DroMOOC

Motion Planning

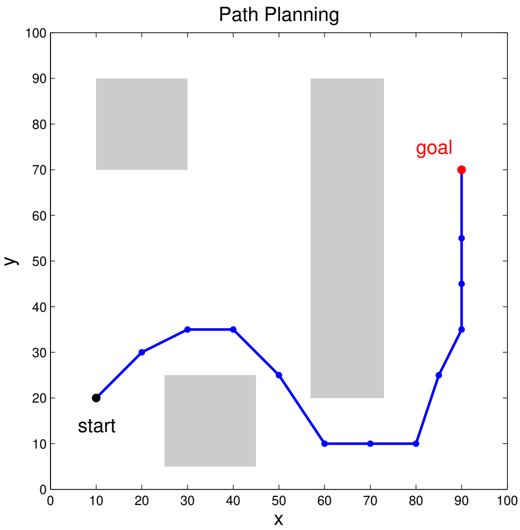
Advanced Level

Methods based on Optimal Control

Bruno Hérissé

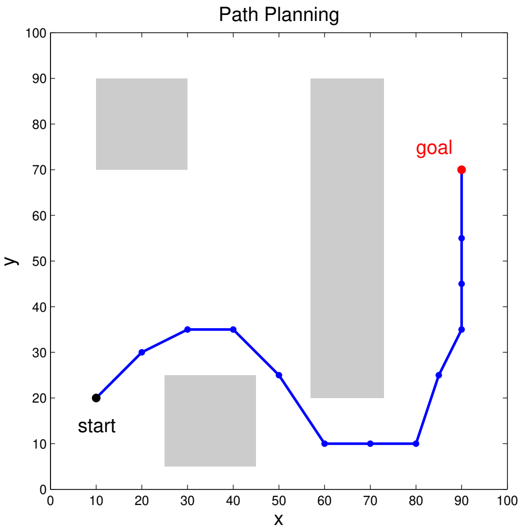
ONERA

Limitations of Path Planning



- Can the path be tracked by the real aerial vehicle?
  - Dynamic constraints
  - State & Control constraints
- Optimality of the trajectory?
  - Minimization of travel time
  - Minimization of control effort (energy)

Limitations of Path Planning



- Can the path be tracked by the real aerial vehicle?
  - Dynamic constraints
  - State & Control constraints
- Optimality of the trajectory?
  - Minimization of travel time
  - Minimization of control effort (energy)

Motion Planning as an Optimal Control Problem

Search an optimal trajectory from initial state x_i to goal set $\mathbb{X}_f$

$$\min \quad C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} f^0(t, x, u)\, dt$$

Subject to dynamic constraints (a model-based approach)

$$\begin{cases} \dot{x} = f(t, x(t), u(t)) \\ x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^m \end{cases}$$

Subject to state constraints (obstacles, etc.)

$$c_i(x) \leq 0,\ \ i = 1, ..., p \quad \text{(inequality constraints)}$$

Example 1 : shortest paths for the Dubins' Vehicle

A very simple model for mobile robots and some aeronautical systems (missile, aircraft) : a vehicle moving forward with bounded curvature

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta), \\ \dot{y} = v\sin(\theta), \\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

where c_m is the maximum curvature (c_m^{-1} is the minimum turning radius) and v the velocity assumed to be constant.

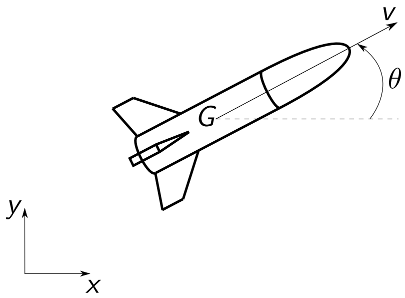


Figure: Dubins' Vehicle

Example 2 : shortest paths for the Double Integrator

A very simple model for quadrotor UAVs or helicopters in the horizontal plane : a vehicle with bounded accelerations

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{\mathbf{x}} = \mathbf{v}, \\ \dot{\mathbf{v}} = a_m \mathbf{u}, \\ \mathbf{x}, \mathbf{v}, \mathbf{u} \in \mathbb{R}^2, ||\mathbf{u}|| \leq 1 \end{cases}$$

where a_m is the maximum acceleration.

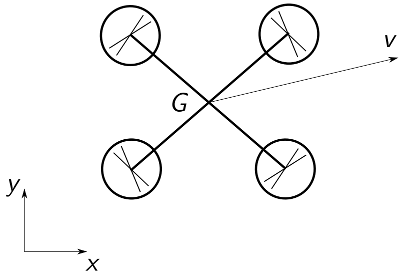


Figure: Quadrotor Vehicle

Pontryagin's Minimum Principle to compute Optimal Control

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} f^0(t,x,u)\,dt \\ \dot{x} = f(t, x(t), u(t)) \\ x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^m, \\ x(0) = x_i, \quad x(t_f) \in \mathbb{X}_f, \quad t_f \in \mathbb{R} \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H(t,x,p,p^0,u) = p^0 f^0(t,x,u) + \langle p, f(t,x,u) \rangle$ with $p^0 \geq 0$
- Define the adjoint vector p verifying $\dot{p} = -\frac{\partial H}{\partial x}(t,x,p,p^0,u)$
- The control u verifies $H(t,x,p,p^0,u) = \min\limits_{v \in \Omega} H(t,x,p,p^0,v)$
- (+ transversality conditions)

Pontryagin's Minimum Principle to compute Optimal Control

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} f^0(t,x,u)\, dt \\ \dot{x} = f(t, x(t), u(t)) \\ \ x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^m, \\ x(0) = x_i, \quad x(t_f) \in \mathbb{X}_f, \quad t_f \in \mathbb{R} \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H(t,x,p,p^0,u) = p^0 f^0(t,x,u) + \langle p, f(t,x,u)\rangle$ with $p^0 \geq 0$
- Define the adjoint vector p verifying $\dot{p} = -\frac{\partial H}{\partial x}(t,x,p,p^0,u)$
- The control u verifies $H(t,x,p,p^0,u) = \min_{v\in\Omega} H(t,x,p,p^0,v)$
- (+ transversality conditions)

Pontryagin's Minimum Principle to compute Optimal Control

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} f^0(t,x,u)\, dt \\ \dot{x} = f(t, x(t), u(t)) \\ x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^m, \\ x(0) = x_i, \quad x(t_f) \in \mathbb{X}_f, \quad t_f \in \mathbb{R} \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H(t, x, p, p^0, u) = p^0 f^0(t, x, u) + \langle p, f(t, x, u) \rangle$ with $p^0 \geq 0$
- Define the adjoint vector p verifying $\dot{p} = -\frac{\partial H}{\partial x}(t, x, p, p^0, u)$
- The control u verifies $H(t, x, p, p^0, u) = \min_{v \in \Omega} H(t, x, p, p^0, v)$
- (+ transversality conditions)

Pontryagin's Minimum Principle to compute Optimal Control

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} f^0(t,x,u)\, dt \\ \dot{x} = f(t, x(t), u(t)) \\ \ x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^m, \\ x(0) = x_i, \quad x(t_f) \in \mathbb{X}_f, \quad t_f \in \mathbb{R} \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H(t,x,p,p^0,u) = p^0 f^0(t,x,u) + \langle p, f(t,x,u)\rangle$ with $p^0 \geq 0$
- Define the adjoint vector p verifying $\dot{p} = -\frac{\partial H}{\partial x}(t,x,p,p^0,u)$
- The control u verifies $H(t,x,p,p^0,u) = \min_{v \in \Omega} H(t,x,p,p^0,v)$
- (+ transversality conditions)

Example 1 : shortest paths for the Dubins' Vehicle in free space

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x = \dot{p}_y = 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t) = 0$ (θ is constant)

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and line segments (u=0).

Example 1 : shortest paths for the Dubins' Vehicle in free space

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x = \dot{p}_y = 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t) = 0$ (θ is constant)

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and line segments (u=0).

Example 1 : shortest paths for the Dubins' Vehicle in free space

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x = \dot{p}_y = 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t) = 0$ (θ is constant)

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and line segments (u=0).

Example 1 : shortest paths for the Dubins' Vehicle in free space

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x = \dot{p}_y = 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t) = 0$ (θ is constant)

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and line segments (u=0).

Example 1 : shortest paths for the Dubins' Vehicle in free space

Optimal Control Problem:

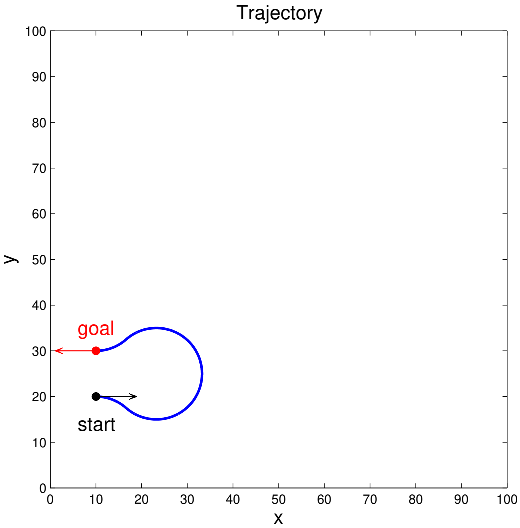
$$\begin{cases} \min \quad C_u(t_f) = t_f \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x = \dot{p}_y = 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t) = 0$ (θ is constant)

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and line segments (u=0).

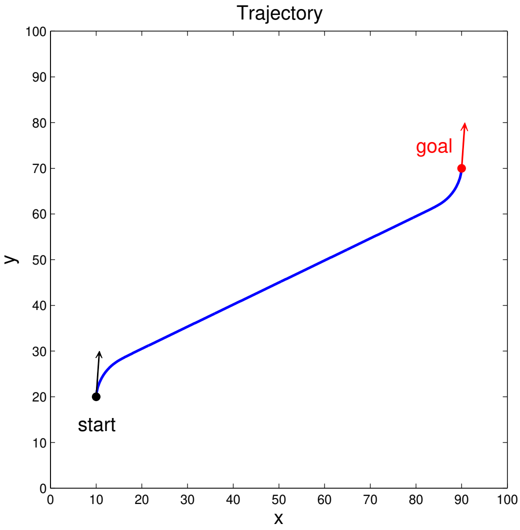
Example 1 : shortest paths for the Dubins' Vehicle in free space



Using Geometric considerations, it can be shown that shortest paths are of two types only:

- Circle-Circle-Circle (CCC type)

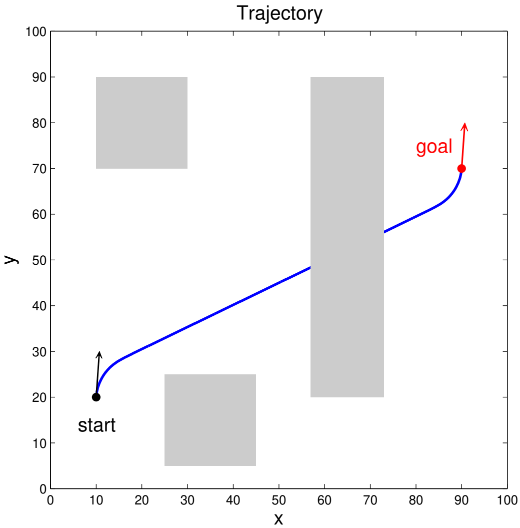
Example 1 : shortest paths for the Dubins' Vehicle in free space



Using Geometric considerations, it can be shown that shortest paths are of two types only:

- Circle-Circle-Circle (CCC type)
- Circle-Segment-Circle (CSC type)

Example 1 : shortest paths for the Dubins' Vehicle with obstacles



With obstacles, solutions are more difficult: CCC and CSC types are not sufficient to describe the solutions) !

Pontryagin's Minimum Principle with state constraints

- In an Optimal Control Problem, obstacles are modelled as state constraints

$$c_i(x) \leq 0,\ i = 1, ..., p \quad \text{(p obstacles).}$$

- Pontryagin's Minimum Principle with state constraints exists but is impossible to use in practice.
- Rather, an approximate solution is searched by adding a penalization in the cost function as follows

$$C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} \left(f^0(t, x, u) + \sum_{i=1}^{p} \phi(c_i(x)) \right) dt$$

where $\phi : \mathbb{R} \to \mathbb{R}_+$ is an increasing C^1 function such that $\phi(-\infty) = 0$.

Pontryagin's Minimum Principle with state constraints

- In an Optimal Control Problem, obstacles are modelled as state constraints

$$c_i(x) \leq 0, \; i = 1, ..., p \quad \text{(p obstacles)}.$$

- Pontryagin's Minimum Principle with state constraints exists but is impossible to use in practice.
- Rather, an approximate solution is searched by adding a penalization in the cost function as follows

$$C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} \left(f^0(t, x, u) + \sum_{i=1}^{p} \phi(c_i(x)) \right) \, dt$$

where $\phi : \mathbb{R} \to \mathbb{R}_+$ is an increasing C^1 function such that $\phi(-\infty) = 0$.

Pontryagin's Minimum Principle with state constraints

- In an Optimal Control Problem, obstacles are modelled as state constraints

$$c_i(x) \leq 0, \; i = 1, ..., p \quad \text{(p obstacles)}.$$

- Pontryagin's Minimum Principle with state constraints exists but is impossible to use in practice.
- Rather, an approximate solution is searched by adding a penalization in the cost function as follows

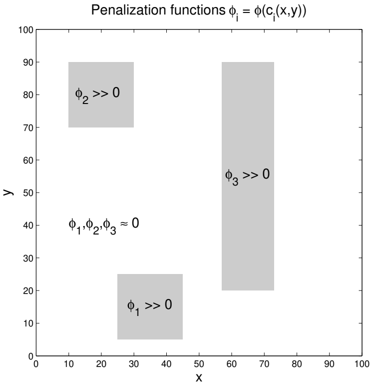
$$C_u(t_f) = g(t_f, x(t_f)) + \int_0^{t_f} \left(f^0(t, x, u) + \sum_{i=1}^{p} \phi(c_i(x)) \right) \, dt$$

where $\phi : \mathbb{R} \to \mathbb{R}_+$ is an increasing C^1 function
such that $\phi(-\infty) = 0$.

Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = v c_m u, \\ |u| \leq 1, \end{cases}$$



Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = v c_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p^0 \sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} + p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x, \dot{p}_y \neq 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t)$ is not necessarily equal to 0 since $\dot{p}_x, \dot{p}_y \neq 0$.

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and complex curves (not necessarily line segments).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = v c_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p^0 \sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} + p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x, \dot{p}_y \neq 0,\ \dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t)$ is not necessarily equal to 0 since $\dot{p}_x, \dot{p}_y \neq 0$.

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and complex curves (not necessarily line segments).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p^0 \sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} + p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x, \dot{p}_y \neq 0$, $\dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t)$ is not necessarily equal to 0 since $\dot{p}_x, \dot{p}_y \neq 0$.

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and complex curves (not necessarily line segments).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p^0 \sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} + p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x, \dot{p}_y \neq 0$, $\dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t)$ is not necessarily equal to 0 since $\dot{p}_x, \dot{p}_y \neq 0$.

$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and complex curves (not necessarily line segments).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles

Optimal Control Problem:

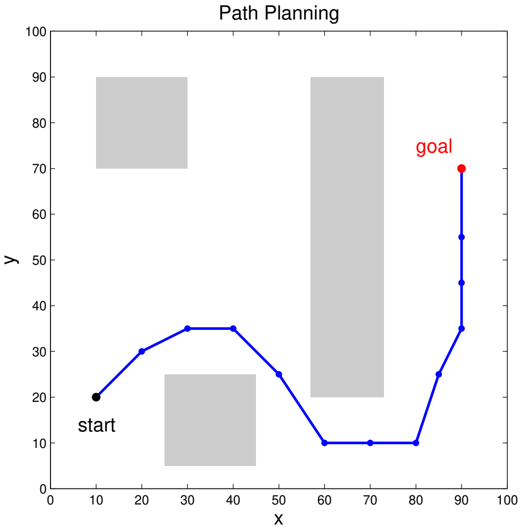
$$\begin{cases} \min \quad C_u(t_f) = t_f + \int_0^{t_f} \left(\sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} \right) \\ \dot{x} = v\cos(\theta),\ \dot{y} = v\sin(\theta),\ \dot{\theta} = vc_m u, \\ |u| \leq 1, \end{cases}$$

Pontryagin's Minimum Principle:

- Define the Hamiltonian $H = p^0 \sum_{i=1}^{p} \frac{1+\tanh(c_i(x,y))}{2} + p_x v\cos(\theta) + p_y v\sin(\theta) + p_\theta v c_m u$
- Then, $\dot{p}_x, \dot{p}_y \neq 0$, $\dot{p}_\theta = p_x v\sin(\theta) - p_y v\cos(\theta)$
- The optimal control u verifies $u = -\frac{p_\theta}{|p_\theta|}$ if $p_\theta \neq 0$
- If $p_\theta(t) = 0$, then $\dot{p}_\theta(t) = p_x v\sin(\theta) - p_y v\cos(\theta) = 0$, then $u(t)$ is not necessarily equal to 0 since $\dot{p}_x, \dot{p}_y \neq 0$.

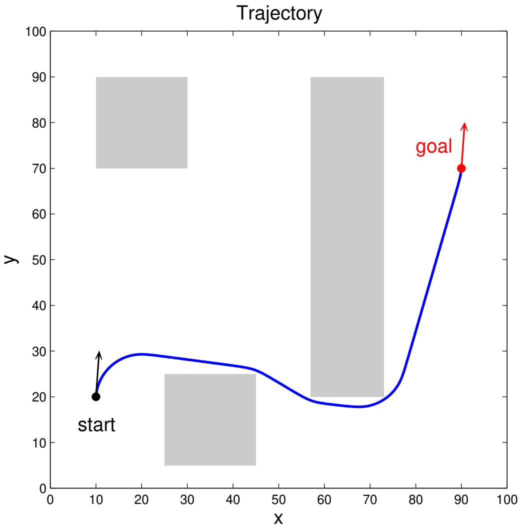
$\rightarrow$ Then, shortest paths are a concatenation of arcs of circle of minimum radius ($u = \pm 1$) and complex curves (not necessarily line segments).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles



- An initial path (obtained using graph-based or sampling-based methods for example) is used as a first guess in a numerical solver.
- The obtained solution is approximatively the true solution, that is a CSCSCSCSC trajectory (4 line segments and 5 arcs of circle).

Example 1 : shortest paths for the Dubins' Vehicle with obstacles



- An initial path (obtained using graph-based or sampling-based methods for example) is used as a first guess in a numerical solver.
- The obtained solution is approximatively the true solution, that is a CSCSCSCSC trajectory (4 line segments and 5 arcs of circle).

Conclusion

- Motion Planning based on Optimal control can handle dynamic constraints and cost functions.
- A numerical solver is necessary to find the solution.
- Initialization of such techniques can be hard, it needs great expertise and mathematical analysis.

Some useful references to go further

- Emmanuel Trélat, *Optimal control and applications to aerospace: some results and challenges*, Journal of Optimization Theory and Applications, 2012.
- J.-D. Boissonnat, A. Cérézo, and J. Leblond, *Shortest paths of bounded curvature in the plane*, Technical report, RR-1503 INRIA, 1991.