DroMOOC

Sensor fusion and state estimation

Basic Level

Observers and Kalman filter (Part I)

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Motivation for observers and Kalman filtering

Example 1 - Quadrotor pitch angle estimation

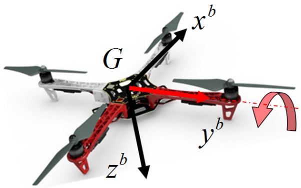
- Physical disturbances on the system (turbulence...)
- Sensor imperfections (noise, bias, drift...)
  - in a single sensor
  - in a set of sensors used in combination

⇒ Need to reconstruct potentially unmeasured variables to control the system

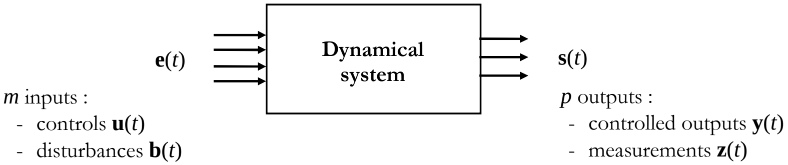
  - **Observers** (deterministic)

⇒ Need to reconstruct accurate real system behavior in presence of ill-known or unmodeled random effects on system and measurements of various characteristics

  - **Kalman filters** (can be seen as a special kind of observers)



State-space representation of systems



General nonlinear case

- Continuous-time system

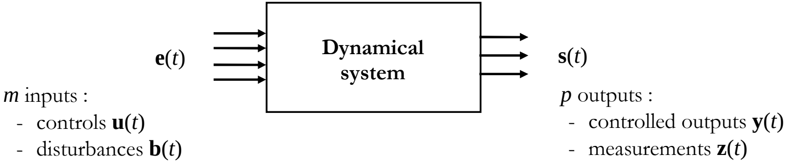
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$
$$\boldsymbol{s}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$

- Discrete-time system

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, \boldsymbol{e}_k, k)$$
$$\boldsymbol{s}_k = \boldsymbol{g}(\boldsymbol{x}_k, \boldsymbol{e}_k, k)$$

- $\boldsymbol{x}$ state vector, dimension n
- $\boldsymbol{e}$ input vector, dimension m
- $\boldsymbol{s}$ output vector, dimension p

State-space representation of LTI systems



Linear Time Invariant (LTI) systems

- Continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\ \mathbf{x}(t) + \mathbf{B}\ \mathbf{e}(t)$$
$$\mathbf{s}(t) = \mathbf{C}\ \mathbf{x}(t) + \mathbf{D}\ \mathbf{e}(t)$$

- Discrete-time system

$$\mathbf{x}_{k+1} = \mathbf{F}\ \mathbf{x}_k + \mathbf{G}\ \mathbf{e}_k$$
$$\mathbf{s}_k = \mathbf{H}\ \mathbf{x}_k + \mathbf{J}\ \mathbf{e}_k$$

- $\mathbf{A},\ \mathbf{F} \in \mathbb{R}^{n\times n}$ state matrix
- $\mathbf{B},\ \mathbf{G} \in \mathbb{R}^{n\times m}$ input matrix
- $\mathbf{C},\ \mathbf{H} \in \mathbb{R}^{p\times n}$ output matrix
- $\mathbf{D},\ \mathbf{J} \in \mathbb{R}^{p\times m}$ direct feedthrough matrix

Local linear model (continuous time)

- Nonlinear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$
$$\boldsymbol{s}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$

- **Equilibrium point:** constant $\boldsymbol{x}_0$, $\boldsymbol{e}_0$, $\boldsymbol{s}_0$ s.t. $\forall t$:

$$(\boldsymbol{S}_0) \quad \begin{cases} 0 = \boldsymbol{f}(\boldsymbol{x}_0, \boldsymbol{e}_0, t) \\ \boldsymbol{s}_0 = \boldsymbol{g}(\boldsymbol{x}_0, \boldsymbol{e}_0, t) \end{cases}$$

- Taylor 1st order approximation around the equilibrium point with $\boldsymbol{x}(t) = \boldsymbol{x}_0 + \tilde{\boldsymbol{x}}(t)$, $\boldsymbol{e}(t) = \boldsymbol{e}_0 + \tilde{\boldsymbol{e}}(t)$, $\boldsymbol{s}(t) = \boldsymbol{s}_0 + \tilde{\boldsymbol{s}}(t)$:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \overset{A}{\boxed{\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]_{(\boldsymbol{x}_0, \boldsymbol{e}_0, t)}}} \tilde{\boldsymbol{x}}(t) + \overset{B}{\boxed{\left[\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{e}}\right]_{(\boldsymbol{x}_0, \boldsymbol{e}_0, t)}}} \tilde{\boldsymbol{e}}(t)$$

$$\tilde{\boldsymbol{s}}(t) = \overset{C}{\boxed{\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right]_{(\boldsymbol{x}_0, \boldsymbol{e}_0, t)}}} \tilde{\boldsymbol{x}}(t) + \overset{D}{\boxed{\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{e}}\right]_{(\boldsymbol{x}_0, \boldsymbol{e}_0, t)}}} \tilde{\boldsymbol{e}}(t)$$

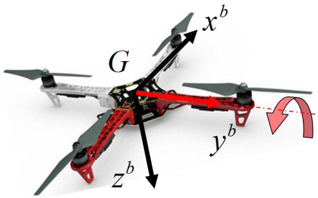
State-space representation of systems

Example 2 - Quadrotor vertical dynamics

$$\dot{p}_z(t) = v_z(t)$$

$$\dot{v}_z(t) = \frac{\cos\theta\cos\varphi}{m}\mathcal{T}(t) + g$$

- Let's consider here only the part of the complete state vector related to the vertical dynamics $\boldsymbol{x}(t) = (p_z(t),\ v_z(t))^T$
- Thrust magnitude is considered as a control input $u(t) = \mathcal{T}(t) = -b\sum_{i=1}^{4}\omega_i(t)^2$
- An output of this system can be defined as $s(t) = p_z(t)$



⚠ Roll and pitch angles $\varphi(t)$ and $\theta(t)$ have their own dynamics. The choice of the state variables $(p_z,\ v_z)$ is therefore incomplete! *For complete nonlinear dynamic modeling of a quadrotor, please refer to the corresponding lecture.*

Local linear model (continuous time)

Example 2 - Quadrotor vertical dynamics

- Consider the equilibrium point representing the **hovering** situation with $\theta_0 \approx 0,\ \varphi_0 \approx 0$. Other variables around this equilibrium are:

$$0 = v_{z_0}$$

$$0 = \cos\theta_0 \cos\varphi_0 \frac{1}{m}\mathcal{T}_0 + g \ \Rightarrow\ \mathcal{T}_0 = -mg$$

- The **linearized equations** of the vertical dynamics around the hovering equilibrium are:

$$\dot{\tilde{p}}_z(t) = \dot{p}_z(t) = v_z(t) = v_{z_0} + \tilde{v}_z(t) \ \Rightarrow\ \dot{\tilde{p}}_z(t) = \tilde{v}_z(t)$$

$$\dot{\tilde{v}}_z(t) = \cos(\theta_0 + \tilde{\theta}(t)) \cos(\varphi_0 + \tilde{\varphi}(t)) \frac{1}{m}(\mathcal{T}_0 + \tilde{\mathcal{T}}(t)) + g$$

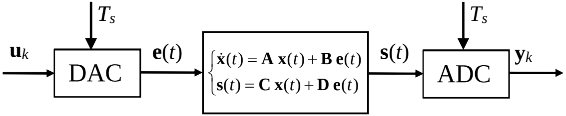
$$\Rightarrow\ \dot{\tilde{v}}_z(t) = \tilde{\mathcal{T}}(t)$$

Linear system

$$\begin{bmatrix} \dot{\tilde{p}}_z \\ \dot{\tilde{v}}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \tilde{p}_z \\ \tilde{v}_z \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} \tilde{\mathcal{T}}$$

$\Rightarrow$ double-integrator system uncoupled from θ and φ

Sampled LTI systems



Discrete-time model of sampled systems

The equivalent model of a continuous-time system sampled with sampling time T_s is:

$$\begin{aligned} \boldsymbol{x}_{k+1} &= \boldsymbol{F}\ \boldsymbol{x}_k + \boldsymbol{G}\ \boldsymbol{u}_k \\ \boldsymbol{y}_k &= \boldsymbol{C}\ \boldsymbol{x}_k + \boldsymbol{D}\ \boldsymbol{u}_k \end{aligned}$$

with

$$\boldsymbol{F} = e^{\boldsymbol{A}T_s}$$

$$\boldsymbol{G} = \int_0^{T_s} e^{\boldsymbol{A}\theta}\boldsymbol{B}d\theta$$

$\Rightarrow$ Solve between $t_k = kT_s$ and $t_{k+1} = (k+1)T_s$ to express $\boldsymbol{x}_{k+1} = \boldsymbol{x}(t_{k+1})$ as function of $\boldsymbol{x}_k = \boldsymbol{x}(t_k)$ and constant input $\boldsymbol{e}(t) = \boldsymbol{u}_k$:

$$\begin{aligned} \boldsymbol{x}(t_{k+1}) = &e^{\boldsymbol{A}(t_{k+1}-t_k)}\boldsymbol{x}(t_k)\dots \\ &+ \int_{t_k}^{t_{k+1}} e^{\boldsymbol{A}(t_{k+1}-\tau)}\boldsymbol{B}\boldsymbol{e}(\tau)d\tau \end{aligned}$$

Sampled LTI systems

Example 2 - Quadrotor vertical dynamics around the hovering equilibrium

- Suppose the control input $\tilde{\mathcal{T}}$ is constant over the sampling period T_s, i.e.:

$$\text{for } t_k = kT_s \leq t \leq t_{k+1} = (k+1)T_s,$$

$$\begin{bmatrix} \dot{\tilde{p}}_z(t) \\ \dot{\tilde{v}}_z(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_z(t) \\ \tilde{v}_z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{\mathcal{T}}_k \text{ with } \tilde{\mathcal{T}}(t) = \tilde{\mathcal{T}}(kT_s) = \tilde{\mathcal{T}}_k$$

- Integrate beween t_k and t_{k+1}:

$$\tilde{v}_z(t_{k+1}) = \tilde{v}_z(t_k) + T_s \tilde{\mathcal{T}}_k$$

$$\tilde{p}_z(t_{k+1}) = \tilde{p}_z(t_k) + T_s \tilde{v}_z(t_k) + \frac{T_s^2}{2} \tilde{\mathcal{T}}_k$$

- Resulting sampled system equations:

$$\begin{bmatrix} \tilde{p}_z[k+1] \\ \tilde{v}_z[k+1] \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}}^{F} \begin{bmatrix} \tilde{p}_z[k] \\ \tilde{v}_z[k] \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix}}^{G} \tilde{\mathcal{T}}_k$$

Stability of the equilibrium point $\boldsymbol{x}_0,\ \boldsymbol{e}_0,\ \boldsymbol{s}_0$

Lyapunov stability

$\forall \varepsilon > 0, \forall t_0, \exists \delta(\varepsilon, t_0)\ s.t.:$

$\|\boldsymbol{x}(t_0) - \boldsymbol{x}_0\| < \delta(\varepsilon, t_0)$

$\Rightarrow \|\boldsymbol{x}(t) - \boldsymbol{x}_0\| < \varepsilon, \forall t \geq t_0$

Asymptotic stability

i. Lyapunov stability

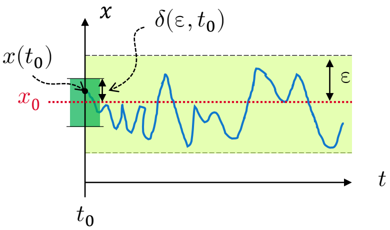
ii. $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \boldsymbol{x}_0\| = 0$

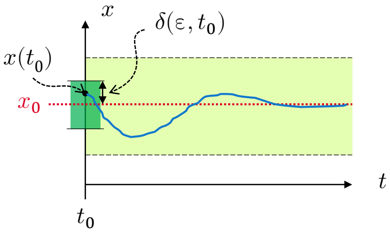
Exponential stability

i. Asymptotic stability

ii. $\exists M, \alpha > 0\ s.t.:$

$\|\boldsymbol{x}(t) - \boldsymbol{x}_0\| \leq M\|\boldsymbol{x}(t_0) - \boldsymbol{x}_0\|e^{-\alpha(t-t_0)}$





Stability of LTI systems (continuous time)

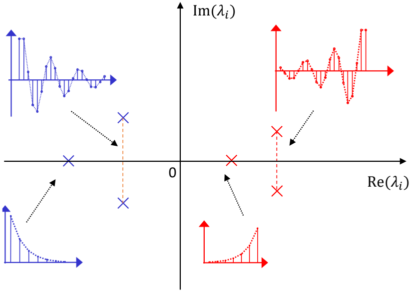
- The free response of the continuous-time LTI system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\ \boldsymbol{x}(t) + \boldsymbol{B}\ \boldsymbol{e}(t)$$
$$\boldsymbol{s}(t) = \boldsymbol{C}\ \boldsymbol{x}(t) + \boldsymbol{D}\ \boldsymbol{e}(t)$$

to initial condition $\boldsymbol{x}(t_0)$ (solution of $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$) is:

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}(t-t_0)}\boldsymbol{x}(t_0) = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \alpha_{ij}\,(t-t_0)^{j-1}\, e^{\lambda_i(t-t_0)}$$

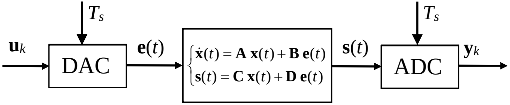
with λ_i $(i = 1..k)$ the i-th eigenvalue of $\boldsymbol{A}$ of order n_i .

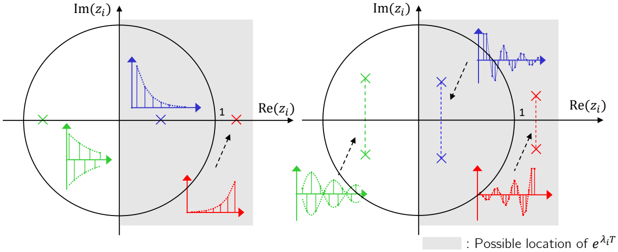


Exponential stability criterion

The real part of all $\boldsymbol{A}$ eigenvalues is strictly negative.

Stability of sampled LTI systems (discrete time)





- The state matrix $\boldsymbol{F} = e^{\boldsymbol{A}T}$ of the sampled system has eigenvalues $p_i = e^{\lambda_i T_s}$, with λ_i $(i = 1..n)$ the eignevalues of $\boldsymbol{A}$
- The free response of $\dot{\boldsymbol{x}}_{k+1} = \boldsymbol{F}\boldsymbol{x}_k$ is $\boldsymbol{x}_k = \boldsymbol{F}^{k-k0}\boldsymbol{x}_{k0}$

Exponential stability criterion

All eigenvalues of $\boldsymbol{F}$ have a modulus strictly inferior to 1.

Stability of LTI systems

Example 2 - Quadrotor vertical dynamics around hovering

- Continuous-time dynamics :

$$\begin{bmatrix} \dot{\tilde{p}}_z(t) \\ \dot{\tilde{v}}_z(t) \end{bmatrix} = \overset{A}{\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}} \begin{bmatrix} \tilde{p}_z(t) \\ \tilde{v}_z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{\mathcal{T}}(t)$$

- Discrete-time dynamics of the sampled system :

$$\begin{bmatrix} \tilde{p}_z[k+1] \\ \tilde{v}_z[k+1] \end{bmatrix} = \overset{F}{\boxed{\begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}}} \begin{bmatrix} \tilde{p}_z[k] \\ \tilde{v}_z[k] \end{bmatrix} + \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix} \tilde{\mathcal{T}}_k$$

Observability

Definition (general case)

The nonlinear continuous-time system with **measurement** vector $\boldsymbol{z}(t)$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$
$$\boldsymbol{z}(t) = \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{e}(t), t)$$

is observable if **for all** $\boldsymbol{x_0}$ s.t. $\boldsymbol{x}(t_0) = \boldsymbol{x_0}$, the observation of $\boldsymbol{z}(t)$ for a **finite** time $t_f - t_0$ allows us to determine $\boldsymbol{x_0}$.

Observability of LTI systems

$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\ \boldsymbol{x}(t) + \boldsymbol{B}\ \boldsymbol{e}(t)$ with $\boldsymbol{x} \in \mathbb{R}^n$

$\boldsymbol{z}(t) = \boldsymbol{C}\ \boldsymbol{x}(t) + \boldsymbol{D}\ \boldsymbol{e}(t)$

is observable $\Leftrightarrow \operatorname{rank} \begin{bmatrix} \boldsymbol{C}^T & (\boldsymbol{CA})^T & \dots & (\boldsymbol{CA^{n-1}})^T \end{bmatrix}^T = n$

Similar definition and criterion hold for discrete-time systems.

Observability of LTI systems

Example 2 - Quadrotor vertical dynamics around hovering

- Continuous-time dynamics :

$$\begin{bmatrix} \dot{\tilde{p}}_z(t) \\ \dot{\tilde{v}}_z(t) \end{bmatrix} = \overset{A}{\boxed{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}} \begin{bmatrix} \tilde{p}_z(t) \\ \tilde{v}_z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{\mathcal{T}}(t) \quad \text{with} \quad \begin{cases} z(t) = \tilde{p}_z(t) = \overset{C_1}{\boxed{\begin{bmatrix} 1 & 0 \end{bmatrix}}} \begin{bmatrix} \tilde{p}_z(t) \\ \tilde{v}_z(t) \end{bmatrix} \\ \text{or} \\ z(t) = \tilde{v}_z(t) = \overset{C_2}{\boxed{\begin{bmatrix} 0 & 1 \end{bmatrix}}} \begin{bmatrix} \tilde{p}_z(t) \\ \tilde{v}_z(t) \end{bmatrix} \end{cases}$$

- Rank of the observability matrices ($n = 2$):

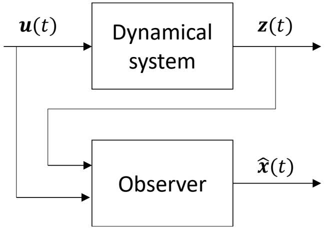
$$\operatorname{rank}\left(\begin{bmatrix} \boldsymbol{C_1} \\ \boldsymbol{C_1 A} \end{bmatrix} \right) = \operatorname{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 2 \quad \textbf{but} \quad \operatorname{rank}\left(\begin{bmatrix} \boldsymbol{C_2} \\ \boldsymbol{C_2 A} \end{bmatrix} \right) = \operatorname{rank}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = 1$$

Deterministic state estimation (continuous time)

- **Problem:** measurements of the state $\boldsymbol{x}(t)$ are usually unavailable (for supervision or control), but $\boldsymbol{z}(t)$ is measured and the state-space model is known:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\ \boldsymbol{x}(t) + \boldsymbol{B}\ \boldsymbol{u}(t)\\ \boldsymbol{z}(t) &= \boldsymbol{C}\ \boldsymbol{x}(t)\end{aligned}$$

$\Rightarrow$ Reconstruction of the state by an observer



Observer (continuous-time)

$$\begin{aligned}\dot{\hat{\boldsymbol{x}}}(t) &= \boldsymbol{A}\ \hat{\boldsymbol{x}}(t) + \boldsymbol{B}\ \boldsymbol{u}(t) + \boldsymbol{K}\left(\boldsymbol{z}(t) - \hat{\boldsymbol{z}}(t)\right)\\ \hat{\boldsymbol{z}}(t) &= \boldsymbol{C}\ \hat{\boldsymbol{x}}(t)\end{aligned}$$

- Dynamics of the reconstruction error $\boldsymbol{\varepsilon} = \boldsymbol{x} - \hat{\boldsymbol{x}}$:

$$\begin{aligned}\dot{\boldsymbol{\varepsilon}} &= \dot{\boldsymbol{x}} - \dot{\hat{\boldsymbol{x}}}\\ &= \boldsymbol{A}\ \boldsymbol{x} + \boldsymbol{B}\ \boldsymbol{u} - \left(\boldsymbol{A}\ \hat{\boldsymbol{x}} + \boldsymbol{B}\ \boldsymbol{u} + \boldsymbol{KC}\left(\boldsymbol{x} - \hat{\boldsymbol{x}}\right)\right)\\ &= \left(\boldsymbol{A} - \boldsymbol{KC}\right)\boldsymbol{\varepsilon}\end{aligned}$$