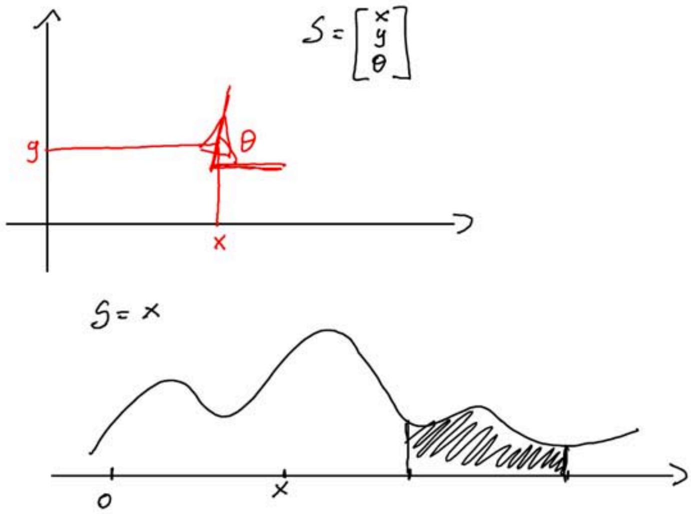
Course Material

Week 2: Bayes & Kalman Filters

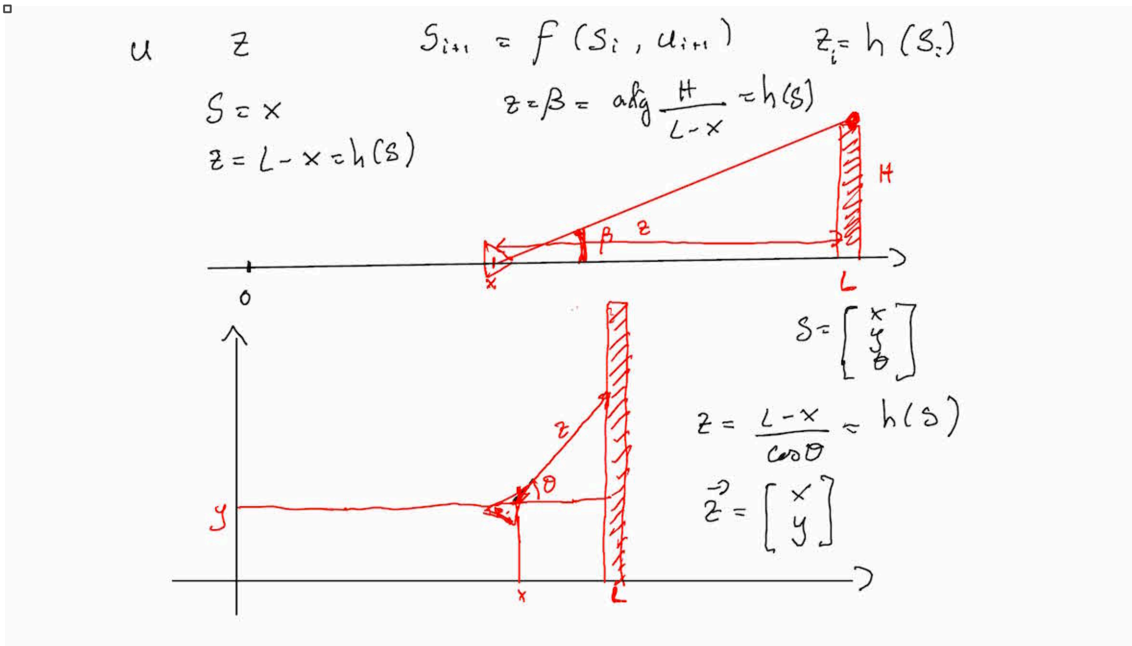
2.1. Localization process in a probabilistic framework: basic concepts



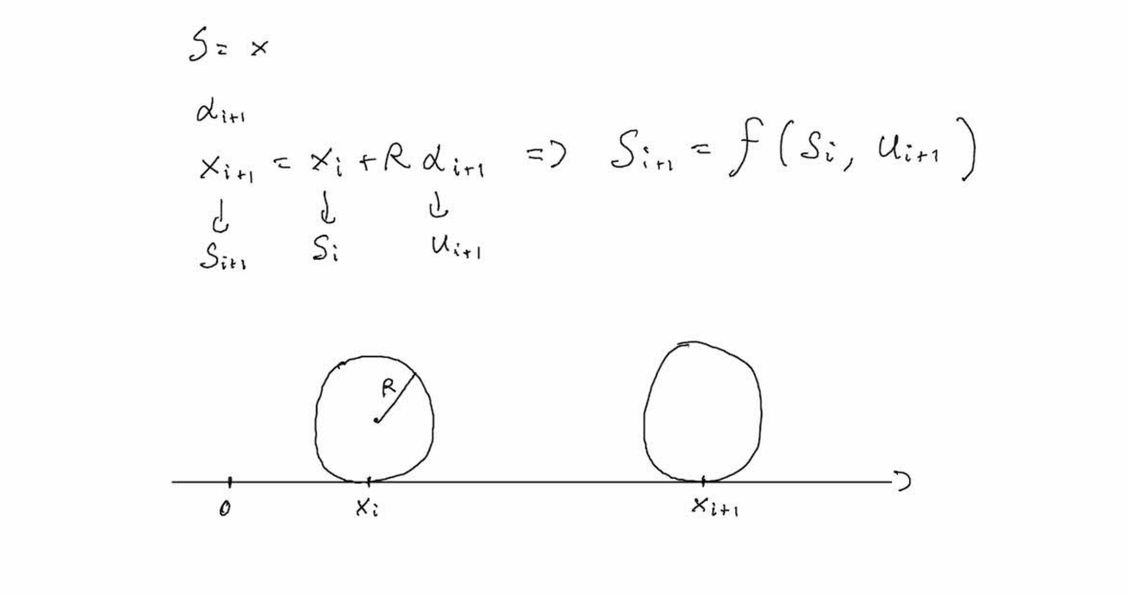
Screen 1: Robot configuration for a wheeled robot (up) and the probability distribution in the case of a robot moving in a 1D space

Screen 2: A qualitative representation of the entire estimation process in 1D

2.2. Characterization of proprioceptive and exteroceptive sensors

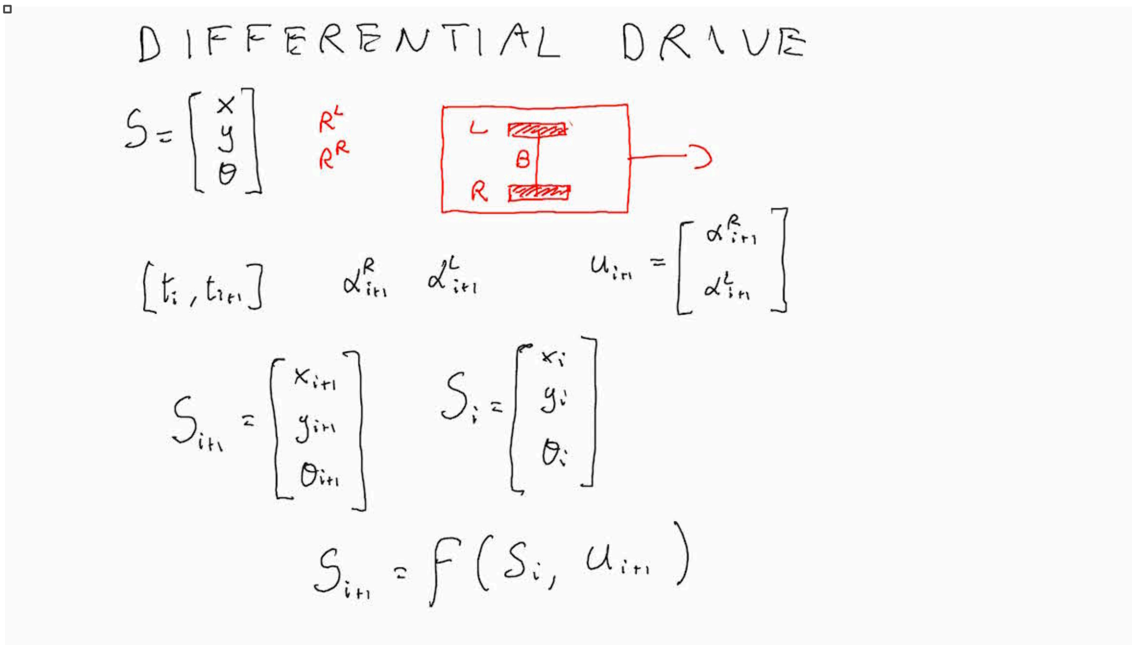


Screen 1: Examples of analytical expressions for exteroceptive sensors

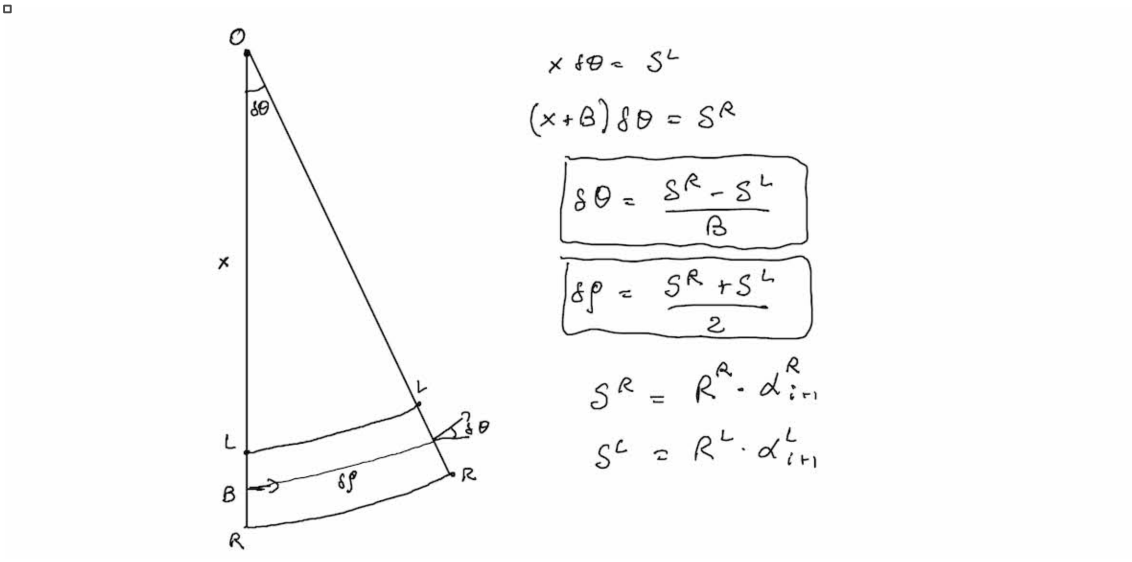


Screen 2: Analytical expression for a proprioceptive sensor in 1D

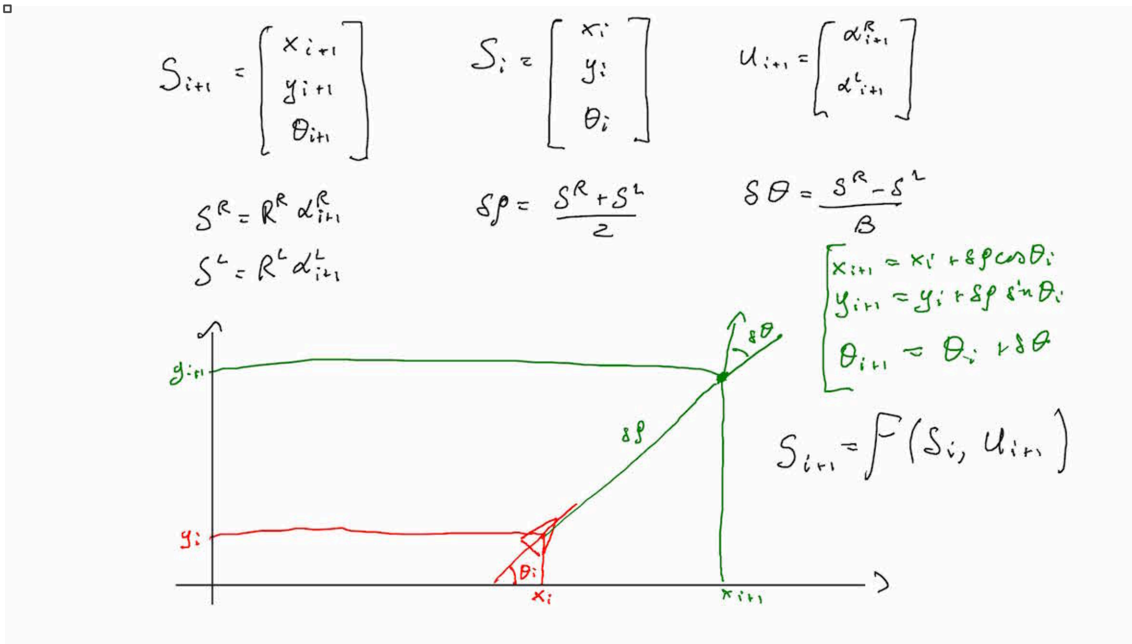
2.3. Wheel encoders for a differential drive vehicle



Screen 1: Differential Drive



Screen 2: Shift and rotation in a differential drive in terms of the rotations of the two wheels



Screen 3: The function that describes the link between the wheel encoders and the robot configuration in a Differential Drive

2.4. Sensor statistical models

$$P(z \mid S) \qquad P(S_{i+1} \mid S_i, u_{i+1})$$

$$z = h(S) + w$$

$$w = N(0, \sigma_w^2) \longrightarrow N\left(\begin{bmatrix}0\\0\\0\end{bmatrix}_n, R\right)$$

$$P(w) = \frac{1}{\sqrt{2\pi}\,\sigma_w} \exp\left(-\frac{w^2}{2\sigma_w^2}\right)$$

$$P(z) = \frac{1}{\sqrt{2\pi}\,\sigma_w} \exp\left(-\frac{[w - h(s)]^2}{2\sigma_w^2}\right)$$

$$P(z) = \frac{1}{(2\pi)^{n/2}\sqrt{\det R}} \exp\left[-\frac{1}{2}[w - h(s)]^T R^{-1} [w - h(s)]\right]$$

Screen 1: Exteroceptive statistical model

$$P(S_{i+1} \mid S_i, u^m_{i+1}) \qquad u = u^m + v$$

$$S_{i+1} = f(S_i, u_{i+1}) \cong f(S_i, u^m_{i+1}) + \frac{\partial f}{\partial u} v$$

$$v = N(0, \sigma_v^2)$$

$$S_{i+1} = N\left(f(S_i, u^m_{i+1}), \left(\frac{\partial f}{\partial u}\right)^2 \sigma_v^2\right)$$

$$v = N\left(\begin{bmatrix}0\\0\\0\end{bmatrix}_n, Q\right) \qquad F_u = \frac{\partial f}{\partial u}$$

$$S_{i+1} = N\left(f(S_i, u^m_{i+1}), F_u Q F_u^T\right)$$

Screen 2: Proprioceptive statistical model

2.5. Reminds on probability

1) MARKOV ASSUMPTION
2) THEOREM OF TOTAL PROBABILITY
3) BAYES THEOREM

$$w_1 \quad w_2 \quad - - - - \quad w_i$$

$$z_i = h(S_i) + w_i$$

$$P(z|S)$$

$$u_i = u_i^m + v_i$$

$$P(S_{i+1} | S_i, u_{i+1}^m)$$

Screen 1: Basic ingredients to derive the Bayes Filter

$$P(A) = \sum_B P(A,B) = \sum_B P(A|B)P(B)$$

$$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

$$P(B|A) = \frac{P(A,B)}{P(A)}$$

$$P(A|B,C) = \frac{P(B|A,C)P(A|C)}{P(B|C)}$$

$$P(A|C) = \sum_B P(A|B,C)P(B|C)$$

Screen 2: Bayes rule and the theorem of total probability

2.6. The Bayes Filter

$$U_i^m = [u_1^m, u_2^m, \ldots, u_i^m] \qquad Z_i = [z_1, \ldots, z_i]$$

$$P(S_i \mid U_i^m, Z_i)$$

FIRST BAYES FILTER EQUATION

$$P(S_{i+1} \mid U_{i+1}^m, Z_i)$$

SECOND BAYES FILTER EQUATION

$$P(S_{i+1} \mid U_{i+1}^m, Z_{i+1})$$

Screen 1: Notation for the Bayes Filter

$$P(A \mid C) = \sum_B P(A \mid B, C)\, P(B \mid C) = \int dB\, P(A \mid B, C)\, P(B \mid C)$$

$$P(S_{i+1} \mid U_{i+1}^m, Z_i) = \int dS_i\, \boxed{P(S_{i+1} \mid S_i, U_{i+1}^m Z_i)}\, P(S_i \mid U_i Z_i),$$

$$P(S_i \mid U_i^m, Z_i)$$

$A \to S_{i+1}$

$$P(S_{i+1} \mid S_i\ u_{i+1}^m)$$

$C \to U_{i+1}^m\ Z_i$

$B \to S_i$

$$P(S_{i+1} \mid U_{i+1}^m\ Z_i) = \int dS_i\, P(S_i \mid U_i^m Z_i)\, P(S_{i+1} \mid S_i, u_{i+1}^m)$$

Screen 2: First equation of the Bayes Filter

$$P(A|B,C) = \frac{P(B|A,C)\,P(A|C)}{P(B|C)}$$

$$P(S_{i+1} | U_{i+1}^m\, Z_{i+1}) = \frac{\boxed{P(z_{i+1} | S_{i+1}\, U_{i+1}^m\, Z_i)}\; P(S_{i+1} | U_{i+1}^m\, Z_i)}{P(z_{i+1} | U_{i+1}^m, Z_i)}$$

$$P(S_{i+1} | U_{i+1}^m, Z_i)$$

$A \to S_{i+1}$

$B \to z_{i+1}$

$C \to U_{i+1}^m\; Z_i$

$$P(z_{i+1} | S_{i+1})$$

$$P(S_{i+1} | U_{i+1}^m\, Z_{i+1}) = \eta\, P(S_{i+1} | U_{i+1}^m\, Z_i)\, P(z_{i+1} | S_{i+1})$$

Screen 3: Second equation of the Bayes Filter

2.7. Grid Localization: an example in 1D

$$P(\ell) = \begin{cases} 0.4 & \text{if } \ell = 3 \\ 0.5 & \text{if } \ell = 4 \\ 0.1 & \text{if } \ell = 5 \\ \emptyset & \text{otherwise} \end{cases} \qquad u^m = 2$$

$$P(S_{i+1} \mid u^m_{i+1}) = \int dS_i \, P(S_i) \, P(S_{i+1} \mid S_i \, u^m_{i+1})$$

$$P(m) = \sum_{\ell} P(\ell) \, P(m \mid \ell, u^m)$$

$$P(m \mid \ell, u^m) = \begin{cases} 0.8 & \text{if } m = \ell + u^m \\ 0.1 & \text{if } m = \ell + u^m + 1 \\ 0.1 & \text{if } m = \ell + u^m - 1 \\ \emptyset & \text{otherwise} \end{cases} \qquad P(m \mid \ell, u^m) = \begin{cases} 1 & \text{if } m = \ell + u^m \\ \emptyset & \text{otherwise} \end{cases}$$

$P(\ell) = \emptyset$ if $\ell \leq 3$

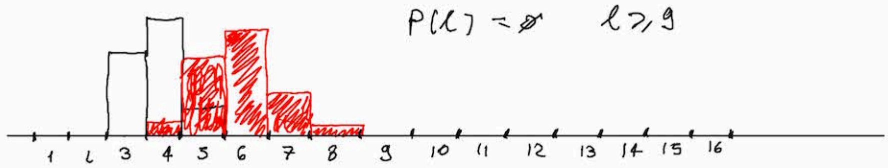
$P(4) = 0.4 \cdot 0.1 = 0.04$

$P(5) = 0.4 \cdot 0.8 + 0.5 \cdot 0.1 = 0.37$

$P(6) = 0.4 \cdot 0.1 + 0.5 \cdot 0.8 + 0.1 \cdot 0.1 = 0.45$

$P(7) = 0.13 \qquad P(8) = 0.01$

$P(\ell) = \emptyset \qquad \ell \geq 9$



Screen 1: Example of grid localization in 1D: Action

$$P(\ell) = \begin{cases} 0 & \ell \leq 3 \,;\ \ell \geq 9 \\ 0.04 & \ell = 4 \\ 0.37 & \ell = 5 \\ 0.45 & \ell = 6 \\ 0.13 & \ell = 7 \\ 0.01 & \ell = 8 \end{cases} \qquad z = 6$$

$$P(S_{i+1} \mid z_{i+1}) = \eta \, P(S_{i+1}) \, P(z_{i+1} \mid S_{i+1})$$

$$P(\ell \mid z) = \eta \, P(\ell) \, P(z \mid \ell)$$

$$P(z \mid \ell) = \begin{cases} 1 & \text{if } \ell = z \\ \emptyset & \text{otherwise} \end{cases}$$

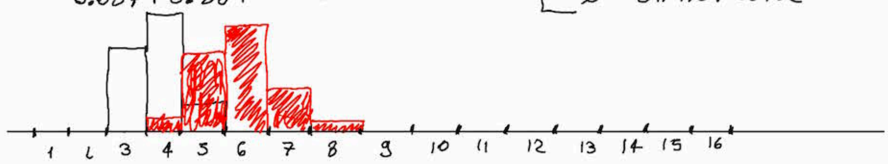
$P(5) = \eta \, 0.37 \cdot 0.1 = \eta \cdot 0.037 \cong 0.09$

$P(6) = \eta \cdot 0.45 \cdot 0.8 = \eta \cdot 0.36 \cong 0.88$

$P(7) = \eta \cdot 0.13 \cdot 0.1 = \eta \cdot 0.013 \cong 0.032$

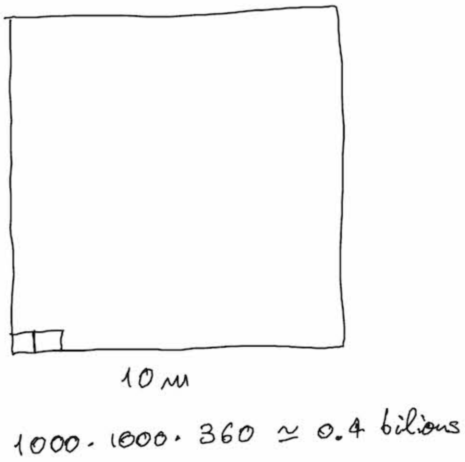
$$P(z \mid \ell) = \begin{cases} 0.8 & \text{if } \ell = z \\ 0.1 & \text{if } \ell = z + 1 \\ 0.1 & \text{if } \ell = z - 1 \\ \emptyset & \text{otherwise} \end{cases}$$

$$\eta = \frac{1}{0.037 + 0.36 + 0.013} = \frac{1}{0.41}$$



Screen 2: Example of grid localization in 1D: Perception

2.8. The Extended Kalman Filter (EKF)



Screen 1: Computational complexity to implement the grid localization

$$S = N(\mu, P) \qquad \text{EXTENDED KALMAN FILTER}$$

$$\mu_a = f(\mu, u^m) \qquad F_x = \left.\frac{\partial f}{\partial S}\right|_{\substack{S=\mu \\ u=u^m}} \qquad F_u = \left.\frac{\partial f}{\partial u}\right|_{\substack{S=\mu \\ u=u^m}}$$

$$P_a = F_x P F_x^T + F_u Q F_u^T$$

$$\mu_a = \mu + P H^T \left[H P H^T + R\right]^{-1} \underbrace{\left(z - \overbrace{h(\mu)}^{\text{PREDICTED OBSERVATION}}\right)}_{\text{ENNOVATION}}$$

$$P_a = P - P H^T \left[H P H^T + R\right]^{-1} H P$$

Screen 2: Equations of the Extended Kalman Filter