Duality in Monoidal Categories

A tale of how details are important sometimes

Based on `arXiv:2301.03545` with Sebastian Halbig.

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The Goal: establishing connections

$$
\begin{array}{c}
\text{Closed monoidal} \\
| \\
\text{Grothendieck–Verdier} \\
| \\
\text{Tensor representable}^{+} \\
| \\
\text{Rigid}
\end{array}
$$

Notions of duality

Closed monoidal categories

Definition

A (strict) monoidal category $(\mathcal{C}, \otimes, 1)$ is called *(left) closed*, if for every $x \in \mathcal{C}$ there exists an adjunction

$$- \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : [x, -]$$

with unit $\eta^x \colon - \Longrightarrow [x, - \otimes x]$ and counit $\varepsilon^x \colon [x, -] \otimes x \Longrightarrow -$.

Example

$(\mathsf{Cat}, \times)$, $(\Bbbk\text{-Mod}, \otimes_\Bbbk)$, $([\mathcal{C}^{\mathrm{op}}, \mathsf{Set}], \star)$, $(\mathbb{S}\text{-Mod}, \circ)$.

(Co)evaluation morphisms in closed categories

Every object x in a closed category C is equipped with natural *(co)evaluation* morphisms

$$\mathrm{coev}_x \coloneqq \eta^x_1 \colon 1 \longrightarrow [x,x], \qquad \mathrm{ev}_x \coloneqq \varepsilon^x_1 \colon [x,1] \otimes x \longrightarrow 1$$

satisfying e.g.,

$$\begin{tikzcd} x \arrow[r, "\eta^x_1 \otimes x"] \arrow[dr, "\mathrm{id}_x"'] & {[x,x] \otimes x} \arrow[d, "\varepsilon^x_x"] \\ & x \end{tikzcd}$$

We need a coherent isomorphism $\phi_x \colon [x,x] \longrightarrow x \otimes [x,1]$.

Rigid monoidal categories

Definition

A closed monoidal category $\mathcal{C}$ is *(left) rigid monoidal* if for every $x \in \mathcal{C}$ there exists a natural isomorphism $\phi^x_y \colon [x, y] \xrightarrow{\sim} y \otimes [x, 1]$, compatible with the evaluation and coevaluation; e.g.,

$$\begin{tikzcd} {[x,x] \otimes x} \arrow[d, "\phi^x_x \otimes x"'] \arrow[dr, "\varepsilon^x_x"] & \\ x \otimes [x,1] \otimes x \arrow[r, "x \otimes \varepsilon^x_1"'] & x \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} x \arrow[r, "\mathrm{coev}_x \otimes x"] \arrow[dr, "\mathrm{id}_x"'] & x \otimes {}^\vee x \otimes x \arrow[d, "x \otimes \mathrm{ev}_x"] \\ & x \end{tikzcd}$$

Example

$(\mathsf{vect}_\Bbbk, \otimes_\Bbbk)$, $\mathsf{Ad}^\infty_{\mathcal{C}} \leq ([\mathcal{C}, \mathcal{C}], \circ)$.

Grothendieck–Verdier categories

Definition

A *Grothendieck–Verdier (also called *-autonomous) category* comprises a monoidal category $\mathcal{C}$ and an object $d \in \mathcal{C}$, such that there exists an equivalence $D\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}$, and a natural isomorphism

$$\mathcal{C}(x \otimes y, d) \cong \mathcal{C}(x, Dy). \qquad \frac{\mathrm{id}\colon Dy \longrightarrow Dy}{\mathrm{ev}_y\colon Dy \otimes y \longrightarrow d}$$

Example

$(\Lambda M, \cap, 0)$, $(\mathsf{Chu}(\mathcal{C}, d), \otimes_{\mathcal{C}}, d)$, $(e\mathcal{C}e, \otimes_{\mathcal{C}}, De)$, $(\mathsf{ban}_{\mathbf{R}}, \otimes_{\mathbf{R}}, \mathbf{R})$.

Tensor representability

Every rigid category yields adjunctions

$$- \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : - \otimes {}^{\vee}x.$$

what can we say about the category $\mathcal{C}$?

Definition

A monoidal category is *tensor representable*, if for every $x \in \mathcal{C}$ there is an adjunction as in Equation (1).

Tensor representability in context

Relationships among different types of duality

$$
\begin{tikzcd}
 & \mathsf{CMCat} & \\
\mathsf{GVCat} \arrow[ur, "{[x,y] := D(x \otimes D^{-1}y)}"] & & \mathsf{TRCat} \arrow[ul] \arrow[ll, dashed, "?"'] \arrow[dl, dashed, bend left, "?"] \\
 & \mathsf{RMCat} \arrow[uu] \arrow[ul, "{}^{\vee}\!- \text{ monoidal}"] \arrow[ur, bend left] &
\end{tikzcd}
$$

Tensor representable categories are Grothendieck–Verdier

Theorem ([HZ23, Theorem 3.2])

If for every $x \in \mathcal{C}$, there exist adjunctions

$$- \otimes Lx \dashv - \otimes x \dashv - \otimes Rx,$$

then $\mathcal{C}$ is Grothendieck–Verdier.

Proof.

The assumption implies that L is a quasi-inverse of R; for example,

$$\mathcal{C}(LRx, y) \cong \mathcal{C}(1, y \otimes Rx) \cong \mathcal{C}(x, y).$$

Set $D := R$. In order to show that $\mathcal{C}$ is a Grothendieck–Verdier category, we need to find some $d \in \mathcal{C}$ such that

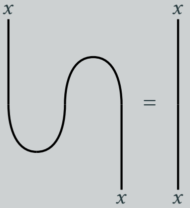
$$\mathcal{C}(y \otimes x, d) \cong \mathcal{C}(y, Dx).$$

$\square$

Is tensor representability and rigidity the same thing?

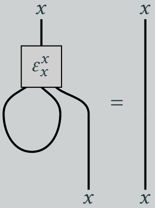
The problem in string diagrams

The rigid case



$(x \otimes \mathrm{ev}_x) \circ (\mathrm{coev}_x \otimes x) = \mathrm{id}_x$

The tensor representable case



$\varepsilon^x_x \circ (\eta^x_1 \otimes x) = \mathrm{id}_x$

A negative result

Theorem ([HZ23, Theorems 2.5 and 4.15])

There exist categories that are tensor representable, but not rigid.

$$\begin{tikzcd}
& \mathsf{CMCat} & \\
\mathsf{GVCat} \arrow[ur] & & \mathsf{TRCat}^{+} \arrow[ul] \arrow[ll] \\
& \mathsf{RMCat} \arrow[uu] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathsf{CMCat} \arrow[d, no head] \\
\mathsf{GVCat} \arrow[d, no head] \\
\mathsf{TRCat}^{+} \arrow[d, no head] \\
\mathsf{RMCat}
\end{tikzcd}$$

Mackey functors

Definition

Let $\Bbbk$ be a field. A *finite-dimensional Mackey functor* M is a coproduct preserving functor $M\colon \mathsf{Sp}_G \longrightarrow \mathsf{vect}_\Bbbk$. such that Mx is finite-dimensional for every $x \in \mathsf{Sp}_G$.

Theorem

The category mky *of finite-dimensional Mackey functors is closed monoidal:*

$$(M \star N)x := \int^{a,b} \Bbbk\mathsf{Sp}_G(a \otimes b, x) \otimes_k Ma \otimes_k Nb,$$

$$[M, N]x := \int_{a,b} \mathsf{vect}_\Bbbk(\Bbbk\mathsf{Sp}_G(x \otimes a, b), \mathsf{vect}_\Bbbk(Ma, Nb)) \cong \mathsf{mky}(M(x^* \otimes -), N).$$

Mackey functors in positive characteristic

Theorem ([HZ23, Theorem 4.15])

The category mky *is tensor representable, but not rigid.*

Proof idea.

1. In $\mathrm{vect}_{\Bbbk}$: $\int_y \int^x \cong \int^x \int_y$.
2. By [Day06]: $\int^x \cong \int_y$.
3. By [PS07, Section 9]: mky is *-autonomous.
4. Calculation: it is even tensor representable.
5. By [Bou05, Lemma 2.2]: mky is rigid iff all short exact sequences split.
6. If the order of G divides the characteristic of $\Bbbk$, then mky is not rigid.

□

Thank you for your attention!

arXiv:2301.03545

[Bou05] Serge Bouc. “**The Burnside dimension of projective Mackey functors**”. English. In: *RIMS Kôkyûroku* 1440 (2005), pp. 107–120. ISSN: 1880-2818.

[Day06] Brian J. Day. “**Compact convolution**”. In: *arXiv e-prints* (2006). arXiv: math/0605463 [math.CT].

[HZ23] Sebastian Halbig and Tony Zorman. “**Duality in Monoidal Categories**”. In: *arXiv e-prints* (2023). arXiv: 2301.03545 [math.CT].

[PS07] Elango Panchadcharam and Ross Street. “**Mackey functors on compact closed categories**”. English. In: *J. Homotopy Relat. Struct.* 2.2 (2007), pp. 261–293. ISSN: 2193-8407.

Spans

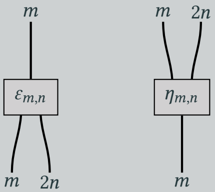
Definition

Let G be a finite group, and write G-set for the category of finite G-sets. The category Sp_G of *spans* of finite G-sets has, as objects, finite G-sets, and as morphisms isomorphism classes of spans

$$\begin{tikzcd} & s \arrow[dl, "l"'] \arrow[dr, "r"] & \\ a & & b \end{tikzcd}$$

The explicit construction

Define a category $\mathcal{D}$ whose objects are natural numbers, with morphisms generated by compositions and tensor products of the identities, as well as



subject to appropriate naturality conditions. Define a *monoidal localisation* $\mathcal{C}$ of $\mathcal{D}$, in which adjunction properties for ε and η hold. By studying strong monoidal functors $\mathcal{C} \longrightarrow \mathsf{vect}_{\Bbbk}$, we find that the snake equation and the identity must lie in different equivalence classes.