Pivotality, twisted centres, and the anti-double of a Hopf monad

A tale of string diagrams, categories, and monads.

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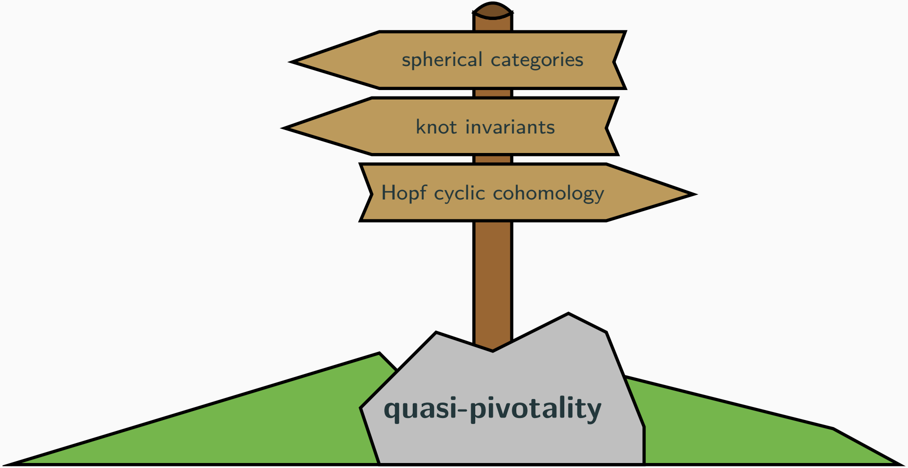
Motivation: Cyclic actions on rigid monoidal categories

Our starting point



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Examples: Set, Vect_k, $[\mathcal{D}, \mathcal{D}]$.

Monoidal categories

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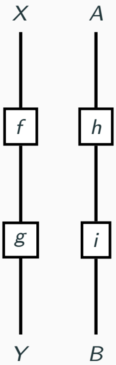
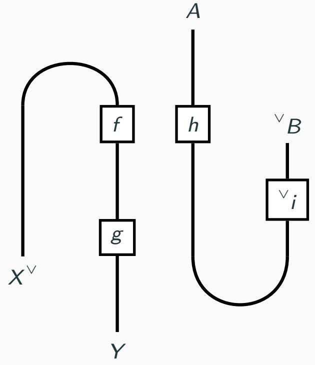


Examples: $(\mathsf{Set}, \times, \{*\})$, $(\mathsf{Vect}_k, \otimes_k, k)$, $([\mathcal{D}, \mathcal{D}], \circ, \mathrm{Id})$.

Rigid categories

We fix a category $\mathcal{C}$ and equip it with the monoidal structure $(\otimes, 1)$, such that *duals* exist.



Examples: $\mathbb{1} \leq (\mathsf{Set}, \times, \{*\})$, $\mathsf{vect}_k \leq (\mathsf{Vect}_k, \otimes_k, k)$, $\mathsf{Ad}^\infty(\mathcal{C}) \leq ([\mathcal{D}, \mathcal{D}], \circ, \mathrm{Id})$.

Module categories

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Bimodule categories

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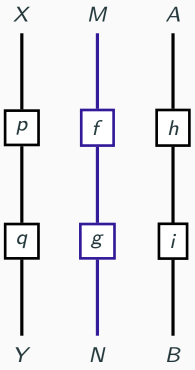
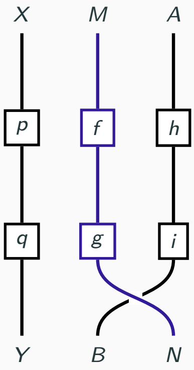


Braided bimodule categories

We consider a second category $\mathcal{M}$ over $\mathcal{C}$ with a left and right action and pass to the *centre* $\mathrm{Z}(\mathcal{M})$.



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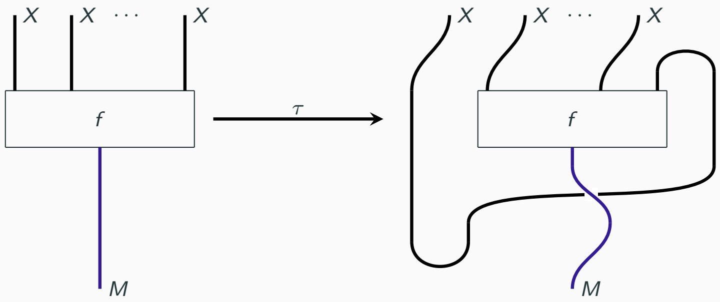
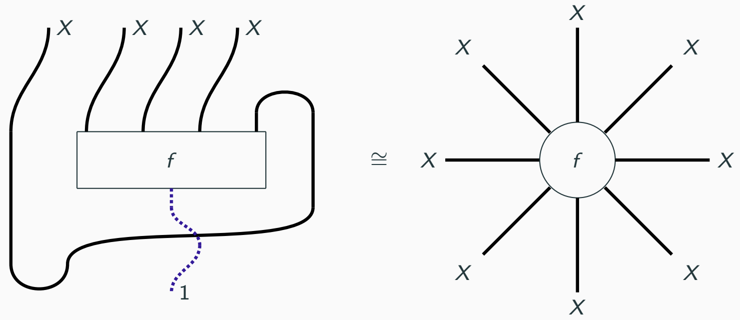
- We can also twist with the bidualising functor on the right.
  $\rightsquigarrow$The *dual of the anti-Drinfeld centre* $\mathrm{A}(\mathcal{C})$.

A cyclic action



The anti-Drinfeld centre induces cyclic permutations.

A cyclic action



If $M = 1$ is 'trivial', we get 'cycles'.

The Hajac–Sommerhäuser theorem

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Theorem (Hajac–Sommerhäuser (unpublished))

For a finite-dimensional Hopf algebra H the following are equivalent:

1. *H is quasi-pivotal,*
2. *The ground field k can be turned into an object of $\mathrm{Q}(H\text{-Mod})$,*
3. *The Hopf and comodule algebras $D(H)$ and $Q(H)$, which parametrise $\mathrm{Z}(H\text{-Mod})$ and $\mathrm{Q}(H\text{-Mod})$, are isomorphic as algebras.*

A proof is given in [Hal21, Theorem 3.4].

Twisted centres and the Hajac–Sommerhäuser theorem for rigid monoidal categories

Quasi-pivotality

Definition

A *quasi-pivotal structure* on a rigid category $\mathcal{C}$ is a pair (β, ρ_β) comprising an invertible object $\beta \in \mathcal{C}$ and a monoidal natural isomorphism

$$\rho_\beta \colon \mathrm{Id}_{\mathcal{C}} \longrightarrow \beta \otimes (-)^{\vee\vee} \otimes \beta^\vee.$$

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We refer to $\mathcal{C}$ as a *quasi-pivotal* category if it admits a quasi-pivotal structure.

The graphical calculus of the anti-Drinfeld centre

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$$\text{red} \cong \mathrm{A}(\mathcal{C}), \qquad \text{black} \cong \mathrm{Z}(\mathcal{C}) \text{ or } \mathcal{C}, \qquad \text{blue} \cong \mathrm{Q}(\mathcal{C}).$$

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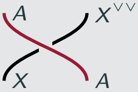
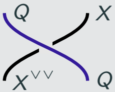
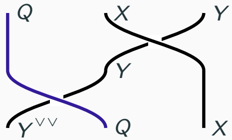
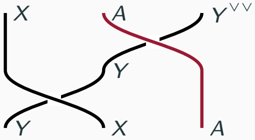
Example

The half-braidings of two objects $A \in \mathrm{A}(\mathcal{C})$ and $Q \in \mathrm{Q}(\mathcal{C})$.

	
The half-braiding $\sigma_{A,X} \colon A \otimes X^{\vee\vee} \to X \otimes A$.	The half-braiding $\sigma_{Q,X} \colon Q \otimes X \to X^{\vee\vee} \otimes Q$.

Gluing of half-braidings

We can glue the half-braidings of objects $X \in \mathrm{Z}(\mathcal{C})$, $A \in \mathrm{A}(\mathcal{C})$, and $Q \in \mathrm{Q}(\mathcal{C})$: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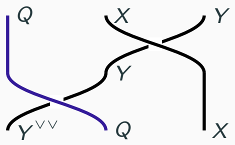	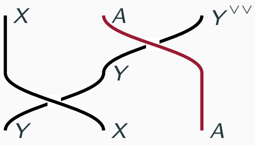
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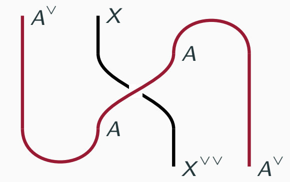
Theorem ([HKRS04, Lemma 2.3], [HZ22, Theorem 4.2])

The tensor product of $\mathcal{C}$ *extends to a left and right action of* $\mathrm{Z}(\mathcal{C})$ *on* $\mathrm{A}(\mathcal{C})$ *and* $\mathrm{Q}(\mathcal{C})$*, respectively.*

The anti-Drinfeld double and duality

The left dual $A^\vee$ of any object $(A, \sigma_{A,-}) \in \mathsf{A}(\mathcal{C})$ is an object of $\mathsf{Q}(\mathcal{C})$ via



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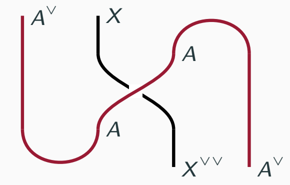

$$\sigma_{A^\vee,X} \colon A^\vee \otimes X \to X^{\vee\vee} \otimes A^\vee.$$

Theorem ([HZ22, Theorem 4.4])

The left dualising functor of $\mathcal{C}$ lifts to an equivalence between $\mathsf{A}(\mathcal{C})$ and $\mathsf{Q}(\mathcal{C})^{\mathrm{op}}$.

Three general observations

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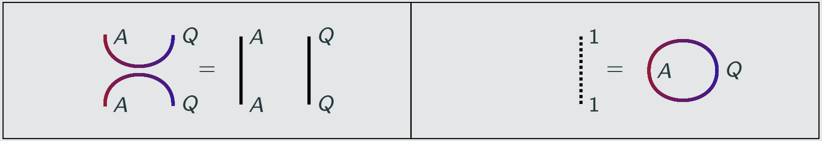
- The functor $- \otimes X$ is an (adjoint) equivalence if and only if X is invertible.

A characterisation of module equivalences $\mathsf{Z}(\mathcal{C}) \longrightarrow \mathsf{A}(\mathcal{C})$

Definition

An object $(A, \sigma_{A,-}) \in \mathsf{A}(\mathcal{C})$ is called $\mathcal{C}$-*invertible* if A is invertible in $\mathcal{C}$.



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Theorem ([HZ22, Theorem 4.6])

Any functor of left $\mathsf{Z}(\mathcal{C})$-modules $F\colon \mathsf{Z}(\mathcal{C}) \longrightarrow \mathsf{A}(\mathcal{C})$ is naturally isomorphic to

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In particular, F is an equivalence if and only if A is $\mathcal{C}$-invertible.

Quasi-pivotality and $\mathcal{C}$-invertible objects of $\mathsf{A}(\mathcal{C})$

Lemma ([HZ22, Lemma 4.11])

The $\mathcal{C}$-invertible elements of $\mathsf{A}(\mathcal{C})$ correspond to quasi-pivotal structures on $\mathcal{C}$.

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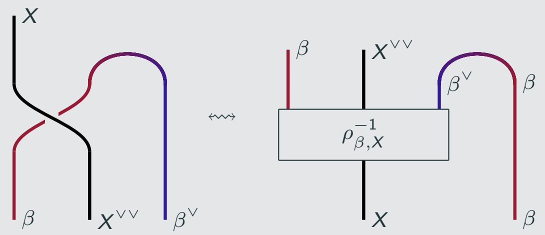
Proof sketch.



Turning $\mathcal{C}$-invertible objects in $\mathsf{A}(\mathcal{C})$ into quasi-pivotal structures on $\mathcal{C}$ and vice-versa.

□

The Hajac–Sommerhäuser theorem for rigid monoidal categories

Theorem ([HZ22, Theorem 4.13])

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Let $\mathcal{C}$ be a strict rigid category. The following are equivalent:

1. *The category $\mathcal{C}$ is quasi-pivotal.*
2. *The class of $\mathcal{C}$-invertible elements of $\mathsf{A}(\mathcal{C})$ is non-empty.*
3. *The categories $\mathsf{Z}(\mathcal{C})$ and $\mathsf{A}(\mathcal{C})$ are equivalent as $\mathsf{Z}(\mathcal{C})$-modules.*

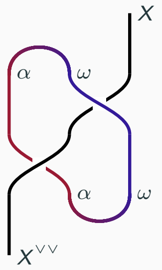
Pivotality arising from pairs in involution

The Picard heap and pivotal structures

Let $A \coloneqq (\alpha, \sigma_{\alpha,-}) \in \mathsf{A}(\mathcal{C})$ be $\mathcal{C}$-invertible.

The Picard heap and pivotal structures

Let $A := (\alpha, \sigma_{\alpha,-}) \in \mathsf{A}(\mathcal{C})$ be $\mathcal{C}$-invertible. We can 'entwine' A with any object $X \in \mathsf{Z}(\mathcal{C})$ in a non-trivial manner, resulting in a morphism from X to its bidual:



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- In fact, this map caries further algebraic information, [HZ22, Theorem 4.22].
- We prove that in general it is neither injective nor surjective, see [HZ22, Remark 4.23, Theorem 4.37].
- These findings are related to the construction of knot invariants via the category of ribbon tangles.

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An important application is the construction of pivotal structures on the Drinfeld centre.

Hopf monads and comodule monads

Diagrammatic bimonads

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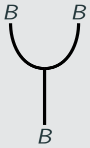	
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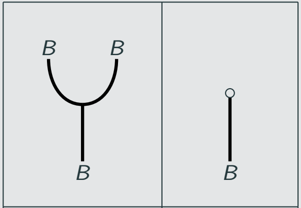
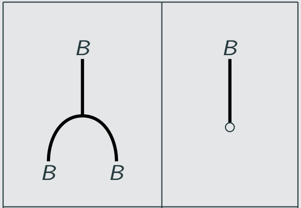


$\mu\colon B^2 \longrightarrow B$ | $\eta\colon \mathrm{Id} \longrightarrow B$



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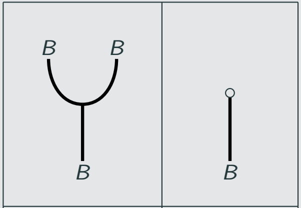

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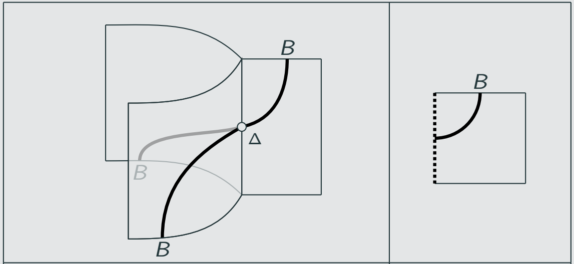
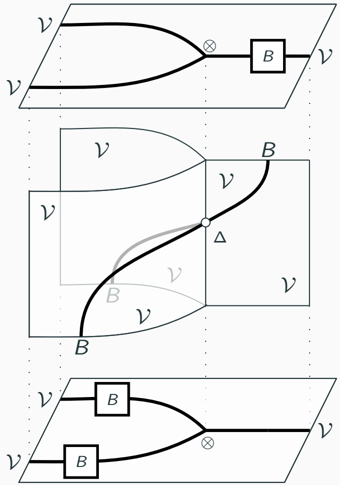


$\Delta\colon B(-\otimes -) \longrightarrow B(-) \otimes B(-)$ $\varepsilon\colon B(1) \longrightarrow 1$

Diagrammatic bimonads: a closer look



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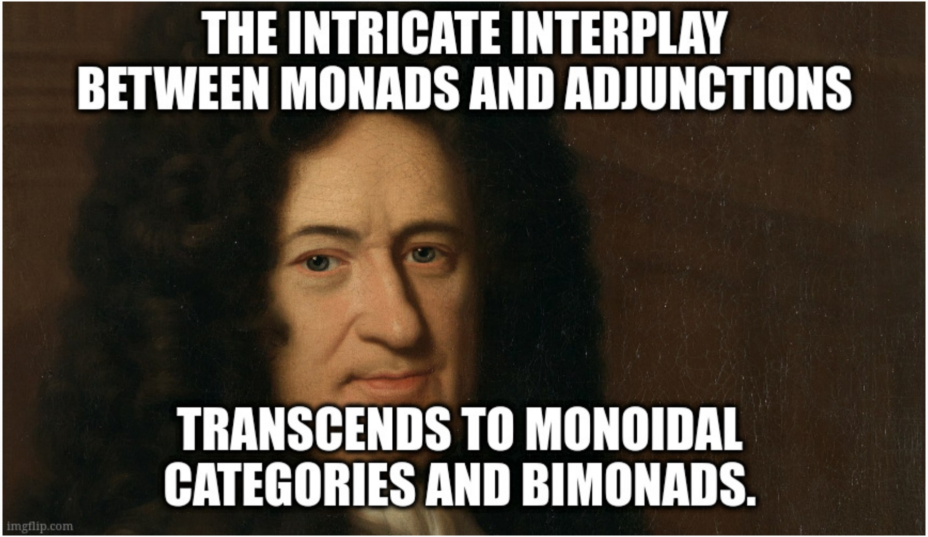
$$
\begin{tikzcd}
& \mathcal{V} \arrow[dl, bend right, "F"'] \arrow[dr, bend left, "F_T"] & \\
\mathcal{D} \arrow[ur, bend right, "U"'] \arrow[rr, bend right, "\Sigma"'] & & \mathcal{V}^T \arrow[ul, bend left, "U_T"]
\end{tikzcd}
$$

Bimonads and adjunctions



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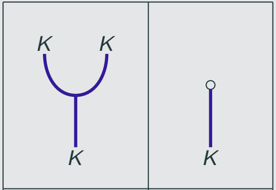
- Any oplax monoidal adjunction $F \colon \mathcal{V} \rightleftarrows \mathcal{D} : U$ induces a bimonad.
- There is a canonical (strict) monoidal comparison functor from $\mathcal{D}$ to $\mathcal{V}^T$.

$$\begin{tikzcd}
& \mathcal{V} \arrow[dl, bend right, "F"'] \arrow[dr, bend left, "F_T"] & \\
\mathcal{D} \arrow[ur, bend right, "U"'] \arrow[rr, bend right, "\Sigma"'] & & \mathcal{V}^T \arrow[ul, bend left, "U_T"]
\end{tikzcd}$$

Comodule monads

Definition

A *comodule monad* $K\colon \mathcal{V} \longrightarrow \mathcal{V}$

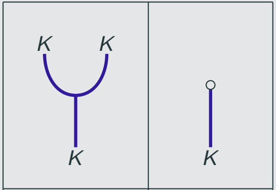


$\mu\colon K^2 \longrightarrow K$	$\eta\colon \mathrm{Id} \longrightarrow K$

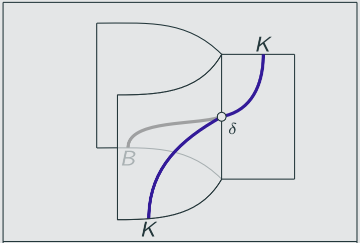
Comodule monads

Definition

A *comodule monad* $K\colon \mathcal{V} \longrightarrow \mathcal{V}$ over a bimonad $B\colon \mathcal{V} \longrightarrow \mathcal{V}$ consists of



$\mu\colon K^2 \longrightarrow K$ | $\eta\colon \mathrm{Id} \longrightarrow K$



$\delta\colon K(- \otimes -) \longrightarrow K(-) \otimes B(-)$

Bimonads and comodule monads are a generalisation of bialgebras and comodule algebras, respectively.

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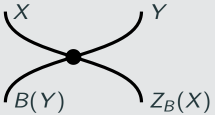
They are intimately related with adjunctions.

The anti-double of a Hopf monad

Centralisable functors and universal coactions

Definition

A *centraliser* of a functor $T\colon \mathcal{V} \longrightarrow \mathcal{V}$ consists of a functor $Z_T\colon \mathcal{V} \longrightarrow \mathcal{V}$ and for all $X, Y \in \mathcal{V}$ a natural transformation called a *universal coaction*



$$\chi_{X,Y}\colon X \otimes Y \longrightarrow B(Y) \otimes Z_B(X),$$

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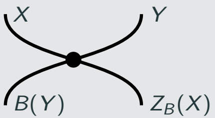

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such that the *extended factorisation property* holds.

Centralisers arise from coends.

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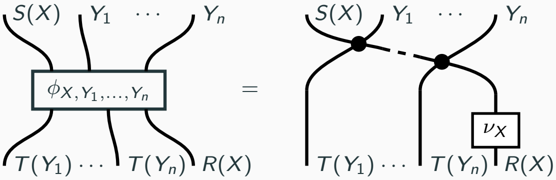

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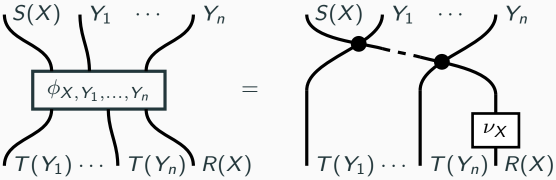

For all functors $S, R\colon \mathcal{D} \longrightarrow \mathcal{V}$, natural numbers $n \in \mathbb{N}$, and natural transformations

$$\phi_{X,Y_1,\dots,Y_n}\colon S(X) \otimes Y_1 \otimes \cdots \otimes Y_n \longrightarrow T(Y_1) \otimes \cdots \otimes T(Y_n) \otimes R(X),$$

there exists a unique natural transformation $\nu\colon Z_T^n(X) \longrightarrow R(X)$ that satisfies the above equation.

Centralisers and the graphical calculus

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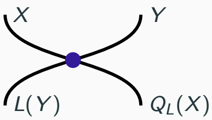
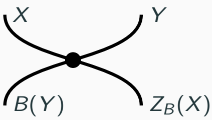
We depict their universal coactions by

	
$\xi_{X,Y}\colon X \otimes Y \to L(Y) \otimes Q_L(X)$,	$\chi_{X,Y}\colon X \otimes Y \to B(Y) \otimes Z_B(X)$.

Bimonad structures on centralisers

Theorem ([BV12, Theorems 5.6 and 5.12, Corollary 5.14])

The centraliser (Z_B, χ) is a bimonad and the centraliser (Q_L, ξ) is a monad.

Bimonad structures on centralisers

Theorem ([BV12, Theorems 5.6 and 5.12, Corollary 5.14])

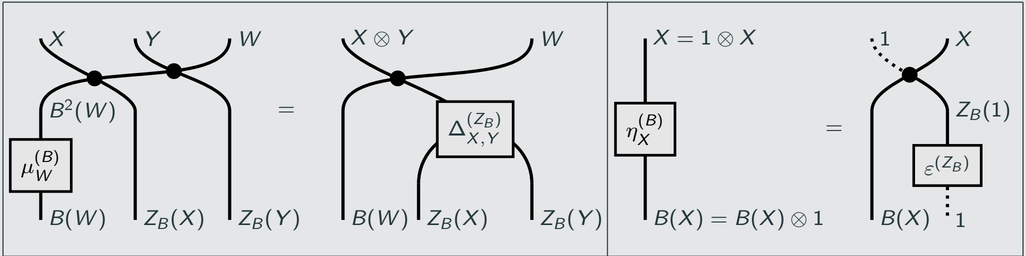
The centraliser (Z_B, χ) *is a bimonad and the centraliser* (Q_L, ξ) *is a monad.*

Their modules are isomorphic to $\mathrm{Z}({}_B\mathcal{V})$ *and* $\mathrm{Z}({}_L\mathcal{V})$ *as monoidal categories and categories, respectively.*

This involves a study of their corresponding comparison functors.

Bimonad structures on centralisers

Proof sketch.



The comultiplication of Z_B	The counit of Z_B

□

Oplax monoidal actions

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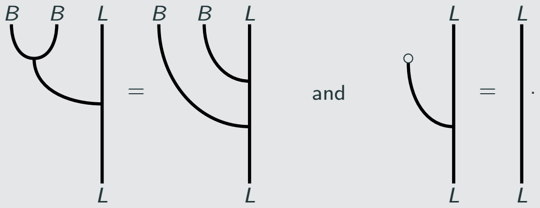

Oplax monoidal actions and module categories

Theorem ([HZ22, Lemma 6.10 and Theorem 6.11])

Let $\alpha\colon LB \longrightarrow L$ *be an oplax monoidal action of* $B\colon \mathcal{V} \longrightarrow \mathcal{V}$ *on* $L\colon \mathcal{V} \longrightarrow \mathcal{V}$.

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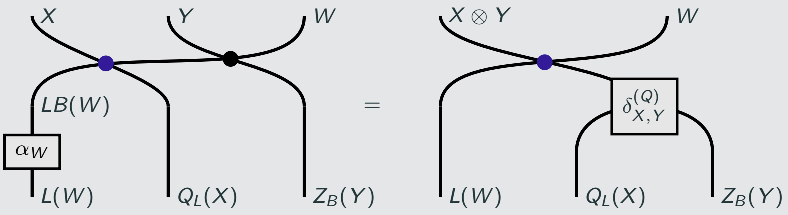
Oplax monoidal actions and module categories

Proof sketch.



The coaction of $Z_B : \mathcal{C} \to \mathcal{C}$ on $Q_L : \mathcal{C} \to \mathcal{C}$

$\square$

An upshot

Given a Hopf monad $H\colon \mathcal{V} \longrightarrow \mathcal{V}$ and suitable choices of $B, L\colon \mathcal{V} \longrightarrow \mathcal{V}$ we can construct:

- A Hopf monad $D(H)\colon \mathcal{V} \longrightarrow \mathcal{V}$, called the *Drinfeld double* of H, which implements the Drinfeld centre of a monoidal category, and

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This involves studying variants of Beck's theory of distributive laws, see [Str72], [BV12, Section 6.2], and [HZ22, Theorem 6.16].

The monadic version of the Hajac–Sommerhäuser theorem

Theorem ([HZ22, Theorem 6.26])

Let $H \colon \mathcal{V} \longrightarrow \mathcal{V}$ be a Hopf monad on a pivotal category that admits a double $D(H)$ and anti-double $Q(H)$. The following statements are equivalent:

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1. *There exists a quasi-pivotal structure on the modules of H.*
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The Drinfeld double and anti-Drinfeld double of a Hopf monad can be used to detect pivotal structures, see [HZ22, Corollary 6.27].

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They are ‘coordinate systems’ of the Drinfeld and anti-Drinfeld centre.

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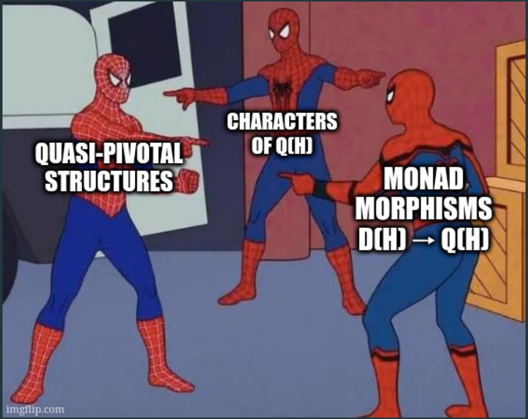

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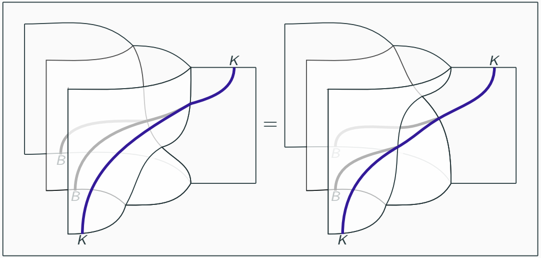

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