Course Material

Week 3: Extended Kalman Filters

3.1. Examples for the Action in the EKF

$$\begin{cases} \mu_a = f(\mu, u^m) \\ P_a = F_x P F_x^T + F_u Q F_u^T \end{cases}$$

$$x = N(\mu, \sigma_x^2) \qquad P = \sigma_x^2$$

$$\alpha = N(\alpha^m, \sigma_\alpha^2) \qquad Q = \sigma_\alpha^2$$

$$S = x$$

$$x_{i+1} = x_i + R\,\underbrace{\alpha_{i+1}}_{u_{i+1}}$$

$$F_x = 1 \qquad F_u = R$$

$$\mu_a = \mu + R\,\alpha^m$$

$$P_a = \sigma_x^2 + R^2 \sigma_\alpha^2$$

Screen 1: Encoders in 1D

$$S = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad S_{i+1} = f(S_i, u_{i+1})$$

$$\begin{cases} x_{i+1} = x_i + \delta\rho \cos\theta_i \\ y_{i+1} = y_i + \delta\rho \sin\theta_i \\ \theta_{i+1} = \theta_i + \delta\theta \end{cases} \qquad \delta\rho = \frac{S_R + S_L}{2} \qquad \delta\theta = \frac{S_R - S_L}{B}$$

$$u^m = \begin{bmatrix} S_R^m \\ S_L^m \end{bmatrix} \qquad u = N\left(\begin{bmatrix} S_R^m \\ S_L^m \end{bmatrix}, Q\right) \qquad Q = \begin{bmatrix} K^R|S_R| & \emptyset \\ \emptyset & K^L|S_L| \end{bmatrix}$$

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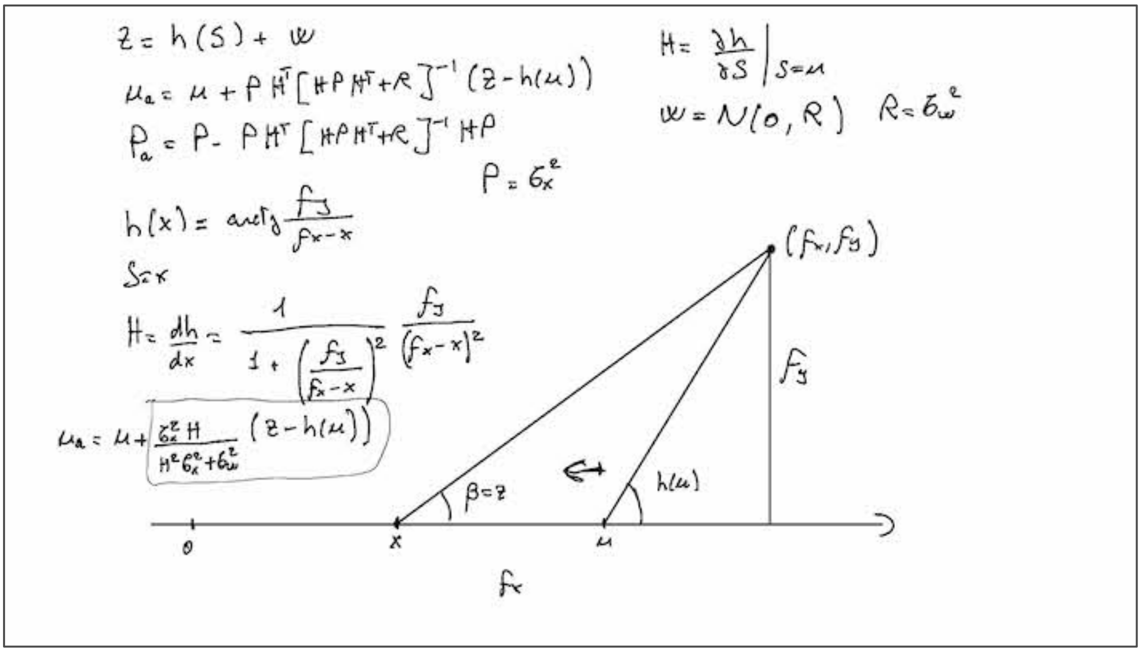
$$\begin{cases} \mu_a = f(\mu, u^m) \\ P_a = F_x P F_x^T + F_u Q F_u^T \end{cases}$$

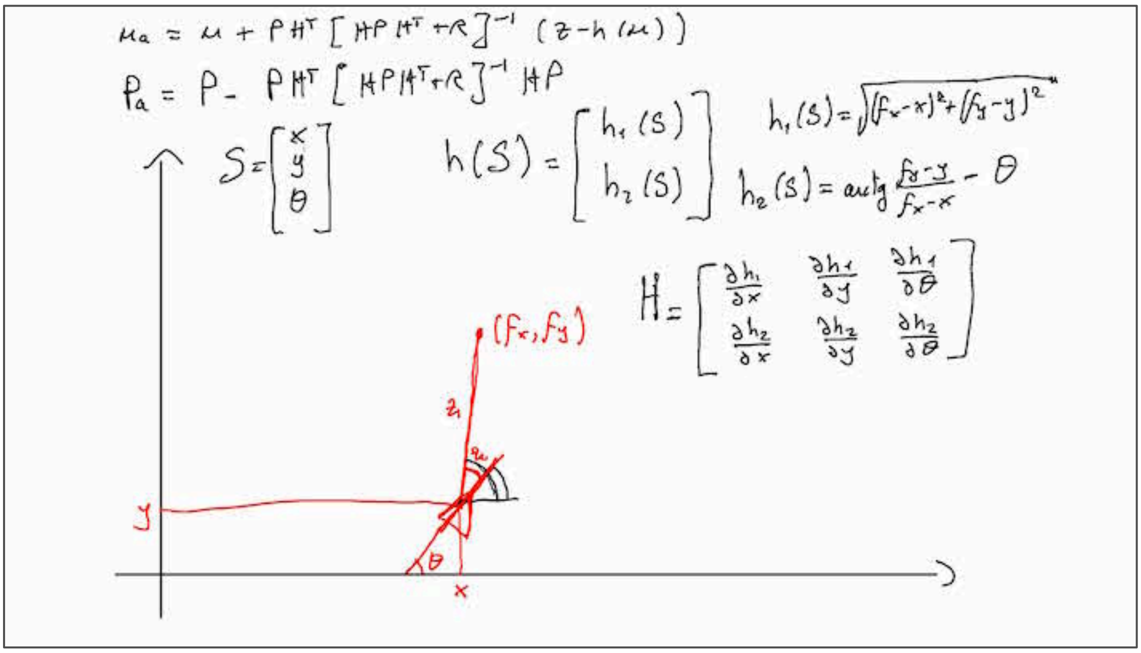
$$F_{x_{3\times3}} = \begin{bmatrix} 1 & 0 & -\delta\rho \sin\theta \\ 0 & 1 & \delta\rho\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \qquad F_{u_{3\times2}} = \begin{bmatrix} \frac{\cos\theta}{2} & \frac{\cos\theta}{2} \\ \frac{\sin\theta}{2} & \frac{\sin\theta}{2} \\ \frac{1}{B} & -\frac{1}{B} \end{bmatrix}$$

Screen 2: Encoders in 2D

3.2. Examples for the Perception in the EKF



Screen 1: Bearing of a point feature when the robot moves in 1D



Screen 2: Bearing and distance of a point features in 2D

3.3. The EKF is a weight mean

$$x \qquad z_1 \qquad z_2 \qquad z_1 = \mathcal{N}(x, \sigma_1^2) \qquad z_2 = \mathcal{N}(x, \sigma_2^2)$$

$$\hat{x} = p_1 z_1 + p_2 z_2 \qquad p_1 + p_2 = 1 \qquad p_1 \propto \frac{1}{\sigma_1^2} \qquad p_2 \propto \frac{1}{\sigma_2^2}$$

$$\boxed{\hat{x} = \frac{\frac{1}{\sigma_1^2} z_1 + \frac{1}{\sigma_2^2} z_2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{\sigma_2^2 z_1 + \sigma_1^2 z_2}{\sigma_1^2 + \sigma_2^2}}$$

$$\hat{\sigma}^2 \qquad \frac{1}{\hat{\sigma}^2} = \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \qquad \boxed{\hat{\sigma}^2 = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

Screen 1: Simple weight mean

$$\mu_a = \mu + P H^T \left[H P H^T + R \right]^{-1} (z - h(\mu))$$

$$P_a = P - P H^T \left[H P H^T + R \right]^{-1} H P$$

$\mu \to z_1$

$z \to z_2$

h = identity function $\quad H = 1$

$P \to \sigma_1^2 \qquad \hat{x} = z_1 + \sigma_1^2 \cdot 1 \left[\sigma_1^2 + \sigma_2^2 \right]^{-1} (z_2 - z_1) =$

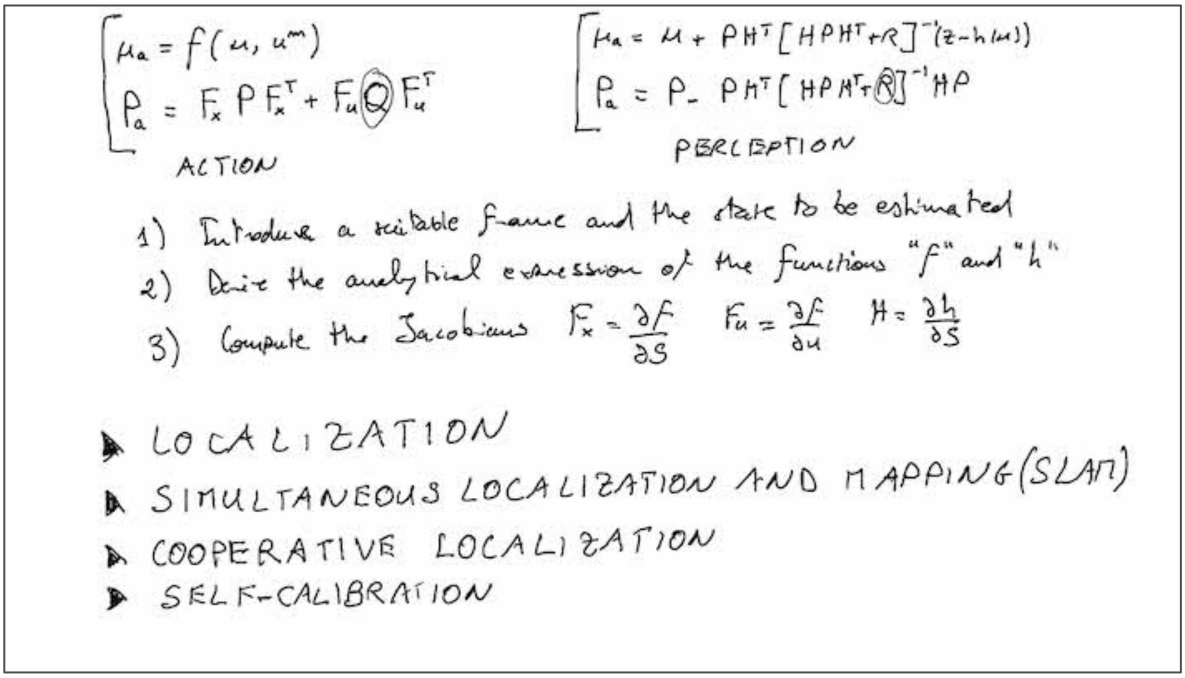
$R \to \sigma_2^2 \qquad z_1 + \dfrac{\sigma_1^2 z_2 - \sigma_1^2 z_1}{\sigma_1^2 + \sigma_2^2} = \dfrac{z_1 \sigma_2^2 + z_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}$

$\mu_a \to \hat{x}$

$P_a \to \hat{\sigma}^2 \qquad \hat{\sigma}^2 = \sigma_1^2 - \sigma_1^2 \left[\sigma_1^2 + \sigma_2^2 \right]^{-1} \sigma_1^2 = \dfrac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$

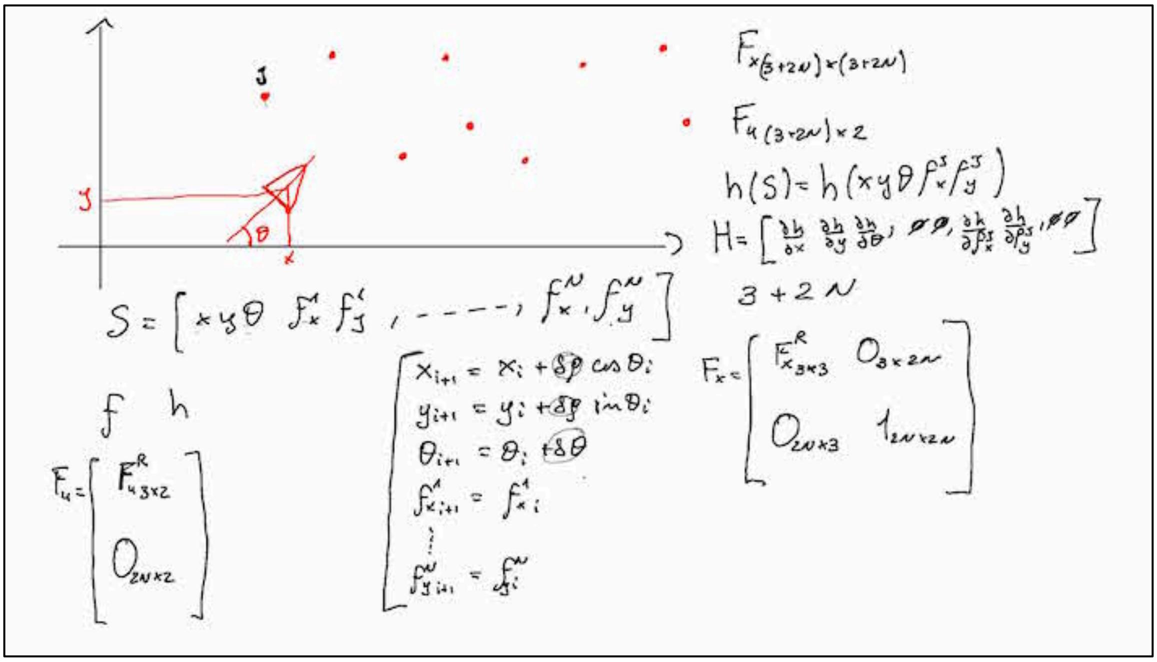
Screen 2: EKF perception for a special case: 1D and h is the identity function

3.4. The use of the EKF in robotics

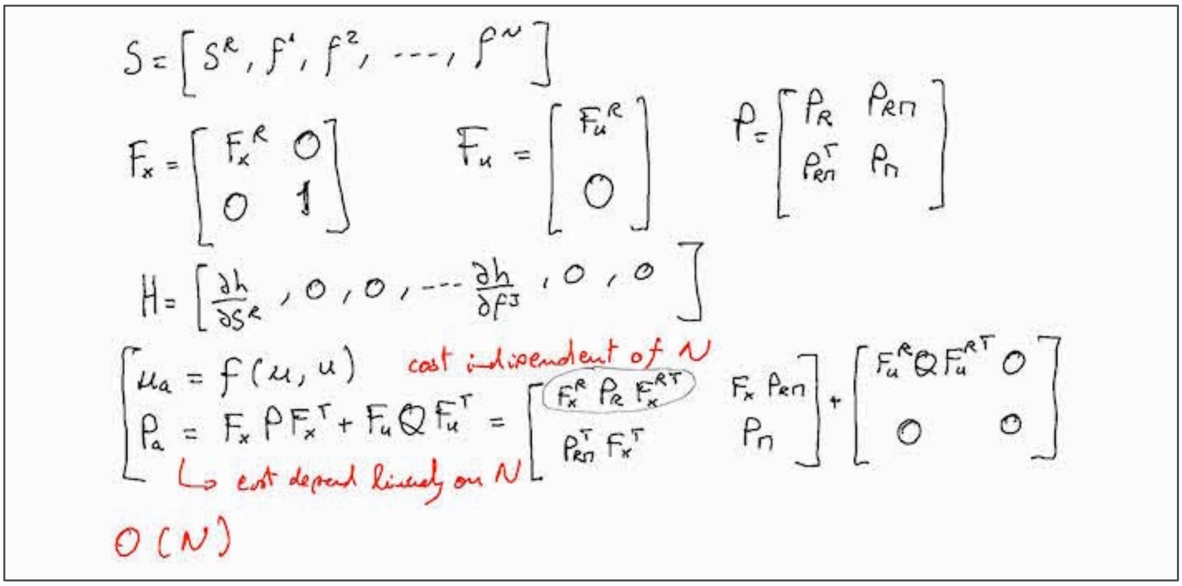


Screen 1: Basic steps to implement an EKF and important estimation problems in robotics

3.5 Simultaneous Localization and Mapping (SLAM)



Screen 2: Basic equations for the 2D case with N point features: the function f and h and their Jacobians



Screen 1: Matrices to implement the EKF and computational cost of the action

$$\mu_a = \mu + PH^T\left[HPH^T + R\right]^{-1}(z - h(\mu)) \quad \text{cost is } O(N)$$

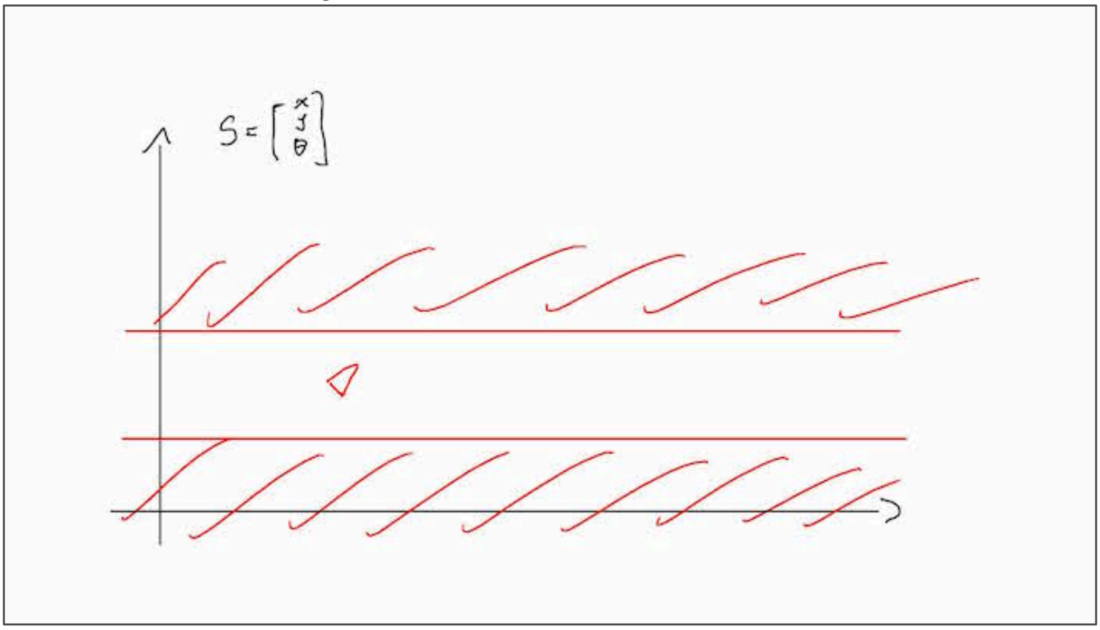
$$P_a = P - PH^T\left[HPH^T + R\right]^{-1}HP \quad \text{cost is } O(N^2)$$

$$H = \left[H^R, \emptyset\ \emptyset, \ldots, H^J, \emptyset\ \emptyset\right]$$

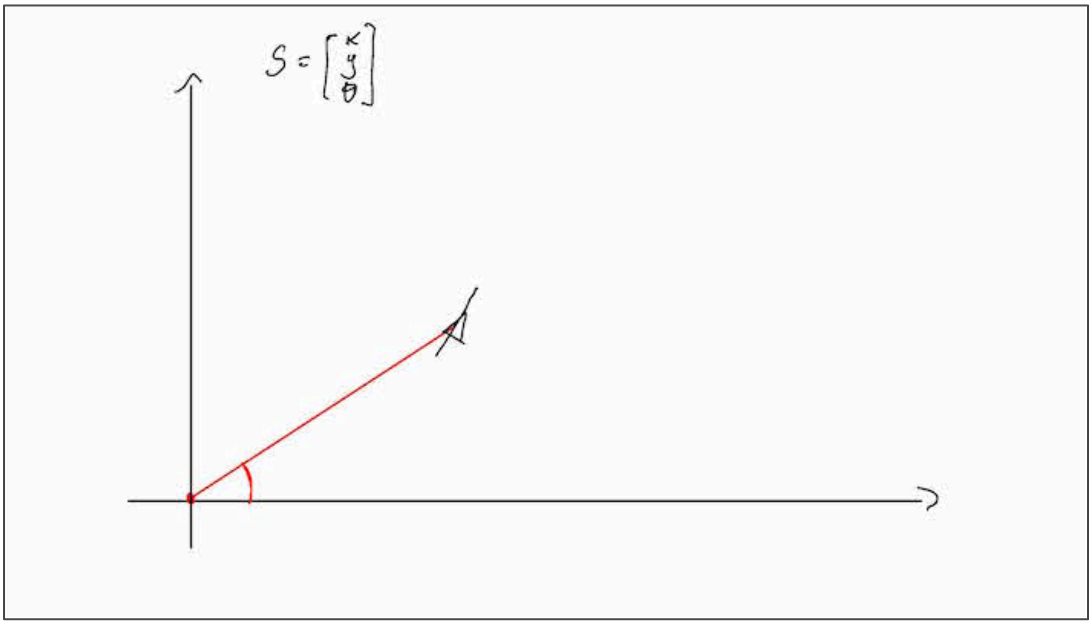
$$\begin{bmatrix} P_R & P_{Rm} \\ P_{Rm}^T & P_m \end{bmatrix} \begin{bmatrix} H^{RT}_r \\ 0 \\ 0 \\ H^{JT}_e \\ 0 \end{bmatrix} \quad O(N)$$

Screen 2: Computational cost of the perception

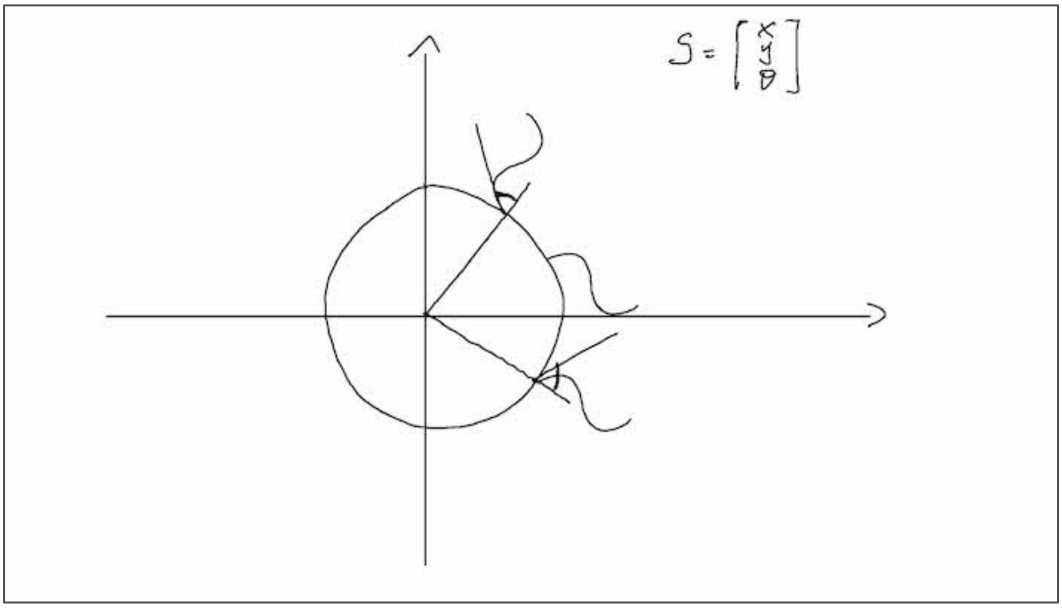
3.6 Observability in robotics



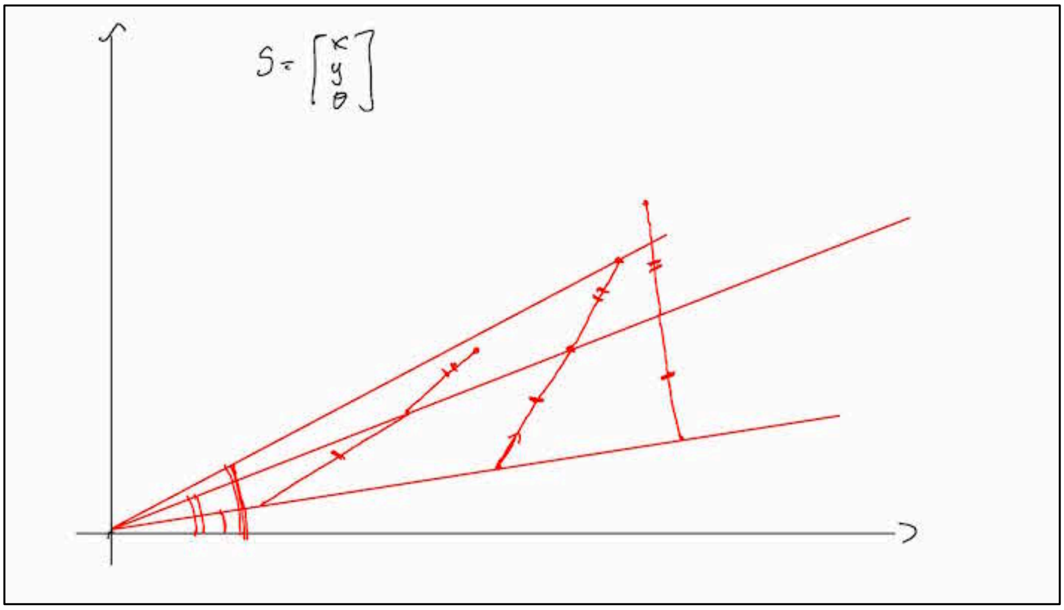
Screen 1: Uniform hallway: intuitively, we cannot estimate the x-coordinate



Screen 2: 2D case with range sensor that provides the distance from the origin



Screen 3: The 2 trajectories (the upper and the lower) provide the same measurements



Screen 4: All the three trajectories provide a different third bearing measurement

3.7. Observability Rank Criterion

OBSERVABILITY RANK CRITERION

HERMANN KRENER 1977

$$\begin{cases} x_{i+1} = x_i + \delta\rho \cos\theta_i \\ y_{i+1} = y_i + \delta\rho \sin\theta_i \\ \theta_{i+1} = \theta_i + \delta\theta \end{cases} \qquad \begin{aligned} \delta\rho &= v\,dt \\ \delta\theta &= \omega\,dt \end{aligned}$$

$$\dot{S} = f_0(S) + \sum_{i=1}^{m} u_i f_i(S)$$

$$\begin{cases} \dot{x} = v\cos\theta \\ \dot{y} = v\sin\theta \\ \dot{\theta} = \omega \end{cases} \qquad f_0(S) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad m = 2 \qquad \begin{aligned} u_1 &= v \\ u_2 &= \omega \end{aligned}$$

$$f_1(S) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \qquad f_2(S) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Screen 1: Continuous time model for a wheeled robot in 2D (the unicycle)

$$\dot{S} = f_0(S) + \sum_{i=1}^{m} u_i f_i(S) \qquad h(S)$$

$$L^0 h = h$$

$m+1$ first order $\qquad L^1_0 h = (\nabla \cdot h)\cdot f_0 \quad --- \quad L^1_i h = (\nabla \cdot h)\cdot f_i$

$(m+1)^2$ second order $\qquad L^2_{ij} h = (\nabla L^1_i h)\cdot f$

$$\Omega^0 = \text{span}\{\nabla h\}$$

Ω^1 = span of the gradients of all the Lie derivatives up to order 1

Screen 2: Lie derivatives and the observable codistribution

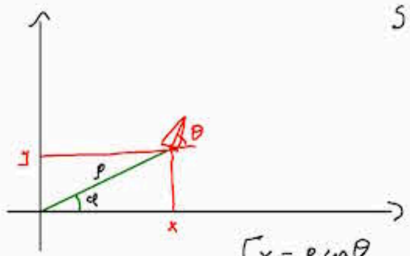
$$\Omega^0 \subset \Omega^1 \subset \Omega^2 \; ---$$

n = dimension of S $\qquad \dim(\Omega^k) \leq n$

$$\dim(\Omega^j) = \dim(\Omega^{j+1}) \leq n$$

Screen 3: Dimension of the observable codistribution

3.8. Applications of the Observability Rank Criterion



$$S=\begin{bmatrix}x\\y\\\theta\end{bmatrix}\qquad\begin{cases}\dot{x}=v\cos\theta\\ \dot{y}=v\sin\theta\\ \dot{\theta}=\omega\end{cases}$$

$$1)\ h(S)=\sqrt{x^2+y^2}=\rho$$

$$2)\ h(S)=\operatorname{atan}\frac{y}{x}=\varphi$$

$$S=\begin{bmatrix}\rho\\\varphi\\\theta\end{bmatrix}\qquad\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}\rightarrow\begin{cases}\rho=\sqrt{x^2+y^2}\\ \theta=\operatorname{atan}\frac{y}{x}\end{cases}$$

$$\dot{S}=f_0(S)+\sum_{i=1}^{m}u_i f_i(S)\qquad u_1=v\quad u_2=\omega$$

$$\begin{cases}\dot{\rho}=v\cos(\theta-\varphi)\\ \dot{\varphi}=\frac{v}{\rho}\sin(\theta-\varphi)\\ \dot{\theta}=\omega\end{cases}\qquad f_0=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad m=2\quad f_1=\begin{bmatrix}\cos(\theta-\varphi)\\\sin(\theta-\varphi)\\0\end{bmatrix}\quad f_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Screen 1: 2D robot (satisfying the unicycle dynamics) in polar coordinates

$$1)\quad f_1=\begin{bmatrix}\cos(\theta-\varphi)\\\frac{\sin(\theta-\varphi)}{\rho}\\0\end{bmatrix}\qquad f_2=\begin{bmatrix}0\\0\\1\end{bmatrix}\qquad h(S)=\rho$$

$$S=\begin{bmatrix}\rho\\\varphi\\\theta\end{bmatrix}\qquad \mathcal{L}^0 h=\rho\qquad\begin{bmatrix}1&0&0\end{bmatrix}$$

$$\mathcal{L}^1_1 h=\cos(\theta-\varphi)\qquad\begin{bmatrix}0,&\sin(\theta-\varphi),&-\sin(\theta-\varphi)\end{bmatrix}$$

$$\mathcal{L}^1_2 h=\emptyset$$

$$\mathcal{L}^2_{11} h=\frac{\sin(\theta-\varphi)^2}{\rho}\qquad\begin{bmatrix}A,&*,&-*\end{bmatrix}$$

$$\mathcal{L}^2_{12} h=-\sin(\theta-\varphi)\qquad \dim(\Omega^1)=2<3$$

$$\dim(\Omega^2)=\dim(\Omega^1)$$

Screen 2: The case when the sensor provides the distance from the origin. The dimension of the observable codistribution is 2<3 and the system is non-observable

2) $f_1 = \begin{bmatrix} \cos(\theta - \varphi) \\ \frac{\sin(\theta - \varphi)}{\rho} \\ 0 \end{bmatrix} \quad f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad h(S) = \varphi$

$$\mathcal{L}^0 h = \varphi \qquad [0 \quad 1 \quad 0]$$

$$\mathcal{L}^1_1 h = \frac{\sin(\theta-\varphi)}{\rho} \qquad \left[-\frac{\sin(\theta-\varphi)}{\rho^2}, \; -\frac{\cos(\theta-\varphi)}{\rho}, \; \frac{\cos(\theta-\varphi)}{\rho} \right]$$

$$\mathcal{L}^1_2 h = \emptyset$$

$$\mathcal{L}^2_{12} h = \frac{\cos(\theta-\varphi)}{\rho} \qquad \left[-\frac{\cos(\theta-\varphi)}{\rho^2}, \; \frac{\sin(\theta-\varphi)}{\rho}, \; -\frac{\sin(\theta-\varphi)}{\rho} \right]$$

$$\dim(\Omega^2) = 3$$

Screen 3: The case when the sensor provides the bearing of the robot. The dimension of the observable codistribution is 3 and the system is observable