

Signal Processing

Reminder

N	continue	discret
1	$\ f\ _1 = \int_{\mathbf{R}} f(t) dt$	$\sum_{n=-\infty}^{+\infty} u_n $
2	$\ f\ _2 = \sqrt{\int_{\mathbf{R}} f(t) ^2 dt}$	$\sqrt{\sum_{n=-\infty}^{+\infty} u_n ^2}$
∞	$\ f\ _{\infty} = \sup_t f(t) $	$\sup_n u_n $
P	$\ f\ _1 = \int_{\mathbf{R}} f(t) ^p dt^{1/p}$	$(\sum_{n=-\infty}^{+\infty} u_n ^p)^{1/p}$

- The uniform convergence, with the ∞ -Norm.
- The absolute convergence, with the 1-Norm
- The convergence in energy, with the 2-Norm

Convolution

Continus time: $y(t) = (h * x)(t) = \int_{\mathbf{R}} h(u)x(t-u)du$
 Discrete time: $y[t] = (h * x)[t] = \sum_{k \in \mathbf{Z}} x[t-k]$
 The convolution is commutative and the neutral element is Dirac distribution $\delta(t)$

The Cross and auto correlation

$(x \star y)(t) = \int_{\mathbf{R}} \overline{x(u)}y(t+u)du = \int_{\mathbf{R}} x(t-u)\overline{y(u)}du$
 The cross correlation is not a commutative operation.
 Let $\tilde{x} = \overline{x(-t)}$, then $(x \star y)(t) = (\tilde{x} \star y)(t)$
 The same think of discret time.

Digital signals are obtained from analog signals by sampling and quantization.

Filter: A filter is a linear time-invariant system. It is perfectly determined by its impulse response h .

Numerical filter with impulse response $h[t]$. The output $y[t]$ of an analog filter with input (also called excitation) $x[t]$ is obtained by :

$$y[y] = (h * x)[t] = \sum_{k \in \mathbf{R}} h[t-k]x[k]$$

Causality: A signal is causal iff its start at (or after) the date $t=0$.

Anti-causality A signal is anti-causal iff its stop a before the date $t=0$.

A signal, or system, which is not causal or anti-causal is acausal.

For systems, causality implies that the output does not depends on the futur of the input.

Stability A signal a stable iff it is summable.

Energy The energy of a signal is its squared 2-norm.

Elementary signals

Numerical Dirac impulse

$$\delta_k[n] = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{otherwise} \end{cases}$$

Heaviside

$$\Theta[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Rectangular window

$$\Pi(t) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Fourier Analysis

Fourier Coefficients(Fourier transform of periodic functions)

$$\hat{f}[n] = c_n(f) = \frac{1}{T} \int_0^T f(t) \exp^{-i\frac{2\pi}{T}nt} dt$$

$\hat{f}[n]$ measure how f is close the the pure frequency $\frac{2\pi n}{T}$.

Spectrum of a periodic signal The spectrum is the set of the squared modulus of the fourier coefficients:
 $spectrum(f) = |\hat{f}[n]|^2$

Propreties

Time shift:

$$\text{for } f_a(t) = f(t+a) \text{ then } \hat{f}_a[n] = \exp^{i\frac{2\pi}{T}na} \hat{f}[n]$$

Derivation:

$$f^{(k)}[n] = (\exp^{i\frac{2\pi}{T}n})^k \hat{f}[n]$$

Fourier Serie

$$S(f)(t) = \sum_{n=-\infty}^{+\infty} \hat{f}[n] \exp^{i\frac{2\pi}{T}nt}$$

Theorem Let $f, g \in L^2([0, T])$ and $\{\hat{f}[n]\}_{n \in \mathbf{Z}}$ and $\{\hat{g}[n]\}_{n \in \mathbf{Z}}$ their respective Fourier coefficients. Then

$$\|f\|^2 = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{+\infty} |\hat{f}[n]|^2$$

$$\langle f, g \rangle = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\bar{g}(t)dt = \sum_{n=-\infty}^{+\infty} \hat{f}[n]\bar{\hat{g}}[n]$$

Fourier Analysis

Coefficients de Fourier trigonométriques

$$a_0(f) = \frac{1}{T} \int_0^T f(t)dt$$

$$a_n(f) = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi}{T}nt\right) dt \quad \forall n \geq 1$$

$$b_n(f) = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi}{T}nt\right) dt \quad \forall n \geq 0$$

$$a_0(f) + \sum_{n=1}^{+\infty} a_n(f) \cos\left(\frac{2\pi}{T}nt\right) + \sum_{n=1}^{+\infty} b_n(f) \sin\left(\frac{2\pi}{T}nt\right)$$

Plancherel-Parseval's Theorem

$$|f|^2 = |a_0(f)|^2 + \frac{1}{2} \sum_{n=1}^{+\infty} |a_n(f)|^2 + \frac{1}{2} \sum_{n=1}^{+\infty} |b_n(f)|^2$$

continuous time

Fourier Transform

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-i2\pi\nu t} dt$$

with ν frequency in Herz and the pulsation $\omega = 2\pi\nu$

Inverse Fourier Transform

$$f(t) = \int_{\mathbf{R}} \hat{f}(\nu)e^{i2\pi\nu t} d\nu$$

Properties

1. Convolution:

$$\widehat{f * g}(\nu) = \hat{f}(\nu)\hat{g}(\nu)$$

2. Multiplication:

$$\widehat{f \cdot g}(\nu) = (\hat{f} * \hat{g})(\nu)$$

3. Translation:

$$\text{let } g_a(t) = f(t-a)$$

$$\hat{g}_a(\nu) = e^{-i2\pi a\nu} \hat{f}(\nu)$$

4. Modulation:

$$\text{let } g_\theta(t) = e^{i2\pi\theta t} f(t)$$

$$\hat{g}_\theta(\nu) = \hat{f}(\nu - \theta)$$

5. Scaling:

$$\text{let } g_s(t) = f(t/s)$$

$$\hat{g}_s(\nu) = |s| \hat{f}(s\nu)$$

6. Time derivation:

$$\widehat{f^{(p)}}(\nu) = (i2\pi\nu)^p \hat{f}(\nu)$$

7. Frequency derivation:

$$\text{let } g_p(t) = (-i2\pi t)^p f(t)$$

$$\hat{g}_p(t) = \hat{f}^{(p)}(\nu)$$

8. Complexe conjugation

$$\widehat{\bar{f}}(\nu) = \overline{\hat{f}(-\nu)}$$

Fourier Analysis

Finite Fourier Transform

Analog Filtering

Bilateral Laplace transform

$$\hat{x}(p) = \int_{-\infty}^{+\infty} x(t)e^{pt} \mathrm{d}t$$

Reference

References