sum_{i=1}^n \hat\varepsilon_i^2$

Theorem:
$$(n-p)\frac{\hat\sigma^2}{\sigma^2} \sim \chi^2_{n-p}$$
$$\hat\beta \perp\!\!\!\perp \hat\sigma^2 \quad \text{(Cochran's theorem)}$$

Significance tests:

① Test whether the j-th explanatory variable is significant in the linear regression ($1\le j\le p$)
② $H_0: \beta_j = 0$ v.s. $H_1: \beta_j \ne 0$
③ If γ_j is the j^{th} diagonal coefficient of $(\mathbb{X}^T\mathbb{X})^{-1}$ ($\gamma_j>0$):
$$\frac{\hat\beta_j - \beta_j}{\sqrt{\hat\sigma^2\gamma_j}} \sim t_{n-p}$$
④ Let $T_n^{(j)} = \frac{\hat\beta_j}{\sqrt{\hat\sigma^2\gamma_j}}$
⑤ Test with non asymptotic level $\alpha\in(0,1)$: $R_{j,\alpha} = \mathbb{1}\{|T_n^{(j)}| > q_{\alpha/2}(t_{n-p})\}$ ← $(1-\alpha/2)$-quantile of t_{n-p}
⇒ p-values...

Bonferroni's test

① Test whether a group of explanatory variables is significant in the linear regression
② $H_0: \beta_j=0, \forall j\in S$ v.s. $H_1: \exists j\in S, \beta_j\ne 0$ where $S\subseteq\{1,\dots,p\}$
③ Bonferroni's test: $R_{S,\alpha} = \bigcup_{j\in S} R_{j,\frac{\alpha}{k}}$ where $k=|S|$
④ This test has nonasymptotic level at most α

Remarks:

- Linear regression exhibits correlation, NOT causality
- Normality of the noise: One can use goodness of fit tests to test whether the residuals $\hat\varepsilon_i = Y_i - \mathbb{X}_i^T\hat\beta$ are Gaussian
- Deterministic design: If $\mathbb{X}$ is not deterministic, all the above can be understood conditionally on $\mathbb{X}$, if the noise is assumed to be Gaussian, conditionally on X.

Unit 7: Generalized linear models

Lecture 21: Introduction to Generalized Linear Models; Exponential Families

Linear model

- A Gaussian linear model assumes:
$$Y|X=x \sim \mathcal{N}(\mu(x), \sigma^2 I)$$
$$\mathbb{E}[Y|X=x] = \mu(x) = x^T\beta$$
($\mu(x)$: regression function)
- Two model components: (to be relaxed in this unit)
  ① Random component: the response variable Y is continuous and $Y|X=x$ is Gaussian with mean $\mu(x)$
  ② Regression function: $\mu(x) = x^T\beta$ (linear)

Generalization

- A generalized linear model (GLM) generalizes normal linear regression models in the following directions:
  ① Random component: $Y|X=x \sim$ some distribution (e.g. Bernoulli, exponential, Poisson)
  ② Regression function: $g(\mu(x)) = x^T\beta$ (g: link function; $\mu(x)$: regression function)

Exponential family

- A family of Distribution $\{\mathbb{P}_\theta: \theta\in\Theta\}$, $\Theta\subset\mathbb{R}^k$ is said to be a k-parameter exponential family on $\mathbb{R}^q$, if there exist real valued functions:
$\begin{cases} \eta_1, \eta_2, \dots, \eta_k \text{ and } B \text{ of } \theta \\ T_1, T_2, \dots, T_k \text{ and } h \text{ of } y\in\mathbb{R}^q \end{cases}$ such that the density function (pmf/pdf) of $\mathbb{P}_\theta$:
$$f_\theta(y) = \exp\Big[\sum_{i=1}^k \eta_i(\theta)T_i(y) - B(\theta)\Big]h(y)$$
($[\eta_1(\theta) \cdots \eta_k(\theta)]$: $\mathbb{R}^k\to\mathbb{R}^k$; $\begin{bmatrix} T_1(y) \\ \vdots \\ T_k(y) \end{bmatrix}$: $\mathbb{R}^q\to\mathbb{R}^k$; B: $\mathbb{R}^k\to\mathbb{R}$; h: $\mathbb{R}^q\to\mathbb{R}$)
* $k=q=1$: $f_\theta(y) = \exp[\eta(\theta)T(y) - B(\theta)]h(y)$

One-parameter canonical exponential family

- Canonical exponential family for $k=1$, $y\in\mathbb{R}$
$$f_\theta(y) = \exp\Big(\frac{y\theta - b(\theta)}{\phi} + c(y,\phi)\Big)$$
(ϕ → dispersion parameter)
for some known function $b(\cdot)$ and $c(\cdot,\cdot)$
  * If ϕ is known: this is a one-parameter exponential family with θ being canonical parameter
  * If ϕ is unknown, this may/may not be a two parameter exponential family
- In this class, we always assume that ϕ is known
$$f_\theta(y) = \exp\Big(\frac{y\theta - b(\theta)}{\phi}\Big)\cdot h(y) \quad \to h(y) = e^{c(y,\phi)}$$

	Normal	Poisson	Bernoulli
Notation	$\mathcal{N}(\mu,\sigma^2)$	$P(\mu)$	$B(p)$
Range of y	$(-\infty,\infty)$	$[0,\infty)$	$\{0,1\}$
ϕ	σ^2	1	1
$b(\theta)$	$\frac{\theta^2}{2}$	e^θ	$\log(1+e^\theta)$
$c(y,\phi)$	$-\frac{1}{2}(\frac{y^2}{\phi}+\log(2\pi\phi))$	$-\log y!$	0

Likelihood

Let $l(\theta) = \log f_\theta(Y)$ denote the log likelihood function. The mean $\mathbb{E}[Y]$ and variance $var(Y)$ can be derived from following:

① First identity $\mathbb{E}[\frac{\partial l}{\partial\theta}] = 0$
Second identity $\mathbb{E}[\frac{\partial^2 l}{\partial\theta^2}] + (\mathbb{E}[\frac{\partial l}{\partial\theta}])^2 = 0$

② Expected value proof: Note that $l(\theta) = \frac{Y\theta - b(\theta)}{\phi} + c(Y;\phi)$
Therefore: $\frac{\partial l}{\partial\theta} = \frac{Y - b'(\theta)}{\phi}$
It yields: $0 = \mathbb{E}[\frac{\partial l}{\partial\theta}] = \frac{E[Y] - b'(\theta)}{\phi}$
Which leads to: $\mathbb{E}[Y] = b'(\theta)$

③ Variance proof: We have: $\frac{\partial^2 l}{\partial\theta^2} + (\frac{\partial l}{\partial\theta})^2 = -\frac{b''(\theta)}{\phi} + \Big(\frac{Y - b'(\theta)}{\phi}\Big)^2$ ($b'(\theta)$: $E[Y]$)
from previous result:
$$\frac{Y - b'(\theta)}{\phi} = \frac{Y - \mathbb{E}(Y)}{\phi}$$
together, with second identity, this yields:
$$0 = -\frac{b''(\theta)}{\phi} + \frac{var(Y)}{\phi^2}$$
which leads to:
$$var(Y) = b''(\theta)\cdot\phi$$

Lecture 22: GLM - Link function and the canonical link function

Link function

- β is the parameter of interest, needs to appear somewhere in likelihood function, to enable usage of maximum likelihood
- A link function g relates the linear predictor $X^T\beta$ to the mean parameter $\mu(x)$
$$X^T\beta = g(\mu(x))$$
- g is required to be monotonically increasing and differentiable
$$\mu(x) = g^{-1}(X^T\beta)$$
- Examples:
  ① Linear model: $g(\cdot)$ = identity
  ② Poisson data. Suppose $Y|X \sim Poisson(\mu(X))$
  $\mu(X) > 0$
  $\log(\mu(X)) = X^T\beta$
  * In general, a link function for count data should map $(0,\infty)$ to $\mathbb{R}$. the log-link is a natural one
  ③ Bernoulli / Binomial data:
  $0<\mu<1$
  g should map $(0,1)$ to $\mathbb{R}$
  2 choices: logit: $\log\big(\frac{\mu(X)}{1-\mu(X)}\big) = X^T\beta$ → natural
  probit: $\Phi^{-1}(\mu(X)) = X^T\beta$ (Φ^{-1} → normal cdf inverse)

Canonical link

- The function g that links the mean $\mu(x)$ to the canonical parameter θ, is called canonical link:
$$g(\mu(x)) = \theta$$
- Since $\mu(x) = b'(\theta)$, the canonical link is given by:
$$g(\mu(x)) = (b')^{-1}(\mu(x))$$
- If $\phi>0$, the canonical link function is strictly increasing
  * Proof: $g\uparrow$ str. $\Leftrightarrow g^{-1}\uparrow$ str. $\Leftrightarrow (g^{-1})'>0 \Leftrightarrow b''>0 \Leftrightarrow var>0, \phi>0$

	$b(\theta)$	$g(\mu)$
Normal	$\theta^2/2$	μ
Poisson	$\exp(\theta)$	$\log\mu$
Bernoulli	$\log(1+e^\theta)$	$\log\frac{\mu}{1-\mu}$
Gamma	$-\log(-\theta)$	$-\frac{1}{\mu}$

Model and notation

① Let $(X_i, Y_i)\in\mathbb{R}^p\times\mathbb{R}$, $i=1,\dots,n$, be independent r.v. pairs s.t. the conditional distribution of Y_i given $X_i=x_i$ has density in the canonical exponential family
$$f_{\theta_i}(y_i) = \exp\Big\{\frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i,\phi)\Big\}$$
② $Y = (Y_1, \dots, Y_n)^T$, $\mathbb{X} = (X_1, \dots, X_n)^T = \begin{pmatrix} -X_1^T- \\ -X_2^T- \\ \vdots \\ -X_n^T- \end{pmatrix} \in\mathbb{R}^{n\times p}$
③ Here the mean $\mu_i = \mathbb{E}[Y_i|X_i]$ is related to the canonical parameter θ_i via:
$$\mu_i = b'(\theta_i)$$
and μ_i depends linearly on the covariates through a link function g:
$$g(\mu_i) = X_i^T\beta$$

Back to β

- Given a link function g, note the following relationship between β and θ:
$$\theta_i = (b')^{-1}(\mu_i) = (b')^{-1}(g^{-1}(X_i^T\beta)) \equiv h(X_i^T\beta)$$
where h is defined as:
$$h = (b')^{-1}\circ g^{-1} = (g\circ b')^{-1}$$
- Remark: if g is canonical link function, h is the identity ($g = (b')^{-1}$)

Log-likelihood

- The log-likelihood is given by:
$$l_n(Y, \mathbb{X}, \beta) = \sum_i \frac{Y_i\theta_i - b(\theta_i)}{\phi} + constant = \sum_i \frac{Y_i h(X_i^T\beta) - b(h(X_i^T\beta))}{\phi} + constant$$
- Note that when using canonical link function, we obtain simpler expression:
$$l_n(Y, \mathbb{X}, \beta) = \sum_i \frac{Y_i X_i^T\beta - b(X_i^T\beta)}{\phi} + constant$$

Strictly concavity:

- The log-likelihood is strictly concave (when $rank(\mathbb{X})=p$) using the canonical function when $\phi>0$
- As a consequence, MLE is unique
- If another parameterization is used, the likelihood function may not be strictly concave, leading to several local maxima

Concluding remarks

- Maximum likelihood for Bernoulli Y and the logit link is called logistic regression
- In general, there is no closed form for the MLE and we have to use optimization algorithms (e.g. gradient descent)
- The asymptotic normality of MLE also applies to GLMs

Recitation: Hypothesis tests for logistic regression

1. Set up:
$Y_i\in\{0,1\}$ (responses), $X_i\in\mathbb{R}^p$ (covariates), (X_i, Y_i) (pairs) independent, $Y_i|X_i=x_i \sim Bernoulli(p_i)$

1.1 Rewrite pmf into canonical exponential
$$\mathbb{P}(Y_i=y_i|p_i) = \begin{Bmatrix} p_i & \text{if } y_i=1 \\ 1-p_i & \text{if } y_i=0 \end{Bmatrix} = p_i^{y_i}(1-p_i)^{1-y_i}$$
$$= \exp(y_i\log p_i + (1-y_i)\log(1-p_i)) = \exp\Big(y_i\log\frac{p_i}{1-p_i} + \log(1-p_i)\Big)$$
($\log\frac{p_i}{1-p_i} = \theta_i$: canonical parameter)
$$\theta_i = \log\frac{p_i}{1-p_i} \Leftrightarrow p_i = \frac{e^{\theta_i}}{1+e^{\theta_i}} \dashrightarrow = \exp(y\theta_i - \log(1+e^{\theta_i}))$$

1.2 Using canonical link
form: $f_{\theta_i}(y_i) = \exp\big(\frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y,\phi)\big)$
$\begin{cases} \text{Canonical link: } g(\mu(x_i)) = \theta_i \\ \text{Since } \mu(x) = b'(\theta),\ g(\mu(x_i)) = (b'^{-1})(\mu(x_i)) \end{cases}$ derive form of canonical link

Link: $X^T\beta = g(\mu(x))$ (General link)
Since $\mu(x_i) = b'(\theta) \Leftrightarrow \theta = (b'^{-1})(\mu(x_i))$
⇓
$\theta = (b'^{-1})(\mu(x_i)) = (b'^{-1})(g^{-1}(X^T\beta))$
⇓ when using canonical link:
$\theta = (b'^{-1})(g^{-1}(X^T\beta)) = X^T\beta$
$g^{-1} = (b'^{-1})^{-1}$

Therefore: $\mathbb{P}(Y_i=y_i|X_i,\beta) = \exp(y_iX_i^T\beta - \log(1+e^{X_i^T\beta}))$

1.3 Test setup updated with the use of canonical link
$X_i \sim D$ iid (some distribution), $Y_i|X_i,\beta \sim Bernoulli\big(\frac{e^{X_i\beta}}{1+e^{X_i\beta}}\big)$
Log Likelihood: $l_n(\beta|X,Y) = \sum_{i=1}^n[Y_iX_i^T\beta - \log(1+e^{X_i^T\beta})]$
$\hat\beta = \underset{\beta}{\operatorname{argmax}}\ l_n(\beta|X,Y)$
Goal: Test $H_0: \beta_j = \beta_j^0$ v.s. $H_1: \beta_j \ne \beta_j^0$