ABSTRACT SCHUR FUNCTORS

Reminder: A SCHUR FUNCTOR is a functor

$$V : \mathrm{vect}_k \longrightarrow \mathrm{vect}_k$$

that looks like

$$V = \bigoplus_{n \geq 0} P_n \otimes_{S_n} (-)^{\otimes n}$$

nat trafos as morphisms

where $P_n = 0$ for all but finitely many

⇝ What is the structure of Schur?

↳ one answer: $[\mathcal{U}, \mathcal{U}]_{ps}$

pseudonat transform.

Some kind of initial object.

For the rest of the talk, fix a field k, char $k = 0$.

Def: A k-LINEAR CATEGORY is a category enriched in vect_k.

— " — FUNCTOR is a functor $F : \mathcal{C} \to \mathcal{D}$ s.t.

$$\mathcal{C}(x,y) \longrightarrow \mathcal{D}(Fx, Fy)$$

is linear.

Idea: Categorify the classical situation of sets and
comm. rings

$$\rightsquigarrow \quad U : \mathrm{CRing} \longrightarrow \mathrm{Set}$$

Any polynomial $P \in \mathbb{Z}[x]$ defines a nat trafo

$$\begin{tikzcd} UR \arrow[r] & UR \\ x \arrow[r, mapsto] & P(x) \end{tikzcd}$$

this is nat by inspection

$\rightarrow \mathbb{Z}[x]$ is the free comm. ring on one generator

$$\Rightarrow \exists_{\text{adj.}} \; \begin{tikzcd} F : \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{CRing} \arrow[l, bend left, "U"] \end{tikzcd}$$

„F" $\mathbb{Z}[-]$

s.t. $F(*) = \mathbb{Z}[x] \longleftarrow$ more or less by def.

$$\Rightarrow UR \cong \mathrm{Set}(*, U(R)) \cong \mathrm{CRing}(\mathbb{Z}[x], R)$$

for all CRings R $\Rightarrow U \cong \mathrm{CRing}(\mathbb{Z}[x], -)$.

$$\Rightarrow U(\mathbb{Z}[x]) \cong \mathrm{CRing}(\mathbb{Z}[x], \mathbb{Z}[x])$$

Yoneda $\longrightarrow$

$$\cong [\mathrm{CRing}(\mathbb{Z}[x], -), \mathrm{CRing}(\mathbb{Z}[x], -)]$$

$$\cong [U, U].$$

Closed under absolute colims

Def: A cat. $\mathcal{L}$ is CAUCHY COMPLETE if it has biproducts and every idempotent splits.

pseudo abelian, Karoubian

$e^2 = e$

$\exists\, s : b \to a$
$r : a \to b$

$$\begin{tikzcd} a \arrow[loop left, "e"] \arrow[r, bend left, "r"] & b \arrow[loop right, "id_b"] \arrow[l, bend left, "s"] \end{tikzcd}$$

Def: A 2-RIG is a symmetric monoidal linear Cauchy complete cat.

$\otimes$ is bilinear

Ex: ·) $vect_k$, f.d. group graded vector spaces

·) bounded chain complexes (f.d. vect)

·) for $k = \mathbb{R}$ or $\mathbb{C}$: f.d. vector bundles over a space

or smooth ones over a manifold

not an abelian category!

Rmk: If $\mathcal{L}$ is a linear cat, its CAUCHY COMPLETION is of interest

$$\mathcal{L} \longrightarrow \overline{\mathcal{L}} .$$

It is formed by the full subcat of $\hat{\mathcal{L}}$ of functors that are direct sums of retracts of representables

$$id : \bigoplus_x \mathcal{L}(-,x) \xrightarrow{i} F \xrightarrow{r} \bigoplus_x \mathcal{L}(-,x)$$

Prop: There is a monoidal equiv.

$$\underbrace{[\mathbb{S}, vect_k]}_{\mathbb{S}\text{-mod.}} \simeq_{\otimes} \underbrace{[k\mathbb{S}, vect_k]^{lin}}_{lin.\ \mathbb{S}\text{-mod}}$$

□: Define a functor

$$\begin{aligned} A : \mathbb{S}\text{-mod} &\longrightarrow \mathbb{S}\text{-mod}^{lin} \\ F &\longmapsto F^{lin} \\ \eta &\longmapsto \eta \end{aligned}$$

$\forall f \in \mathbb{S}(a,b)$

$F^{lin}_x = Fx$
$F^{lin}_f = Ff$

⤳ A is the identity on morphisms, so it's faithful

nat trafos between $\mathbb{S}$-modules F and G can naturally be endowed with a linear struct. coming from $Hom_k(Fx, Gx)$:

For $\alpha, \beta : F \Rightarrow G$ and $\lambda \in k$, we have

$$\begin{aligned} (\lambda\alpha + \beta)_x : Fx &\longrightarrow Gx \\ v &\longmapsto \lambda\alpha_x v + \beta_x v \end{aligned}$$

⇒ For $f \in \mathbb{S}(n,m)$ we have

$$((\lambda\alpha + \beta)_m \circ Ff)(v) = \lambda(\alpha_m(Ff(v))) + \beta_m(Ffv)$$

$$= \lambda(Gf\,\alpha_n(v)) + Gf\beta_n(v) = Gf(\lambda\alpha_n + \beta_n)(v)$$

⤳ A is full: Let $\alpha^{lin} : F^{lin} \Rightarrow G^{lin}$ be a nat tr.

⤳ obtain α by pointwise restricting to the canonical bases.

⤳ $A\alpha = \alpha^{lin}$

⤳ A is e.s.: having $F^{lin} : k\mathbb{S} \longrightarrow vect_k$ we can restrict to bases on morphisms to obtain F

$\Rightarrow AF = F^{lin}$.

For the monoidal str. just compare

$$F * G\,(k) := \int^{n,m} \mathbb{S}(n+m,k) \cdot Fn \otimes_k Gm$$

$k\mathbb{S}(n+m,k) \otimes_k Fn$

and

$$F^{lin} * G^{lin}\,(k) := \int^{n,m} k\mathbb{S}(n+m,k) \otimes_k Fn^{lin} \otimes_k Gm^{lin}$$

■

It is a general fact that given an equivalence

$$F : \mathcal{L} \longrightarrow \mathcal{U}$$

with $\mathcal{L}$ monoidal, one can equip $\mathcal{U}$ with a mon str.

↳ $x \otimes_{\mathcal{U}} y := F(F^{-1}x \otimes F^{-1}y)$, $1 := F^{-1}1$

Prop: As categories, polynomial species are equiv to the Cauchy compl. $\overline{k\mathbb{S}}$ of $k\mathbb{S}$

⇒ The above equiv restricts to "the finite case"

□: A polynomial species looks like

$$\bigoplus_{i=0}^{n} P_i \quad , \quad P_i(j) = 0 \quad \text{for } i \neq j$$

Maschke: each P_i is the retract of a finite sum of $\underbrace{k[S_i]}_{}$

$$\Leftrightarrow k\mathbb{S}(-,i)$$

so $\overline{k\mathbb{S}}$ really are just linear species.

■

In fact, by the same trick as above, we have

$$\overline{k\mathbb{S}} \simeq Poly$$

as (linear) monoidal cats, again by Day conv.

objects: $f : x \to y$

morphs: $$\begin{tikzcd} x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$$

Def: A 2-CATEGORY $\mathbb{B}$ comprises...

- A collection of objects ob $\mathbb{B}$
- for $x, y \in$ ob $\mathbb{B}$, a hom-category $\mathbb{B}(x,y)$
- for $x, y, z \in$ ob $\mathbb{B}$, a composition (horizontal)
$$\otimes_y : \mathbb{B}(y,z) \times \mathbb{B}(x,y) \longrightarrow \mathbb{B}(x,z)$$
- for $x \in$ ob $\mathbb{B}$, an identity 1-cell $\mathbb{1}_x \in \mathbb{B}(x,x)$

such that assoc. and unitality axioms hold on the nose.

Alternatively we want coherent nat isos

e.g. $\alpha : P \otimes_y (Q \otimes_x M) \longleftarrow (P \otimes_y Q) \otimes_x M$

in $\mathbb{B}(w,z)$

satisfying a pentagon!

$$\begin{tikzcd} & x \arrow[r, "Q"] & y \arrow[dr, "M"] & \\ w \arrow[ur, "P"] \arrow[dr, "P"'] & & \Downarrow \alpha & z \\ & x \arrow[r, "Q"'] & y \arrow[ur, "M"'] & \end{tikzcd}$$

$\rightsquigarrow$ A 2-cat is something like a multi object monoidal cat.

↳ Indeed, if $\mathbb{B}$ is a 2-caty then $(\mathbb{B}(x,x), \otimes_x, \mathbb{1}_x)$ is a mon caty for all $x \in$ ob $\mathbb{B}$.

One can now try to develop all of the fun concepts from 1-caty theory in this context!

↳ e.g., a monad in $\mathbb{B}$ is a monoid in $(\mathbb{B}_x^x, \otimes_x, \mathbb{1}_x)$

2-cat

Ex: (i) Cat in the usual way → MonCat → OpMonCat   ModCat

(ii) For a caty $\mathcal{L}$ with pullbacks, $\mathrm{Sp}\mathcal{L}$ is a "real" bicaty

bicat

- $\mathrm{ob}\,\mathrm{Sp}\mathcal{L} = \mathrm{ob}\,\mathcal{L}$
- $x \rightsquigarrow y$ = $$\begin{tikzcd} & S \in \mathcal{L} \arrow[dl, "\alpha"'] \arrow[dr] & \\ x & & y \end{tikzcd}$$
  ↳ Composition is by pullback
- $$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[dd] & \\ x \arrow[r, phantom, "\circlearrowleft"] & \arrow[r, phantom, "\circlearrowleft"] & y \\ & t \arrow[ul] \arrow[ur] & \end{tikzcd}$$ 2-cells.

to turn this into a 1-cat one has to form isoclasses

as strict as $\mathcal{V}$

(iii) If $\mathcal{V}$ is a mon caty with coproducts and $\otimes$ pres. them in each var., we can form matrices over $\mathcal{V}$:

- obj. : (finite) sets
- $I \longrightarrow J$: a functor $I \times J \longrightarrow \mathcal{V}$
  ↳ composition of $$I \xrightarrow{M} J \xrightarrow{N} K$$ is $$(N \otimes_J M) := \sum_{j \in J} N_{jk} \otimes M_{ij}$$
- 2-cells are nat trafos

In fact: $\mathrm{Sp}(\mathrm{FinSet}) \simeq \mathrm{Mat}(\mathrm{FinSet})$

bicat

(iv) $\mathbb{B}\mathrm{im}$: ·) objects are rings

- $R \xrightarrow{M} S \iff M \in R\text{-mod-}S$
  ↳ comp. is tensor product of modules
- 2-cells are module morphisms

bicat

(v) More generally, take categories as objects, functors $\mathcal{L}^{op} \times \mathcal{V} \to \mathrm{Set}$ as morphs and nat trafos as 2-cells
↳ Day convolution is composition.

as str. as $\mathcal{L}$

(vi) Any mon caty can be seen as a 2-caty via delooping

2-cat

(vii) Any caty is a 2-caty with identity 2-cells

2-cat

(viii) For $\mathcal{V}$ symm. closed monoidal bicomplete
↳ $\mathcal{V}$-Cat is a 2-category
↳ e.g. ordinary catys ($\mathcal{V}$ = Set), 2-categories ($\mathcal{V}$ = Cat), dg-cats ($\mathcal{V}$ = chain compls), preorders ($\mathcal{V} = \cdot \longrightarrow \cdot$)

(ix) For $\mathcal{V}$ having at least finite limits, $\mathbb{C}\mathrm{at}(\mathcal{V})$ = internal catys form a 2-caty.

(x) $\mathbb{R}\mathrm{el}$: ·) objects are sets

bicat

- $\mathcal{X} \nrightarrow \mathcal{Y}$ = mono $R \rightarrowtail \mathcal{X} \times \mathcal{Y}$ $\subseteq$ ← we are in Set !
- $R \Rightarrow S \iff R \subseteq S$ in $\mathcal{X} \times \mathcal{Y}$ as a subobject
  ↳ $$\begin{tikzcd} & R \arrow[dl] \arrow[dr] \arrow[dd] & \\ \mathcal{X} \arrow[r, phantom, "\circlearrowleft"] & \arrow[r, phantom, "\circlearrowleft"] & \mathcal{Y} \\ & S \arrow[ul] \arrow[ur] & \end{tikzcd}$$ ⇝ at most one 2-cell per R and S.
- Composition is given by $$x(R \circ S)y \iff \exists y\; xRySz$$

Def: Let $\mathbb{B}$ and $\mathbb{C}$ be 2-categories. A 2-FUNCTOR

$$(F, F_2, F_0) : \mathbb{B} \longrightarrow \mathbb{C}$$

comprises an object assignment $F : \mathrm{ob}\,\mathbb{B} \to \mathrm{ob}\,\mathbb{C}$ and a fam. of functors

$$F_{xy} : \mathbb{B}(x,y) \longrightarrow \mathbb{C}(Fx, Fy)$$

and $$F^2_{MN} : FM \otimes_{Fy} FN = F(M \otimes_y N)$$

$$F^0_x : 1_{Fx} = F 1_x$$

$(M : y \to z,\ N : x \to y \quad 1_x : x \to x)$

⤳ The more general notion of bifunctor (pseudofunctor) where F^2 and F^0 are lax (invertible) and coherent also exists.

↳ coherence includes lax associativity, as well as left and right unitality.

such that

e.g.:

$$\begin{tikzcd}
FM \otimes_{Fy} FN \otimes_{Fz} FL \arrow[r] \arrow[d] & FM \otimes_{Fy} F(N \otimes_z L) \arrow[d] \\
F(M \otimes_y N) \otimes_{Fz} FL \arrow[r] & F(M \otimes_y N \otimes_z L)
\end{tikzcd}$$

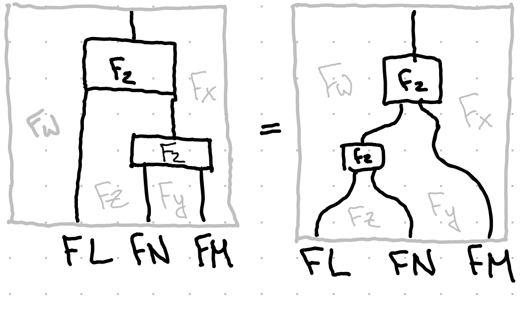
or, as a pasting diagram

$$\begin{tikzcd}
Fz \arrow[r, "FN"] \arrow[dr] & Fy \arrow[d, "FM"] \arrow[dl, phantom, "\Swarrow F_2"] \\
Fw \arrow[u, "FL"] \arrow[r] \arrow[r, bend right=50, "\Downarrow \alpha"'] \arrow[ur, phantom, "\Downarrow F_2"] & Fx
\end{tikzcd}
=
\begin{tikzcd}
Fz \arrow[r, "FN"] \arrow[dr, phantom, "\Searrow F_2"] & Fy \arrow[d, "FM"] \\
Fw \arrow[u, "FL"] \arrow[r] \arrow[ur] & Fx \arrow[ul, phantom, "\Downarrow F_2"]
\end{tikzcd}$$

Thm (Pasting): Every pasting diagram has a uniqe composite.

↗ So two pastings can be equal by being so as 2-cells.

Better yet, we have string diagrams



⤳ In bicats, you need to indicate the parens somehow.

⟹ monoidal bicats become fun.

⤳ 3 dimensional string diags

Def: Suppose that $F, G : \mathbb{B} \longrightarrow \mathbb{C}$ are 2-funs an ICON comprises

Identity component oplax natural

- the assertion (really) that $Fx = Gx$ on objects,
- a nat. transformation

$$\phi_{xy} : \mathbb{B}(x,y) \longrightarrow \mathbb{C}(Fx, Fy) = \mathbb{C}(Gx, Gy)$$

satisfying e.g.

$$\begin{tikzcd}
FM \otimes_{Fy} FN \arrow[r, "\phi_M \otimes \phi_N"] \arrow[d, "F_{2,M,N}"'] & GM \otimes_{Gy} GN \arrow[d, "G_{2,M,N}"] \\
F(M \otimes_y N) \arrow[r, "\phi_{M \otimes_y N}"'] & G(M \otimes N)
\end{tikzcd}$$

plus some unitality cond.

⇝ Why ?

⇒ There is a 2-category of

- bicatys
- bifunctors
- Icons

} Study 2-categories 2-categorically.

Internal things become global defs

Def: Suppose that $F, G : \mathbb{B} \longrightarrow \mathbb{C}$ are 2-functors A PSEUDONATURAL TRANSFORMATION

$$\phi : F \Rightarrow G$$

comprises a family $\{\phi_x : Fx \longrightarrow Gx\}_{x \in \mathbb{B}}$

$$\begin{tikzcd}
Fx \arrow[r, "\phi_x"] \arrow[d, "FM"'] & Gx \arrow[d, "GM"] \arrow[dl, phantom, "\swarrow \phi_M"] \\
Fy \arrow[r, "\phi_y"'] & Gy
\end{tikzcd}$$

s.t. e.g.

$$\begin{tikzcd}
& Fx \arrow[r] \arrow[dl] & Fy \arrow[dl] \arrow[dr] \arrow[d, phantom, "\Leftarrow"] & & \\
Gx \arrow[r] \arrow[drr, bend right] & Gy \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \arrow[r, phantom, "\Leftarrow"] & Fz \arrow[dl] \\
& \, & Gz &
\end{tikzcd}$$

$$=$$

$$\begin{tikzcd}
Fx \arrow[r] \arrow[drr] \arrow[dd] & Fy \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
& \, \arrow[d, phantom, "\Downarrow"] & Fz \arrow[dl] \\
Gx \arrow[r] & Gz &
\end{tikzcd}$$

and some other axioms.

on components

$$\begin{tikzcd} Fx \arrow[r, bend left=50, "\eta_x"] \arrow[r, bend right=50, "\sigma_x"'] \arrow[r, phantom, "\Downarrow \Gamma_x"] & Gx \end{tikzcd}$$

Def: A MODIFICATION is

$$\begin{tikzcd} \mathbb{B} \arrow[rr, bend left=40, "F"] \arrow[rr, bend right=40, "G"'] \arrow[rr, phantom, "\eta \Downarrow \overset{\Gamma}{\Rrightarrow} \Downarrow \sigma"] & & \mathbb{C} \end{tikzcd}$$

such that

$$\begin{tikzcd} Fx \arrow[r, "FM"] \arrow[d, "\eta_x"'] \arrow[dr, phantom, "\overset{\sim}{\Rightarrow}\ \eta_M"] & Fy \arrow[d, bend right=50, "\eta_y"'] \arrow[d, bend left=50, "\sigma_y"] \arrow[d, phantom, "\overset{\Gamma_y}{\Rrightarrow}"] \\ Gx \arrow[r, "GM"'] & Gy \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, "FM"] \arrow[d, bend right=50, "\eta_x"'] \arrow[d, bend left=50, "\sigma_x"] \arrow[d, phantom, "\overset{\Gamma_x}{\Rrightarrow}"] \arrow[dr, phantom, "\overset{\sim}{\Rightarrow}\ \sigma_M"] & Fy \arrow[d, "\sigma_y"] \\ Gx \arrow[r, "GM"'] & Gy \end{tikzcd}$$

Can even be a bicat

needs to be 2-cat for the whole thing to be.

⇝ Why?

↳ There is a 2-catg Bicat$(\mathbb{B}, \mathbb{C})$

↳ lax functors, lax trafos, and mods.

↳ There is a tricategory $\mathbb{B}$icat but it's very hard to handle.

Thm: There exists a 2-categorical Yoneda emb.

$$\begin{tikzcd} \mathbb{B} \arrow[r, hook, "\text{よ}_{\mathbb{B}}"] & \mathrm{Str}(\mathbb{B}^{op}, \mathbb{C}\mathrm{at}) \\ x \arrow[r, mapsto] & \mathbb{B}(-, x) \end{tikzcd}$$

Pseudofunctor for bicats
2-functor for 2-catys

$$\begin{tikzcd} \mathbb{B}(M,x) \otimes \underline{\mathbb{B}(N,x)} \arrow[d] \\ \mathbb{B}(M \otimes N, x) \end{tikzcd}$$

$a \xrightarrow{N} b \xrightarrow{M} c$

strict/strong transformations also exist, 2-cells again come from assoc.

$\mathbb{B}(M,x) : \mathbb{B}(c,x) \longrightarrow \mathbb{B}(b,x)$
$f \longmapsto f \otimes M$

$\mathbb{B}(N,x) : \mathbb{B}(b,x) \longrightarrow \mathbb{B}(a,x)$
$g \longmapsto g \otimes N$

$\mathbb{B}(M \otimes N, x) : \mathbb{B}(c,x) \longrightarrow \mathbb{B}(a,x)$
$h \longmapsto h \otimes M \otimes N$

every よ_{xy} is

Further, よ is locally ff and e.s

$\Rightarrow$ locally an equivalence

↳ i.e. $\text{よ}_{xy} : \mathbb{B}(x,y) \xrightarrow{\sim} \mathrm{Str}(\mathbb{B}(x,z), \mathbb{B}(y,z))$

Thm (Yoneda Lemma): Let $F : \mathbb{B}^{op} \longrightarrow \mathbb{C}\mathrm{at}$ be a pseudofunctor. Then

$$e : \mathrm{Str}(\text{よ}, F) \Longrightarrow F$$

is an invertible pseudonat. transformation.

Comprising

$e_x : \mathrm{Str}(\text{よ}_x, F) \longrightarrow Fx$ equiv. of cats

$$\begin{tikzcd} e_f : & \mathrm{Str}(\text{よ}_x, F) \arrow[r, "(\text{よ}_f)^*"] \arrow[d, "e_x"'] \arrow[dr, phantom, "\overset{}{\Rightarrow}\ e_f"] & S(\text{よ}_y, F) \arrow[d, "e_y"] \\ & Fx \arrow[r, "Ff"'] & Fy \end{tikzcd}$$

R = dg-alg

Construct 2-Cat $\mathbb{C}$

- ·) obj. R-moduln
- ·) 1-cells: morphs of dg modules
- ·) 2-cells: homotopies

Now M = dg-bimod. $^{R-R}$

↳ $M : \mathbb{C} \longrightarrow \mathbb{C}$
$x \longmapsto M \otimes_R x$
total grading

EXAMPLE MODIFICATION

$$\begin{tikzcd} \mathbb{B} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow \eta"] & \mathbb{B} \end{tikzcd}$$

$M : \mathbb{C} \longrightarrow \mathbb{C}$

$f : M \Rightarrow M$

$f_x : M_x \longrightarrow M_x$

$g_x : M_x \longrightarrow M_x$

Consider the 2-caty 2-Rig: strong!

- obj = small 2-rigs
- symmetric mon k-linear functors
- Symm. mon k-linear nat transformations

strong?

Our goal is to study the forgetful 2-Functor

$$\mathcal{U} : 2\text{-}\mathrm{Rig} \longrightarrow \mathrm{Cat}$$

Def: For any polynomial species $\rho : \mathbb{S} \longrightarrow \mathrm{vect}_k$ and any 2-rig $\mathcal{R}$ there is a functor

$$F_{\rho,\mathcal{R}}(x) := \bigoplus_{n\geq 0} \rho(n) \otimes_{S_n} x^{\otimes n}$$

monoid in vect_k

right $k[S_n]$ module

$$\rightsquigarrow F_{\rho,\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{R}$$

$\mathbb{S}_\rho^{\mathcal{R}}$ in script

How does this make sense?

Prop (Prop. VI.6.1): vect_k is the initial 2-rig.

$\hookrightarrow k \longmapsto \mathbb{1} \in \mathcal{R}$

$\Rightarrow k^n \longmapsto \underbrace{\mathbb{1} \oplus \cdots \oplus \mathbb{1}}_{n}$

(Sym. mon since $\mathbb{1}^{\oplus n} \otimes \mathbb{1}^{\oplus m} \cong \underbrace{\mathbb{1}^{\oplus n \cdot m}}_{\mathbb{1}\otimes\mathbb{1}\cong\mathbb{1}}$)

Prop (Tensor Cats p. 35): There is a coeq. diagram

$$A \otimes k[S_n] \otimes x^{\otimes n} \rightrightarrows A \otimes x^n \twoheadrightarrow A \otimes_{S_n} x^{\otimes n}$$

for A f.g. right $k[S_n]$-module, treated as a right mod in $\mathcal{R}$ via the unique functor $\mathrm{vect}_k \longrightarrow \mathcal{R}$.

We identify $k[S_n]$ with $\bigoplus_{\sigma\in S_n} 1 \in \mathcal{R} \rightsquigarrow x^{\otimes n}$ is a module over this $k[S_n]$.

Prop (Tensor Cats Theorem VI.7): For a poly species ρ and a symm. monoidal linear fun.

$$G : \mathcal{R} \longrightarrow \mathcal{L}$$

there is a nat iso.

$$\Phi : G \circ F_{\rho,\mathcal{R}} \cong F_{\rho,\mathcal{L}} \circ G$$

$$F_{\rho,\mathcal{R}}(Gx) \overset{\text{Def}}{=} \bigoplus_n \rho_n \otimes_{S_n} (Gx)^{\otimes n}$$

$$\cong \bigoplus_n \rho_n \otimes_{S_n} G(x^{\otimes n})$$

unique fun $\mathrm{vect} \to \mathcal{R}$

$$\cong \bigoplus_n G\rho_n \otimes_{S_n} G(x^{\otimes n})$$

$$\cong \bigoplus_n G(\rho_n \otimes_{S_n} x^{\otimes n})$$

absolute colimit

$$\cong G\left(\bigoplus_n \rho_n \otimes_{S_n} x^{\otimes n}\right)$$

$\rightsquigarrow$ Investigate how important Φ really is for Schur functors.

$\Rightarrow$ these are the only properties that distinguish Schur functors?

Def: An ABSTRACT SCHUR FUNCTOR is a pseudonat. trans

$$S : \mathcal{U} \Rightarrow \mathcal{U}$$

A MORPHISM between abstract Schur functors is a modification between pseudonat trafos.

Define $\mathrm{Schur} := [\mathcal{U}, \mathcal{U}]_{ps}$

Cool: These Schur functors are automatically closed under composition.

Prop: The functor $F_{\rho,-}$ is an abstract Schur functor

$\rightsquigarrow$ This follows from the previous observations

A morphism of species $\rho, \sigma : \mathbb{S} \longrightarrow \mathrm{vect}_k$ is a nat trafo

$\Rightarrow$ This lifts to $F_{\rho,-} \Rightarrow F_{\sigma,-}$

A modification in this setting:

$$\begin{tikzcd}[column sep=huge] \text{2-Rig} \arrow[r, bend left=60, "\mathcal{U}", ""{name=A, below}] \arrow[r, bend right=60, "\mathcal{U}"', ""{name=B}] & \mathrm{Cat} \arrow[from=A, to=B, Rightarrow, shift right=6, "F_\rho"'] \arrow[from=A, to=B, Rightarrow, shift left=6, "F_\sigma"] \arrow[from=A, to=B, phantom, "\overset{\Gamma}{\Rrightarrow}"] \end{tikzcd}$$

which more explicitly means components

$$\rightsquigarrow \begin{tikzcd} \mathcal{L} \arrow[r, bend left, "F_{\rho,\mathcal{L}}", ""{name=A, below}] \arrow[r, bend right, "F_{\sigma,\mathcal{L}}"', ""{name=B}] & \mathcal{L} \arrow[from=A, to=B, Rightarrow] \end{tikzcd} \quad \left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} \text{of 2-cells} \\ = \text{nat trafo} \end{array}$$

such that

$$\begin{tikzcd} \mathcal{L} \arrow[r, "G"] \arrow[d, "F_{\rho,\mathcal{L}}"'] \arrow[dr, phantom, "\overset{\sim}{\Rrightarrow}"] & \mathcal{D} \arrow[d, bend right=50, "F_{\rho,\mathcal{D}}"', ""{name=A}] \arrow[d, bend left=50, "F_{\sigma,\mathcal{D}}", ""'{name=B}] \arrow[from=A, to=B, Rightarrow] \\ \mathcal{L} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{L} \arrow[r, "G"] \arrow[d, bend right=50, "F_{\rho,C}"', ""{name=A}] \arrow[d, bend left=50, "F_{\sigma,C}", ""'{name=B}] \arrow[from=A, to=B, Rightarrow] \arrow[dr, phantom, "\overset{\sim}{\Rrightarrow}"] & \mathcal{D} \arrow[d, "F_{\sigma,D}"] \\ \mathcal{L} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

⌜ ⌝

$$\begin{tikzcd} G \circ F_{\rho,\mathcal{L}} \arrow[r, "\sim"] \arrow[d] & F_{\rho,\mathcal{D}} \circ G \arrow[d] \\ G \circ F_{\sigma,\mathcal{L}} \arrow[r] & F_{\sigma,\mathcal{D}} \circ G \end{tikzcd}$$

$$G\Big(\bigoplus_n \rho_n \otimes_{S_n} x^n\Big) \cong \bigoplus_n \rho_n \otimes_{S_n} (Gx)^n$$

$$\longrightarrow \bigoplus_n \sigma_n \otimes_{S_n} (Gx)^n$$

$$G\Big(\bigoplus_n \rho_n \otimes_{S_n} x^n\Big) \longrightarrow G\Big(\bigoplus_n \sigma_n \otimes_{S_n} x^n\Big)$$

$$\cong \bigoplus_n \sigma_n \otimes_{S_n} (Gx)^n$$

⌞ ⌟

Nice.

Thm: $\mathrm{Poly} \simeq \mathrm{Schur}$ via $\rho \longmapsto F_{\rho,-}(-)$

Strategy:

1) We have already seen that

$$\mathrm{Poly} \simeq \overline{k\mathbb{S}} \quad \Rightarrow \mathcal{U}\overline{k\mathbb{S}} \text{ as cats}$$

2) $\mathcal{U}$ can be decomposed as

Paper says 2 adjunctions, but probably biad.

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "\mathrm{Sym}"] \arrow[r, phantom, "\perp"] & \mathrm{SMCat} \arrow[l, bend left, "U_0"] \arrow[r, bend left, "k-"] \arrow[r, phantom, "\perp"] & \mathrm{SMLin} \arrow[l, bend left, "U_1"] \arrow[r, bend left, "\overline{(-)}"] \arrow[r, phantom, "\perp"] & 2\text{-}\mathrm{Rig} \arrow[l, bend left, "U_2"] \end{tikzcd}$$

2-adj: $F : \mathbb{B} \rightleftarrows \mathbb{C} : U$ s.t. $\mathbb{C}(Fa,b) \cong \mathbb{B}(a,Ub)$ is a nat iso!
$\Rightarrow$ biadj. is just an equiv that is pseudonat in both vars

$\overline{(-)}$ is the Cauchy completion

$k-$ comes from the CHANGE OF ERRICHMENT functor

$$(-)_k \; \mathrm{Cat} \longrightarrow \mathrm{LinCat} = \mathrm{vect}_k\text{-}\mathrm{Cat}$$

$\rightsquigarrow \mathrm{ob}\,\mathcal{L}_* = \mathrm{ob}\,\mathcal{L}$

$\mathcal{L}_*(x,y) = k\mathcal{L}(x,y)$

$\mathcal{L}_*(y,z) \otimes_k \mathcal{L}_*(x,y) \cong k(\mathcal{L}(y,z) \times \mathcal{L}(x,y)) \longrightarrow \mathcal{L}_*(x,z)$

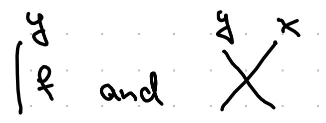
2-cells are the same.

Since $k- : \mathrm{Set} \longrightarrow \mathrm{Vect}_k$ is strong sym. mon., the induced fun sends pseudomons to pseudomons.

Sym is the standard free sym. mon cat

- obj: $(x_1, \ldots, x_n) \quad x_i \in \mathcal{L}$

Morphs: "generated" by



and $\Rightarrow$ permutations, labeled by f

Subject to the usual conditions

- Functors are taken to

Notice:

(i) $\mathrm{Sym}\,\mathbb{1} \simeq \mathbb{S}$ as sym mon cat.

$$\Rightarrow \underbrace{x : \mathbb{1} \longrightarrow U_0 C}_{\text{picks out } x \in C} \iff x' : \mathbb{S} \longrightarrow C \qquad n \longmapsto x^{\otimes n}$$

sym mon fun

(ii) $k\mathbb{S} \simeq k\mathbb{S}$

$$\Rightarrow \underbrace{x' : \mathbb{S} \longrightarrow U_1 C}_{\text{sym. mon. fun.}} \iff \underbrace{x'' : k\mathbb{S} \longrightarrow C \quad n \longmapsto x^{\otimes n}}_{\text{sym. mon. lin. fun.}}$$

(iii) $\overline{k\mathbb{S}} \simeq \overline{k\mathbb{S}}$

$$\Rightarrow \underbrace{x'' : k\mathbb{S} \longrightarrow C}_{\text{SML fun}} \iff \overline{x}'' : \overline{k\mathbb{S}} \longrightarrow C \qquad \rho \longmapsto F_{\rho,C}$$

3) Once we have this it is immediate that

$$2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}}, -) \cong \mathrm{SMLin}(k\mathbb{S}, U_2)$$
$$\cong \mathrm{SMCat}(\mathbb{S}, U_1U_2)$$
$$\cong \underbrace{\mathrm{Cat}(\mathbb{1}, U)}$$
$$\cong U$$

pseudonatural equivalences $\quad \| \quad F(1)$

$\phi \mapsto \phi(1)$

by Cart closedness

probably 1/2-cats here?

4) Proof the theorem as in the easy case, using the 2-cat. Yoneda lemma instead.

$$\mathrm{Poly} \simeq \mathcal{U}\overline{k\mathbb{S}} \simeq 2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}}, \overline{k\mathbb{S}})$$
$$\simeq \mathrm{Str}(2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}},-), 2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}},-))$$
$$\simeq \mathrm{Str}(\mathcal{U},\mathcal{U}) = \mathrm{Schur}.$$

These are all real equivs of (enriched) Catys

More explicitly:

$$\rho \longmapsto \tilde{\rho} \in \mathcal{U}(\overline{k\mathbb{S}}) \text{ OR } \tilde{\rho} : \mathbb{1} \longrightarrow \mathcal{U}\overline{k\mathbb{S}}$$
$$\longmapsto \overline{\rho} \cdot \overline{k\mathbb{S}} \longrightarrow \overline{k\mathbb{S}}$$
$$\longmapsto \overline{\rho}^* : 2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}},-) \overset{\sim}{\Longrightarrow} 2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}},-)$$
$$\overline{\rho}^*_R : \phi \longmapsto \phi \circ \overline{\rho}$$

The fact that $\mathcal{U}$ is repr. by $\overline{k\mathbb{S}}$ via

$$\phi \longmapsto \phi(1)$$

gives: for $x \in R$, pass it through $R \simeq 2\text{-}\mathrm{Rig}(\overline{k\mathbb{S}}, R)$ to obtain

$$\overline{x} : \overline{k\mathbb{S}} \longrightarrow R$$

$$\rightsquigarrow \overline{\rho}^*_R(\overline{x}) = \overline{k\mathbb{S}} \xrightarrow{\overline{\rho}} \overline{k\mathbb{S}} \xrightarrow{\overline{x}} R$$

$$\overset{\text{ev.}}{\Rightarrow} 1 \xrightarrow{\eta} \overline{k\mathbb{S}} \xrightarrow{\overline{\rho}} \overline{k\mathbb{S}} \xrightarrow{\overline{x}} R$$

$$\|$$

$$\underbrace{1 \xrightarrow{\rho} \overline{k\mathbb{S}} \xrightarrow{\overline{x}} R}$$

$= F_{\rho,R}$ by previous observation.

Transfer of structure means that we have monoidal equivs

$$([\mathcal{V},\mathcal{V}],\circ) \simeq (\mathcal{Z}\text{-}\mathbb{R}\mathrm{ig}(\overline{kS},\overline{kS}),\circ)$$
$$\simeq (\underbrace{\mathcal{V}\overline{kS}}_{\mathrm{Poly}},\bullet)$$

which gives exactly the substitution product on Poly.

$$\rho\bullet\tau = \bigoplus_n \rho_n\otimes_{S_n}\tau^{\otimes n}$$

unit is $\overline{kS}(-,1): S^{op}\longrightarrow \mathrm{vect}_k$

Why? if we put $R=\overline{kS}$ in what we said and $x=\tau$

we get
$$\overline{kS}\xrightarrow{\bar\rho}\overline{kS}\xrightarrow{\bar\tau}\overline{kS}$$
= the composite.

evaluation (as before) gives
$$\bar\tau = \bigoplus_n \rho_n\otimes_{S_n}\tau^{\otimes n}.$$

We already know that
$$\mathrm{Poly}\simeq\overline{kS}$$
wrt Day convolution ⇝ transfer this to Schur!

⇒ we obtain the ptwise tensor product

⇝ $F, G\in$ Schur, define
$$(F\otimes G)_R(x) = F_R x\otimes G_R x$$

$$\hookrightarrow (F\otimes G)_R(x) = \int^k (\rho*\tau)(k)\otimes_{S_n} x^{\otimes k}$$
$$= \int^{k,l,p} kS(l+p,k)\otimes\rho_l\otimes\tau_p\otimes x^{\otimes k}$$
$$= \int^{l,p}\rho_l\otimes\tau_p\otimes x^{\otimes l+p}$$
$$= F_R x\otimes G_R x$$