A Tale of Two Markets for High Penetration of Zero-Marginal-Cost Renewable Resources

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***Abstract*—This article studies two kinds of solar energy markets at power distribution level: real-time trading market and plan-ahead rental market. A real-time distribution-level market operates in a similar way as that at the transmission-level and it can maximize the social welfare. However, a crucial difference at the distribution-level is that solar energy has zero marginal cost. We demonstrate that a real-time market has potential weaknesses of either high price volatility, or price-fixing issues with disadvantages of unpredictable outcomes and unfair benefit allocation between firms and consumers. To avoid those potential weaknesses while still keeping the maximum social welfare, we then propose a novel plan-ahead rental market, where firms lease PV panels to consumers in advance. We use game theory to prove that, as long as there are substantial renewable uncertainty and each supplier is small, this rental market has an unique pure Nash equilibrium that achieves perfect-competition. As a result, both price-volatility and price-fixing will be eliminated.**

I. Introduction

As the deployment of solar generation at the distribution level continues to rise [6], [36], there are significant interests in developing a new distribution-level electricity market that allows solar-energy producers and electricity consumers to directly trade renewable energy [26]. In today's power systems, solar generation at the distribution level has to be either consumed locally, or sold to the utility according to a combination of net-metering/feed-in-tariff, connection charges, and/or peak-based demand charges. At the same time, consumers are charged by the utility via a separate set of retail prices. These prices and tariffs are often decided by a consortium of utility, government regulators, and/or consumer interest groups. However, since their financial interests are often conflicting with each other, determining the "right" price/charges has already become a fierce political fight [12], [24]. With the introduction of new distribution-level markets, the solar producers and electricity consumers will be able to directly trade energy with each other [7], [15], [21], [23], [34], [37]. The hope is that, by its "invisible hand", a well-designed market may be more effective in discovering the "right" valuation for solar energy based on the market condition.

However, we should also keep in mind that *market is not a panacea* [5]: One may look at the Enron example [9] to see how producers can exert market power, drive up price, and significantly hurt consumers. Since distribution-level markets are still under development, there is thus a pressing need to develop a comprehensive understanding of different market designs, so that an informed choice can be made when actually deploying such markets. In particular, note that most studies of distributed-level markets (including the recent pilot program in [21]) have focused on real-time markets [7], [15], [23], [34], [37], which replicate the real-time markets at the transmission-level. Such markets set prices based on discovering the lowest "marginal cost" of generation to meet demand. Such prices can be shown to maximize the total social surplus of the system, and is thus "efficient". However, from a market designer's point of view, efficiency is usually not the only consideration. In fact, as we will show shortly, due to the much higher penetration of zero-marginal-cost solar generation, real-time markets at the distribution level tend to produce multiple equilibrium prices, all of which are efficient. Thus, it is no longer clear why pricing based on the marginal cost is the most appealing option. Other potentially equally-important considerations could be: (i) Is the market outcome fair? In other words, how is the social surplus split between consumer surplus and producer surplus? (ii) Is the market outcome unique and predictable? and (iii) Is the price stable given that solar generation can be highly uncertain? To the best of our knowledge, there have been no studies in the literature that analyze distribution-level markets under these more comprehensive lenses.

Real-time markets: In this paper, we aim to answer the above questions by first studying the outcome of real-time markets. We focus on the setting where solar generation can be predicted quite accurately at the beginning of each time-interval of real-time markets [11], [33], the marginal cost of solar producers is zero (which reflects that fact that solar generation incurs high investment cost but low production cost), and the consumer (if she wishes) can always buy electricity from the utility at a reserved price π_g. Under similar settings, prior studies have shown that, assuming that solar energy producers do not withhold energy generation, the market prices will fluctuate wildly between 0 and π_g, depending on whether the total solar generation is higher or lower than the total electricity demand [18], [30]. We advance this line of study by further studying how such price volatility will incentivize solar producers to withhold supply. In particular, when the total supply is higher than the demand, the solar producers will earn zero revenue since the price becomes 0. As a result, they will have a strong incentive to withhold their supply and raise the price. Indeed, in this paper we show that, once the solar producers can withhold supply (see discussions in Section IV-B where we argue that such withholding is quite realistic), significantly different outcomes will arise: (i) there will emerge multiple market equilibria that are all efficient in maximizing the social surplus, which makes it difficult to predict which market outcome will be realized; and (ii) one of the market equilibria will correspond to price always equal to π_g, which means that the consumer surplus is always zero

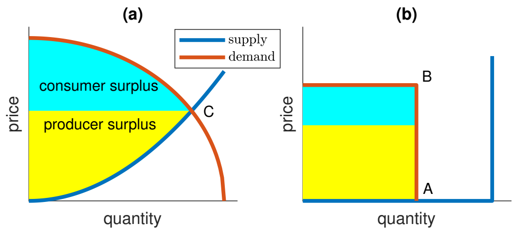

Fig. 1. (a) Classical market design theory of maximizing social surplus. (b) When the marginal cost of the supply is zero and the demand is fixed. both the supply and demand curves become highly-inelastic. Any points between A and B maximize the social surplus.

and the producers get all the surplus (thanks to the elevated price). Thus, even though the market outcome may still be "efficient," they can be (i) quite unpredictable and (ii) quite unfair, which is shown in Table I.

The role of zero marginal cost and inelastic demand: The underlying reason for the above highly-undesirable outcome is the combination of (i) zero marginal cost of solar producers, and (ii) inelasic demand. In classical market design theory, one would expect a supply curve with an upward slope and a demand curve with a downward slope (see Fig. 1(a)). As a result, there is only one point where the demand meets the supply, which maximizes the social surplus. However, when the marginal cost of the supply is zero and the demand is fixed, both the supply and demand curves become highly-inelastic (see Fig. 1(b)). Note that although the intersection point A of the supply and demand curves still maximizes the social surplus, any points between A and B will also do! Further, the suppliers have every incentive to drive the equilibrum to B, so that they earn the entire social surplus and drive the consumer surplus to zero.

Rental markets: In view of the above issues of real-time markets, we then propose an alternate form of distribution-level markets, which we refer to as rental markets. In such a rental market, consumers rent a certain amount of PV panels from solar producers in advance. Then, in real time, the consumer gets to use the electricity generated from the rented panels at no addition costs. We then study the strategic behavior of solar producers and the resulting rental price, and show the following desirable results. First, the rental price is naturally *stable* (i.e., non-volatile) as it is not immediately affected by real-time conditions. Second, as long as the solar generation is variable, the rental demand function becomes elastic (i.e., having a downward slope as in Fig. 1(a)). Thus, once the number of producers is larger than a threshold (which is function of the elasticity of the rental demand function), a *unique* outcome arises, which is always equal to the outcome of perfect competition (i.e, the point C in Fig. 1(a)) [32]. Third, at this unique market outcome, both the consumer surplus and producer surplus are not zero. Last but not least, this unique outcome also maximizes the social surplus, and is thus *efficient*. In summary, our analysis suggests that the rental market can potentially produce much more desirable market outcomes than real-time markets, once we consider not only efficiency, but also price volatility, fairness of surplus division, and predictability of outcome.

The role of solar variability: A key insight revealed from our analysis is how the variability of solar generation will affect the market outcome under different market rules. Recall that in real-time markets, since we assume that solar generation can be accurately predicted for the immediately next time-interval, there is no solar variability *within* each time-interval of real-time markets. Instead, this variability manifests as high price-volatility *across* time-intervals [18], [30]. In contrast, rental markets operate over a longer time horizon of many time-intervals. Thus, the variability directly enters into the strategic consideration of the market participants. Our main result in Section V shows that a higher level of solar variability contributes to higher elasticity of the rental demand function. Indeed, if there was no variability, we can show that the rental demand function would have also become inelastic, and then the renal market will exhibit the same pitfalls as the real-time markets. This suggests that, while the variability and uncertainty of solar generation is often considered a *detrimental* factor for real-time markets (e.g., it may lead to price-volatility [30]), it becomes a *beneficial* factor for rental markets (e.g., it produces demand elasticity and lowers the bar for perfectly-competitive outcome to arise.)

We summarize the main contribution of the paper as follows:

- We perform the first comparative study of two market designs (i.e., real-time markets and rental markets) not only in terms of efficiency, but also in terms of fairness, predictability and price volatility,
- We uncover new market equilibria for real-time markets when the producers can withhold supply. In particular, we show that multiple market equilibria can arise and one of them could lead to high price and zero consumer surplus.
- We then propose a new form of rental market, and show that, under suitable conditions on the variability of solar generaiton and the number of producers, the market outcome is unique and equals to that of perfect competititon. Further, not only is the social surplus maximized, both the producer surplus and the consumer surplus are positive.
- Our analysis reveals new insights on how solar variability impacts the market outcome under different market designs. Specifically, with high penetration of zero-marginal-cost resources, solar variability can contribute to demand elasticity in rental markets, and thus produce more favorable outcomes than real-time markets.

The rest of this paper is organized as follows. In Section III, we introduce the general model of solar energy market at the power distribution level. In Section IV, we study the real-time solar-energy trading market and demonstrate its pitfalls. In Section V, we introduce the concept of PV panel renting, and show that it produces desirable market outcomes, and is as efficient as real-time markets in terms of social surplus. Additional numerical results are provided in Section VI, and then we conclude.

II. Related work

There is significant interest in understanding how providers of uncertain renewable energy should participate in the en-

TABLE I
FEATURES COMPARISON OF DIFFERENT MARKETS.

features / markets	Social surplus	Price volatility	Fairness of surplus division	Predictability of outcome
Real-time markets (single-price bid, unrealistic)	Maximized	Highest	Relatively fair	Unique outcome (for most of the time)
Real-time markets (price-quantity bid)	Maximized	Zero	Extremely unfair (consumer surplus is 0)	Multiple Nash equilibria
Rental markets	Maximized	Zero	Relatively fair	Unique outcome (with modest assumptions)

ergy market. Existing work can be broadly divided into two categories, according to the assumption on the market price.

The first category assumes that prices are exogenously given, and studies optimal bidding strategies when market participants are price-takers [7], [15], [23]. In [23], authors study the optimal bidding strategy for renewable producers as price-takers, under the assumption of deterministic market prices known in advance. In [7], authors analyze the problem of optimizing the offering of a power contract in the day-ahead market with a fixed and known price, where uninstructed deviations from the contract in real time will be penalized. In [15], the price of distributed energy resource (DER) is given by the aggregator, and each firm can only bid the supply quantity. However, this line of work fails to capture the impact of the market participants' bidding strategies on the price.

The second category explicitly consider how the market price is formed from the agents' bidding. A significant body of related work assume traditional generators at the transmission level with significant production costs [14], [29]. In contrast, renewable generators (such as PV panels) at the distribution level have nearly-zero marginal cost. There are only limited studies on the market equilibrium with zero-marginal-cost generation, for the settings of two producers [34], with storage [18], or along with the investment game [20]. However, these results all assume that the suppliers can only change the bidding prices, but cannot withhold the supply quantiy. In contrast, our work is the first to study the setting where renewable energy suppliers can vary both price and quantity in their bids, and reveal the emergence of price-fixing behavior in real-time energy markets. Further, all of these studies assume a real-time energy market, while our work is the first to study alternative market designs in the form of rental markets.

Finally, since we allow the suppliers to bid both price and quantity, our study is also related to the literature of supply function equilibrium (SFE) game [3], [4], [8], [16], [19]. The prevailing approach to analyze the SFE is based on the assumption that the supply function is differentiable or even linear. In contrast, our main result in Theorem 1 corresponds to a supply function that is neither differentiable nor linear, and thus cannot be predicted by standard results in SFE (see further discussins in Appendix C).

III. System Model

We consider a distribution-level power system with one utility, N consumers, and M firms (i.e., suppliers).

The *utility* not only manages the physical power distribution grid, but also operates a distribution-level energy market. Further, it provides a reliable source of energy external to the distribution-level market at a fixed retail price π_g (see further discussions on consumer model below).

Firms own solar panels and sell solar energy to the distribution-level market. Note that a key attribute of solar generation is that it incurs high investment cost but low operation cost. Thus, we assume that the marginal cost of each firm is zero. For simplicity, we assume that a firm does not receive revenue for any solar energy that is not sold in the distribution-level market. This assumption corresponds to the case where utility either does not buy back electricity from the distribution level or imposes a feed-in tariff of zero. (See remark below for possible extensions.) Let C_i denote the size (in m^2) of firm i's solar panel. Let $G(t)$ denote the energy generated by unit size of solar panels at time t, which is assumed to be a random variable. We assume that all solar panels have the same $G(t)$. Therefore, the amount of solar energy available from firm i is $C_i G(t)$. Let $C = \sum_{i=1}^{M} C_i$ denote the total size of panels of all firms.

Consumers (e.g., households) consume but do not generate electricity. We let $L_n(t)$ denote the real-time electricity demand of consumer n at time t, which is a continuous random variable. Let $L(t)$ denote the total electricity demand of all consumers at time t, i.e., $L(t) = \sum_{n=1}^{N} L_n(t)$. We assume that the real-time demand is inelastic, i.e., no demand-response, which reflects the practical setting where demand elasticity is low [2]. We assume that consumers can always buy electricity from the utility at the fixed retail price π_g. On the other hand, consumers would be interested in buying the solar energy from the distribution-level market if the price is lower.

The objective of the distribution-level market is to determine the price and quantity with which firms and consumers can directly trade solar energy, based on the bids submitted by them. We are particularly interested in how the equilibirum market price is formed due to the strategic behaviors of the participants. As we discussed in the introduction, we will study not only the efficiency (i.e., whether the social surplus is maximized), but also the questions of (i) fairness of surplus division between consumers and firms; (ii) the uniqueness and predictability of the market outcome; and (iii) price volatility.

Remark: We briefly comment on some of the simplifying assumption made earlier. We assume that firms and consumers are separate, i.e., one market participant cannot be both a firm and a consumer at the same time. In reality, a firm may consume energy by herself. In that case, it is common for the firm to first use the solar energy for her own demand, and then sell the remaining solar energy to the market. Note that such firms can be equivalently viewed as with a lower generation efficiency $G(t)$. It is possible to extend our analysis to the setting with varying generation efficiency $G(t)$ across firms. Indeed, our results for real-time markets will still hold, and readers can refer to Appendix E on how an "effective panel size" can be calculated for each firm in the rental market to adjust for different $G(t)$. It is also possible to generalize our result to the setting where utility offers non-zero feed-in tariffs, without changing the main conclusions qualitatively. For example, price volatility and price-fixing in real-time markets will still hold, with the only difference being that the price will vary between π_g and the feed-in-tariff (instead of between π_g and 0).

IV. Price-volatility and Price-Fixing in Real-time markets

In this section, we will first consider real-time distribution-level energy markets, which operate in a way similar to how renewable suppliers bid in the existing transmission-level real-time energy markets [22]. We fix a time-instant t, and consider an instance of the real-time market at time t. Thus, we will drop the index t when there is no source of confusion. Let $q_i^0 = C_i G(t)$ be the actual amount of renewable electricity generated by firm i at time t, which is determined exogeneously (by solar irradiance). We assume that each firm knows the value of q_i^0 when she submits her bid to the real-time market, which is reasonable as short-term prediction of solar generation can be quite accurate.

A. Price-volatility in a single-price-bid market

We first consider a single-price-bid system where each firm i can only vary the bidding price p_i, while the supply quantity q_i is fixed at q_i^0 and cannot be varied. Without loss of generality, we assume that the bidding price p_i is within $[0, \pi_g]$. Under similar assumptions, some earlier studies [20], [34] have shown that this type of real-time markets with single-price-bids will produce high price volatility. Below, we report a similar result but for uniform prices.

Market clearing: After the market receives the bids of all firms, i.e., their bidding price p_i and actual generation amount q_i^0, the market stacks all the bids together to compute the supply curve, i.e., the total available quantity from all bids at or below each price point p. Then, the market clearing price π_{eq} is given by the lowest price such that the total available quantity exceeds the total demand $L(t)$.

The sold amount s_i of each firm i is then determined as follows. All bids with price lower than π_{eq} clear their entire quantity q_i^0, i.e., $s_i = q_i^0$. All bids with price higher than π_{eq} clear zero quantity, i.e., $s_i = 0$. For those bids with price exactly equal to π_{eq}, we assume that the sold/cleared amount is assigned proportionally to q_i^0 to split the left-over demand, i.e.,

$$s_i = \min\left\{ \frac{q_i^0 \left(L(t) - \sum_{\{j:\ p_j < \pi_{\text{eq}}\}} q_j^0 \right)}{\sum_{\{j:\ p_j = \pi_{\text{eq}}\}} q_j^0},\ q_i^0 \right\}. \quad (1)$$

Under such a market rule, the whole system can be viewed as a game where each firm chooses her bidding price. We are then interested in the Nash equilibrium of the bidding prices and the market-clearing price. We have the following result.

Proposition 1. *At each time-instant t, there are three cases:*

Case 1. *(Limited Supply) When $\sum_{i=1}^M q_i^0 \le L(t)$, the market clearing price is $\pi_{eq} = \pi_g$ no matter how the firms bid. Further, the bidding strategy where every firm bids π_g is a Nash equilibrium.*

Case 2. *(Abundant Supply) When $\sum_{i \ne j} q_i^0 > L(t)$ for all j, then the strategy that all firms bid zero price is a Nash equilibrium. Additionally, if we also have $\sum_{i \in S} q_i^0 \ne L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, then at any Nash equilibrium the market clearing price must be $\pi_{eq} = 0$.*

Case 3. *(Borderline Supply) Consider the situation where $\sum_{i=1}^M q_i^0 > L(t)$, $\sum_{i \ne j} q_i^0 < L(t)$ for some j, and $\sum_{i \in S} q_i^0 \ne L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$. Let $\mathcal{I} = \left\{ j :\ \sum_{i \ne j} q_i^0 < L(t) \right\}$, which is non-empty by the earlier assumption. Then, there exists at least one Nash equilibrium. Further, at any Nash equilibrium, the market clearing price must be $\pi_{eq} = \pi_g$, and one firm in $\mathcal{I}$ bids at the price π_g while other firms bid below the price π_g.*

Prop. 1 indicates that real-time markets with single-price bids will experience significant price-volatility: as the real-time solar generation fluctuates above or below the demand, the market price will jump between 0 and π_g. These results are intuitive. In Case 1, the supply is much lower than the demand. Thus, all firms will bid the highest price π_{eq}. In Case 2, the supply is above the demand. Thus, the competition drives the bidding price to zero. In Case 3, the firms in $\mathcal{I}$ have a high market power. Specifically, without any firm in $\mathcal{I}$, the system changes from "too much supply" to "not enough supply". Thus, they will be the ones that set the market price to π_g. Note that similar results on price-volatility have been reported in [1], [34], but they allow a different market clearing price for each firm. There, for Case 3 there exists no pure Nash equilibrium, which suggests even higher price-volatility. Our model assumes uniform price for all firms, which is more similar to the way current transmission-level energy markets work.

B. Price-fixing in markets allowing price-quantity bids

In this paper, we advance this line of study of real-time markets by furthering considering the impact of price-volatility on firms' strategic behavior. Specifically, whenever the total solar generation exceeds the demand (Case 2 in Prop. 1), firms will receive zero revenue because the market price is driven to zero. Intuitively, there will then be a strong incentive for firms to withhold supply, which will likely lead to very different equilibrium dynamics compared to Section IV-A.

We note that existing studies in the literature [20], [34] assume that renewable suppliers cannot vary their bidding quantity, partly because renewable generation (unlike fossil-fuel generation) is usually considered uncontrollable. However, in practice it is actually quite easy for a solar-energy supplier to adjust her supply. For example, a firm can shut down part of her solar panels. Further, because solar generation is inherently uncertain, it would be difficult for the utility to tell whether the lower supply is due to withholding or due to lack of solar irradiance. Therefore, it is practially important to consider the possibility of withholding supply.

Supply withholding leads to the following model for a real-time market with price-quantity bids. Recall that $q_i^0 = C_i G(t)$ is the real solar generation available to firm i at time t. Now, each firm can submit a bid that varies both the price $p_i \in [0, \pi_g]$ and the quantity $q_i \in [0, q_i^0]$. The market clearing mechanism remains the same as in Section IV-A, except that the supply curve is computed based on the declared quantity q_i instead of q_i^0. The following result shows that the equilibrium market outcome changes drastically once firms can submit price-quantity bids.

Proposition 2. *A Nash equilibrium with $\pi_{eq} = \pi_g$ always exists, regardless of the demand or the generation. Specifically, any bidding strategy that satisfies the following conditions is a Nash equilibrium:*

$$\begin{cases} \sum_{i=1}^{M} q_i = \min\left\{L(t),\ \sum_{i=1}^{M} q_i^0\right\}, \\ p_i = 0 \text{ for all } i. \end{cases} \tag{2}$$

Prop. 2 shows that real-time markets with price-quantity bids always have a Nash equilibrium where the market price is π_g. Such an equilibrium is highly unfair to the consumers because the consumer surplus will always be zero. Indeed, the entire social surplus is earned by the firms as the supplier surplus. As a result, consumers will be disincentivized to participate in the market. Compared to Case 2 of Proposition 1, a key difference in Prop. 2 is that, when the total solar generation exceeds the supply (i.e., $L(t) < \sum_{i=1}^{M} q_i^0$), some firms do understate their generation at the new Nash equilibrium (2), which then drives the price back up to π_g. We note that in practice it is also easy for each firm to reach such an equilibrum strategy. Assume that solar generation does not vary significantly from time-instant to time-instant. If the market price was zero in the previous time-instant due to over-supply, each firm can simply withhold her supply at the next time-instant to be slightly below her sold/cleared-amount from the previous time-instant.

Another consequence of the above analysis is that there exist multiple equilibrium market outcomes. In particular, it is not hard to verify that, in addition to the equilibrium stated in Prop. 2, the outcome predicted in Prop. 1 may also be a Nash equilibrium for the setting with price-quantity bids. Note that the equilibrum stated in Prop. 2 is Pareto optimal, i.e., no firms can gain higher payoff without hurting other firms, and thus may be more favorable. However, because (2) has multiple solutions, there will be multiple such Pareto-optimal equilibria too. The existence of multiple equilibria means that it is difficult to predict the outcome of the market.

We summarize the outcomes of real-time markets under the four considerations of interest, i.e., efficiency, fairness of surplus division, predictability of outcome, and price volatility. It is not hard to verify that the oucomes reported in both Sections IV-A and IV-B are efficient in maximizing the total social surplus. The reason that different prices can all maximize the total social surplus is because (as shown in Fig. 1(b)) both the supply curve and the demand curve in our real-time markets are inelastic due to firms' zero marginal cost and consumers' fixed reserved price π_g. Therefore, the social surplus is maximized at any price in $[0, \pi_g]$. In the single-price-bid system, the outcome is predictable. Both the supplier surplus and the consumer surplus may be positive (under Cases 1 & 3 and Case 2, respectively, of Prop. 1). However, prices exhibit high volatility. On the other hand, in the price-quantity-bid system, while the prices may no longer be volatile, the consumer surplus may always be zero, and the market outcome may become unpredictable due to multiple equilibria. It thus remains an open question whether we can design distribution-level energy markets that produce desirable behaviors under all four aspects of considerations.

V. Solar-panel rental markets

In view of the issues of price-volatility and pricing-fixing in real-time markets, in this section we propose an alternative form of distribution markets that avoid these issues and lead to desirable outcomes in terms of all four considerations of efficiency, stable price, fairness of surplus division, and predictability of outcome. The key ideas of this new market are two-fold. First, instead of trading in real-time, this market trades once over a time-period of length T (e.g., T could correspond to a month). Second, instead of trading energy, this market trades the usage right for a certain size of solar panels. Specifically, consumers lease a certain size of solar panels from the firms T time ahead and can then use all the electricity generated by the rented solar panels in real-time. Therefore, we refer to this type of markets as *rental markets*. Note that if the real-time demand of a consumer exceeds the generation of her rented panels, she still has to buy the deficit part from the utility at the grid price π_g.

We will show that rental markets can eliminate both price-volatility and price-fixing. First, the price in the rental market is determined once over the time-period of length T. Thus, real-time price-volatility is eliminated by design. Second and more importantly, we will show soon that solar variability naturally induces elasticity of the demand function in such a rental market. Recall that the inelasticity of the demand was one of the main reasons for price-fixing in real-time markets. In contrast, we will show that this variability-induced demand elasticity will help the rental market avoid price-fixing.

A. Demand Function and Its Elasticity in Rental Markets

Towards this end, we first derive the demand function in rental markets, which corresponds to the size (in m^2) of the solar panel that consumer n wishes to rent at a unit rental price of π (in $\$/m^2$, normalized to one time-instant). Recall that the real-time demand of consumer n is $L_n(t)$. Suppose that the consumer n has rented c_n unit of solar panels. Then, the generation at time t is $G(t)c_n$. If $L_n(t) \leq G(t)c_n$, the consumer does not need to buy any electricity from the utility. Otherwise, she needs to buy the deficit $L_n(t) - G(t)c_n$ at the price π_g. Thus, when the market price of renting unit-size solar panel for a unit-time is π, the time-average expected cost of the consumer n is given by

$$J_n(\pi, c_n) = \pi c_n + \frac{\pi_g}{T}\int_0^T \mathbb{E}[L_n(t) - G(t)c_n]^+ dt. \tag{3}$$

We take the demand function $d_n(\pi)$ of customer n as the largest c_n^* that minimizes $J_n(\pi, c_n)$ over c_n, i.e.,

$$d_n(\pi) = \sup\left\{c_n^* \middle|\ J_n(\pi, c_n^*) \leq J_n(\pi, c_n),\ \text{for all } c_n \geq 0\right\}. \tag{4}$$

In Eq. (4), we take the maximum of all minimizers as the demand function value when there are multiple minimizers. One reason is that when there is no enough total supply to satisfy demand of all consumers, there should be a rule to allocate the supply to each consumer. For fairness, one possibility is that the actual amount that each consumer can get should be proportional to the her submitted demand. In this case, a consumer prefer to choose the maximum of all minimizers as her demand to get more share in case there is no enough total supply. Another reason is that by choosing the demand as Eq. 4, we do not need to worry about special case of the non-existence of Nash equilibrium of the competition of suppliers at the discontinuity point of the total demand in Theorem. 1, which will be introduced in the next subsection.

It is easy to see that, if $L_n(t)$ and $G(t)$ were both constant over t, i.e., $L_n(t) = L$ and $G(t) = G$, then the demand function $d_n(\pi)$ would be equal to $\frac{L}{G}$ when $\pi \leq \pi_g$, and 0, otherwise. In other words, the demand function would still be inelastic. In contrast, the following result shows that, if the solar generation is sufficiently random, then the demand function will become elastic. Towards this end, we introduce the following assumption on the randomness of solar generation.

Assumption 1. *Suppose that for any $t \in [0, T]$, $L_n(t)$ and $G(t)$ are non-negative continuous random variables with a joint probability density function (pdf) $f^t_{G,L_n}(x, y) = f^t_{G|L_n}(x|y) \cdot f^t_{L_n}(y)$, where $f^t_{G|L_n}(x|y)$ denotes the conditional pdf of $G(t)$ conditioned on $L_n(t)$, and $f^t_{L_n}(y)$ denotes the marginal pdf of $L_n(t)$. We assume that there is a finite $\gamma > 0$ such that*

$$x^2 f^t_{G|L_n}(x|y) \leq \gamma \text{ for all } x, y, \text{ and } t. \tag{5}$$

Intuitively, Assumption 1 states that the probability distribution of solar generation should not be too concentrated (e.g., there should not be any atom in the distribution) and its tail should not be too heavy. Assumption 1 holds for many common continuous random distributions, such as Gaussian distribution and any sub-Gaussian distributions [10]. We now define the total demand function $D(\pi)$ as the sum of all consumers' demand functions, i.e.,$D(\pi) = \sum_{n=1}^N d_n(\pi)$. We then have the following result, which relates the elasticity of $D(\pi)$ to the parameter γ in Assumption 1. Recall that elasticity corresponds to the ratio between the relative change of demand and the relative change of price, and is defined as $\eta(\pi) := |\frac{\pi}{D(\pi)}\frac{dD(\pi)}{d\pi}|$.

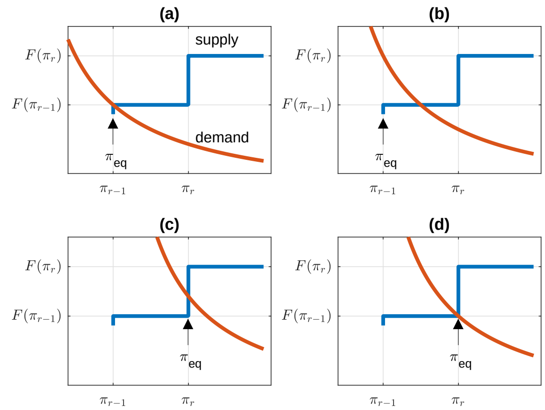

Fig. 2. Determination of the market clearing price π_{eq} in rental markets, depending on the different ways that the supply curve intersects the demand curve. The supply curve $F(\pi)$ is given by as $F(\pi) = \sum_{\{i,k:\ p_{i,k} \leq \pi\}} q_{i,k}$.

Proposition 3. *For any π such that $D(\pi) > 0$, the elasticity of the total demand function $D(\pi)$ is bounded by*

$$\eta(\pi) \geq \frac{\pi}{\pi_g \gamma}. \tag{6}$$

(Note that since $D(\pi)$ may be discontinuous, $\frac{\partial D(\pi)}{D(\pi)}$ may be $+\infty$. Nonetheless, Eq. (6) still holds.)

Prop. 3 shows that the demand elasticity in rental markets is bounded from below by a function of γ. Note that according to (5), as γ decreases, the distribution of random solar generation is even less concentrated (i.e., more random). Prop. 3 then shows that the demand elasticity will also be higher. In this sense, while the variability and uncertainty of solar generation is often regarded as a detrimental factor for energy systems, in our proposed rental markets it becomes a beneficial factor in contributing to demand elasticity.

B. Outcomes of Rental Markets

Next, we show that, thanks to the demand elasticity reported in Prop. 3, rental markets will be able to eliminate price-fixing and produce much more desirable outcomes than real-time markets. Recall from Section IV-B that price-fixing in real-time markets arises when each firm is allowed to vary both her price and quantity. Thus, for a fair comparison, we will also allow such price-quantity bids in rental markets. Further, note that in today's transmission-level energy markets, a generator can even bid multiple blocks of price-quantity pairs, which is more general than a single price-quantity bid. Our result below (that rental markets can eliminate price-fixing) applies to such a more general setting with multi-block bids, and hence is even stronger.

The operation of rental markets under multi-block bids can be described in a similar way as Section IV-B. with the key difference that the traded product is solar panel instead of electrical energy. Suppose that the firm i makes a multi-block bid containing K_i sub-bids. Denote the price and the quantity (i.e., solar panel size) of her k-th sub-bid as $p_{i,k}$ and $q_{i,k}$, respectively. Without loss of generality, assume that

$p_{i,1} < p_{i,2} < \cdots < p_{i,K_i}$, $q_{i,k} > 0$ for all $k \in \{1, \cdots, K_i\}$, and $\sum_{k=1}^{K_i} q_{i,k} \leq C_i$. Firm i's bid can then be described by $\vec{p}_i = [p_{i,1}\ p_{i,2}\ \cdots\ p_{i,K_i}]$ and $\vec{q}_i = [q_{i,1}\ q_{i,2}\ \cdots\ q_{i,K_i}]$. The market collects the bids from all firms, as well as the demand functions $d_n(\pi)$ from all consumers. As the demand of each individual consumer is usually small and may have very little power to change the market outcome, we regard consumers' bidding are not strategic. In other words, consumers just honestly bid their demand functions. The market then stacks all the sub-bids together to compute the supply curve (similar to real-time markets in Section IV) for the total available solar-panel size at each price point. Similarly, the demand function $d_n(\pi)$ is added together to form the total demand curve $D(\pi)$. The market clearing price is then determined by the intersection of the supply curve and the demand curve. See Fig. 2 for different ways that one of the bidding prices is chosen as the market-clearing price, depending on whether the demand curve intersects the vertical or horizontal part of the supply curve. Denote $s_{i,k}$ as the cleared/sold amount of the k-th sub-bid of firm i. For those sub-bids with price lower than (or higher than) π_{eq}, the sold amount is equal to the bidding quantity (or zero). For those sub-bids with price equal to π_{eq}, the sold amount is assigned proportionally to the bidding quantity (similar to (1)).

The following main result shows that rental market will not suffer from price-fixing under suitable conditions, even with multi-block bids.

Theorem 1. *If $\frac{\max_i C_i}{C} \leq \frac{D^{-1}(C)}{\pi_g \gamma}$, then the rental market allowing multi-block bids will possess at least one Nash equilibrum where each firm bids a single price-quantity pair. Further, at any Nash equilibrium, all solar panels from all firms are cleared, i.e., $\sum_{k=1}^{K_i} s_{i,k} = \sum_{k=1}^{K_i} q_{i,k} = C_i$ for all i; and the market clearing price must be $\pi_{eq} = D^{-1}(C)$.*

We comment on the highly-desirable features of the market outcome predicted by Theorem 1. Note that $D^{-1}(C)$ can be viewed as the price under perfect competition [28]. Theorem 1 thus states that the market clearing price π_{eq} is always identical to perfect competition. Indeed, since all solar panels from all firms are cleared, there is no price-fixing or supply-withholding. Further, since one of the equilibria corresponds to each firm bidding a single price-quantity pair, the same conclusion will hold if we allow each firm to always bid a single price-quantity pair only.

The condition $\frac{\max_i C_i}{C} \leq \frac{D^{-1}(C)}{\pi_g \gamma}$ in Theorem 1 can be interpreted as follows. Since $D^{-1}(C) = \pi_{\text{eq}}$, according to Proposition 3, the right-hand-side of the condition is simply a lower bound on the demand elasticity. Thus, Theorem 1 captures precisely the importance of demand elasticity, which is induced by the randomness of solar generation via Prop. 3. Specifically, price-fixing is eliminated as long as no single firm dominates, i.e., the supply of each firm is smaller than a corresponding fraction of the total supply. This fraction is exactly equal to the demand elasticity. Thus, the more random the solar generation is (i.e., the smaller γ is), the higher the demand elasticity, and the larger each firm can be without the worry of price-fixing.

The rest of this section is devoted to sketching the proof of Theorem 1. During the proof, we will frequently let a firm or some firms deviate to another bidding strategy and analyze the outcome after the deviation. We will use $(\cdot)'$ to denote the new value after the deviation. For example, π'_{eq} denotes the new market price, $q'_{i,k}$ denotes the new quantity of the firm i's k-th bid, and K'_i denotes the new number of bids that the firm i has. We will also need the following definition and supporting proposition to simplify our proof.

Definition 1. *Two Nash equilibriums $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$ are **outcome-equivalent** when $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ for all i, and $\pi'_{eq} = \pi_{eq}$.*

Theorem 2. *For any Nash equilibrium $(\mathbf{p}, \mathbf{q})$ in the multi-block-bid system, there must exist an outcome-equivalent Nash equilibrium $(\mathbf{p}', \mathbf{q}')$ such that each firm only makes two bids, and the price of any bid is either 0 or π_{eq}.*

We omit the relatively easy part of verifying a Nash equilibrium, while focus on sketching the main steps of proving the uniqueness of a Nash equilibrium. Suppose on the contrary that at a Nash equilibrium $(\mathbf{p}, \mathbf{q})$, there exists at least one firm with an unsold/partly-sold bid. By applying Theorem 2 and some other lemmas, among all firms with an unsold/partly-sold bid, we can always find a certain firm j that satisfies several properties as follows. ① $\frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \leq \frac{C_j}{C}$, i.e., the proportion of firm j's sold amount in the demand is less or equal than the proportion of her panel size in the total panel size. ② $\pi'_{\text{eq}} = \pi^*$ after the firm j deviates to another strategy ($K'_j = 1$, $p'_{j,1} = \pi^*$, and $q'_{j,1} = C_j$), as long as π^* is in a certain left neighbourhood of π_{eq}. In other words, if the firm j deviates by bidding all at π^* that undercuts π_{eq}, then the market price will become π^*. ③ $s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \geq D(\pi^*) - D(\pi_{\text{eq}})$ after that deviation, i.e., the increasing part of the firm j's sold amount will be at least the increasing part of the total demand.

We briefly explain why ①②③ are true. For ①, notice that the total sold amount must be less than or equal to the demand. Thus, if every firm has unsold/partly sold bids, then the sum of the left-hand-side of ① over all j is less than or equal to 1, and the corresponding sum of the right-hand-side equals to 1. Therefore, at least one of the firm j satisfies ①. For the case that not every firm has unsold/partly sold bids, we can still get ① by first eliminate those fully-sold firms' capacity out of the total demand, and then applying the similar method like the case that every firm has unsold/partly sold bids. For ②, notice that the firm j has an unsold/partly-sold bid, then the firm j's undercutting will lower the market price. For ③, because only the firm j changes her bid, the increasing part of the demand (notice that $D(\pi)$ is strictly monotone decreasing with respect to π) will be assigned to the firm j only. It is also possible that the firm j can "rob" other firms' sold amount. Therefore, the increasing part of the firm j's sold amount will be at least the increasing part of the total demand.

Then, we establish the contradiction. Because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, the firm j cannot get more profit by deviating to another bidding strategy. Applying ② and ③, we can get the following inequality by comparing the firm j's profit before and after her deviation:

$$\left|\frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})}\right| \le \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})}.$$

By ① and $\frac{C_j}{C} \le \frac{D^{-1}(C)}{\pi_{\text{eq}}\gamma}$ (the condition of this theorem), we then have

$$\left|\frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})}\right| < \left|\frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}}\right| \frac{\pi_{\text{eq}}}{\pi_g \gamma},$$

which contradicts the elasticity of $D(\cdot)$ stated in Proposition 3.

We conclude by summarizing the outcome of rental markets under the four considerations of interest introduced in Section III. First, Theorem 1 shows that the outcome (in terms of market price) of rental markets is unique and easily predictable. Second, by design rental markets also do not incur price-volatility in real-time. Third, since the equilibrium market price $\pi_{\text{eq}} = D^{-1}(C)$ of rental markets is a function of the total supply quantity C and is in general lower than π_g, unless C is very large or very small, neither the supplier surplus nor the consumer surplus will be zero. Therefore, the rental market also divide the social surplus more fairly between firms and consumers.

The last aspect, i.e., efficiency, warrants some further discussion. Note that the competitive price $\pi_{\text{eq}} = D^{-1}(C)$ is known to maximize the social surplus based on the demand- and supply-curves of rental markets [27]. In addition, one may be interested in comparing the social surplus of rental markets with that of real-time markets. In particular, since the demand/supply-curves in rental markets are defined differently from those in real-time markets (i.e., in terms solar panel size instead of energy), the respective maximum social surplus may still differ. To see this, note that after a consumer leases certain size of solar panels, she may not be able to fully use the solar generation in real-time if her demand is low. As a result, even if the total solar generation in the system is higher than the total demand, some consumers may still have to buy electricity from the utility at the price π_g. This suggests that rental markets may lose some efficiency in terms of the social surplus.

However, the following result shows that, if we include the utility's profit in defining the total *social welfare* of the system, then rental markets will attain the same maximum total social welfare as real-time markets. Specifically, we define the social welfare SW as

$$\text{SW} = \text{Util}_c - \text{Cost}_c + \text{Rev}_f - \text{Cost}_f + \text{Rev}_u - \text{Cost}_u$$

where Util_c and Cost_c are the consumers' utility value and cost, Rev_f and Cost_f are firms's revenue and cost, and Rev_u and Cost_u are utility company's revenue and cost for meeting the consumers' remaining electricity needs not met by the distribution-level market. We make the following assumption on how the solar generation from the rented panels will be fed back to the grid when it is not utilized by the consumer.

Assumption 2. *When a consumer's demand $L_n(t)$ is below the solar generation $G(t)c_n$ of the rented panels, the surplus generation $G(t)c_n - L_n(t)$ will be fed back to the grid.*

This assumption is reasonable because, after solar panels are already traded in the rental market, neither the firm nor the consumer has the incentive to curtail the surplus generation. The following result then shows that the total social welfare is independent of the clearing price and quantity in the rental market.

Proposition 4. *Under Assumption 2, the rental markets are as efficient as real-time markets in terms of* SW*, regardless of the clearing price and quantity in the rental markets.*

Proof. The insensitivity reported in Proposition 4 is again due to the inelasticity of consumers' real-time demand and the zero marginal-costs of the firms. First, the consumer's utility value Util_c is a constant since the real-time energy demand is inelastic. Second, the firms' cost Cost_f is always zero due to zero marginal costs. Third, the payment of the consumers must equal to the revenue of the firms plus the revenue of the utility company, i.e., $\text{Cost}_c = \text{Rev}_f + \text{Rev}_u$. Thus, the only term that may change the social welfare is the utility company's cost, which is true for both the rental markets and the real-time markets. However, under Assumption 2 the amount of electricity that the utility needs to procure from the transmission level is fixed at $[L(t) - G(t)C]^+$. Therefore, the total social welfare is independent of the outcome of the market. Note that using a similar argument, we can show that the social welfare in real-time markets must also be equal to the same value. (The only difference is that the utility company's revenue Rev_u will also be independent of the real-time market's clearing price, because the amount of electricity sold by the utility company is also fixed at $[L(t) - G(t)C]^+$.) Thus, under Assumption 2 rental markets are as efficient as real-time markets in terms of social welfare. □

In summary, rental markets produce desirable market outcomes under all four considerations of interest. Readers familiar with transmission-level energy markets will recognize rental markets as a form of *forward* markets, similar to the design of day-ahead markets in current transmission-level energy markets. A key difference, however, is that, in traditional day-ahead markets, generators make forward commitments in energy. Such forward commitments are difficult for solar generation due to its uncertainty. Instead, in our rental markets, firms make forward commitment on the future *usage rights* for solar panels, which is a lot easier.

Notice that our main conclusions on the rental market can be easily extended to heterogeneous $G(t)$ for different firms, as we have mentioned in Section III. Readers can refer to Appendix E on how an "effective panel size" can be calculated for each firm in the rental market to adjust for different $G(t)$.

VI. Simulations

We will verify features of different markets that shown in Table I by simulation with solar generation and energy consumption data. The solar generation data are from a PV farm located near Purdue University (latitude: $40.45°$, longitude: $86.85°$). The data were taken every five minutes during the whole year 2006 (provided by NREL [35]). The load data are from two residential houses, one at Purdue University,

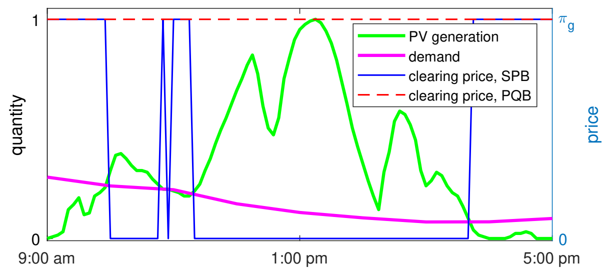

Fig. 3. Real-time single-price-bid market experiences high price volatility, while real-time price-quantity-bid market experiences price fixing.

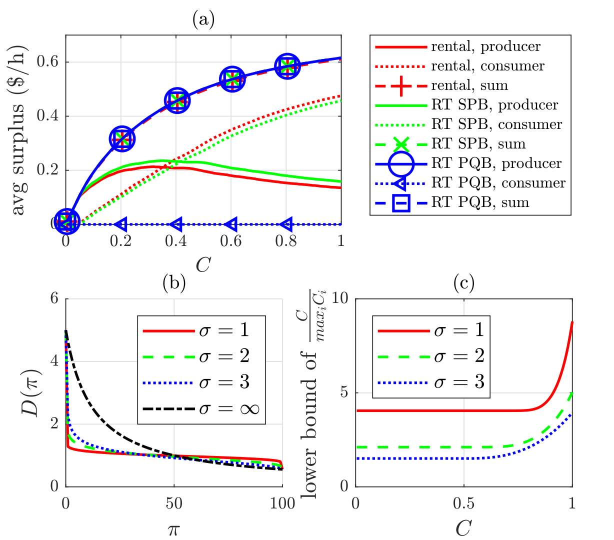

Fig. 4. (a) Producer surplus, consumer surplus, utility surplus and social surplus for different markets under different amount of solar panels. (b) Demand function with different amount of uncertainty for PV generation. (c) The requirement firms' size in Theorem 1 with different amount of solar panels.

the other in Indianapolis. The data were taken hourly during a typical meteorological year (provided by EERE [25]). We scale the generation and load data to simulate different solar energy penetration ratio (defined as the ratio between the sum of total solar generation and the sum of total load during the whole year). The retail price of electricity in West Lafayette is equal to $\pi_g = 0.1029\$/\mathrm{kWh}$.

We first show price-volatility in real-time single-price-bid (SPB) markets, and price-fixing in real-time price-quantity-bid (PQB) markets. In Fig. 3, we can find that the clearing price in real-time single-price-bid markets (represented by the blue curve) jumps between 0 and π_g. For real-time price-quantity-bid markets, we can observe that the clearing price stays at π_g.

Then, we look into the fairness of division and the efficiency. In Fig. 4(a), we plot the consumer surplus, producer surplus, and their sum for different markets with different amount of total solar panels. The first thing we find is that for each kind of markets, the sum of consumer surplus and producer surplus are almost the same[1], which verifies that the rental market is as efficient as real-time markets in terms of social surplus maximization. We find that for real-time price-quantity-bid markets, the consumer surplus equals to zero, while the producer takes all of the total surplus. In contrast, the rental markets, as well as real-time single-price-bid markets, have comparable value of consumer surplus and producer surplus, which is fairer than those in the real-time price-quantity-bid markets. Another interesting phenomenon is that for rental markets and real-time single-price-bid markets, although the curve of consumer surplus keeps increasing, the curve of producer surplus increases first and then decrease. The reason is that when the total amount of solar panels are not too large, the competition between firms are not intensive. Therefore, the curve of producer surplus increases first. As there are more and more solar panels in the market, the competition between firms becomes intensive, which makes the producer surplus decreases.

As we mentioned earlier, for rental markets, elasticity of the demand function increases with uncertainty of solar generation, which determines whether the rental market has desired outcome. In order to evaluate how the rental market behaves as the uncertainty in the system varies, we next turn to a synthetic setting so that we can vary the level of uncertainty easily. We then use synthetic data to reveal some interesting observations of the rental market by Figs. 4(b) and 4(c). We consider the situation that the total time duration is of unit length, $T = 1$. The generation $G(t)$ for all $t \in [0,1]$ is equal to a common random variable G. We assume that G follows a truncated normal distribution $\mathcal{N}_{\text{truncated}}(\mu, \sigma^2, \underline{G}, \overline{G})$, i.e., a normal distribution $\mathcal{N}(\mu, \sigma^2)$ restricted to the interval $[\underline{G}, \overline{G}]$, where $0 \leq \underline{G} < \overline{G}$. In the following numerical simulation, we let $\underline{G} = 2$ and $\overline{G} = 18$. Let $\mu = \frac{\underline{G}+\overline{G}}{2} = 10$. We let the load profile for each consumer be flat, i.e., $L_n(t) = L_n$ for all $t \in [0,1]$ where L_n is a constant. Thus, the total load is also a constant, which is denoted by $L \triangleq \sum_{n=1}^{N} L_n$. We let $L = 10$, $\pi_g = 10$. In this kind of setting, large σ means high uncertainty of solar generation.

In Fig. 4(b), we verify that the elasticity of the demand function $D(\cdot)$ increases with uncertainty of solar generation (which is indicated by σ in the figure). Since $D(\pi)$ is affected by σ, in the following discussions, we denote $D(\pi)$ by $D(\pi|\sigma)$ to emphasize this dependency. In Fig. 4(b), we find that, as σ increases, $D(\pi|\sigma)$ increases with σ if π is small, but decreases with σ if π is high. We can explain this phenomenon as follows. When σ increases, the solar generation becomes more uncertain, which implies that the probability of low G and high G are both higher. When G is low, consumers run the risk of having to purchase more expensive energy from the grid. When G is high, the consumers run the risk of wasting excess solar generation. Based on the price π, consumers may emphasize one risk over the other. Specifically, if π is low (i.e., panels are relatively cheap), the consumers will not incur much cost even when solar energy is wasted. Therefore, as σ

[1]In this figure, the curve of the sum surplus of the rental markets is a bit lower than those of real-time markets. This is because we do not include utility surplus in the sum, and difference of utility surplus of rental markets and that of real-times markets differ slightly in our simulation.

increases, consumers prefer to rent more panels so that they can still use cheap solar energy when G is low. In contrast, if π is high, the cost for wasting solar energy become more significant. Therefore, as σ increases, consumers prefer to rent fewer panels so that they will not waste too much solar energy when G is high.

We also find that all demand curves in Fig. 4(b) intersect at $\pi = 0^+$ and $\pi = 100$. Specifically, $D(0^+) \triangleq \lim_{\pi\to 0^+} D(\pi) = 5$ for all σ (notice that $D(0) \neq D(0^+)$ in this case[2]). This is because, when renting solar panel is free, consumers will rent as much as possible to cover their load L. Since the solar generation is at least $\underline{G}$, the consumers will need to rent at most $D(0^+) = \frac{L}{\underline{G}} = 5$. We also find that all curves have the same value at $\pi = 100$. This is because, when the average solar energy cost equals to π_g (i.e., very expensive), consumers do not want to waste any solar energy. Since the solar generation is at most $\overline{G}$, the consumers will only rent $D(100) = \frac{L}{\overline{G}} = 0.5556$.

Next, we use Fig. 4(b) to show that large uncertainty helps rental markets get desired outcome by examining whether the condition in Theorem 1 holds with different values of σ and C. By Fig. 4(c), we can infer that the condition in Theorem 1 is not hard to satisfy. For example, when $C = 1$ (i.e., the average load of all consumers is equal to the average solar energy produced by all panels), the condition holds if $\frac{C}{\max_i C_i} \geq 6$ when $\sigma = 2,\ 4$ or 6. In other words, if every firm has a similar amount of panels, then we only need 6 firms, which is quite likely in reality.

We also find that all three curves in Fig. 4(c) are flat when $C \in (0, 0.5556]$. The reason is that, for all $C \in (0, 0.5556] = (0, D(100)]$, we always have $D^{-1}(C) = 100$ which does not change with C (notice that $D(\pi)$ is discontinuous at $\pi = 100$). By Theorem 1, we know that the lower bound of $\frac{C}{\max_i C_i}$ is $\frac{\pi_g \gamma_s}{D^{-1}(C)}$. Therefore, the lower bound of $C/\max_i C_i$ does not change with C when $C \in (0, 0.5556]$.

VII. Conclusion

The key message of the paper is that the real-time market may not be suitable for the solar energy exchange at the distribution level, and the rental market of PV panel maybe a better option. Specifically, we show that the real-time market has the severe pitfall of price-fixing/supply-withholding, so that consumers get no benefits by participating in the market. In contrast, the proposed rental market eliminates price-fixing under certain mild conditions. Therefore, rental market may be more desirable.

The rental market is inspired by the structure of other markets with zero marginal costs, in particular markets for telecommunication services [17]. For example, for mobile wireless service providers, while base stations and backbone networks are very expensive to build, the cost of providing service for one phone call or one KB of data is often negligible. Although dynamic pricing based on real-time demand levels was discussed at various times, what prevails today are often fixed-price monthly contracts (e.g., unlimited voice and data for 50 dollars per month). Such a fixed-price contract has been found to reduce the financial uncertainty to the suppliers, and provide them with the guaranteed revenue for future network expansion. While the concept of rental markets is inspired by fixed-price contracts, our study is complementary to these discussions since we focus on how rental markets eliminates market manipulation and supply withholding. Therefore, they provide new insights why rental market can be more favorable for the distribution level than real-time energy markets.

There are many interesting directions for future work. We have studied the strategy behavior of the supply side in this paper. However, whether consumers will truthfully report their demand function in the rental market also deserves a careful study. Another direction is to introduce the storage into the power distribution level. This brings an extra dimension on market design. There are several interesting questions such as 1) whether the markets of PV panels and battery should operate together or separately; 2) how consumers bid their optimal choice on both products.

[2] Based on our definition of $D(\cdot)$ in Section V, we always have $D(0) = \infty$. As a result, in this case, we do have a discontinuity at $\pi = 0$.

References

[1] Acemoglu, D., Bimpikis, K., and Ozdaglar, A. Price and capacity competition. *Games and Economic Behavior 66*, 1 (2009), 1–26.

[2] Albadi, M. H., and El-Saadany, E. F. A summary of demand response in electricity markets. *Electric power systems research 78*, 11 (2008), 1989–1996.

[3] Baldick, R., Grant, R., and Kahn, E. Theory and application of linear supply function equilibrium in electricity markets. *Journal of regulatory economics 25*, 2 (2004), 143–167.

[4] Baldick, R., and Hogan, W. W. Stability of supply function equilibria implications for daily versus hourly bids in a poolco market. *Journal of Regulatory Economics 30*, 2 (2006), 119–139.

[5] Bator, F. M. The anatomy of market failure. *The quarterly journal of economics 72*, 3 (1958), 351–379.

[6] Bazilian, M., Onyeji, I., Liebreich, M., MacGill, I., Chase, J., Shah, J., Gielen, D., Arent, D., Landfear, D., and Zhengrong, S. Re-considering the economics of photovoltaic power. *Renewable Energy 53* (2013), 329–338.

[7] Bitar, E. Y., Rajagopal, R., Khargonekar, P. P., Poolla, K., and Varaiya, P. Bringing wind energy to market. *IEEE Transactions on Power Systems 27*, 3 (2012), 1225–1235.

[8] Bolle, F. Supply function equilibria and the danger of tacit collusion: the case of spot markets for electricity. *Energy economics 14*, 2 (1992), 94–102.

[9] Bratton, W. W. Enron and the dark side of shareholder value. *Tul. L. Rev. 76* (2001), 1275.

[10] Buldygin, V. V., and Kozachenko, Y. V. Sub-gaussian random variables. *Ukrainian Mathematical Journal 32*, 6 (1980), 483–489.

[11] Chow, S. K., Lee, E. W., and Li, D. H. Short-term prediction of photovoltaic energy generation by intelligent approach. *Energy and Buildings 55* (2012), 660–667.

[12] Dastrup, S. R., Zivin, J. G., Costa, D. L., and Kahn, M. E. Understanding the solar home price premium: Electricity generation and "green" social status. *European Economic Review 56*, 5 (2012), 961–973.

[13] Donsig, A. P., and Davidson, K. R. *Real Analysis with Real Applications*. Prentice Hall, 2002.

[14] Fan, L., Wang, J., Jiang, R., and Guan, Y. Min-max regret bidding strategy for thermal generator considering price uncertainty. *IEEE Transactions on Power Systems 29*, 5 (2014), 2169–2179.

[15] Gharesifard, B., Başar, T., and Domínguez-García, A. D. Price-based coordinated aggregation of networked distributed energy resources. *IEEE Transactions on Automatic Control 61*, 10 (2015), 2936–2946.

[16] Green, R. Increasing competition in the british electricity spot market. *The journal of Industrial economics* (1996), 205–216.

[17] Júnior, S. L. F., and Pizzolato, N. D. Pricing and costing practices in telecommunications: the brazilian experience and challenges. *Brazilian Journal of Operations & Production Management 10*, 1 (2013), 7–24.

[18] Kalathil, D., Wu, C., Poolla, K., and Varaiya, P. The sharing economy for the electricity storage. *IEEE Transactions on Smart Grid* (2017).

[19] Klemperer, P. D., and Meyer, M. A. Supply function equilibria in oligopoly under uncertainty. *Econometrica: Journal of the Econometric Society* (1989), 1243–1277.

[20] Li, P., Sekar, S., and Zhang, B. A capacity-price game for uncertain renewables resources. *arXiv preprint arXiv:1804.06496* (2018).

[21] Mengelkamp, E., Gärttner, J., Rock, K., Kessler, S., Orsini, L., and Weinhardt, C. Designing microgrid energy markets: A case study: The brooklyn microgrid. *Applied Energy 210* (2018), 870–880.

[22] Midcontinent Independent System Operator, Inc. *MISO Business Practices Manuals*, accessed May 9, 2019.

[23] Morales, J. M., Conejo, A. J., Madsen, H., Pinson, P., and Zugno, M. *Integrating renewables in electricity markets: operational problems*, vol. 205. Springer Science & Business Media, 2013.

[24] Mufson, S. The hottest fight in american politics? arizona's smackdown over solar power. *The Washington Post* (2018).

[25] Office of Energy Efficiency & Renewable Energy (EERE). Commercial and residential hourly load profiles for all tmy3 locations in the united states. https://openei.org/doe-opendata/dataset/commercial-and-residential-hourly-load-profiles-for-all-tmy3-locations-in-the-united-states. Accessed: 2019-07-28.

[26] Peck, M. E., and Wagman, D. Energy trading for fun and profit buy your neighbor's rooftop solar power or sell your own-it'll all be on a blockchain. *IEEE Spectrum 54*, 10 (October 2017), 56–61.

[27] Rader, T. *Theory of microeconomics*. Academic Press, 2014.

[28] Robinson, J. *The economics of imperfect competition*. Springer, 1969.

[29] Rodriguez, C. P., and Anders, G. J. Bidding strategy design for different types of electric power market participants. *IEEE Transactions on Power Systems 19*, 2 (2004), 964–971.

[30] Roozbehani, M., Dahleh, M. A., and Mitter, S. K. Volatility of power grids under real-time pricing. *IEEE Transactions on Power Systems 27*, 4 (2012), 1926–1940.

[31] Stein, E. M., and Shakarchi, R. *Real analysis: measure theory, integration, and Hilbert spaces*. Princeton University Press, 2009.

[32] Stigler, G. J. Perfect competition, historically contemplated. *Journal of political economy 65*, 1 (1957), 1–17.

[33] Su, Y., Chan, L.-C., Shu, L., and Tsui, K.-L. Real-time prediction models for output power and efficiency of grid-connected solar photovoltaic systems. *Applied energy 93* (2012), 319–326.

[34] Taylor, J. A., and Mathieu, J. L. Strategic bidding in electricity markets with only renewables. In *2016 American Control Conference (ACC)* (2016), IEEE, pp. 5885–5890.

[35] The National Renewable Energy Laboratory (NREL). Solar power data for integration studies. https://www.nrel.gov/grid/solar-power-data.html. Accessed: 2019-07-28.

[36] Timilsina, G. R., Kurdgelashvili, L., and Narbel, P. A. Solar energy: Markets, economics and policies. *Renewable and sustainable energy reviews 16*, 1 (2012), 449–465.

[37] Tushar, W., Yuen, C., Mohsenian-Rad, H., Saha, T., Poor, H. V., and Wood, K. L. Transforming energy networks via peer-to-peer energy trading: the potential of game-theoretic approaches. *IEEE Signal Processing Magazine 35*, 4 (2018), 90–111.

Appendix A
Proofs in Section IV

We provide the following two algorithms to indicate how real-time markets compute market price and sold amount according to bidding and demand. Notice that those algorithms work for both the single-price-bid market and markets allowing price-quantity bids.

Remark: In Eq. (7) of Algorithm 2, bids at the market price π_{eq} split the sold amount in proportion to their generation amount. Here is an example. After clearing the supply below the price π_{eq}, the remaining demand is 10, firm 1 has a bid at the price π_{eq} with the amount 60, firm 2 has a bid at the price π_{eq} with the amount 40. There is no other bid at the price π_{eq}.

Algorithm 1: Compute market price π_{eq}.

Sort elements in $\{p_i: \text{ for all } i\}$ as $\pi_1 < \pi_2 < \cdots < \pi_R$ (note that all π_r's are distinct);
$Q \leftarrow 0$;
$\pi_{\text{eq}} \leftarrow \pi_g$;
for $r = 1 : R$ **do**
  $Q \leftarrow Q + \sum_{\{i:\ p_i = \pi_r\}} q_i$;
  if $L(t) < Q$ *and* $\pi_r < \pi_g$ **then**
    $\pi_{\text{eq}} \leftarrow \pi_r$;
    break;

Algorithm 2: Decide sold amount s_i.

$Q \leftarrow 0$;
$s_i \leftarrow 0$, for all i;
for $i = 1 : M$ **do**
  if $p_i < \pi_{eq}$ **then**
    $s_i \leftarrow q_i$;
    $Q \leftarrow Q + q_i$;
Denote those bids with the price π_{eq} as $q_{a_1},\ q_{a_2}, \cdots, q_{a_h}$.
for $i = 1 : h$ **do**

$$s_{a_i} \leftarrow \min\left\{ \frac{q_{a_i}(L(t) - Q)}{\sum_{l=1}^{h} q_{a_l}},\ q_{a_i} \right\}. \tag{7}$$

Note that in this case the market price will indeed equal to π_{eq} according to Algorithm 1. Then, firm 1 sells the amount 6 and the firm 2 sells the amount 4, i.e., they split the remaining demand 10 in proportion to their bidding amount.

We immediately get the following lemma that indicates some properties out of those two algorithms.

Lemma 1. *(a) If $\sum_{i=1}^{M} q_i > L(t)$, then we must have $\pi_{eq} = p_j$ for all j that satisfies*

$$\sum_{i:\ p_i \le p_j} q_i > L(t),$$
$$\sum_{i:\ p_i < p_j} q_i \le L(t),$$

and this kind of j must exists.

(b) If $s_i = 0$, then we must have $p_i \ge \pi_{eq}$.

(c) If $p_i > \pi_{eq}$, then $s_i = 0$. If $p_i < \pi_{eq}$, then $s_i = q_i$.

(d) If $0 < s_i < q_i$, then we must have $p_i = \pi_{eq}$. Further, if $0 < s_i < q_i$ and $s_j = q_j$, then we must have $p_j < p_i = \pi_{eq}$.

(e) When $L(t) \le \sum_i q_i$, we have

$$L(t) = \sum_{i:\ p_i < \pi_{eq}} q_i + \sum_{i:\ p_i = \pi_{eq}} s_i.$$

Remark: These results are intuitive. Part (a) states that at π_{eq}, supply and demand are close. Part (b) states that if one bid gets no sell, then this bid must have the price higher than π_{eq}. Part (c) states that bidding above π_{eq} gets no sell, while bidding below π_{eq} sells all. Part (d) states that a partly sold bid must have the price π_{eq}. Part (e) states that when the supply is enough, the total sell amount equals to the demand.

Proof. (a) Because $\sum_{i=1}^{M} q_i > L(t)$, the condition stated in Line 6 of Algorithm 1 must be met at some iteration. Thus, we can directly get the result of this Lemma by Line 5∼7 of Algorithm 1.

(b) By Algorithm 2, we know that if $p_i < \pi_{\text{eq}}$, then $s_i = p_i > 0$. Thus, if $s_i = 0$, then we must have $p_i \geq \pi_{\text{eq}}$.

(c) This result is directly derived from Algorithm 2.

(d) Due to Line 5 of Algorithm 2, we get the first statement that $p_i = \pi_{\text{eq}}$ if $0 < s_i < q_i$. Next, we prove the second statement. Because $s_j = q_j > 0$, we know that $p_j \leq \pi_{\text{eq}}$ (otherwise $s_j = 0$ from part (c)). Thus we only need to prove that $p_j = \pi_{\text{eq}}$ is impossible. We prove by contradiction. Suppose in contrary that $p_j = \pi_{\text{eq}}$. Since $0 < s_i < p_i$, from Eq. (7) we know that $L(t) - Q > \sum_{l=1}^{h} q_{a_l}$. Thus, we must have $s_j < q_j$ by Eq. (7). This contradicts with $s_j = q_j$. Thus, we conclude that $p_j \neq \pi_{\text{eq}}$. As a result, we must have $p_j < \pi_{\text{eq}}$. Finally, using the result of the first statement, we can get the result of the second statement that $p_j < p_i = \pi_{\text{eq}}$.

(e) This result directly follows from Algorithm 2. □

A. *Proof of Proposition 1*

We first give the following lemma about any Nash equilibrium in the single-price-bid market, where every firm bids all generation amount, i.e., $q_i = q_i^0$ for all $i \in \{1, 2, \cdots, M\}$.

Lemma 2. *Suppose $\sum_{i \in S} q_i^0 \neq L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$ and $\sum_{i=1}^{M} q_i^0 > L(t)$. At a Nash equilibrium, if $\pi_{eq} > 0$, then there must exist one and only one firm j that satisfies $0 < s_j < q_j^0$, and $s_i = q_i^0$ for all other firms $i \neq j$.*

Proof. We split the proof by several steps as below.

Step 1: we prove that, for any bidding strategy (not necessarily a Nash equilibrium), we must have $s_i > 0$ for all $p_i = \pi_{\text{eq}}$. To see this, by Line 5 of Algorithm 1, we know that Q must equal to the sum of the elements in some subset of $\{q_1^0, q_2^0, \cdots, q_M^0\}$. In other words, we can write $Q = \sum_{i \in S_0} q_i^0 \neq L(t)$, where S_0 consists of the indices of firms that bid prices less than π_{eq}. Obviously, we have $S_0 \subseteq \{1, 2, \cdots, M\}$. Because $\sum_{i \in S} q_i^0 \neq L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, in Eq. (7) we must have $L(t) \neq Q$. Thus, by Eq. (7), we must have $s_i > 0$ for all $p_i = \pi_{\text{eq}}$.

Step 2: we prove that $s_i > 0$ for all i. We prove by contradiction. Suppose that there exists a firm i^* such that $s_{i^*} = 0$. Thus, the payoff of the firm i^* equals to $s_{i^*}\pi_{\text{eq}} = 0$. Because $s_{i^*} = 0$, by Lemma 1(b), we have $p_{i^*} \geq \pi_{\text{eq}}$. Further, considering the result in step 1, we must have $p_{i^*} > \pi_{\text{eq}}$. By Lemma 1(a), we have

$$\sum_{i:\ p_i \leq \pi_{\text{eq}}} q_i^0 > L(t), \tag{8}$$

$$\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \leq L(t). \tag{9}$$

Now, let the firm i^* deviate to another bidding strategy with $p'_{i^*} = \pi_{\text{eq}}$. Then, Eq. (8) and Eq. (9) still hold. By Lemma 1(a), we know that the new market price will not change, i.e., $\pi'_{\text{eq}} = \pi_{\text{eq}}$. By the result of step 1, we have $s'_{i^*} > 0$. Thus, the new payoff of the firm i^* equals to $s'_{i^*}\pi'_{\text{eq}} = s'_{i^*}\pi_{\text{eq}} > 0$, which is larger than the previous payoff of zero. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we have proven that $s_i > 0$ for all i.

Step 3: we prove that there exists one only one firm j that satisfies $s_j < q_j^0$. Because $\sum_{i \in S} q_i^0 \neq L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, Lemma 1(e) implies that at least one firm j satisfies $0 < s_j < q_j^0$. Now, we prove by contradiction that no other firms satisfy $s_j < q_j^0$. Suppose that there exists another firm $k \neq j$ such that $0 < s_k < q_k^0$. By Lemma 1(d), we have $p_j = p_k = \pi_{\text{eq}}$. The payoff of the firm j equals to $s_j\pi_{\text{eq}}$. By Lemma 1(a) we have

$$\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \leq L(t).$$

By Lemma 1(e) we have

$$L(t) = \sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 + \sum_{i:\ p_i = \pi_{\text{eq}}} s_i \geq \left(\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \right) + s_j + s_k. \tag{10}$$

Now, let the firm j deviate to another bidding strategy that $p'_j = \pi_{\text{eq}} - \epsilon$ where

$$\epsilon = \frac{\pi_{\text{eq}} \min\{s_k,\ q_j^0 - s_j\}}{2 \min\{s_j + s_k,\ q_j^0\}}.$$

Because $\min\{s_k,\ q_j^0 - s_j\} < \min\{s_k,\ q_j^0 - s_j\} + s_j = \min\{s_j + s_k,\ q_j^0\}$, we have $0 < \epsilon < \frac{\pi_{\text{eq}}}{2}$. The other firms' bidding prices do not change, i.e., $p'_i = p_i$, for all $i \neq j$. Then, we must have $\{i:\ p_i < \pi_{\text{eq}} - \epsilon\} = \{i:\ p'_i < \pi_{\text{eq}} - \epsilon\}$, $\{i:\ p_i = \pi_{\text{eq}} - \epsilon\} = \{i:\ p'_i = \pi_{\text{eq}} - \epsilon,\ i \neq j\}$, and

$$\sum_{i:\ p'_i < \pi_{\text{eq}} - \epsilon} q_i^0 = \sum_{i:\ p_i < \pi_{\text{eq}} - \epsilon} q_i^0 \leq \sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \leq L(t),$$

where the last inequality follows from Eq. (10). By Lemma 1(a), it implies that the new market price must satisfy $\pi'_{\text{eq}} \geq \pi_{\text{eq}} - \epsilon$. There exist two possible cases. Case 1: $\pi'_{\text{eq}} > \pi_{\text{eq}} - \epsilon$. By Lemma 1(c), we have $s'_j = q_j^0$. Case 2: $\pi'_{\text{eq}} = \pi_{\text{eq}} - \epsilon$. We have

$$\begin{aligned}
&\sum_{i:\ p_i < \pi_{\text{eq}} - \epsilon} q_i^0 + \left(\sum_{i:\ p_i = \pi_{\text{eq}} - \epsilon} s'_i \right) + s'_j \\
&= \sum_{i:\ p'_i < \pi_{\text{eq}} - \epsilon} q_i^0 + \sum_{i:\ p'_i = \pi_{\text{eq}} - \epsilon} s'_i \\
&\qquad \text{(because only the firm } j \text{ deviates)} \\
&= L(t) \quad \text{(by Lemma 1(e))} \\
&\geq \left(\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 \right) + s_j + s_k \ \text{(by Eq. (10))}.
\end{aligned}$$

Moving the first two terms of the left-hand side to the right-hand side, we have

$$\begin{aligned}
s_j' &\geq s_j + s_k + \left(\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i < \pi_{\text{eq}} - \epsilon} q_i^0 - \sum_{i:\ p_i = \pi_{\text{eq}} - \epsilon} s_i' \right) \\
&\geq s_j + s_k + \left(\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i < \pi_{\text{eq}} - \epsilon} q_i^0 - \sum_{i:\ p_i = \pi_{\text{eq}} - \epsilon} q_i^0 \right) \\
&= s_j + s_k + \left(\sum_{i:\ p_i < \pi_{\text{eq}}} q_i^0 - \sum_{i:\ p_i \leq \pi_{\text{eq}} - \epsilon} q_i^0 \right) \\
&\geq s_j + s_k.
\end{aligned}$$

In summary, in both cases, we always have $s_j' \geq \min\{s_j + s_k,\ q_j^0\}$. Thus, the new payoff $\pi_{\text{eq}}' s_j'$ of the firm j satisfies

$$\begin{aligned}
\pi_{\text{eq}}' s_j' &\geq (\pi_{\text{eq}} - \epsilon) \min\{s_j + s_k,\ q_j^0\} \\
&= \pi_{\text{eq}} \min\{s_j + s_k,\ q_j^0\} - \epsilon \min\{s_j + s_k,\ q_j^0\} \\
&= \pi_{\text{eq}} \min\{s_j + s_k,\ q_j^0\} - \frac{\pi_{\text{eq}} \min\{s_k,\ q_j^0 - s_j\}}{2} \\
&> \pi_{\text{eq}} \big(\min\{s_j + s_k,\ q_j^0\} - \min\{s_k,\ q_j^0 - s_j\} \big) \\
&= \pi_{\text{eq}} s_j.
\end{aligned}$$

That means that the firm j gets more payoff after deviating, which contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we have proven that there exists one and only one firm j that satisfies $0 < s_j < q_j^0$.

Finally, recall from step 2 that for all firm i must satisfy $s_i > 0$. Thus, the result of step 3 also implies that $s_i = q_i^0$ for all $i \neq j$. □

1) Proof of Case 1:

Proof. When $\sum_{i=1}^{M} q_i^0 \leq L(t)$, Line 6 of Algorithm 1 will never be reached. Thus, the market price must be $\pi_{\text{eq}} = \pi_g$. Next, we show that every firm bids at π_g is a Nash equilibrium. First, when every firm bids at π_g, because $\pi_{\text{eq}} = \pi_g$, we have $a_i = i$ and $Q = 0$ in Eq. (7). Thus, we have $s_i = q_i^0$ in Eq. (7) since $L(t) - Q = L(t) > \sum_{i=1}^{M} q_i^0$. As a result, the profit of the firm i equals $q_i^0 \pi_g$. Since at any circumstance the market price cannot exceed π_g and the firm i cannot sell more than q_i^0, the firm i cannot earn more profit than $q_i^0 \pi_g$ by choosing any other bidding strategy. Thus, the situation that every firm bids at π_g is Nash equilibrium. □

2) Proof of Case 2:

Proof. First, we prove that the strategy that all firms bid zero price is a Nash equilibrium. When every firm i bids zero price $p_i = 0$, because $\sum_{i=1}^{M} q_i^0 \geq \sum_{i \neq j} q_i^0 > L(t)$ for all j, the condition in Line 6 of Algorithm 1 must be met at some time. Thus, we must have $\pi_{\text{eq}} = 0$, and the payoff for every firm i equals to $q_i^0 \pi_{\text{eq}} = 0$. If one firm j^* bids differently (i.e., $p_{j^*}' > 0$), since $\sum_{i \neq j^*} q_i^0 > L(t)$, we still have $\pi_{\text{eq}}' = 0$ by Algorithm 1. Because $p_{j^*}' > 0 = \pi_{\text{eq}}'$, by Algorithm 2, we have $s_{j^*}' = 0$. As a result, the new payoff of the firm j^* equals to $\pi_{\text{eq}}' s_{j^*}' = 0$, i.e., the firm j^* cannot get more benefits. Thus, we have proven that the strategy that all firms bid zero price is a Nash equilibrium.

Then, we prove the second statement of this proposition by contradiction. Suppose on the contrary that, at a Nash equilibrium, we have $\pi_{\text{eq}} > 0$. By Lemma 2, we have one firm j^* such that $0 < s_{j^*} < q_{j^*}$ and $s_i = q_i^0$ for all other firms $i \neq j^*$. By Lemma 1(d), we know $p_{j^*} = \pi_{\text{eq}}$ and $p_i < \pi_{\text{eq}}$ for all $i \neq j^*$. Thus, by Lemma 1(e), we have

$$L(t) = \left(\sum_{i \neq j^*} q_i^0 \right) + s_{j^*} > \sum_{i \neq j^*} q_i^0.$$

This contradicts the assumption that $\sum_{i \neq j} q_i^0 > L(t)$ for all j. Thus, we have proven that at any Nash equilibrium, we must have $\pi_{\text{eq}} = 0$. □

3) Proof of Case 3:

Proof. First, we illustrate that the following bidding strategy is a Nash equilibrium. Let one firm $j \in \mathcal{I}$ bid the price π_g, and let all other firms bid the price zero. Since $\sum_{i \neq j} q_i^0 < L(t)$, we have $\pi_{\text{eq}} = \pi_g$ because Line 6 of Algorithm 1 will never be met. By Algorithm 2, any firm $i \neq j$ sells all, i.e., $s_i = q_i^0$. Thus, any firm $i \neq j$ has already achieved its maximum possible payoff $q_i^0 \pi_g$ and none of them has an incentive to deviate. Consider the firm j. Since all firms except the firm j bid zero price, the firm j cannot sell more unless bidding the price zero. Thus, if the firm j bids any positive price less than π_g, its payoff will be lower. If the firm j bids zero price, by Algorithm 1 we know the market price will be zero and thus its payoff will be zero because $\sum_{i=1}^{M} q_i^0 > L(t)$. In summary, we conclude that the firm j cannot get more payoff by deviating. Thus, we have shown the strategy that $p_j = \pi_{\text{eq}},\ p_i \Big|_{i \neq j} = 0$ is a Nash equilibrium.

Then, we prove that at any Nash equilibrium we must have $\pi_{\text{eq}} = \pi_g$ and only one firm in $\mathcal{I}$ bids at π_g. We split the proof by several steps as follows.

Step 1: we show that if any firm $j \in \mathcal{I}$ bids the price $p_j = \pi_g$, then $\pi_{\text{eq}} = \pi_g$. At the beginning of this subsection, we have already made the assumption that a legitimate bid should satisfy $p_i \in [0, \pi_g]$. Thus, we have

$$\sum_{i:\ p_i \leq \pi_g} q_i^0 = \sum_{i=1}^{M} q_i^0 > L(t).$$

Further, if any firm $j \in \mathcal{I}$ bids the price $p_j = \pi_g$, by the assumption of this Proposition, we then have

$$\sum_{i:\ p_i < \pi_g} q_i^0 \leq \sum_{i \neq j} q_i^0 < L(t).$$

By Lemma 1(a), we know $\pi_{\text{eq}} = \pi_g$. Thus, we have proven that if $p_j = \pi_g$ for some $j \in \mathcal{I}$, then $\pi_{\text{eq}} = \pi_g$.

Step 2: based on the result of step 1, in the rest of the proof, we only need to show that, at any Nash equilibrium, there must exist one firm $j \in \mathcal{I}$ that bids $p_j = \pi_g$. Towards this end, we will first show that $\pi_{\text{eq}} > 0$ at any Nash equilibrium, based on which we can then apply Lemma 2. We prove by contradiction. Suppose that $\pi_{\text{eq}} = 0$ at a Nash equilibrium. Consider any firm $j \in \mathcal{I}$. By step 1, we know $p_j \neq \pi_g$. Note that since $\pi_{\text{eq}} = 0$, the payoff of the firm j equals to 0. Let the firm j deviate its bidding strategy to $p_j' = \pi_g$. By step 1, we know that the new market price equals to $\pi_{\text{eq}}' = \pi_g$. Because $\sum_{i\in S} q_i^0 \neq L(t)$ for all $S \subseteq \{1, 2, \cdots, M\}$, by Eq. (7) we must have $s_j' > 0$. Thus, the new payoff of the firm j equals to $s_j'\pi_{\text{eq}}' > 0$, which is larger than the payoff of its previous bidding strategy. This contradicts the assumption that the previous bidding strategy is a Nash equilibrium. Thus, we have proven that $\pi_{\text{eq}} > 0$ at any Nash equilibrium.

Step 3: we prove $\pi_{\text{eq}} = \pi_g$ at any Nash equilibrium. We prove by contradiction. Suppose in the contrary that $\pi_{\text{eq}} < \pi_g$. By the result of step 2, we must have $\pi_{\text{eq}} > 0$. By Lemma 2, there exists one and only one firm j such that $0 < s_j < q_j^0$ and $s_i = q_i^0$ for all $i \neq j$. By Lemma 1(e), we have

$$L(t) = \left(\sum_{i\neq j} q_i^0\right) + s_j.$$

Thus, $\sum_{i\neq j} q_i^0 < L(t)$, i.e., we must have $j \in \mathcal{I}$. By Lemma 1(d), we have $p_j = \pi_{\text{eq}} < \pi_g$ and $p_i < p_j$ for all $i \neq j$. By Lemma 1(e), we have $s_j = L(t) - \sum_{i\neq j} q_i^0$. The payoff of the firm j thus equals to $\pi_{\text{eq}} s_j$. Now, let the firm j deviate to another bidding strategy that $p_j' = \pi_g$. By step 1, we know that the new market price equals to $\pi_{\text{eq}}' = \pi_g$. Because $p_i < p_j = \pi_{\text{eq}} < \pi_g = \pi_{\text{eq}}'$ for all $i \neq j$, by Lemma 1(e), we have $s_j' = L(t) - \sum_{i\neq j} q_j^0 = s_j$. Thus, the new payoff of the firm j equals $\pi_{\text{eq}}' s_j' = \pi_g s_j' > \pi_{\text{eq}} s_j$. This contradicts the assumption that the previous bidding strategy is a Nash equilibrium. Thus, we have proven that $\pi_{\text{eq}} = \pi_g$.

In summary, we conclude that, at any Nash equilibrium, we must have $\pi_{\text{eq}} = \pi_g$ and one firm in $\mathcal{I}$ bids at the price π_g while other firms bid the price below π_g. □

B. Proof of Proposition 2

Proof. First, we prove that, if $L(t) > \sum_{i=1}^M q_i^0$, then there must exist a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Specifically, we want to show the bidding strategy that $q_i = q_i^0, p_i = \pi_{\text{eq}}$ for all i is a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Because $L(t) > \sum_{i=1}^M q_i^0 = \sum_{i=1}^M q_i$, by Algorithm 1, we have $\pi_{\text{eq}} = \pi_g$. By Algorithm 2, we have $s_i = q_i$. Because $q_i^0 = q_i$, any firm i sells all of its generation, i.e., $s_i = q_i^0$. Recall that, by Algorithm 1, regardless of the bids, we always have $\pi_{\text{eq}} \leq \pi_g$. Thus, each firm i has already achieved its maximum possible profit $\pi_g q_i^0$. In other words, no bidding strategy can make any firm get more profit. Thus, the bidding strategy $q_i = q_i^0,\ p_i = \pi_{\text{eq}}$ for all i is a Pareto-optimal Nash equilibrium.

Then, we prove that if $L(t) \leq \sum_{i=1}^M q_i^0$, then there must exist a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Specifically, we will show that any bidding strategy that satisfies the following conditions is such a Nash equilibrium:

$$\begin{cases} L(t) = \sum_{i=1}^M q_i, \\ p_i = 0 \text{ for all } i. \end{cases} \tag{11}$$

Because $L(t) \leq \sum_{i=1}^M q_i^0$, we can always find such $(q_0, q_1, \cdots)$ that satisfies Eq. (11). By Algorithm 1 (especially Line 6), we know that $\pi_{\text{eq}} = \pi_g$ in this situation[3]. By Algorithm 2, we know $s_i = p_i$ for all i. Thus, the profit of any firm j equals $p_j\pi_g$. Now, suppose that the firm j deviates to an arbitrary bidding strategy $(q_j',\ p_j')$, while the bids of the other firms $i \neq j$ remain the same, i.e., $p_i' = p_i,\ q_i' = q_i$. There are three different cases.

Case 1: $p_j' > 0$. By Algorithm 2, we know that $s_i' = q_i' = q_i$ for all $i \neq j$, because all other firms $i \neq j$ bid lower than the firm j and $\sum_{i\neq j} q_i' = \sum_{i\neq j} q_i = L(t) - q_j < L(t)$. Thus, $s_j' \leq L(t) - \sum_{i\neq j} q_i = q_j$. As a result, the profit of the firm j becomes $s_j'\pi_{\text{eq}} \leq q_j\pi_g$, which is not greater than the original payoff.

Case 2: $p_j' = 0$ and $q_j' \leq q_j$. The payoff of the firm j equals to $s_j'\pi_{\text{eq}} \leq q_j'\pi_g \leq q_j\pi_g$, which is also not greater than the original profit.

Case 3: $p_j' = 0$ and $q_j' > q_j$. By Algorithm 1, we have that the new market price now equals to $\pi_{\text{eq}}' = 0$. Thus, the payoff of the firm j becomes zero, which cannot be greater than its original payoff.

In all three cases, the firm j cannot get more payoff by deviating to another bidding strategy. As a result, we conclude that the original bidding strategy is a Nash equilibrium. Now, we prove that this Nash equilibrium is Pareto optimal. The total payoff of all firms equals to $\pi_{\text{eq}} \sum_{i=1}^M s_i = \pi_g \sum_{i=1}^M q_i = \pi_g L(t)$. By Algorithm 1, we know that the market price cannot exceed π_g. By Algorithm 2, we know that the total sold amount cannot exceed $L(t)$. Thus, $\pi_g L(t)$ is the maximum total payoff to the firms as a whole. It implies that this Nash equilibrium is Pareto optimal. The result of this proposition thus holds. □

Remark on how the marginal price is determined: In Algorithm 1, the marginal price (i.e., market price π_{eq}) is defined as the cost for one *additional* unit of demand. There is an alternative way of defining the market price as the cost of the *last* unit of demand. Specifically, we may change the condition in Line 6 in Algorithm 1 as "if $L(t) \leq Q$ and $\pi_r < \pi_g$". For markets allowing only single-price bids (which we study in subsection IV-A), since we assume

$$\mathbf{Pr}\left\{\sum_{i\in S} q_i \neq L(t) \text{ for all } S \subseteq \{1, 2, \cdots, M\}\right\} = 1,$$

the above change makes no difference to the calculation of $\mathbb{E}[\pi_{\text{eq}}]$ in subsection IV-A. For markets allowing price-quantity bids (which we study in this subsection), as we explain below, similar outcomes as Proposition 2 will arise even when the market price is defined as the cost of the last unit of demand. In this case, we made the additional assumption that the quantity of a bid must be a multiple of some smallest-possible unit δ

[3] Readers may be surprised why the market price is π_g even though every firm bids at zero price. Note that by Algorithm 2, the market price is the marginal cost for one *additional* unit of demand. With the bids in Eq. (11), the demand is equal to the total bidding quantity. Thus, the marginal price that consumers have to pay for one *additional* unit of electricity is π_g. See the remark at the end of this subsection on what happens if the market price is defined as the marginal cost for the *last* unit of demand.

(e.g., 1 kW). Then, we can change the bidding strategy in Eq. (2) to

$$\begin{cases}\sum_{i=1}^{M} q_i = \min\left\{\left\lfloor \frac{L(t)}{\delta}\right\rfloor\cdot\delta,\ \sum_{i=1}^{M} q_i^0\right\},\\ p_i=0,\ \text{both } q_i \text{ and } q_i^0 \text{ are a multiple of } \delta,\ \text{for all } i.\end{cases} \tag{12}$$

We can use similar methods as in the proof of Proposition 2 to show that Eq. (12) is a Pareto-optimal Nash equilibrium with $\pi_{\text{eq}} = \pi_g$. Detailed proof is omitted here.

Appendix B
Proofs in Section V

Let $\mathbf{x}_t = (L_n(t),\ G(t)) \in \mathbb{R}^2$, and $H_n(\mathbf{x}_t, c_n) = L_n(t) - G(t)c_n$. Then, $[H_n(\mathbf{x}_t, c_n)]^+$ is the amount of extra electricity that the consumer n needs to buy from the grid at time t, where $[a]^+ = \max\{0, a\}$. Let $[0, T]$ denote the range of time[4].

A. Existence of $d_n(\pi)$

Lemma 3. *The function $J_n(\pi, c_n)$ is continuous with respect to both π and c_n.*

Proof. To verify the continuity of J_n, note that

$$\begin{aligned}
&\lim_{\Delta\pi\to 0,\Delta c_n\to 0}\left(J_n(\pi+\Delta\pi, c_n+\Delta c_n) - J_n(\pi, c_n)\right)\\
=&\lim_{\Delta\pi\to 0,\Delta c_n\to 0}\left((\pi+\Delta\pi)(c_n+\Delta c_n) - \pi c_n\right)\\
&+\frac{\pi_g}{T}\lim_{\Delta c_n\to 0}\int_0^T \mathbb{E}\big[[L_n(t)-G(t)(c_n+\Delta c_n)]^+\\
&-[L_n(t)-G(t)c_n]^+\big]dt\\
=&0+\frac{\pi_g}{T}\int_0^T \mathbb{E}\lim_{\Delta c_n\to 0}\big[[L_n(t)-G(t)(c_n+\Delta c_n)]^+\\
&-[L_n(t)-G(t)c_n]^+\big]dt\\
=&0+\frac{\pi_g}{T}\int_0^T \mathbb{E}\ 0\ dt = 0.
\end{aligned}$$

Thus, J_n is continuous with respect to both π and c_n. Note that in the second equality, we exchange the order between the limit and the expectation. This exchange of order is valid by the Dominated Convergence Theorem [31], since $\left|\int_0^T \mathbb{E}[L_n(t)]dt\right| < \infty$ and $|[L_n(t) - G(t)(c_n + \Delta c_n)]^+| \le L_n(t)$ for all Δc_n. □

Lemma 4. *The minimizer of $J_n(\pi, c_n)$ over c_n must exist, i.e., there exists c_n^* such that*

$$J_n(\pi, c_n^*) \le J_n(\pi, c_n) \text{ for all } c_n \ge 0.$$

Proof. We first prove that, for any $\pi \ge 0$, there exists $\bar{c} > 0$ such that

$$J_n(\pi, 1) \le J_n(\pi, c_n) \text{ for all } c_n \ge \bar{c}. \tag{13}$$

To see this, note that because $H_n(\mathbf{x}_t, c_n) \le L(t)$, we have

$$\begin{aligned}
J_n(\pi,1) &\le \pi + \frac{\pi_g}{T}\int_0^T L(t)dt\\
&= \pi\left(1+\frac{\pi_g}{\pi T}\int_0^T L(t)dt\right).
\end{aligned}$$

We let $\bar{c} = 1 + \frac{\pi_g}{\pi T}\int_0^T L(t)dt$. Thus, we have

$$\begin{aligned}
J_n(\pi,1) &\le \pi\left(1+\frac{\pi_g}{\pi T}\int_0^T L(t)dt\right)\\
&= \pi\bar{c}\\
&\le J_n(\pi, c_n),\ \text{for all } c_n \ge \bar{c}.
\end{aligned}$$

Further, by the Extreme Value Theorem [13], over the closed and bounded interval $[0, \bar{c}]$, the continuous function $J_n(\pi, c_n)$ must have a minimum c_n^*, i.e.,

$$J_n(\pi, c_n^*) \le J_n(\pi, c_n) \text{ for all } c_n \le \bar{c}.$$

Combining with Eq. (13), the result of the lemma then follows. □

B. Calculation of the threshold price $\overline{\pi}_n$

Intuitively, when the price is higher than a threshold value defined as $\overline{\pi}_n$, the consumer n chooses to rent zero PV panels. In the following lemma, we give the exact value of $\overline{\pi}_n$ and prove $d_n(\pi) = 0$ when $\pi > \overline{\pi}_n$.

Lemma 5. *If $\pi > \overline{\pi}_n \triangleq \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt$, then $J_n(\pi, c_n) > J_n(\pi, 0)$ for all $c_n > 0$, and thus $d_n(\pi) = 0$.*

Proof.

$$\begin{aligned}
&\pi > \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt \text{ and } c_n > 0\\
\Longrightarrow\ &\pi c_n > \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)c_n\mathbb{1}_{\{L_n(t)>0\}}]dt\\
\Longrightarrow\ &\pi c_n > \frac{\pi_g}{T}\int_0^T \mathbb{E}\big[\left(L_n(t) - [L_n(t)-G(t)c_n]^+\right)\\
&\qquad\qquad\cdot\mathbb{1}_{\{L_n(t)>0\}}\big]dt\\
\Longrightarrow\ &\pi c_n + \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[[L_n(t)-G(t)c_n]^+\right]dt\\
&\quad > \frac{\pi_g}{T}\int_0^T \mathbb{E}[L_n(t)]dt\\
\Longrightarrow\ &J_n(\pi, c_n) > J_n(\pi, 0).
\end{aligned}$$

□

In Lemma 5, the condition $\pi > \overline{\pi}_n \triangleq \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt$ can be rewritten as

$$\frac{\pi}{\frac{1}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt} > \pi_g.$$

Notice that the left hand side represents the equivalent price of one unit amount of solar energy that is useful for the consumer n (i.e., when her load is positive). Lemma 5 thus states that, when this equivalent price is higher than the retail price, the consumer n will not rent any PV panels, i.e., $d_n(\pi) = 0$ if $\pi > \overline{\pi}_n$.

[4] Although we adopt a continuous-time model here, the result also applies to the discrete-time model.

C. Monotonicity of demand function

Lemma 6. *Define* $h(c_n) = \int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt$ *where* $\mathcal{A}(c_n) = \{(t,\ \mathbf{x}_t) : \ H_n(\mathbf{x}_t, c_n) > 0\}$. *Then, under Assumption 1,* $h(c_n)$ *is continuous when* $c_n > 0$. *Further, for all* $c_{i,2} > c_{i,1} > 0$, *we have* $0 \leq h(c_{i,1}) - h(c_{i,2}) \leq \gamma T \ln\left(\frac{c_{i,2}}{c_{i,1}}\right) \leq \gamma T \frac{c_{i,2}-c_{i,1}}{c_{i,1}}$.

Proof. Suppose that $c_{i,2} > c_{i,1} > 0$. Because $H_n(\mathbf{x}_t, c_{i,2}) \leq H_n(\mathbf{x}_t, c_{i,1})$, we have $\mathcal{A}(c_{i,2}) \subseteq \mathcal{A}(c_{i,1})$. Thus, we have $h(c_{i,1}) - h(c_{i,2}) \geq 0$. By the definition of $h(\cdot)$, we have

$$
\begin{aligned}
& h(c_{i,1}) - h(c_{i,2}) \\
=& \int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_{i,1})}]dt - \int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_{i,2})}]dt \\
=& \int_0^T \int_0^\infty \int_0^{\frac{y}{c_{i,1}}} x f^t_{G,L_n}(x,y)dxdydt \\
& - \int_0^T \int_0^\infty \int_0^{\frac{y}{c_{i,2}}} x f^t_{G,L_n}(x,y)dxdydt \\
=& \int_0^T \int_0^\infty \int_{\frac{y}{c_{i,2}}}^{\frac{y}{c_{i,1}}} x f^t_{G|L_n}(x|y)\cdot f^t_{L_n}(y)dxdydt \\
\leq& \int_0^T \int_0^\infty \int_{\frac{y}{c_{i,2}}}^{\frac{y}{c_{i,1}}} \frac{\gamma}{x}\cdot f^t_{L_n}(y)dxdydt \quad \text{(by Assumption 1)} \\
=& \int_0^T \int_0^\infty \gamma \ln\left(\frac{c_{i,2}}{c_{i,1}}\right)\cdot f^t_{L_n}(y)dxdydt \\
=& \gamma T \ln\left(\frac{c_{i,2}}{c_{i,1}}\right) \\
\leq& \gamma T \frac{c_{i,2}-c_{i,1}}{c_{i,1}} \ \text{(because } \ln(x) \leq x-1 \text{ for } x \geq 1).
\end{aligned}
$$

It also implies that

$$\lim_{c_{i,2}\to c_{i,1}} (h(c_{i,2}) - h(c_{i,1})) = 0.$$

As a result, we conclude that $h(c_n)$ is continuous when $c_n > 0$. □

Lemma 7. *Under Assumption 1, for a fixed* π, *the function* $J_n(\pi, c_n)$ *is differentiable, and*

$$\frac{\partial J_n(\pi, c_n)}{\partial c_n} = \pi - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt, \tag{14}$$

where $\mathcal{A}(c_n) = \{(t,\ \mathbf{x}_t) : \ H_n(\mathbf{x}_t, c_n) > 0\}$.

Proof. Since πc_n, the first term of $J_n(\pi, c_n)$, is differentiable, we only need to prove that $\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt$ is differentiable. We have

$$
\begin{aligned}
& \lim_{\Delta c\to 0}\frac{1}{\Delta c}\bigg(\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n+\Delta c)]^+\right]dt \\
& \qquad - \int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt\bigg) \\
=& \lim_{\Delta c\to 0}\frac{1}{\Delta c}\bigg(\int_0^T \mathbb{E}\left[H_n(\mathbf{x}_t, c_n+\Delta c)\mathbb{1}_{\mathcal{A}(c_n+\Delta c)}\right]dt \\
& \qquad - \int_0^T \mathbb{E}\left[H_n(\mathbf{x}_t, c_n)\mathbb{1}_{\mathcal{A}(c_n)}\right]dt\bigg) \\
=& \lim_{\Delta c\to 0}\frac{1}{\Delta c}\Big(\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)})] \\
& dt + \int_0^T \mathbb{E}[(H_n(\mathbf{x}_t, c_n+\Delta c) - H_n(\mathbf{x}_t, c_n))\cdot\mathbb{1}_{\mathcal{A}(c_n)}]dt\Big) \\
=& \lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)\left(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right)]dt}{\Delta c} \\
& + \lim_{\Delta c\to 0}\frac{-\int_0^T \mathbb{E}[G(t)\Delta c\cdot\mathbb{1}_{\mathcal{A}}]dt}{\Delta c}.
\end{aligned}
$$

For the first limit, we have

$$
\begin{aligned}
& \left|\lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[H_n(\mathbf{x}_t, c_n+\Delta c)\left(\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right)]dt}{\Delta c}\right| \\
\leq& \lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}[|H_n(\mathbf{x}_t, c_n+\Delta c)|\cdot\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right|]dt}{|\Delta c|} \\
\leq& \lim_{\Delta c\to 0}\frac{\int_0^T \mathbb{E}\left[|G(t)\Delta c|\cdot\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt}{|\Delta c|} \qquad (*) \\
=& \lim_{\Delta c\to 0}\int_0^T \mathbb{E}\left[G(t)\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt \\
=& \int_0^T \mathbb{E}\left[G(t)\lim_{\Delta c\to 0}\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right|\right]dt \\
& \text{(by the Dominated Convergence Theorem)} \\
=& 0.
\end{aligned}
$$

(The reason of the inequality $(*)$ is as follows. If $\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)} = 1$, then $H_n(\mathbf{x}_t, c_n+\Delta c) > 0$ and $H_n(\mathbf{x}_t, c_n) \leq 0$. Thus, we have $0 < H_n(\mathbf{x}_t, c_n+\Delta c) = H_n(\mathbf{x}_t, c_n) + G(t)\Delta c \leq G\Delta c$. If $\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)} = -1$, then $H_n(\mathbf{x}_t, c_n+\Delta c) \leq 0$ and $H_n(\mathbf{x}_t, c_n) > 0$. Thus, we have $0 \geq H_n(\mathbf{x}_t, c_n+\Delta c) = H_n(\mathbf{x}_t, c_n) + G(t)\Delta c > G(t)\Delta c$. In conclusion, when $\left|\mathbb{1}_{\mathcal{A}(c_n+\Delta c)} - \mathbb{1}_{\mathcal{A}(c_n)}\right| = 1$, we must have $|H_n(\mathbf{x}_t, c_n+\Delta c)| \leq |G(t)\Delta c|$.)

Thus, we have

$$
\begin{aligned}
& \lim_{\Delta c\to 0}\frac{1}{\Delta c}\bigg(\int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n+\Delta c)]^+\right]dt \\
& \qquad - \int_0^T \mathbb{E}\left[[H_n(\mathbf{x}_t, c_n)]^+\right]dt\bigg) \\
=& 0 + \lim_{\Delta c\to 0}\frac{-\int_0^T \mathbb{E}[G(t)\Delta c\cdot\mathbb{1}_{\mathcal{A}}]dt}{\Delta c} \\
=& -\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(c_n)}\right]dt.
\end{aligned}
$$

Thus, we have proven the differentiability of $J_n(\pi, c_n)$, and it follows that

$$\frac{\partial J_n(\pi, c_n)}{\partial c_n} = \pi - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(c_n)}]dt.$$

□

Lemma 8. *The demand function $d_n(\pi)$ is monotone decreasing with respect to π.*

Proof. It suffices to prove that any $c > d_n(\pi_0)$ cannot be the minimizer of $J_n(\pi_1, c)$ for all $\pi_1 > \pi_0$. In other words, it suffices to show that

$$J_n(\pi_1, c) > J_n(\pi_1, d_n(\pi_0)), \quad \text{for all } c > d_n(\pi_0).$$

To prove this, we have

$$\begin{aligned}
& J_n(\pi_1, c) - J_n(\pi_1, d_n(\pi_0)) \\
=& \pi_1(c - d_n(\pi_0)) + \frac{\pi_g}{T}\int_0^T \Big(\mathbb{E}\left[[L_n(t) - G(t)c]^+\right] \\
& - \mathbb{E}\left[[L_n(t) - G(t)d_n(\pi_0)]^+\right]\Big)dt \\
=& (\pi_1 - \pi_0)(c - d_n(\pi_0)) + \pi_0(c - d_n(\pi_0)) \\
& + \frac{\pi_g}{T}\int_0^T \Big(\mathbb{E}\left[[L_n(t) - G(t)c]^+\right] \\
& \qquad - \mathbb{E}\left[[L_n(t) - G(t)d_n(\pi_0)]^+\right]\Big)dt \\
=& (\pi_1 - \pi_0)(c - d_n(\pi_0)) + J_n(\pi_0, c) - J_n(\pi_0, d_n(\pi_0)) \\
\geq& (\pi_1 - \pi_0)(c - d_n(\pi_0)) \\
& \text{(because } J_n(\pi_0, c) - J_n(\pi_0, d_n(\pi_0)) \geq 0 \\
& \text{by the definition of } d_n(\pi_0)) \\
>& 0.
\end{aligned}$$

The result of the lemma thus holds. □

Lemma 9. *Under Assumption 1, the demand function $d_n(\pi)$ is strictly monotone decreasing with respect to π when $\pi \in (0, \overline{\pi}_n)$.*

Proof. Consider two prices $0 < \pi_1 < \pi_0 < \overline{\pi}_n$. By Lemma 8, we already have $d_n(\pi_0) \geq d_n(\pi_1)$. Thus, we only need to prove that $d_n(\pi_0) \neq d_n(\pi_1)$. Towards this end, it is sufficient to prove that such $d_n(\pi_1)$ must not be the minimizer of $J_n(\pi_0, c_n)$.

Because $\pi_1 < \overline{\pi}_n$, we have

$$\pi_1 < \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\{L_n(t)>0\}}\right]dt.$$

Applying Lemma 7, we have

$$\left.\frac{\partial J_n(\pi_1, c_n)}{\partial c_n}\right|_{c_n=0} = \pi_1 - \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\{L_n(t)>0\}}]dt < 0.$$

This implies that $d_n(\pi_1) > 0$. By the first-order condition, we have

$$\left.\frac{\partial J_n(\pi_1, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_1)} = 0.$$

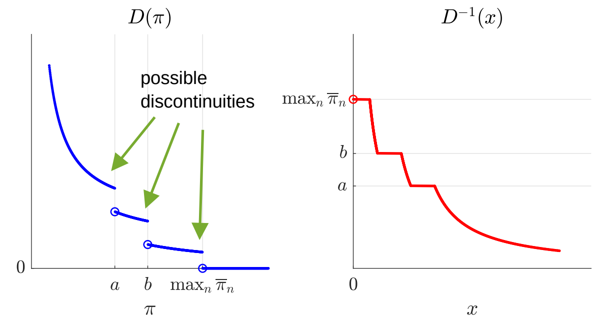

Fig. 5. Demand function $D(\pi)$ and the corresponding inverse function $D^{-1}(x)$.

Similarly, we have

$$\left.\frac{\partial J_n(\pi_0, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_0)} = 0.$$

As a result, we have

$$\begin{aligned}
& \left.\frac{\partial J_n(\pi_0, c_n)}{\partial c_n}\right|_{c_n=d_n(\pi_1)} \\
=& \pi_0 - \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_1))}\right]dt \\
& \quad \text{(where } \mathcal{A}(\cdot) \text{ is defined in Lemma 7)} \\
=& (\pi_0 - \pi_1) + \left(\pi_1 - \frac{\pi_g}{T}\int_0^T \mathbb{E}\left[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_1))}\right]dt\right) \\
=& (\pi_0 - \pi_1) \neq 0.
\end{aligned}$$

Therefore, we conclude that $d_n(\pi_1)$ is not the minimizer of $J_n(\pi_0, c_n)$. The result of this lemma then holds. □

D. Inverse function of $D(\cdot)$

Based on the properties of $d_n(\pi)$, we know $D(\pi)$ is strictly monotone decreasing with respect to π when $\pi \in [0, \max_n \overline{\pi}_n]$, and $D(\pi)$ equals to zero when $\pi > \max_n \overline{\pi}_n$. We depict $D(\pi)$ in Fig. 5. Then, we can define the inverse function $D^{-1}(x)$ on $x > 0$. Notice that even though $d_n(\cdot)$ and $D(\cdot)$ may have discontinuities, $D^{-1}(x)$ is well-defined for all $x > 0$ thanks to the strict monotonicity of $D(\cdot)$. Further, we can verify that $D^{-1}(x) := \max\{\pi : D(\pi) \geq x\}$. Actually, because $D(\cdot)$ is strictly monotone decreasing, we still have $D^{-1}(D(\pi)) = \pi$. However, $D(D^{-1}(x)) = x$ is not always true. Instead, we have $D(D^{-1}(x)) \geq x$. Fig. 5 shows what $D(\pi)$ and corresponding $D^{-1}(x)$ look like.

E. Proof of Proposition 3

Proof. Let $\pi_a > \pi_b$. According to Lemma 7, we have

$$\begin{aligned}
\frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_a))}]dt = \pi_a, \\
\frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_b))}]dt = \pi_b.
\end{aligned}$$

Thus, by Lemma 6, we have

$$\begin{aligned}
\pi_a - \pi_b =& \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_a))}]dt \\
&- \frac{\pi_g}{T}\int_0^T \mathbb{E}[G(t)\mathbb{1}_{\mathcal{A}(d_n(\pi_b))}]dt \\
\leq& \pi_g\gamma\frac{d_n(\pi_b)-d_n(\pi_a)}{d_n(\pi_a)} \\
\implies d_n(\pi_a) \leq& \pi_g\gamma\frac{d_n(\pi_b)-d_n(\pi_a)}{\pi_a-\pi_b}.
\end{aligned}$$

Fix π_a and let π_b approach π_a. We have

$$d_n(\pi) \leq \pi_g\gamma\left|\frac{\partial d_n(\pi)}{\partial\pi}\right|.$$

Summing this inequality over all consumers, we then have

$$\begin{aligned}
&D(\pi) \leq \pi_g\gamma\left|\frac{\partial D(\pi)}{\partial\pi}\right| \\
\implies &\frac{\left|\frac{\partial D(\pi)}{D(\pi)}\right|}{\left|\frac{\partial\pi}{\pi}\right|} \geq \frac{\pi}{\pi_g\gamma}.
\end{aligned}$$

□

F. Corollaries of Proposition 3

Solving the differential inequality in Proposition 3, we have the following corollary.

Corollary 1. *Let π and π_0 be two arbitrary prices such that $\pi > \pi_0$. We then have*

$$D(\pi) \leq D(\pi_0)e^{-\frac{\pi-\pi_0}{\pi_g\gamma}}.$$

Proof. Notice that $D(\pi)$ is strictly monotone decreasing. We have

$$\begin{aligned}
&\left|\frac{\partial D(\pi)}{D(\pi)}\right| \geq \frac{\pi}{\pi_g\gamma}\left|\frac{\partial\pi}{\pi}\right| \\
\implies &\left|\int_{D(\pi_0)}^{D(\pi)}\frac{1}{D(a)}\partial D(a)\right| \geq \left|\int_{\pi_0}^{\pi}\frac{1}{\pi_g\gamma}\partial\pi\right| \\
\implies &\left|\ln\frac{D(\pi)}{D(\pi_0)}\right| \geq \left|\frac{\pi-\pi_0}{\pi_g\gamma}\right| \\
\implies &\ln\frac{D(\pi_0)}{D(\pi)} \geq \frac{\pi-\pi_0}{\pi_g\gamma} \\
\implies &\ln\frac{D(\pi)}{D(\pi_0)} \leq -\frac{\pi-\pi_0}{\pi_g\gamma} \\
\implies &D(\pi) \leq D(\pi_0)e^{-\frac{\pi-\pi_0}{\pi_g\gamma}}.
\end{aligned}$$

□

Replacing π_0 by $D^{-1}(x)$ and replacing $D(\pi_0)$ by x, we have a more general conclusion stated in the following corollary.

Corollary 2. *Let $\pi > D^{-1}(x)$. We then have*

$$D(\pi) \leq xe^{-\frac{\pi-D^{-1}(x)}{\pi_g\gamma}}.$$

Proof. If $x = D(D^{-1}(x))$, then the result of this corollary is obviously true by applying Corollary 1. If $x \neq D(D^{-1}(x))$, then $D(\cdot)$ must be discontinuous at $D^{-1}(x)$, which implies that $D(y) < x < D(D^{-1}(x))$ for all $y > D^{-1}(x)$. Thus, for all y such that $D^{-1}(x) < y < \pi$, we have

$$D(\pi) \leq D(y)e^{-\frac{\pi-y}{\pi_g\gamma}} < xe^{-\frac{\pi-y}{\pi_g\gamma}}.$$

Let y approach $D^{-1}(x)$, we have

$$D(\pi) \leq \lim_{y\to D^{-1}(x)} xe^{-\frac{\pi-y}{\pi_g\gamma}} = xe^{-\frac{\pi-D^{-1}(x)}{\pi_g\gamma}}.$$

□

G. Multi-block-bid mechanism

Algorithm 3: Compute market price π_{eq}.

1 Sort elements in $\{p_{i,k}: \text{ for all } i,k\}$ in an ascending order as $\pi_1 < \pi_2 < \cdots < \pi_R$ (note that all π_r's are distinct);
2 $Q \leftarrow 0$;
3 **for** $r = 1:R$ **do**
4   **if** $D(\pi_r) < Q$ **then**
5         $\pi_{\text{eq}} \leftarrow \pi_{r-1}$; (cases (a) and (b) in Fig. 2)
6   exit;
7     $Q \leftarrow Q + \sum_{\{i,k:\ p_{i,k}=\pi_r\}} q_{i,k}$;
8   **if** $D(\pi_r) < Q$ **then**
9         $\pi_{\text{eq}} \leftarrow \pi_r$; (cases (c) and (d) in Fig. 2)
10   exit;
11 $\pi_{\text{eq}} \leftarrow \pi_R$;

Algorithm 4: Decide sold amount $s_{i,k}$.

$Q \leftarrow 0$
$s_{i,k} \leftarrow 0$, for all i,k;
for $i = 1:M$ **do**
    for $k = 1:K_i$ **do**
        if $p_{i,k} < \pi_{eq}$ **then**
            $s_{i,k} \leftarrow q_{i,k}$;
            $Q \leftarrow Q + q_{i,k}$;
Denote those bids with the price π_{eq} as $q_{a_1,b_1}, q_{a_2,b_2}, \cdots, q_{a_h,b_h}$.
for $i = 1:h$ **do**

$$s_{a_i,b_i} \leftarrow \min\left\{\frac{q_{a_i,b_i}\left(D(\pi_{\text{eq}})-Q\right)}{\sum_{l=1}^{h} q_{a_l,b_l}},\ q_{a_i,b_i}\right\}. \tag{15}$$

As readers can see, the main difference between Algorithms 1 and 3 (and between Algorithms 2 and 4) is that the former assumes a fixed demand, while the latter assumes a demand function $D(\pi)$ that decreases with the price π. As a result, in Algorithm 3 there are multiple places (in Line 4, Line 8 and some other corner cases) where the market price is determined. That is because, when the demand is not fixed, there are multiple ways the supply curve and the demand curve intersect, which is depicted in Fig. 2.

Remark on Algorithm 3 and 4: In the above model, we have assumed that the multiple bids of each firm have different prices, i.e., $p_{i,k_1} \neq p_{i,k_2}$ when $k_1 \neq k_2$. This assumption is without loss of generality because, if a firm has two or more bids at the same price, we can merge them into on bid, and both the market price price and the firm's profit remains the same under Algorithms 3 and 4. That is because we adopt the uniform price policy (i.e., all sold part gets paid at the common market price π_{eq}), and the proportional assignment Eq. (15) (that all bids at the market price π_{eq} are assigned sales in proportion to the bidding quantity). The above property also implies that, even if a firm divides its equity into two firms that bid cooperatively in the market, the outcome of the market would be the same as the firm bids as a single entity.

1) Preparation for the proof of Theorem 2: We first provide some useful definitions, lemmas, and corollaries.

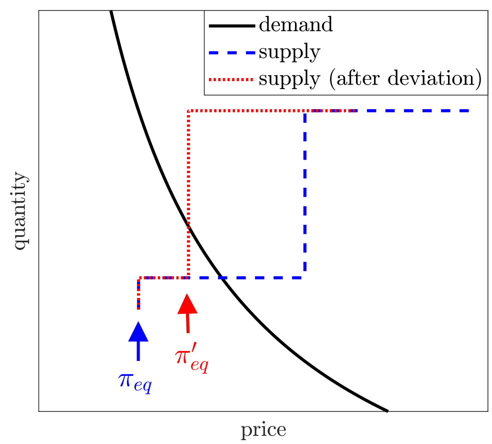

Fig. 6. An example of $\pi'_{\text{eq}} > \pi_{\text{eq}}$ when $\{p'_{i,k} : \text{ for all } k\} \not\subseteq \{p_{j,k} : \text{ for all } j,k\} \cup \{0\}$.

Definition 2. *In the multi-bid system, suppose a firm i changes its bid from $(\vec{p_i},\ \vec{q_i})$ to $(\vec{p_i}',\ \vec{q_i}')$. we say that the firm i **bids more aggressively** (in the new bid $(\vec{p_i}',\ \vec{q_i}')$) if*

$$\begin{cases} \sum_{\{k:\ p'_{i,k}<\pi\}} q'_{i,k} \geq \sum_{\{k:\ p_{i,k}<\pi\}} q_{i,k}, & \textit{for all } \pi \in \mathbb{R}, \\ \{p'_{i,k} : \textit{ for all } k\} \subseteq \{p_{j,k} : \textit{ for all } j,k\} \cup \{0\}. \end{cases}$$

Remark on Definition 2: The first condition states that, when a firm bids more aggressively, her total bidding quantity below any price π becomes larger. Later, we will show that, when a firm bids more aggressively, the market price should not increase (see Lemma 11(b)). However, for this to be true, the second condition in this definition becomes necessary, i.e., the new prices must be from the set of prices in the original bids (possibly by another firm j). Fig. 6 shows an counter-example where market price actually increases after one firm increases her bidding quantity at certain prices, without this constraint.

Lemma 10. *For real numbers a, b, x, y that $a \geq 0$, $b > 0$, $x \geq 0$, $y \geq 0$, $a - x \geq 0$, $b - x + y > 0$, we must have*

$$\min\left\{\frac{a-x}{b-x+y},\ 1\right\} \leq \min\left\{\frac{a}{b},\ 1\right\}.$$

Proof. If $a \geq b$, then we have

$$\min\left\{\frac{a-x}{b-x+y},\ 1\right\} \leq 1 = \min\left\{\frac{a}{b},\ 1\right\}.$$

If $a < b$, then we only need to prove

$$\frac{a-x}{b-x+y} \leq \frac{a}{b},$$

which is true because

$$\frac{a-x}{b-x+y} \leq \frac{a-x}{b-x} \leq \frac{a}{b}.$$

This lemma thus holds. $\square$

Lemma 11. *(a)*

$$\pi_{eq} = \max\left\{p_{j,l} : \sum_{\{i,k:\ p_{i,k}<p_{j,l}\}} q_{i,k} \leq D(p_{j,l})\right\}.$$

(b) If any firm i^* bids more aggressively and other firms do not change, then $\pi'_{eq} \leq \pi_{eq}$.

(c) $D(\pi_{eq}) \geq \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$. Further, if $D(\pi_{eq}) > \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}$, we have $s_{i,k} = q_{i,k}$ for all i,k such that $p_{i,k} \leq \pi_{eq}$.

(d) Consider two different bidding strategies $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}')$. If

$$\sum_{\{i,k:\ p'_{i,k}<\pi_{eq}\}} q'_{i,k} = \sum_{\{i,k:\ p_{i,k}<\pi_{eq}\}} q_{i,k}, \textit{ and}$$
$$\sum_{\{i,k:\ p'_{i,k}=\pi_{eq}\}} q'_{i,k} \geq \sum_{\{i,k:\ p_{i,k}=\pi_{eq}\}} q_{i,k},$$

then $\pi'_{eq} \geq \pi_{eq}$.

(e) If $s_{i^*,k^*} = 0$ and $p_{i^*,k^*} = \pi_{eq}$, then $s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{eq}$, and $D(\pi_{eq}) = \sum_{\{i,k:\ p_{i,k}<\pi_{eq}\}} q_{i,k}$.

Remark on Lemma 11: These results are intuitive. Part (a) states that π_{eq} is roughly the point where the demand and the supply intersect. Part (b) states that bidding more aggressively only makes the market price lower. Part (c) states that the total sold amount is always less than or equal to the demand. Further, if the total sold amount is less than the demand, then there are no partly sold bids. Part (d) states that bidding larger amount at or below the market price will never make the market price decrease. Part (e) states that if a bid with the price π_{eq} sells zero amount, then any bid with the price π_{eq} must also sell zero amount.

Proof. (a) We examine the outcome of Algorithm 3 in all possible situations. Define $F(\pi)$ as $F(\pi) = \sum_{\{i,k:\ p_{i,k}\leq\pi\}} q_{i,k}$. Define $S(\pi)$ as $S(\pi) = \sum_{\{i,k:\ p_{i,k}<\pi\}} q_{i,k}$. Obviously, $F(\pi)$ and $S(\pi)$ is monotone increasing. Suppose that the bidding prices are ranked as $\pi_1 < \pi_2 < \cdots < \pi_R$. We have $S(\pi_{a+1}) = F(\pi_a)$ for any pair of adjacent prices $(\pi_a,\ \pi_{a+1})$.

We also have $S(\pi) \leq F(\pi)$ for all π. What we need to prove can be written as

$$\pi_{\text{eq}} = \max_i \{\pi_i : \; S(\pi_i) \leq D(\pi_i)\}. \tag{16}$$

We consider three cases (i.e., Case 1 to 3 below).

Case 1: $D(\pi_1) < F(\pi_1)$. Then $\pi_{\text{eq}} = \pi_1$ as Algorithm 3 exits on Line 8. We have $S(\pi_1) = 0 \leq D(\pi_1)$, and $S(\pi_2) = F(\pi_1) > D(\pi_1)$. Eq. (16) thus follows.

Case 2: $D(\pi_R) \geq F(\pi_R)$. Then $\pi_{\text{eq}} = \pi_R$ as Algorithm 3 does not exit on Line 4 or Line 8, i.e., π_{eq} is determined by Line 11. We have $S(\pi_{\text{eq}}) \leq F(\pi_{\text{eq}}) \leq D(\pi_{\text{eq}})$. Eq. (16) thus follows.

Case 3: $D(\pi_1) \geq F(\pi_1)$ and $D(\pi_R) < F(\pi_R)$. Then, we can always find π_r such that $D(\pi_{r-1}) \geq F(\pi_{r-1})$ and $D(\pi_r) < F(\pi_r)$ (notice that $D(\pi) - F(\pi)$ is monotone decreasing). We consider two sub-cases (i.e., Case 3.1 and 3.2) below.

Case 3.1: $D(\pi_{r-1}) = F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_{r-1}$ as Algorithm 3 exits on Line 4. See Fig. 2(a). We have $S(\pi_{r-1}) \leq F(\pi_{r-1}) = D(\pi_{r-1})$, and $S(\pi_r) = F(\pi_{r-1}) = D(\pi_{r-1}) > D(\pi_r)$. Eq. (16) thus follows.

Case 3.2: $D(\pi_{r-1}) \neq F(\pi_{r-1})$. Because $D(\pi_{r-1}) \geq F(\pi_{r-1})$ in Case 3, we now have $D(\pi_{r-1}) > F(\pi_{r-1})$. We then consider three further sub-cases (i.e., Case 3.2.1 to 3.2.3).

Case 3.2.1: $D(\pi_r) < F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_{r-1}$ as Algorithm 3 exits on Line 4. See Fig. 2(b). We have $S(\pi_{r-1}) \leq F(\pi_{r-1}) \leq D(\pi_{r-1})$, and $S(\pi_r) = F(\pi_{r-1}) > D(\pi_r)$. Eq. (16) thus follows.

Case 3.2.2: $D(\pi_r) > F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_r$ as Algorithm 3 exits on Line 8. See Fig. 2(c). We have $S(\pi_r) = F(\pi_{r-1}) < D(\pi_r)$, and $S(\pi_{r+1}) = F(\pi_r) > D(\pi_r)$. Eq. (16) thus follows.

Case 3.2.3: $D(\pi_r) = F(\pi_{r-1})$. Then $\pi_{\text{eq}} = \pi_r$ as Algorithm 3 exits on Line 8. See Fig. 2(d). We have $S(\pi_r) = F(\pi_{r-1}) = D(\pi_r)$, and $S(\pi_{r+1}) = F(\pi_r) > D(\pi_r)$. Eq. (16) thus follows.

(b) Because only one firm deviates, by the definition of bidding more aggressively, we have

$$\sum_{\{i,k:\; p'_{i,k} < \pi\}} q'_{i,k} \geq \sum_{\{i,k:\; p_{i,k} < \pi\}} q_{i,k}, \text{ for all } \pi \in \mathbb{R}, \text{ and}$$
$$\{p'_{i,k} : \text{ for all } i,k\} \subseteq \{p_{i,k} : \text{ for all } i,k\} \cup \{0\}.$$

As a result, we have

$$\left\{ p'_{j,l} : \sum_{\{i,k:\; p'_{i,k} < p'_{j,l}\}} q'_{i,k} \leq D(p'_{j,l}) \right\} \subseteq \left\{ p_{j,l} : \sum_{\{i,k:\; p_{i,k} < p_{j,l}\}} q_{i,k} \leq D(p_{j,l}) \right\} \cup \{0\}.$$

By (a), we have $\pi'_{\text{eq}} \leq \pi_{\text{eq}}$.

(c) Obviously, $s_{i,k} = q_{i,k}$ for all i,k such that $p_{i,k} < \pi_{\text{eq}}$. We now consider two cases. (i) If $D(\pi_{\text{eq}}) - \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} \leq \sum_{\{i,k:\; p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}$, by Eq. (15), we have

$$s_{j,l} = \frac{q_{j,l} \left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} \right)}{\sum_{\{i,k:\; p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}}$$

for all j,l such that $p_{j,l} = \pi_{\text{eq}}$.

Summing this equation over all such j and l, we have

$$\sum_{\{i,k:\; p_{i,k} = \pi_{\text{eq}}\}} s_{i,k} = D(\pi_{\text{eq}}) - \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}$$
$$\implies D(\pi_{\text{eq}}) = \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k}.$$

(ii) If $D(\pi_{\text{eq}}) - \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} > \sum_{\{i,k:\; p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}$, by Eq. (15), we have $s_{i,k} = q_{i,k}$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Thus, $s_{i,k} = p_{i,k}$ for all i,k such that $p_{i,k} \leq \pi_{\text{eq}}$. As a result, $D(\pi_{\text{eq}}) > \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} + \sum_{\{i,k:\; p_{i,k} = \pi_{\text{eq}}\}} q_{i,k} = \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k}$. The result of part (c) then follows.

(d) By (a), we have $\sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} < D(\pi_{\text{eq}})$. By the given relationship between $(\mathbf{p}, \mathbf{q})$ and $(\mathbf{p}', \mathbf{q}')$, we have $\sum_{\{i,k:\; p'_{i,k} < \pi_{\text{eq}}\}} q'_{i,k} = \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} < D(\pi_{\text{eq}})$. Thus, we have

$$\pi_{\text{eq}} \in \left\{ p'_{j,l} : \sum_{\{i,k:\; p'_{i,k} < p'_{j,l}\}} q'_{i,k} \leq D(p'_{j,l}) \right\}$$
$$\implies \max \left\{ p'_{j,l} : \sum_{\{i,k:\; p'_{i,k} < p'_{j,l}\}} q'_{i,k} \leq D(p'_{j,l}) \right\} \geq \pi_{\text{eq}}$$
$$\implies \pi'_{\text{eq}} \geq \pi_{\text{eq}} \text{ (applying (a))}.$$

(e) Because $s_{i^*,k^*} = 0$, by Eq. (15), we have $D(\pi_{\text{eq}}) - Q = 0$, which implies $s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Therefore, we must have $D(\pi_{\text{eq}}) = \sum_{\{i,k:\; p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}$. □

Lemma 12. *Consider any bidding strategy* $(\mathbf{p}, \; \mathbf{q})$*, which is not necessarily a Nash equilibrium. Suppose that a firm* i^* *bids more aggressively and other firms do not change their bids. Then, we must have* $\pi'_{eq} \leq \pi_{eq}$ *and* $s'_{i,k} \leq s_{i,k}$*, for all* $i \neq i^*$*, and for all* k*. Consequently, any other firm* $i \neq i^*$ *will not earn more profit.*

Proof. By Lemma 11(b), we have $\pi'_{\text{eq}} \leq \pi_{\text{eq}}$. It only remains to show that $s'_{i,k} \leq s_{i,k}$, for all $i \neq i^*$ and for all k. There are two possible cases, $\pi'_{\text{eq}} < \pi_{\text{eq}}$ or $\pi'_{\text{eq}} = \pi_{\text{eq}}$. We discuss them separately as follows.

Case 1: $\pi'_{\text{eq}} < \pi_{\text{eq}}$. By Algorithm 4, we have

$$s'_{i,k} = s_{i,k} = q_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} < \pi'_{\text{eq}}, \; i \neq i^*,$$
$$s'_{i,k} \leq q_{i,k} = s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} = \pi'_{\text{eq}}, \; i \neq i^*,$$

and

$$s'_{i,k} = 0 \leq s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} > \pi'_{\text{eq}}, \; i \neq i^*.$$

Thus, we have shown that $s'_{i,k} \leq s_{i,k}$, for all $i \neq i^*$ and for all k.

Case 2: $\pi'_{\text{eq}} = \pi_{\text{eq}}$. We have

$$s'_{i,k} = q_{i,k} = s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} < \pi_{\text{eq}}, \; i \neq i^*,$$

and

$$s'_{i,k} = 0 = s_{i,k}, \text{ for all } i,k \text{ such that } p_{i,k} > \pi_{\text{eq}}, \ i \neq i^*.$$

Further, let

$$x = \sum_{\{j,l:\ p'_{j,l} < \pi_{\text{eq}}\}} q'_{j,l} - \sum_{\{j,l:\ p_{j,l} < \pi_{\text{eq}}\}} q_{j,l} \geq 0,$$
$$y = \sum_{\{j,l:\ p'_{j,l} \leq \pi_{\text{eq}}\}} q'_{j,l} - \sum_{\{j,l:\ p_{j,l} \leq \pi_{\text{eq}}\}} q_{j,l} \geq 0.$$

Then, for all i,k such that $p_{i,k} = \pi_{\text{eq}}, \ i \neq i^*$, we have

$$\begin{aligned} s'_{i,k} &= \min\left\{\frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p'_{j,l} < \pi_{\text{eq}}\}} q'_{j,l}\right)}{\sum_{\{j,l:\ p'_{j,l} = \pi_{\text{eq}}\}} q'_{j,l}},\ q_{i,k}\right\} \\ &= \min\left\{\frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p_{j,l} < \pi_{\text{eq}}\}} q_{j,l} - x\right)}{\sum_{\{j,l:\ p_{j,l} = \pi_{\text{eq}}\}} q_{j,l} - x + y}, q_{i,k}\right\} \\ &\leq \min\left\{\frac{q_{i,k}\left(D(\pi_{\text{eq}}) - \sum_{\{j,l:\ p_{j,l} < \pi_{\text{eq}}\}} q_{j,l}\right)}{\sum_{\{j,l:\ p_{j,l} = \pi_{\text{eq}}\}} q_{j,l}},\ q_{i,k}\right\} \\ &\quad \text{(applying Lemma 10)} \\ &= s_{i,k}. \end{aligned}$$

Thus, we have also shown that $s'_{i,k} \leq s_{i,k}$, for all $i \neq i^*$ and for all k. The conclusion of this lemma thus follows. □

Lemma 13. *Suppose that the original bidding strategy $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Consider a new bidding strategy where an arbitrary firm i^* deviates from this Nash equilibrium and bids more aggressively, and another firm $j^* \neq i^*$ also deviates from $(\vec{p_{j^*}}, \vec{q_{j^*}})$ in arbitrary way. Assume that no other firm $k \neq i^*, j^*$ changes its bid. Then, no matter how the firm j^* changes its bid, its profit under $(\mathbf{p}', \mathbf{q}')$ cannot increase compared to its profit at the original Nash equilibrium $(\mathbf{p}, \mathbf{q})$.*

Proof. Because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, no matter how the firm j^* changes its strategy to $(\vec{p_{j^*}}', \vec{q_{j^*}}')$, its profit cannot increase. After the firm j^* deviates, suppose now the firm i^* bids more aggressively. By Lemma 12, this change of the firm i^* cannot make the profit of the firm j^* higher. Thus, firm j^* still cannot make more profit. The conclusion of this lemma thus follows. □

Corollary 3. *Assume that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Consider a new bidding strategy $(\mathbf{p}', \mathbf{q}')$ where the firm i^* bids more aggressively and other firms do not deviate. If the market outcome under $(\mathbf{p}', \mathbf{q}')$ satisfies $\pi'_{eq} = \pi_{eq}$ and $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ for all i, then the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ must also be a Nash equilibrium.*

Proof. Because $\pi'_{\text{eq}} = \pi_{\text{eq}}$ and $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$ for all i, we know every firm's profit does not change under $(\mathbf{p}', \mathbf{q}')$ compared with that under $(\mathbf{p}, \mathbf{q})$. Then, we check whether any firm can get more profit by deviating to $(\mathbf{p}'', \mathbf{q}'')$ from $(\mathbf{p}', \mathbf{q}')$.

First, we consider the case where the firm i^* deviates. In this case, the firm i^* is the only firm that changes its bid from the Nash equilibrium $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{p}'', \mathbf{q}'')$. By the definition of the Nash equilibrium $(\mathbf{p}, \mathbf{q})$, the firm i^* cannot get more profit under $(\mathbf{p}'', \mathbf{q}'')$ than that under $(\mathbf{p}, \mathbf{q})$, which is also equal to its profit under $(\mathbf{p}', \mathbf{q}')$. Hence, we conclude that the firm i^* cannot get more profit by deviating from $(\mathbf{p}', \mathbf{q}')$.

Second, we consider the case where another firm $i \neq i^*$ deviates. The whole deviation process from $(\mathbf{p}, \mathbf{q})$ to $(\mathbf{p}'', \mathbf{q}'')$ is the same as what described in Lemma 13. As a result, the profit of the firm i under $(\mathbf{p}'', \mathbf{q}'')$ is not more than that under $(\mathbf{p}, \mathbf{q})$, which is also equal to its profit under $(\mathbf{p}', \mathbf{q}')$. This means that any firm $i \neq i^*$ cannot get more profit by deviating from $(\mathbf{p}', \mathbf{q}')$. The conclusion of this corollary thus follows. □

Lemma 14. *For real numbers a, b, x, y such that $b \geq a > 0$, $x \geq 0$, and $y > 0$, we must have*

$$\min\left\{\frac{ay}{b},\ a\right\} \leq \min\left\{\frac{(a+x)y}{b+x},\ a+x\right\},$$

where equality holds if and only if $\underline{x=0}$ or $\underline{a=b \text{ and } y \leq b}$.

Proof. Since $a \leq a + x$ (equality holds when $x = 0$), it only remains to show that $\frac{ay}{b} \leq \frac{(a+x)y}{b+x}$. We have

$$\begin{aligned} &\frac{ay}{b} \leq \frac{(a+x)y}{b+x} \\ \iff &\frac{a}{b} \leq \frac{a+x}{b+x} \\ \iff &a(b+x) \leq b(a+x) \\ \iff &(a-b)x \leq 0. \end{aligned}$$

Because $a \leq b$ and $x \geq 0$, we do have $(a - b)x \leq 0$ (equality holds if and only if $a = 0$ or $x = 0$).

Now, we check the condition of $\min\left\{\frac{ay}{b},\ a\right\} = \min\left\{\frac{(a+x)y}{b+x},\ a+x\right\}$. When $x = 0$, equality obviously holds. When $x \neq 0$, equality holds if and only if $\frac{ay}{b} = \frac{(a+x)y}{b+x}$ and $\frac{ay}{b} \leq a$, which implies $a = b$ and $y \leq b$. This lemma thus follows. □

Proposition 5. *Assume that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose that a firm i^* has at least one bid with the bidding price higher or equal than π_{eq}. Let $k^* = \min\{k :\ p_{i^*,k} \geq \pi_{eq}\}$. Consider the new bidding strategy $(\mathbf{p}', \mathbf{q}')$ when the firm i^* bids as follows*

$$\begin{cases} K'_{i^*} = k^*, \\ p'_{i^*,k^*} = \pi_{eq}, \quad q'_{i^*,k^*} = \sum_{k \geq k^*} q_{i^*,k}, \\ p'_{i^*,k} = p_{i^*,k}, \quad q'_{i^*,k} = q_{i^*,k}, \quad \textit{for all } k < k^*, \end{cases}$$

and other firms' bids do not change. Then, the bidding strategy $(\mathbf{p}', \mathbf{q}')$ is an outcome-equivalent Nash equilibrium.

Proof. From the assumptions, the firm i^* bids more aggressively. Note that π_{eq} must belong to $\{p_{j,k} :\ \text{for all } j,k\} \cup \{0\}$, and thus the new bid satisfies Definition 2. Specifically, we have

$$\sum_{\{i,k:\ p'_{i,k} < \pi_{\text{eq}}\}} q'_{i,k} = \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k},$$
$$\sum_{\{i,k:\ p'_{i,k} = \pi_{\text{eq}}\}} q'_{i,k} \geq \sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}.$$

By Lemma 11(b) and 11(d), we have $\pi'_{\text{eq}} = \pi_{\text{eq}}$. It only remains to show that $\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{k=1}^{K_i} s_{i,k}$, for all i. First, we have $s'_{i,k} = q_{i,k} = s_{i,k}$ for all (i,k) such that $p'_{i,k} < \pi_{\text{eq}}$. Second, we prove $\sum_{\{k:\ p_{i,k}=\pi_{\text{eq}}\}} s_{i,k} = \sum_{\{k:\ p'_{i,k}=\pi'_{\text{eq}}\}} s'_{i,k}$ for all i by three steps [5].

Step 1: We prove that, if there exists no l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i^*,k^*} = 0$ and $s'_{i,k} = s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$. Towards this end, note that because $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium, the firm i^*'s profit under $(\mathbf{p}',\mathbf{q}')$ should be less than or equal to its profit under $(\mathbf{p},\mathbf{q})$. Thus, we have

$$
\begin{aligned}
&\pi'_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\ p'_{i^*,k}<\pi'_{\text{eq}}\}} s'_{i^*,k}\right) \\
&\quad \le \pi_{\text{eq}} \sum_{\{k:\ p_{i^*,k}<\pi_{\text{eq}}\}} s_{i^*,k} \\
\Longrightarrow\ & s'_{i^*,k^*} = 0 \\
\Longrightarrow\ & s'_{i,k} = 0 \text{ for all } i,k \text{ such that } p'_{i,k} = \pi'_{\text{eq}}, \\
&\text{and thus } D(\pi'_{\text{eq}}) = \sum_{i,k:\ p'_{i,k}<\pi'_{\text{eq}}} q'_{i,k} \\
&\text{(because of Lemma 11(e))} \\
\Longrightarrow\ & D(\pi_{\text{eq}}) = \sum_{i,k:\ p_{i,k}<\pi_{\text{eq}}} q_{i,k} \\
&\text{(because } \pi_{\text{eq}} = \pi'_{\text{eq}} \text{ and } q_{i,k} = q'_{i,k} \text{ for all } i,k \\
&\ \text{such that } p_{i,k} < \pi_{\text{eq}}) \\
\Longrightarrow\ & \sum_{i,k:\ p_{i,k}<\pi_{\text{eq}}} s_{i,k} = \sum_{i,k:\ p_{i,k}<\pi_{\text{eq}}} q_{i,k} = D(\pi_{\text{eq}}) \\
&\ge \sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k} \text{ (by Lemma 11(c))} \\
\Longrightarrow\ & s_{i,k} = 0 \text{ for all } i,k \text{ such that } p_{i,k} = \pi_{\text{eq}}.
\end{aligned}
$$

In conclusion, we have $s'_{i^*,k^*} = 0$ and $s'_{i,k} = s_{i,k} = 0$ for all i,k such that $p_{i,k} = \pi_{\text{eq}}$.

Step 2: We prove that, if there exists l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i^*,k^*} = s_{i^*,l}$. Since only one block bid of user i^*'s bid is at price π_{eq}, we must have $l = k^*$. To prove the conclusion of this step, define $x = q'_{i^*,k^*} - q_{i^*,k^*}$. Note that $x \ge 0$ because $q'_{i^*,k^*} = \sum_{k\ge k^*} q_{i^*,k}$. By Eq. (15), we have

$$
\begin{aligned}
&s'_{i^*,k^*} \\
&= \min\left\{\frac{q'_{i^*,k^*}\left(D(\pi'_{\text{eq}}) - \sum_{\{i,k:\ p'_{i,k}<\pi'_{\text{eq}}\}} q'_{i,k}\right)}{\sum_{\{i,k:\ p'_{i,k}=\pi'_{\text{eq}},\ i\neq i^*\}} q'_{i,k} + q'_{i^*,k^*}},\ q'_{i^*,k^*}\right\} \\
&= \min\left\{\frac{q'_{i^*,k^*}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}},\ i\neq i^*\}} q_{i,k} + q'_{i^*,k^*}},\ q'_{i^*,k^*}\right\} \\
&\quad \text{(since } \pi'_{\text{eq}} = \pi_{\text{eq}}, \text{ and } q_{i,k} = q'_{i,k} \text{ for all } i \neq i^*) \\
&= \min\left\{\frac{(q_{i^*,l}+x)\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k} + x},\right. \\
&\qquad\qquad \left. q_{i^*,l} + x\right\} \\
&\ge \min\left\{\frac{q_{i^*,l}\left(D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}} q_{i,k}},\ q_{i^*,l}\right\} \\
&\quad \text{(applying Lemma 14)} \\
&= s_{i^*,l}.
\end{aligned}
$$

Then, it only remains to show $s'_{i^*,k^*} \le s_{i^*,l}$. Because $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium, the firm i^*'s profit under $(\mathbf{p}',\mathbf{q}')$ should be less than or equal to its profit under $(\mathbf{p},\mathbf{q})$. Thus, we have

$$
\begin{aligned}
&\pi'_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\ p'_{i^*,k}<\pi'_{\text{eq}}\}} s'_{i^*,k}\right) \\
&\le \pi_{\text{eq}}\left(s_{i^*,l} + \sum_{\{k:\ p_{i^*,k}<\pi_{\text{eq}}\}} s_{i^*,k}\right) \\
\Longrightarrow\ & \pi_{\text{eq}}\left(s'_{i^*,k^*} + \sum_{\{k:\ p_{i^*,k}<\pi_{\text{eq}}\}} q_{i^*,k}\right) \\
&\le \pi_{\text{eq}}\left(s_{i^*,l} + \sum_{\{k:\ p_{i^*,k}<\pi_{\text{eq}}\}} q_{i^*,k}\right) \\
\Longrightarrow\ & s'_{i^*,k^*} \le s_{i^*,l}.
\end{aligned}
$$

Thus, we must have $s'_{i,k^*} = s_{i^*,l}$.

Step 3: we prove that, if there exists l such that $p_{i^*,l} = \pi_{\text{eq}}$, then $s'_{i,k} = s_{i,k}$ for all i,k such that $p'_{i,k} = \pi'_{\text{eq}}$. By the definition of k^*, we have $l = k^*$. By Step 2, we have $s'_{i^*,k^*} = s_{i^*,k^*}$. Let $x = q'_{i^*,k^*} - q_{i^*,k^*} \ge 0$. Define $y = D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k}$. If $y = 0$, by Lemma 11(c), we have $s'_{i,k} = s_{i,k} = 0$ for all i,k such that $p'_{i,k} = \pi'_{\text{eq}}$. Thus, it only remains to consider the situation of $y > 0$. From the conclusion of Step 2, we have

$$
s_{i^*,k^*} = s'_{i^*,k^*},
$$

[5] Note that for every firm i, both $\{k:\ p_{i,k} = \pi_{\text{eq}}\}$ and $\{k:\ p'_{i,k} = \pi'_{\text{eq}}\}$ have at most one element, and (one or both) could be empty for some firms.

which implies

$$\min\left\{\frac{q_{i^*,k^*}\left(D(\pi_{\text{eq}})-\sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}}q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}},\ q_{i^*,k^*}\right\}$$

$$=\min\left\{\frac{q'_{i^*,k^*}\left(D(\pi'_{\text{eq}})-\sum_{\{i,k:\ p'_{i,k}<\pi'_{\text{eq}}\}}q'_{i,k}\right)}{\sum_{\{i,k:\ p'_{i,k}=\pi'_{\text{eq}}\}}q'_{i,k}},\ q'_{i^*,k^*}\right\},$$

i.e.,

$$\min\left\{\frac{q_{i^*,k^*}\left(D(\pi_{\text{eq}})-\sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}}q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}},\ q_{i^*,k^*}\right\}$$

$$=\min\left\{\frac{(q_{i^*,k^*}+x)\left(D(\pi_{\text{eq}})-\sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}}q_{i,k}\right)}{\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}+x},\right.$$
$$\left.q_{i^*,k^*}+x\right\}.$$

Applying Lemma 14, we have

$$x=0 \text{ or } q_{i^*,k^*}=\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}.$$

If $x=0$, then $(\mathbf{p},\mathbf{q})$ and $(\mathbf{p}',\mathbf{q}'))$ are exactly the same. If $q_{i^*,k^*}=\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}$, then (i^*,k^*) is the only pair (i,k) such that $p_{i,k}=\pi_{\text{eq}}$. In both cases, we always have $s'_{i,k}=s_{i,k}$ for all i,k such that $p'_{i,k}=\pi'_{\text{eq}}$.

By Step 1 to 3, we have $\sum_{\{k:\ p_{i,k}=\pi_{\text{eq}}\}}s_{i,k}=\sum_{\{k:\ p'_{i,k}=\pi'_{\text{eq}}\}}s'_{i,k}$ for all i. Notice that $\sum_{\{k:\ p_{i,k}>\pi_{\text{eq}}\}}s_{i,k}=0=\sum_{\{k:\ p'_{i,k}>\pi'_{\text{eq}}\}}s'_{i,k}$ for all i. Thus, we have $\sum_{k=1}^{K'_i}s'_{i,k}=\sum_{k=1}^{K_i}s_{i,k}$ for all i. By Corollary 3, we conclude that $(\mathbf{p}',\mathbf{q}')$ is an outcome-equivalent Nash equilibrium. □

Proposition 6. *Assume that $(\mathbf{p},\mathbf{q})$ is a Nash equilibrium. Suppose that a firm i^* has at least one bid with the bidding price lower than π_{eq} and higher than 0. Let $k^*=\min\{k: 0<p_{i^*,k}<\pi_{eq}\}$[6]. Consider the new bidding strategy $(\mathbf{p}',\mathbf{q}')$ when the firm i^* bids as follows*

$$\begin{cases}K'_{i^*}=K_{i^*}-(k^*-1),\\ p'_{i^*,1}=0,\quad q'_{i^*,1}=\sum_{k\in\{1\}\cup\{k^*\}}q_{i^*,k},\\ p'_{i^*,k}=p_{i^*,k+k^*-1},\ q'_{i^*,k}=q_{i^*,k+k^*-1},\ \textit{for all } k\geq 2.\end{cases}$$

and other firms' bids do not change. Then, the new bidding strategy $(\mathbf{p}',\mathbf{q}')$ is an outcome-equivalent Nash equilibrium.

Proof. By assumptions, the firm i^* bids more aggressively and we have

$$\sum_{\{i,k:\ p'_{i,k}<\pi_{\text{eq}}\}}q'_{i,k}=\sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}}q_{i,k},$$
$$\sum_{\{i,k:\ p'_{i,k}=\pi_{\text{eq}}\}}q'_{i,k}=\sum_{\{i,k:\ p_{i,k}=\pi_{\text{eq}}\}}q_{i,k}.$$

[6]Note that, because there can be only one bid at price zero, k^* can only take values of either 1 or 2. Specifically, $k^*=2$ when $p_{i^*,1}=0$, and $k^*=1$ when $p_{i^*,1}>0$.

Thus, by Lemma 11(b)(d), we have $\pi'_{\text{eq}}=\pi_{\text{eq}}$. For any firm i, we have

$$\sum_{\{k:\ p'_{i,k}<\pi_{\text{eq}}\}}q'_{i,k}=\sum_{\{k:\ p_{i,k}<\pi_{\text{eq}}\}}q_{i,k},$$
$$q'_{i,k}\Big|_{k:\ p'_{i,k}=\pi_{\text{eq}}}=q_{i,l}\Big|_{l:\ p_{i,l}=\pi_{\text{eq}}}.$$

Thus, by Algorithm 4, for any firm i, we have

$$\sum_{k=1}^{K'_i}s'_{i,k}=\sum_{k=1}^{K_i}s_{i,k}.$$

By Corollary 3, the new bidding strategy $(\mathbf{p}',\mathbf{q}')$ is an outcome-equivalent Nash equilibrium. □

H. Proof of Theorem 2

Proof. At any Nash equilibrium $(\mathbf{p},\mathbf{q})$ in the multi-block-bid system, if a firm has more than 1 bid with the bidding price lower than π_{eq}, then we can repeatedly apply Proposition 6 to combine all such bids with prices below π_{eq} to one bid with the price 0. If a firm has any bid with the price higher than or equal to π_{eq}, then we can use Proposition 5 to merge such bids into one bid at the price π_{eq}. Each step in those changes produces an outcome-equivalent Nash equilibrium. At the end, each firm only has at most two bids, one at the price π_{eq} and another at the price 0. The result of this theorem thus follows. □

I. Preparation for the proof of Theorem 1

By Theorem 2, when analyzing the Nash equilibrium $(\mathbf{p},\mathbf{q})$ of the market, we can restrict our attention to $K_i\leq 2,\ p_{i,k}\in\{0,\ \pi_{\text{eq}}\}$ for all i,k.

Lemma 15. $D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i}s_{i,k}\right)=\pi_{eq}$ *at any Nash equilibrium $(\mathbf{p},\mathbf{q})$.*

Proof. We prove by contradiction. Suppose on the contrary that $D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i}s_{i,k}\right)\neq\pi_{\text{eq}}$. Since $D^{-1}(D(\pi_{\text{eq}}))=\pi_{\text{eq}}$, it implies that we must have $D(\pi_{\text{eq}})\neq\sum_{i=1}^{M}\sum_{k=1}^{K_i}s_{i,k}$. By Lemma 11(c), we must have $D(\pi_{\text{eq}})>\sum_{i=1}^{M}\sum_{k=1}^{K_i}s_{i,k}$. Recall from Theorem 2 that we can restrict our attention to outcome-equivalent Nash equilibrium such that $p_{i,k}\in\{0,\pi_{\text{eq}}\}$ for all i and for all k. By Lemma 11(c), we must have $s_{i,k}=q_{i,k}$ for all i,k. Thus, the profit of the firm i equals $\pi_{\text{eq}}\sum_{k=1}^{K_i}q_{i,k}$. We also have

$$D(\pi_{\text{eq}})>\sum_{i=1}^{M}\sum_{k=1}^{K_i}s_{i,k}$$
$$\Longrightarrow D(\pi_{\text{eq}})>\sum_{i=1}^{M}\sum_{k=1}^{K_i}q_{i,k}$$
$$\Longrightarrow D(\pi_{\text{eq}})>C\ (\text{recall that }\sum_{k=1}^{K_i}q_{i,k}=C_i).$$

We now show that there must exist π_0 such that $\pi_0>\pi_{\text{eq}}$ and $D(\pi_{\text{eq}})>D(\pi_0)\geq C$. To see this, suppose on the contrary that $D(\pi) < C$ for all $\pi > \pi_{\text{eq}}$. Then, because we have shown that $D(\pi_{\text{eq}}) > C$, $D(\cdot)$ must be discontinuous at π_{eq}. Thus, we have $\pi_{\text{eq}} = D^{-1}(C) = D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right)$, which contradicts our initial assumption that $D^{-1}\left(\sum_{i=1}^{M}\sum_{k=1}^{K_i} s_{i,k}\right) \neq \pi_{\text{eq}}$. Thus, there must exist π_0 such that $\pi_0 > \pi_{\text{eq}}$ and $D(\pi_{\text{eq}}) > D(\pi_0) \geq C$. Now, let a firm i^* deviate to another bidding strategy $(\mathbf{p}', \mathbf{q}')$ that $K'_{i^*} = 1$, $p'_{i^*,1} = \pi_0$, $q'_{i^*,1} = C_i$ (i.e., bidding all its amount at the price π_0). Since $\pi_0 > \pi_{\text{eq}}$ and $p'_{i,k} = p_{i,k} \in \{0, \pi_{\text{eq}}\}$ for all $i \neq i^*$ and for all k, we have $p'_{i,k} \leq \pi_0$ for all i and for all k. Thus, $\sum_{\{i,k:\ p'_{i,k} \leq \pi_0\}} q'_{i,k} = \sum_{i=1}^{M}\sum_{k=1}^{K'_i} q'_{i,k} = C$ (recall that $\sum_{k=1}^{K'_i} q'_{i,k} = C_i$). We then have $D(\pi_0) > \sum_{\{i,k:\ p'_{i,k} \leq \pi_0\}} q'_{i,k}$. By Lemma 11(a), we have $\pi'_{\text{eq}} \geq \pi_0$. Since only the firm i^* bids at the price π_0 and other firms bids at π_{eq} or 0, by Algorithm 3, we have $\pi'_{\text{eq}} = \pi_0$. Because $D(\pi_0) > \sum_{\{i,k:\ p'_{i,k} \leq \pi_0\}} q'_{i,k} \geq \sum_{i,k} s'_{i,k}$, by Lemma 11(c), we have $s'_{i^*,1} = q'_{i^*,1} = C_i$. Thus, the profit of the firm i^* under $(\mathbf{p}', \mathbf{q}')$ equals to $\pi'_{\text{eq}} C_i = \pi_0 C_i > \pi_{\text{eq}} \sum_{k=1}^{K_i} q_{i^*,k}$, i.e., the new profit is greater than the profit of the original bidding strategy $(\mathbf{p}, \mathbf{q})$. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. The conclusion of this lemma thus follows. $\square$

Lemma 15 implies that, in the outcome described in Theorem 1, the third description automatically holds if the first and the second description hold. Therefore, we only need to check the first and the second description during the proof of the desired outcome.

Proposition 7. *If $C \leq D(\max_n \overline{\pi}_n)$, then the outcome of the market could only be the outcome described in Theorem 1. Further, at any Nash equilibrium, we must have $\pi_{eq} = \max_n \overline{\pi}_n$.*

Proof. Step 1: We prove that the bidding strategy $(\mathbf{p}, \mathbf{q})$ defined as $K_i = 1$, $q_{i,1} = C_i$, $p_{i,1} = \max_n \overline{\pi}_n$ for all i is a Nash equilibrium. By Algorithm 3, we have $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. By Algorithm 4, we have $s_{i,1} = q_{i,1} = C_i$ for all i. Thus, the profit of the firm i is $C_i \max_n \overline{\pi}_n$. Because $D(\pi) = 0$ for all $\pi > \max_n \overline{\pi}_n$, we know the firm i has already gotten the maximum profit compared to any bidding strategy. Thus, the bidding strategy $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

Step 2: We prove that, if at any Nash equilibrium there exists a firm i^* whose bid is $K_{i^*} = 1$, $p_{i^*,1} = \max_n \overline{\pi}_n$, and $q_{i^*,1} = C_{i^*}$, then the market outcome is $\pi_{\text{eq}} = \max_n \overline{\pi}_n$ and $s_{i^*,1} = C_{i^*}$. The result will then be used in Step 3 to show that, at any Nash equilibrium, the market price π_{eq} must be $\max_n \overline{\pi}_n$. Because $D(\pi) = 0$ for all $\pi > \max_n \overline{\pi}_n$ and $D(\max_n \overline{\pi}_n) > C$, by Lemma 11(a), we have $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. Now, it remains to show that $s_{i^*,1} = C_{i^*}$. We have

$$
\begin{aligned}
& D(\pi_{\text{eq}}) \geq C \\
\Longrightarrow\ & D(\pi_{\text{eq}}) \geq \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k} + \sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k} \\
\Longrightarrow\ & \frac{D(\pi_{\text{eq}}) - \sum_{\{i,k:\ p_{i,k} < \pi_{\text{eq}}\}} q_{i,k}}{\sum_{\{i,k:\ p_{i,k} = \pi_{\text{eq}}\}} q_{i,k}} \geq 1.
\end{aligned}
$$

By Eq. (15), we have $s_{i^*,1} = q_{i^*,1} = C_{i^*}$.

Step 3: We prove that at any Nash equilibrium, we must have $s_{i,k} = q_{i,k}$ for all i, k, and $\pi_{\text{eq}} = \max_n \overline{\pi}_n$. We prove by contradiction. Suppose on the contrary that, at a Nash equilibrium, there exist i^*, k^* such that $s_{i^*,k^*} < q_{i^*,k^*}$ or $\pi_{\text{eq}} \neq \max_n \overline{\pi}_n$. Then, according to the conclusion of Step 2, the firm i^*'s profit is $s_{i^*,k^*}\pi_{\text{eq}} < C_i \max_n \overline{\pi}_n$. Now, let the firm i^* deviates to the bidding strategy described in Step 2. Then, the firm i^*'s profit under the new bidding strategy is $C_i \max_n \overline{\pi}_n$, which is larger than its profit under the original bidding strategy. This contradicts the assumption that the original bidding strategy is a Nash equilibrium. Thus, we must have $s_{i,k} = q_{i,k}$ for all i, k and $\pi_{\text{eq}} = \max_n \overline{\pi}_n$ at any Nash equilibrium. The result of the proposition thus follows. $\square$

Proposition 7 shows that if the total panel area is scarce, then every firm leases out all of her solar panels, and thus the market outcome must be the outcome described in Theorem 1.

By Proposition 7, when we prove Theorem 7, we only need to consider the case when $C > D(\max_n \overline{\pi}_n)$, i.e., the total panel area is plentiful.

J. *Proof of Theorem 1*

We first prove the following lemma.

Lemma 16. *Let a, b, c, r, s be five positive real numbers. If $\frac{s}{r} \leq \min\left\{\frac{a}{c},\ 1\right\}$ and $b > a$, then we must have*

$$r - \frac{b-a}{b}s \geq re^{-\frac{b-a}{c}}.$$

Proof. We consider two cases.

Case 1: $\frac{a}{c} \leq 1$. Then, we have $\frac{s}{r} \leq \min\left\{\frac{a}{c},\ 1\right\} = \frac{a}{c}$. Since

$$\frac{b}{a} - \frac{b-a}{c} \geq \frac{b}{a} - \frac{b-a}{a} = 1,$$

we have

$$\frac{\frac{b-a}{c}}{\frac{b}{a} - \frac{b-a}{c}} \leq \frac{b-a}{c}.$$

Then,

$$
\begin{aligned}
e^{\frac{b-a}{c}} \geq 1 + \frac{b-a}{c} \geq 1 + \frac{\frac{b-a}{c}}{\frac{b}{a} - \frac{b-a}{c}} &= \frac{\frac{b}{a}}{\frac{b}{a} - \frac{b-a}{c}} \\
&= \frac{1}{1 - \frac{a}{c}\frac{b-a}{b}},
\end{aligned}
$$

which implies

$$re^{-\frac{b-a}{c}} \leq r - r\frac{a}{c}\frac{b-a}{b} \leq r - s\frac{b-a}{b} \ (\text{since } \frac{s}{r} \leq \frac{a}{c}).$$

Case 2: $\frac{a}{c} > 1$. Then, we have $\frac{s}{r} \leq \min\left\{\frac{a}{c},\ 1\right\} = 1$. Let $c' = a > c$. We have

$$e^{-\frac{b-a}{c'}} \geq e^{-\frac{b-a}{c}}.$$

Thus, it is sufficient to show that

$$r - \frac{b-a}{b}s \geq re^{-\frac{b-a}{c'}} \text{ for all } \frac{s}{r} \leq 1.$$

This follows from case 1 because $\frac{a}{c'} = 1$.

In conclusion, the result of this lemma thus follows. $\square$

Now, we start to prove Theorem 1.

Proof. Before entering the main part of our proof, we first show that $\pi_{\text{eq}} > 0$ for any Nash equilibrium. Because $\frac{\max_i C_i}{C} \le \frac{D^{-1}(C)}{\pi_g \gamma}$, we must have $D^{-1}(C) > 0$. By Lemma 15, we then have $\pi_{\text{eq}} = D^{-1}(\sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k}) \ge D^{-1}(C) > 0$.

We divide the proof into two parts according to the definition of the desired outcome. In Part 1, we prove the first description. In Part 2, we prove the second description. (Recall that due to Lemma 15, the third description follows once we meet the first and the second description.)

Part 1: we show that the bidding strategy $(\mathbf{p}, \mathbf{q})$ defined as $K_i = 1$, $p_{i,1} = D^{-1}(C)$, $q_{i,1} = C_i$ for all i is a Nash equilibrium. After running Algorithm 3 and 4, we must have $\pi_{\text{eq}} = D^{-1}(C)$ and $s_{i,1} = C_i$ for all i. For any firm j, the profit is $\pi_{\text{eq}} C_j$. We now prove by contradiction that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Suppose on the contrary that the current bidding strategy is not a Nash equilibrium. Then, the firm j can deviate to another bidding strategy $(\mathbf{p}', \mathbf{q}')$ to increase her profit. Thus, we must have $\pi'_{\text{eq}} > \pi_{\text{eq}}$ because the new sold amount of the firm j cannot exceed C_j. For any other firm $i \ne j$, since $p'_{i,1} = p_{i,1} = \pi_{\text{eq}} < \pi'_{\text{eq}}$, by Algorithm 4, we must have $s'_{i,1} = q_{i,1}$. Thus, by Lemma 11(c), we have

$$\begin{aligned} \sum_{k=1}^{K'_j} s'_{j,k} &\le D(\pi'_{\text{eq}}) - \sum_{i \ne j} s'_{i,1} \\ &= D(\pi'_{\text{eq}}) - \sum_{i \ne j} q_{i,1} \\ &= D(\pi'_{\text{eq}}) - C + C_j. \end{aligned}$$

Since the new profit of the firm j is larger than its old profit, we then have

$$\begin{aligned} & \pi'_{\text{eq}} \sum_{k=1}^{K'_j} s'_{j,k} > \pi_{\text{eq}} q_{j,1} \\ \Longrightarrow\ & \pi'_{\text{eq}} \left(D(\pi'_{\text{eq}}) - C + C_j \right) - \pi_{\text{eq}} C_j > 0 \\ \Longrightarrow\ & (\pi'_{\text{eq}} - \pi_{\text{eq}}) C_j + \pi'_{\text{eq}} \left(D(\pi'_{\text{eq}}) - C \right) > 0 \\ \Longrightarrow\ & D(\pi'_{\text{eq}}) > C - \frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi'_{\text{eq}}} C_j. \end{aligned} \tag{17}$$

On the other hand, by Corollary 2, we have

$$D(\pi'_{\text{eq}}) \le C e^{-\frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi_g \gamma}}.$$

Because $\frac{C_j}{C} \le 1$ and $\frac{C_j}{C} \le \frac{\max_i C_i}{C} \le \frac{\pi_{\text{eq}}}{\pi_g \gamma}$, we must have $\frac{C_j}{C} \le \min\left\{ \frac{\pi_{\text{eq}}}{\pi_g \gamma}, 1 \right\}$. Applying Lemma 16, we have

$$D(\pi'_{\text{eq}}) \le C - \frac{\pi'_{\text{eq}} - \pi_{\text{eq}}}{\pi'_{\text{eq}}} C_j.$$

This contradicts Eq. (17). Thus, we have proven that the original bidding strategy $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium.

Part 2: we prove that, at any Nash equilibrium, we must have $s_{i,k} = q_{i,k}$ for all i, k. We prove by contradiction. Suppose on the contrary that at a Nash equilibrium $(\mathbf{p}, \mathbf{q})$, there exists at least one firm with an unsold/partly-sold bid, i.e.,

$$\left\{ i^* : \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*} \right\} \ne \varnothing.$$

In the following, we will consider another bidding strategy $(\mathbf{p}', \mathbf{q}')$ that a carefully-chosen firm j deviates from the original bidding strategy $(\mathbf{p}, \mathbf{q})$ to another strategy with $K'_j = 1$, $p'_{j,1} = \pi^*$, and $q'_{j,1} = C_j$. We find j and π^* through the following steps 1 and 2. We then get some useful properties in steps 3, 4 and 5. In the end, we establish the contradiction to complete the proof.

Step 1: We prove that there exists a firm j such that $\sum_{k=1}^{K_j} s_{j,k} < C_j$ and $\frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \le \frac{C_j}{C}$. By Lemma 11(c), we must have

$$\begin{aligned} D(\pi_{\text{eq}}) \ge \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k} = & \left(\sum_{\left\{ i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*} \right\}} \sum_{k=1}^{K_{i^*}} s_{i^*,k} \right) \\ & + \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i, \end{aligned}$$

which implies that

$$\sum_{\left\{ i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*} \right\}} \frac{\sum_{k=1}^{K_{i^*}} s_{i^*,k}}{D(\pi_{\text{eq}}) - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i} \le 1.$$

On the other hand, we have

$$\begin{aligned} & C = \sum_{\left\{ i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*} \right\}} C_{i^*} + \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i \\ \Longrightarrow\ & \sum_{\left\{ i^*:\ \sum_{k=1}^{K_{i^*}} s_{i^*,k} < C_{i^*} \right\}} \frac{C_{i^*}}{C - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i} = 1. \end{aligned}$$

As a result, there must exist a firm j such that $\sum_{k=1}^{K_j} s_{j,k} < C_j$ and

$$\begin{aligned} & \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}}) - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i} \\ & \le \frac{C_j}{C - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i} \\ \Longrightarrow\ & \left(\sum_{k=1}^{K_j} s_{j,k} \right) \left(C - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i \right) \\ & \le C_j \left(D(\pi_{\text{eq}}) - \sum_{\left\{ i:\ \sum_{k=1}^{K_i} s_{i,k} = C_i \right\}} C_i \right) \\ \Longrightarrow\ & \left(\sum_{k=1}^{K_j} s_{j,k} \right) C \le C_j D(\pi_{\text{eq}}) \ (\text{because } \sum_{k=1}^{K_j} s_{j,k} < C_j) \\ \Longrightarrow\ & \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \le \frac{C_j}{C}. \end{aligned}$$

Step 2: We let the firm j found in step 1 change her bids in the way that we describe earlier (i.e., $K'_j = 1$, $p'_{j,1} = \pi^*$, and $q'_{j,1} = C_j$). We now prove that for all $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$, the market outcome must satisfy $s'_{j,1} < C_j$. We prove by contradiction. Suppose on the contrary that there exists a price $\pi \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$ such that $s'_{j,1} = C_j$ when the firm bids $p'_{j,1} = \pi$. The firm j's profit with $(\mathbf{p}', \mathbf{q}')$ must then be equal to or greater than πC_j. Since $\pi > \frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j}$, we then have

$$\pi C_j > \pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k},$$

i.e., the firm j's profit under $(\mathbf{p}', \mathbf{q}')$ is larger than that under $(\mathbf{p}, \mathbf{q})$. This contradicts the assumption that $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium. Thus, we have proven that, for all $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$, the market outcome under the new bid must satisfy $s'_{j,1} < C_j$.

Step 3: We now let firm j found in step 1 change her bid in the way that we describe earlier, with $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$. We next prove that the market outcome must satisfy $\pi'_{\text{eq}} = \pi^*$. By Step 2, we have $s'_{j,1} < C_j = q'_{j,1}$. Thus, we have $\pi'_{\text{eq}} \le p'_{j,1} = \pi^*$. Because $p'_{i,k} \in \{0, \pi^*, \pi_{\text{eq}}\}$ for all (i,k) (notice that $\pi_{\text{eq}} > \pi^* > \frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j} \ge 0$ and we have restrict our attention to two-block bids), we have

$$\begin{aligned}
\sum_{\{i,k:\ p'_{i,k} < \pi^*\}} q'_{i,k} &= \sum_{\{i,k:\ p'_{i,k}=0\}} q'_{i,k} \\
&= \sum_{\{i,k:\ p_{i,k}=0,\ i\neq j\}} q_{i,k} \\
&\le \sum_{\{i,k:\ p_{i,k}=0\}} q_{i,k} \\
&\le \sum_{\{i,k:\ p_{i,k}<\pi_{\text{eq}}\}} q_{i,k} \text{ (because } \pi_{\text{eq}} > 0) \\
&\le D(\pi_{\text{eq}}) \text{ (by Lemma 11(a))} \\
&\le D(\pi^*).
\end{aligned}$$

Applying Lemma 11(a), we have $\pi'_{\text{eq}} \ge \pi^*$. As a result, we conclude that $\pi'_{\text{eq}} = \pi^*$.

Step 4: Following Step 3, we prove that $s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \ge D(\pi^*) - \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k}$. By step 2, we know that $s'_{j,1} < C_j$. Thus, by Lemma 11(c), we must have $D(\pi^*) = \sum_{i=1}^{M} \sum_{k=1}^{K'_i} s'_{i,k}$. As a result, we have

$$\begin{aligned}
D(\pi^*) - \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k} &= \sum_{i=1}^{M} \sum_{k=1}^{K'_i} s'_{i,k} - \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k} \\
&= s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} + \sum_{i\neq j} \left(\sum_{k=1}^{K'_i} s'_{i,k} - \sum_{k=1}^{K_i} s_{i,k} \right).
\end{aligned}$$

Because $\pi'_{\text{eq}} = \pi^* < \pi_{\text{eq}}$ and the bidding strategy of any firm $i \neq j$ does not deviate, we have

$$\sum_{k=1}^{K'_i} s'_{i,k} = \sum_{\{k:\ p'_{i,k}=0\}} q'_{i,k} \le \sum_{k=1}^{K_i} s_{i,k} \text{ for all } i \neq j.$$

Thus, we have $s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \ge D(\pi^*) - \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k}$.

Step 5: We prove that $\pi_{\text{eq}} > D^{-1}(C)$ by contradiction. Suppose on the contrary that $\pi_{\text{eq}} \le D^{-1}(C)$. Thus, we have $D(\pi_{\text{eq}}) \ge D\left(D^{-1}(C)\right) \ge C > \sum_{i=1}^{M} \sum_{k=1}^{K_i} q_{i,k}$. By Lemma 11(c), we have $s_{i,k} = q_{i,k}$ for all i,k (because $p_{i,k} \in \{0, \pi_{\text{eq}}\}$). That contradicts the assumption that the firm j has an unsold/partly-sold bid. Thus, we have proven that $\pi_{\text{eq}} > D^{-1}(C)$.

Step 6: We establish the contradiction for the initial assumption of the whole Part 2 that there exists at least one firm with an unsold/partly-sold bid at $(\mathbf{p}, \mathbf{q})$. Following the strategy $(\mathbf{p}', \mathbf{q}')$ stated in Step 3, because $(\mathbf{p}, \mathbf{q})$ is a Nash equilibrium, we have

$$\begin{aligned}
&\pi^* s'_{j,1} \le \pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k} \\
&\Longrightarrow \pi^* \left(s'_{j,1} - \sum_{k=1}^{K_j} s_{j,k} \right) \le (\pi_{\text{eq}} - \pi^*) \sum_{k=1}^{K_j} s_{j,k} \\
&\Longrightarrow \pi^* \left(D(\pi^*) - \sum_{i=1}^{M} \sum_{k=1}^{K_i} s_{i,k} \right) \le (\pi_{\text{eq}} - \pi^*) \sum_{k=1}^{K_j} s_{j,k} \\
&\qquad \text{(by Step 4)} \\
&\Longrightarrow \pi^* \left(D(\pi^*) - D(\pi_{\text{eq}}) \right) \le (\pi_{\text{eq}} - \pi^*) \sum_{k=1}^{K_j} s_{j,k} \\
&\qquad \text{(by Lemma 11(c))} \\
&\Longrightarrow \frac{D(\pi^*) - D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})} \le \frac{\pi_{\text{eq}} - \pi^*}{\pi^*} \cdot \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})}.
\end{aligned}$$

Note that the last inequality holds for any $\pi^* \in \left(\frac{\pi_{\text{eq}} \sum_{k=1}^{K_j} s_{j,k}}{C_j},\ \pi_{\text{eq}}\right)$. Letting π^* approach π_{eq}, we have

$$\begin{aligned}
\left| \frac{\partial D(\pi_{\text{eq}})}{D(\pi_{\text{eq}})} \right| &\le \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{\sum_{k=1}^{K_j} s_{j,k}}{D(\pi_{\text{eq}})} \\
&\le \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{C_j}{C} \text{ (by Step 1)} \\
&\le \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{D^{-1}(C)}{\pi_g \gamma} \\
&< \left| \frac{\partial \pi_{\text{eq}}}{\pi_{\text{eq}}} \right| \frac{\pi_{\text{eq}}}{\pi_g \gamma} \text{ (by Step 5),}
\end{aligned}$$

which contradicts Proposition 3.

In conclusion, the result stated in this theorem thus holds. □

Appendix C
Supply Function Equilibrium Explanation

Readers familiar with Supply Function Equilibrium [19] will notice that the multi-block bids that we studied in Section

V can also be viewed as a form of supply function. However, there are a number of major differences between our results and the SFE results in the literature, as we explain below.

The supply functions studied in the literature are often assumed to be continuous and differentiable. In contrast, the multi-block bidding strategy described in Theorem 1 corresponds to a supply function that is neither continuous nor differentiable. Second, the way that SFEs are computed is usually by taking the derivative of the payoff function and set the derivative to zero. Not only does this approach require differentiability, it also only holds if the payoff function has no constraints. As we will elaborate below, due to this requirement, the SFEs thus derived occur at different setting compared to the equilibrium described in Theorem 1.

To see this, we use a setup similar to [19]. Consider a demand function $\mathcal{D}(\pi)$ that is twice differentiable, monotone decreasing, and concave. We let all firms' cost be zero. Suppose that the market adopts the uniform price. Each firm can provide her own supply function $\mathcal{S}_i(\pi)$ that is twice differentiable. Note that because π_{eq} is the market price, we must have $\mathcal{D}(\pi_{\text{eq}}) = \sum_{i=1}^{M} \mathcal{S}_i(\pi_{\text{eq}})$ (recall that M denotes the number of firms). If firm i deviates to another supply function $\mathcal{S}_i^0(\cdot)$, then the market price becomes π_{eq}^0 such that $\mathcal{D}(\pi_{\text{eq}}^0) = \mathcal{S}_i(\pi_{\text{eq}}^0) + \sum_{j\neq i} \mathcal{S}_j(\pi_{\text{eq}}^0)$. Thus, firm i's profit becomes $\pi_{\text{eq}}^0 \mathcal{S}_i^0(\pi_{\text{eq}}^0) = \pi_{\text{eq}}^0 \left(\mathcal{D}(\pi_{\text{eq}}^0) - \sum_{j\neq i} \mathcal{S}_j(\pi_{\text{eq}}^0)\right)$. Apparently, firm i wants to maximize her profit. Therefore, to make π_{eq} be the market price at a Nash equilibrium, we must have

$$\pi_{\text{eq}} = \arg\max_{\pi_{\text{eq}}^0} \pi_{\text{eq}}^0 \left(\mathcal{D}(\pi_{\text{eq}}^0) - \sum_{j\neq i} \mathcal{S}_j(\pi_{\text{eq}}^0)\right). \tag{18}$$

The literature of SFE then postulates that an SFE arises if the first-order derivative of the profit with respect to π_{eq} must be zero and the second-order derivative is negative. The second-order condition is easy to verify, and thus next we focus on the first-order condition, i.e.,

$$\mathcal{D}(\pi_{\text{eq}}) - \sum_{j\neq i} \mathcal{S}_j(\pi_{\text{eq}}) + \pi_{\text{eq}} \left(\mathcal{D}'(\pi_{\text{eq}}) - \sum_{j\neq i} \mathcal{S}_j'(\pi_{\text{eq}})\right) = 0,$$

and thus

$$\mathcal{S}_i(\pi_{\text{eq}}) + \pi_{\text{eq}} \left(\mathcal{D}'(\pi_{\text{eq}}) - \sum_{j\neq i} \mathcal{S}_j'(\pi_{\text{eq}})\right) = 0. \tag{19}$$

Let us now focus on the symmetric case such that, at the Nash equilibrium, the sold amount is split evenly among all firms, which implies that $\mathcal{S}_i(\pi_{\text{eq}}) = \mathcal{S}_j(\pi_{\text{eq}}) = \frac{\mathcal{D}(\pi_{\text{eq}})}{M}$ for all i and j. As a result, since all firms need to satisfy Eq. (19), we must have $\mathcal{S}_i'(\pi_{\text{eq}}) = \mathcal{S}_j'(\pi_{\text{eq}}) \triangleq \beta$ for all i and j. Notice that a legitimate supply function also needs to be monotone increasing. Thus, we need $\beta \geq 0$. Using the first order condition Eq. (19), we have

$$\frac{\mathcal{D}(\pi_{\text{eq}})}{M} + \pi_{\text{eq}} \left(\mathcal{D}'(\pi_{\text{eq}}) - (M-1)\beta\right) = 0$$
$$\iff (M-1)\beta = \frac{\mathcal{D}(\pi_{\text{eq}})}{\pi_{\text{eq}} M} - \mathcal{D}'(\pi_{\text{eq}}).$$

Therefore, the SFE thus derived will only exist if $\beta \geq 0$, i.e.,

$$\frac{|\mathcal{D}'(\pi_{\text{eq}})|}{\mathcal{D}(\pi_{\text{eq}})} \pi_{\text{eq}} \leq \frac{1}{M}. \tag{20}$$

(Notice that in oligopoly we must have $M \geq 2$.)

A. Nash equilibrium in Theorem 1

We now show that the Nash equilibrium in Theorem 1 arises in settings opposite to Eq.(20). In the proposed rental market, suppose every firm has the same quantity, i.e., $C_i = \frac{C}{M}$ for all i. Then, the condition in Theorem 1 becomes

$$\frac{D^{-1}(C)}{\pi_g \gamma} \geq \frac{1}{M}.$$

Recall that, when the condition of Proposition 3 holds, we have

$$\frac{|D'(\pi)|}{D(\pi)} \geq \frac{1}{\pi_g \gamma}.$$

Thus, our Nash equilibrium exists when

$$\frac{|D'(\pi)|}{D(\pi)} D^{-1}(C) = \frac{|D'(\pi)|}{D(\pi)} \pi_{\text{eq}} \geq \frac{1}{M}.$$

(Notice that in our Nash equilibrium, we have $\pi_{\text{eq}} = D^{-1}(C)$.) The setting of

$$\frac{|D'(\pi_{\text{eq}})|}{D(\pi_{\text{eq}})} \pi_{\text{eq}} \geq \frac{1}{M}$$

is exactly the opposite of Eq. (20). In other words, the Nash equilibrium stated in our results occurs at different settings from that of SFE. At close inspection, we find that one of the reasons that SFE cannot predict the equilibrium in Theorem 1 is because, at our equilibrium, the derivative of firm i's profit with respect to the market price is not even zero. Instead, it is negative. To see this, we can check the derivative of firm i's profit as

$$\begin{aligned}
&\frac{\partial\ \pi_{\text{eq}} \left(\mathcal{D}(\pi_{\text{eq}}) - \sum_{j\neq i} \mathcal{S}_j(\pi_{\text{eq}})\right)}{\partial \pi_{\text{eq}}} \\
=&\frac{\mathcal{D}(\pi_{\text{eq}})}{M} - \pi_{\text{eq}} \left(|\mathcal{D}'(\pi_{\text{eq}})| + \sum_{j\neq i} \mathcal{S}_j'(\pi_{\text{eq}})\right) \\
\leq&\frac{\mathcal{D}(\pi_{\text{eq}})}{M} - \pi_{\text{eq}} |\mathcal{D}'(\pi_{\text{eq}})| \\
&\quad \text{(because } \mathcal{S}_j'(\pi) \geq 0 \text{ for all } j \text{ and for all } \pi) \\
=&\mathcal{D}(\pi_{\text{eq}}) \left(\frac{1}{M} - \frac{|\mathcal{D}'(\pi_{\text{eq}})|}{\mathcal{D}(\pi_{\text{eq}})} \pi_{\text{eq}}\right) < 0.
\end{aligned}$$

However, this still corresponds to the maximum profit in Eq.(18) due to other constraints, i.e., the total panel area is limited by C and thus the market price cannot be lower than $D^{-1}(C)$.

Finally, in addition to the above technical difference, we also would like to emphasize that the SFEs studied in [19] are only one of the possible equilibrium. In contrast, Theorem 1 characterizes the property of *all* possible equilibriums. In this sense, the result of Theorem 1 is also stronger.

$$d_n(\pi)=\begin{cases}\max\left\{c:\ \frac{\pi}{\pi_g}=\frac{\mu\left(\Phi\left(\frac{\frac{L_n}{c}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)\right)+\sigma\left(\phi\left(\frac{\underline{G}-\mu}{\sigma}\right)-\phi\left(\frac{\frac{L_n}{c}-\mu}{\sigma}\right)\right)}{\Phi\left(\frac{\overline{G}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)}\right\} & ,\quad \pi\le\overline{\pi}_n,\\ 0 & ,\quad \pi>\overline{\pi}_n.\end{cases} \tag{24}$$

$$D(\pi)=\begin{cases}\max\left\{c:\ \frac{\pi}{\pi_g}=\frac{\mu\left(\Phi\left(\frac{\frac{L}{c}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)\right)+\sigma\left(\phi\left(\frac{\underline{G}-\mu}{\sigma}\right)-\phi\left(\frac{\frac{L}{c}-\mu}{\sigma}\right)\right)}{\Phi\left(\frac{\overline{G}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)}\right\} & ,\quad \pi\le 100,\\ 0 & ,\quad \pi>100.\end{cases} \tag{25}$$

APPENDIX D
DERIVATION IN SECTION VI

We calculate the demand function $D(\pi)$ under this set up. Let $\phi(\cdot)$ be the probability density function of the standard normal distribution and $\Phi(\cdot)$ be its cumulative distribution function. By Eq. (4) and Lemma 7, consumer n's demand is

$$d_n(\pi)=\begin{cases}\max\left\{c:\ \pi=\pi_g\int_{\underline{G}}^{\frac{L_n}{c}}xf(x)dx\right\} & ,\ \pi\le\overline{\pi}_n,\\ 0 & ,\ \pi>\overline{\pi}_n,\end{cases}$$

where $\overline{\pi}_n=\frac{\pi_g}{T}\int_0^T\mathbb{E}[G(t)\mathbb{1}_{\{L_n>0\}}]dt=\frac{\underline{G}+\overline{G}}{2}\pi_g=100$. After integration, we get the following result of Eq. (24). Since we assume that any consumer n's load profile is flat, it is obvious that $\frac{L_n}{d_n(\pi)}=\frac{L}{D(\pi)}$. Thus, we have Eq. (25) Recall that σ represents the uncertainty of solar generation. In the following simulation, besides finite values of σ like $\sigma=1$ or 2, we are also interested in the extreme case that $\sigma=\infty$, where the uncertainty is maximized. When $\sigma=\infty$, the truncated normal distribution becomes uniform distribution in $[\underline{G},\ \overline{G}]$. In this situation, we have

$$\phi\left(\frac{G-\mu}{\sigma}\right)-\phi\left(\frac{\frac{L}{c}-\mu}{\sigma}\right)=0,\ \text{and}$$
$$\frac{\Phi\left(\frac{\frac{L}{c}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)}{\Phi\left(\frac{\overline{G}-\mu}{\sigma}\right)-\Phi\left(\frac{\underline{G}-\mu}{\sigma}\right)}=\frac{\frac{L}{c}-\underline{G}}{\overline{G}-\underline{G}}.$$

Thus, when $\sigma=\infty$, the demand function can be simplified as

$$D(\pi)\Big|_{\sigma=\infty}=\begin{cases}\frac{L}{\underline{G}+\frac{\pi}{\mu\pi_g}(\overline{G}-\underline{G})}=\frac{10}{2+0.16\pi} & ,\quad \pi\le 100,\\ 0, & ,\quad \pi>100.\end{cases}$$

APPENDIX E
SITUATION OF DIFFERENT $G(t)$

Now we discuss about the situation that $G(t)$ is not the same for all PV panels. It occurs when PV panels are not built in the same location, or they are not of the same kind. When this happens, the market operator can introduce a normalization procedure based of the evaluation of different panels before running the market. Specifically, the market runs in following steps.

1) The market operator accesses the efficiency of each firm's PV panels based on history data. Let $G_i(t)$ denote the generation of the firm i's unit area of panels at time t, which is a random variable. The efficiency e_i of the firm i's PV panel is defined as the average generation of an unit area of PV panels, i.e.,
$$e_i=\frac{1}{T}\int_0^T\mathbb{E}[G_i(t)]dt.$$
2) The market operator normalizes the panel area of each firm by efficiency. Specifically, the market operator declares a standard efficiency e_0 in advance. Denote the firm i's real panel area as C_i^r. The normalized panel area should be
$$C_i=\frac{e_i}{e_0}C_i^r.$$
The market operator publishes the distribution of aggregate generation with one unit normalized area of PV panels, i.e., the distribution of
$$G(t)=\frac{\sum_{i=1}^M G_i(t)C_i^r}{\sum_{i=1}^M C_i}.$$
This step guarantees that the total generation calculated by using normalized panel area (i.e., $G(t)\sum_{i=1}^M C_i$) is still equal to the actual total generation amount $\sum_{i=1}^M G_i(t)C_i^r$.
3) Firms submit their bid to the market operator.
4) consumers provide their demand curve to the market operator.
5) The market operator determines the market price and the transaction amount of each firm and consumer.

Remark on Steps 1 and 2: In reality, to get C_i, the market operator can directly use

$$C_i=\frac{\text{average total generation from firm } i\text{'s all PV panels}}{e_0}$$
$$=\frac{C_i^r\int_0^T\mathbb{E}[G_i(t)]dt}{Te_0},$$

instead of applying Steps 1 and 2 separately. The value of $C_i^r\int_0^T\mathbb{E}[G_i(t)]dt$ is directly from the history data. In this way, the market operator no longer relies on the reported data (e.g., C_i^r) by firms.

Alternative definitions of efficiency in Step 1: The current definition of efficiency in Step 1 is quite straightforward. However, we can also define the efficiency as the average generation of an unit area of PV panels when the total generation less than or equal to the total load, i.e.,

$$e_i=\frac{1}{T}\int_0^T\mathbb{E}\left[G_i(t)\mathbb{1}_{\{\sum_{i=1}^M G_i(t)C_i^r\le L(t)\}}\right]dt.$$

We can also use a more complicated definition that is base on the useful generation part, i.e.,

$$e_i = \frac{1}{T}\int_0^T \mathbb{E}\Big[G_i(t)\mathbb{1}_{\{\sum_{i=1}^M G_i(t)C_i^r \le L(t)\}} + \frac{G_i(t)L(t)}{\sum_{i=1}^M G_i(t)C_i^r}\mathbb{1}_{\{\sum_{i=1}^M G_i(t)C_i^r > L(t)\}}\Big]dt.$$

There may be other reasonable ways of defining the efficient of PV panels. No matter which one is used, the operation and the behavior of the market keeps unaffected. However, the definition does affect the investment phase of building new PV panels. Firms tend to build PV panels that have high efficiency. As a result, depending on which type of solar generation that the market operator (or the society) wants, maybe a better definition should be used. For example, when the transmission line capacity or the voltage control is critical in the distribution level, the market operator may prefer to add some penalty (or reward) term in the definition of e_i based on the firm i's effect on those issues.