Signal Processing [1]

Reminder

N	continue	discret
1	$\|f\|_1 = \int_{\mathbf{R}} \|f(t)\|dt$	$\sum_{n=-\infty}^{+\infty} \|u_n\|$
2	$\|f\|_2 = \sqrt{\int_{\mathbf{R}} \|f(t)\|^2 dt}$	$\sqrt{\sum_{n=\infty}^{\infty} \|u_n\|^2}$
∞	$\|f\|_\infty = sup_t \|f(t)\|$	$sup_n \|u_n\|$
P	$\|f\|_1 = \int_{\mathbf{R}} \|f(t)\|^p dt^{1/p}$	$\left(\sum_{n=\infty}^{\infty} \|u_n\|^p\right)^{1/p}$

- The uniform convergence, with the ∞-Norm.
- The absolute convergence, with the 1-Norm
- The convergence in energy, with the 2-Norm

Convolution

Continus time: $y(t) = (h * x)(t) = \int_{\mathbf{R}} h(u)x(t-u)du$
Discrete time: $y[t] = (h * x)[t] = \sum_{k \in \mathbf{Z}} x[t-k]$
The convolution is commutative ans the neutral element is Dirac distribution $\delta(t)$

The Cross and auto correlation

$(x \star y)(t) = \int_{\mathbf{R}} \overline{x(u)} y(t+u)du = \int_{\mathbf{R}} x(t-u)\overline{y(u)}du$
The cross correlation is not a commutative operation.
Let $\tilde{x} = \overline{x(-t)}$, then $(x \star y)(t) = (\tilde{x} \star y)(t)$
The same think of discret time.
Digital signals are obtained from analog signals by sampling and quantization.
Filter: A filter is a linear time-invariant system. It is perfectly determined by its impulse response h.
Numerical filter with impulse response $h[t]$. The output $y[t]$ of an analog filter with input (also called excitation) $x[t]$ is obtained by :
$y[y] = (h * x)[t] = \sum_{k \in \mathbf{R}} h[t-k]x[k]$
Causality: A signal is causal iff its start at (or after) the date t=0.
Anti-causality A signal is anti-causal iff its stop a before the date t=0.
A signal, or system, which is not causal or anti-causal is acausal.
For systems, causality implies that the output does not depends on the futur of the input.
Stability A signal a stable iff it is summable.
Energy The energy of a signal is its squared 2-norm.

Elementary signals

Numerical Dirac impulse

$$\delta_k[n] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Heaviside

$$\Theta[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Rectangular window

$$\Pi(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Fourier Analysis

Fourier Coefficients(Fourier transform of periodic functions)
$\hat{f}[n] = c_n(f) = \frac{1}{T}\int_0^T f(t)\exp^{-i\frac{2\pi}{T}nt} dt$
$\hat{f}[n]$ measure how f is close the the pure frequency $\frac{2\pi n}{T}$.
Spectrum of a periodic signal The spectrum is the set of the squared modulus of the fourier coefficients: $spectrum(f) = |\hat{f}[n]|^2$
Propreties
Time shift:
for $f_a(t) = f(t+a)$ then $\hat{f}_a[n] = \exp^{i\frac{2\pi}{T}na}\hat{f}[n]$
Derivation:
$f^{(\hat{k})}[n] = (\exp^{i\frac{2\pi}{T}n})^k \hat{f}[n]$
Fourier Serie

$$S(f)(t) = \sum_{n=-\infty}^{+\infty} \hat{f}[n]\exp^{i\frac{2\pi}{T}nt}$$

Theorem Let $f, g \in L^2([0,T])$ and $\{\hat{f}[n]\}_{n\in\mathbb{Z}}$ and $\{\hat{g}[n]\}_{n\in\mathbb{Z}}$ their respective Fourier coefficients. Then

$$\|f\|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 \mathrm{d}t = \sum_{n=-\infty}^{+\infty} |\hat{f}[n]|^2$$

$$\langle f, g\rangle = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\bar{g}(t)\mathrm{d}t = \sum_{n=-\infty}^{+\infty} \hat{f}[n]\bar{\hat{g}}_n$$

Fourier Analysis

Coefficients de Fourier trigonométriques

$$a_0(f) = \frac{1}{T}\int_0^T f(t)\mathrm{d}t$$
$$a_n(f) = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi}{T}nt\right)\mathrm{d}t \quad \forall n \geq 1$$
$$b_n(f) = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi}{T}nt\right)\mathrm{d}t \quad \forall n \geq 0$$

$$a_0(f) + \sum_{n=1}^{+\infty} a_n(f)\cos\left(\frac{2\pi}{T}nt\right) + \sum_{n=1}^{+\infty} b_n(f)\sin\left(\frac{2\pi}{T}nt\right)$$

Plancherel-Parseval's Theorem
$|f|^2 = |a_0(f)|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}|a_n(f)|^2 + \frac{1}{2}\sum_{n=1}^{+\infty}|b_n(f)|^2$

continuous time

Fourier Transform

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi\nu t}\mathrm{d}t$$

with ν frequency in Herz and the pulsation $\omega = 2\pi\nu$

Inverse Fourier Transform

$$f(t) = \int_{\mathbb{R}} \hat{f}(v)e^{i2\pi vt}\mathrm{d}v$$

Properties

1. Convolution:
$$\widehat{f * g}(\nu) = \hat{f}(\nu)\hat{g}(\nu)$$
2. Multiplication:
$$\widehat{f \cdot g}(\nu) = (\hat{f} * \hat{g})(\nu)$$
3. Translation:
let $g_a(t) = f(t-a)$
$$\hat{g}_a(\nu) = e^{-i2\pi a\nu}\hat{f}(\nu)$$
4. Modulation:
let $g_\theta(t) = e^{i2\pi\theta t}f(t)$
$$\hat{g}_\theta(\nu) = \hat{f}(\nu - \theta)$$
5. Scaling:
let $g_s(t) = f(t/s)$
$$\hat{g}_s(\nu) = |s|\hat{f}(s\nu)$$
6. Time derivation:
$$\widehat{f(p)}(\nu) = (i2\pi\nu)^p\hat{f}(\nu)$$
7. Frequency derivation:
let $g_p(t) = (-i2\pi t)^p f(t)$
$$\hat{g}_p(t) = \hat{f}^{(p)}(\nu)$$
8. Complexe conjugation
$$\hat{\bar{f}}(\nu) = \overline{\hat{f}(-\nu)}$$

Numerical Filtering

Z transform Let $x = \{x_n\}_{n \in \mathbb{Z}}$ be a numerical signal. The z transform of s is the series

Reference

References

[1] Signal Processing Matthieu Kowalski. http://hebergement.universite-paris-saclay.fr/mkowalski/.