

DUALITY IN MONOIDAL CATEGORIES

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A symmetric monoidal category $(C, \otimes, 1)$ is called **closed** if there exists a tensor-hom adjunction

$$- \otimes x : C \rightleftarrows C : [x, -] \quad \text{for all } x \in C.$$

For any $x \in C$, we write $x^* := [x, 1]$ and define

$$\phi_x := x \otimes x^* \xrightarrow{\eta} [x, x \otimes x^* \otimes x] \xrightarrow{[x, x \otimes \epsilon]} [x, x],$$

using the unit and counit of the above adjunction.

The category C is **rigid** if ϕ_x is invertible for all objects $x \in C$. This implies the existence of an ambidextrous adjunction

$$- \otimes x^* \dashv \vdash - \otimes x \dashv \vdash - \otimes x^*.$$

To investigate if rigidity can be characterised by these adjunctions, we first promote the previous observations to a definition.

The internal hom $[-, -]: C^{\text{op}} \times C \rightarrow C$ is said to be **tensor representable** if for all $x \in C$

$$- \otimes x^* \dashv \vdash - \otimes x \dashv \vdash - \otimes x^*.$$

In this case, there is a natural isomorphism

$$\zeta_{x,y}: [x, y] \rightarrow y \otimes x^*.$$

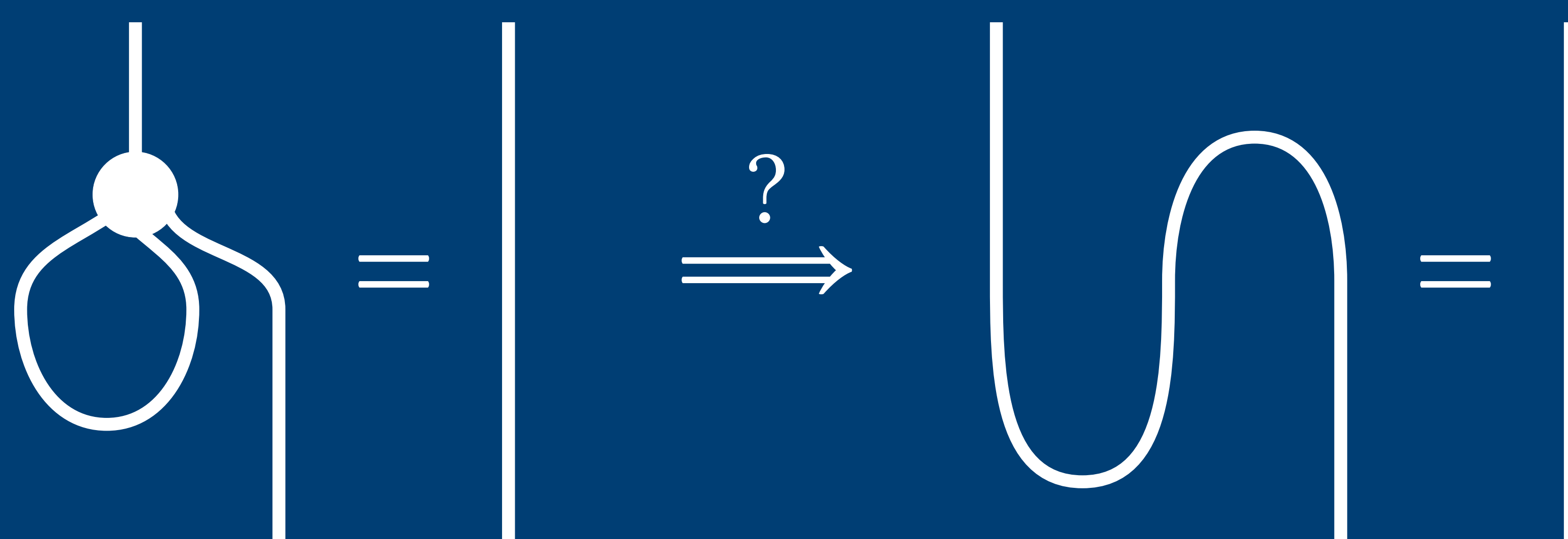
All of this motivates the question:

Given a closed monoidal category, does the tensor representability of its internal hom imply rigidity?

In many (algebraic) contexts the answer is: yes; given a commutative ring R and an R -module M , the following are equivalent:

1. M admits a dual in the rigid sense,
2. $- \otimes_R M \dashv \vdash - \otimes_R \text{Hom}_R(M, R)$, and
3. M is finitely generated and projective.

Do adjunctions characterise rigidity?



There exists a **strict** ordering

Closed



*-autonomous/Grothendieck–Verdier



Tensor representable



Rigid



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Fix a finite group G and a field k . The category of finite-dimensional k -valued **Mackey functors** of G is defined as the functor category

$$[\text{Sp}_{G\text{-set}}, \text{vect}_k]_+ =: \text{mky}_G;$$

that is, additive functors from spans of G -sets into finite-dimensional k -vector spaces.

Day convolution endows mky_G with the structure of a closed symmetric monoidal category.

Theorem (HZ). The internal hom of mky_G is tensor representable. It is rigid if and only if the characteristic of k does not divide the order of G .

An illuminating characterisation of Mackey functors is due to Peter Webb:

A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation mappings in group representation theory.

In line with this description, examples of Mackey functors of G include representations of G , the Grothendieck group of $\text{Rep}_k(G)$, the Burnside ring of G , as well as many K -theoretic and homological constructions.

Although tensor representability does not imply rigidity, it is related to another duality concept.

A ***-autonomous** or **Grothendieck–Verdier** category is a closed monoidal category C endowed with a fixed object $d \in C$, the **dualising object**, such that there is an anti-equivalence

$$D := [-, d]: C \rightarrow C^{\text{op}}.$$

Theorem (HZ). Every tensor representable category is *-autonomous, with the monoidal unit as the dualising object.