DroMOOC

Sensor fusion and state estimation

Basic Level

Observers and Kalman filter (Part II)

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State estimation in presence of random disturbances (discrete time)

Problem

The linear system (possibly non invariant) is affected by noise:

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}\ \boldsymbol{x}_k + \boldsymbol{G}\ \boldsymbol{u}_k + \boldsymbol{v}_k$$

$$\boldsymbol{z}_k = \boldsymbol{C}\ \boldsymbol{x}_k + \boldsymbol{w}_k$$

with $\boldsymbol{v}_k,\ \boldsymbol{w}_k$ white centered gaussian non correlated noise:

$$E\left\{\boldsymbol{v}_k\right\} = 0 \quad E\left\{\boldsymbol{v}_k \boldsymbol{v}_j^T\right\} = \boldsymbol{Q}(t)\delta(k-j),\ \boldsymbol{Q} = \boldsymbol{Q}^T \geq 0$$

$$E\left\{\boldsymbol{w}_k\right\} = 0 \quad E\left\{\boldsymbol{w}_k \boldsymbol{w}_j^T\right\} = \boldsymbol{R}(t)\delta(k-j),\ \boldsymbol{R} = \boldsymbol{R}^T > 0$$

Objective

The goal is to reconstruct $\boldsymbol{x}_k$ from $\boldsymbol{u}_k$ and $\boldsymbol{z}_k$

- without bias on the estimation error
- with minimal variance of the estimation error

Solution - discrete-time Kalman filter

Discrete-time Kalman filter

$$\hat{\boldsymbol{x}}_{k+1|k+1} = \boldsymbol{F}\ \hat{\boldsymbol{x}}_{k|k} + \boldsymbol{G}\ \boldsymbol{u}_k + \boldsymbol{K}_{k+1}\left(\boldsymbol{z}_{k+1} - \hat{\boldsymbol{z}}_{k+1|k}\right)$$

$$\hat{\boldsymbol{x}}_{k_0|k_0} = E\left\{\boldsymbol{x}_{k_0}\right\}$$

$\Rightarrow$ this is an observer which performs an *estimation* of the state $\hat{\boldsymbol{x}}_{k|k}$, with gain $\boldsymbol{K}_k$ varying in time (even in case of LTI systems).

Notations

- Predicted state $\hat{\boldsymbol{x}}_{k+1|k}$
  - Prediction error $\boldsymbol{\varepsilon}_{k+1|k} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k}$
    - Variance of the prediction error $\boldsymbol{\Sigma}_{k+1|k} = E\left\{\boldsymbol{\varepsilon}_{k+1|k}\boldsymbol{\varepsilon}_{k+1|k}^T\right\}$
- Estimated state $\hat{\boldsymbol{x}}_{k+1|k+1}$
  - Estimation error $\boldsymbol{\varepsilon}_{k+1|k+1} = \boldsymbol{x}_{k+1} - \hat{\boldsymbol{x}}_{k+1|k+1}$ $\Rightarrow$ **without bias**: $E\left\{\boldsymbol{\varepsilon}_{k+1|k+1}\right\} = 0$
    - Variance of the estimation error $\boldsymbol{\Sigma}_{k+1|k+1} = E\left\{\boldsymbol{\varepsilon}_{k+1|k+1}\boldsymbol{\varepsilon}_{k+1|k+1}^T\right\}$ $\Rightarrow$ **minimized**

Discrete-time Kalman filter recursive equations

Initialization

- Initial state estimate $\hat{\boldsymbol{x}}_{0|0}$
- Initial prediction error variance $\boldsymbol{\Sigma}_{0|0}$

Prediction (before measurement $k+1$)

1. $\hat{\boldsymbol{x}}_{k+1|k} = \boldsymbol{F}\,\hat{\boldsymbol{x}}_{k|k} + \boldsymbol{G}\boldsymbol{u}_k$
2. $\boldsymbol{\Sigma}_{k+1|k} = \boldsymbol{F}\boldsymbol{\Sigma}_{k|k}\boldsymbol{F}^T + \boldsymbol{Q}$
3. $\hat{\boldsymbol{z}}_{k+1|k} = \boldsymbol{C}\hat{\boldsymbol{x}}_{k+1|k}$

Estimation (update after measurement $k+1$)

4. $\boldsymbol{K}_{k+1} = \boldsymbol{\Sigma}_{k+1|k}\boldsymbol{C}^T\left(\boldsymbol{C}\boldsymbol{\Sigma}_{k+1|k}\boldsymbol{C}^T + \boldsymbol{R}\right)^{-1}$
5. $\hat{\boldsymbol{x}}_{k+1|k+1} = \hat{\boldsymbol{x}}_{k+1|k} + \boldsymbol{K}_{k+1}\left(\boldsymbol{z}_{k+1} - \hat{\boldsymbol{z}}_{k+1|k}\right)$
6. $\boldsymbol{\Sigma}_{k+1|k+1} = \left(\boldsymbol{I}_n - \boldsymbol{K}_{k+1}\boldsymbol{C}\right)\boldsymbol{\Sigma}_{k+1|k}$

Tuning

- $\hat{\boldsymbol{x}}_{0|0}$ initial guess
- $\boldsymbol{\Sigma}_{0|0} \propto 1/\text{confidence in } \hat{\boldsymbol{x}}_{0|0}$ value
- $\boldsymbol{Q} \propto 1/\text{confidence in state equations}$
- $\boldsymbol{R} \propto 1/\text{confidence in measurements}$

Convergence and steady-state Kalman filter for LTI systems

Assumptions

A0 The system is LTI, i.e. $\boldsymbol{F}$, $\boldsymbol{G}$, $\boldsymbol{C}$, $\boldsymbol{Q}$, $\boldsymbol{R}$ are constant matrices

A1 If $(\boldsymbol{C}, \boldsymbol{F})$ is detectable, and $(\boldsymbol{F}, \boldsymbol{H})$ is stabilisable with $\boldsymbol{H}$ s.t. $\boldsymbol{Q} = \boldsymbol{H}\boldsymbol{H}^T$, the filter is asymptotically stable.

Results

1. $\Sigma_{k+1|k}$ tends to a constant matrix Σ_p, unique positive definite solution of the Riccati algebraic matrix equations:

$$\boldsymbol{\Sigma_p} = \boldsymbol{F}\boldsymbol{\Sigma_p}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{\Sigma_p}\boldsymbol{C}^T\left(\boldsymbol{C}\boldsymbol{\Sigma_p}\boldsymbol{C}^T + \boldsymbol{R}\right)^{-1}\boldsymbol{C}\boldsymbol{\Sigma_p}\boldsymbol{F}^T + \boldsymbol{Q}$$

2. $\boldsymbol{K}_k$ tends to a constant matrix:

$$\boldsymbol{K} = \boldsymbol{\Sigma_p}\boldsymbol{C}^T\left(\boldsymbol{C}\boldsymbol{\Sigma_p}\boldsymbol{C}^T + \boldsymbol{R}\right)^{-1}$$

3. $\Sigma_{k|k}$ tends to $\Sigma_e = (\boldsymbol{I}_n - \boldsymbol{K}\boldsymbol{C})\,\Sigma_p$
4. In steady state, the estimation error becomes white noise if the model is perfect.

The Kalman filter then becomes an observer which gain $\boldsymbol{K}$ is constant and entirely determined by the choice of matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$.

Illustration of theoretical results

Example 3 - sine wave with noise

Consider a simple integrator system with v_k and w_k state and measurement noise of variances Q_{real} and R_{real}, and u_k an sinusoidal input:

$$x_{k+1} = x_k + u_k + v_k$$

$$z_k = x_k + w_k$$

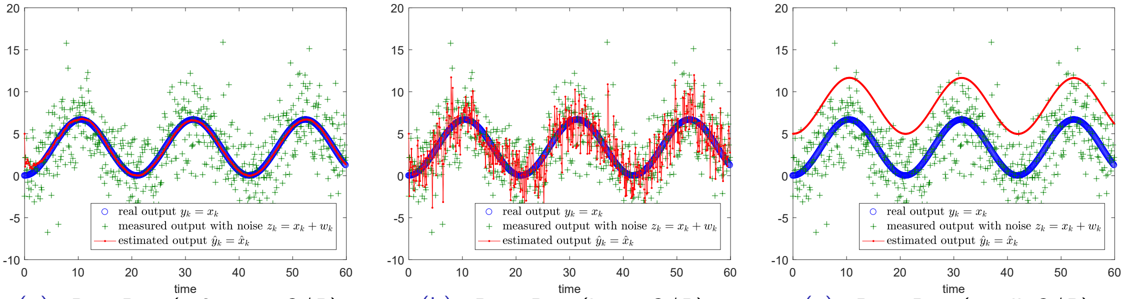
1. $\hat{x}_{k+1|k} = \hat{x}_{k|k} + u_k$
2. $\Sigma_{k+1|k} = \Sigma_{k|k} + Q$
3. $\hat{z}_{k+1|k} = \hat{x}_{k+1|k}$
4. $K_{k+1} = \Sigma_{k+1|k}\left(\Sigma_{k+1|k} + R\right)^{-1}$
5. $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\left(z_{k+1} - \hat{z}_{k+1|k}\right)$
6. $\Sigma_{k+1|k+1} = (1 - K_{k+1})\,\Sigma_{k+1|k}$

Steady state ($\hat{x}_k = \hat{x}_{k|k}$)

$$\hat{x}_{k+1} = \hat{x}_k + u_k + K(z_k - \hat{x}_k - u_k)$$

$$K = \frac{1}{2}\left(-\frac{Q}{R} + \sqrt{\left(\frac{Q}{R}\right)^2 + 4\frac{Q}{R}}\right)$$

Illustration of theoretical results (example 3)



(a) $R = R_{real}$ (reference Q/R) (b) $R \ll R_{real}$ (large Q/R) (c) $R \gg R_{real}$ (small Q/R)

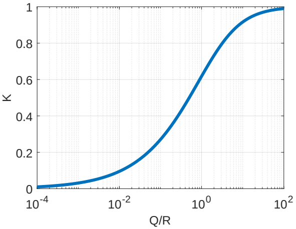


Figure: Influence of R as tuning parameter (with $Q = Q_{real}$)

Back to Example 1 - Quadrotor pitch angle estimation

Problem formulation

- **Available measurements to reconstruct pitch angle**
  1. Gyro measurements of pitch rate $\dot{\theta}^m_{Gyr}(t) = GYR^m_y(t)$
  2. Accelerometer measurements of pitch angle
     $\theta^m_{Acc}(t) = \arctan\left(\frac{-ACC^m_x}{\sqrt{(ACC^m_y)^2+(ACC^m_z)^2}}\right)$
- **Towards a state-space formulation**
  1. Gyro measurements $\dot{\theta}^m_{Gyr}$ represent real pitch rate $\dot{\theta}$ affected by a white noise $v_{Gyr} \sim \mathcal{N}(0,\ \sigma^2_{Gyr})$ and an almost constant bias b_{Gyr} with $v_b \sim \mathcal{N}(0,\ \sigma^2_b)$:
     $$\dot{\theta}^m_{Gyr}(t) = \dot{\theta}(t) + b_{Gyr}(t) + v_{Gyr}(t)$$
     $$\dot{b}_{Gyr}(t) = v_b(t)$$
  2. Accelerometer measurements $\theta^m_{Acc}(t)$ represent real pitch angle θ affected by a measurement noise $w_\theta \sim \mathcal{N}(0, \sigma^2_{\theta_{acc}})$:
     $$\theta^m_{Acc}(t) = \theta(t) + w_\theta(t)$$

State space

- State: $\boldsymbol{x}(t) = [\theta,\ b_{Gyr}]^T$
- Input: $u(t) = \dot{\theta}^m_{Gyr}$
- Output: $z(t) = \theta^m_{Acc}$
- State noise: $\boldsymbol{v} = [v_{Gyr},\ v_b]^T$
- Measurement noise: w_θ

Back to Example 1 - Quadrotor pitch angle estimation

Continuous-time state-space model

$$\begin{cases} \dot{\theta}(t) = \dot{\theta}^m_{Gyr}(t) - b_{Gyr}(t) - v_{Gyr}(t) \\ \dot{b}_{Gyr}(t) = v_b(t) \\ \theta^m_{Acc}(t) = \theta(t) + w_\theta(t) \end{cases}$$

$$\Leftrightarrow \begin{cases} \dot{\boldsymbol{x}}(t) = \overset{A}{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}} \boldsymbol{x}(t) + \overset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} u(t) + \boldsymbol{v}(t) \\ z(t) = \overset{C}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \boldsymbol{x}(t) + w_\theta(t) \end{cases}$$

Discrete-time state-space model

$$\boldsymbol{F} = \begin{bmatrix} 1 & -T_s \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{G} = \begin{bmatrix} T_s \\ 0 \end{bmatrix}$$

State space

- State: $\boldsymbol{x} = [\theta, \ b_{Gyr}]^T$
- Input: $u = \dot{\theta}^m_{Gyr}$
- Output: $z = \theta^m_{Acc}$
- State noise: $\boldsymbol{v} = [v_{Gyr}, \ v_b]^T$
- Measurement noise: w_θ

Numerical values

- $T_s = 0.01$s
- $\hat{\boldsymbol{x}}_{0|0} = ?$
- $\boldsymbol{\Sigma}_{0|0} = ?$
- $\boldsymbol{Q} = ?$
- $R = ?$

Noise variance

$$\boldsymbol{Q} = E\left\{\boldsymbol{v}(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})^{\boldsymbol{T}}\right\} = \begin{bmatrix} \sigma^2_{Gyr} & 0 \\ 0 & \sigma^2_b \end{bmatrix}$$

$$R = E\left\{w_\theta(t) w_\theta(t)^T\right\} = \sigma^2_{\theta_{acc}}$$

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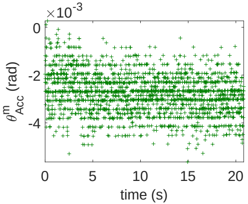
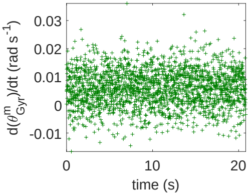


(a) $z_k = \theta^m_{Acc}$ (rad)



(b) $u_k = \dot{\theta}^m_{Gyr}$ (rad/s)

Figure: Analysis of sensor readings w/o mvt

Tuning

First tuning using sensor readings without movement ($\theta = 0$):

- Noise variances
  - Using $u_k = \dot{\theta}^m_{Gyr}$
    - $\sigma^2_{Gyr} = E\left\{(\dot{\theta}^m_{Gyr})^2\right\} = 4.3 \cdot 10^{-5}$
    - $\sigma^2_b = 10^{-9}$ (we suppose that the model is quite accurate)
  - Using $z_k = \theta^m_{Acc}$
    - $\sigma^2_{\theta_{acc}} = E\left\{(\theta^m_{Acc})^2\right\} = 7 \cdot 10^{-7}$
- Initialization
  - $\hat{\mathbf{x}}_{0|0} = \begin{bmatrix} E\{\theta\} & E\{b_\theta\} \end{bmatrix}^T = \begin{bmatrix} 0 & 6 \cdot 10^{-3} \end{bmatrix}^T$
  - $\mathbf{\Sigma}_{0|0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ This initial tuning can be adjusted to provide satisfactory results.

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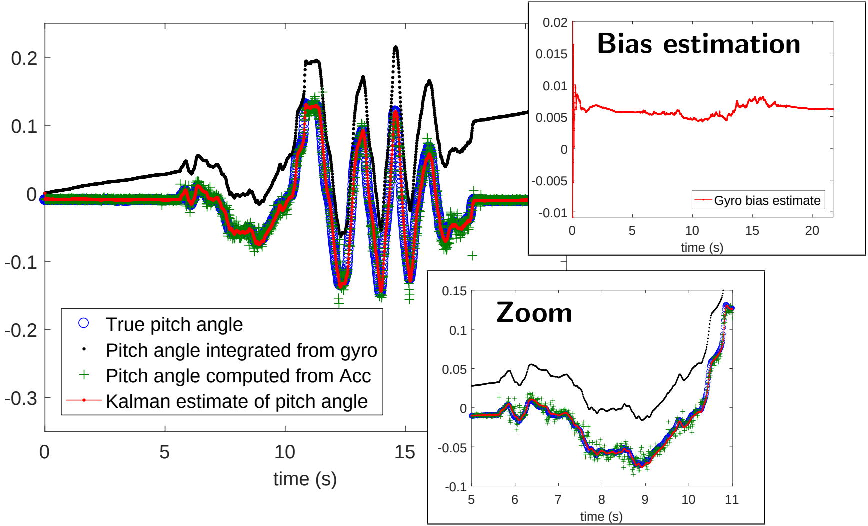

Figure: Pitch angle estimation - estimated state

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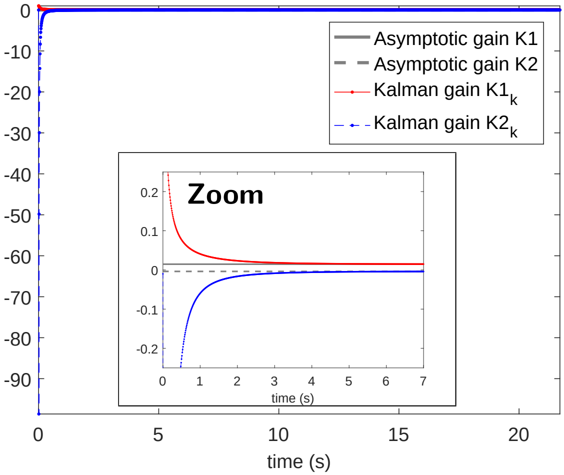

Figure: Pitch angle estimation - Kalman filter gains