Mackey Functor Seminar

Ivan Bartulović Sebastian Halbig Manuel Martins Edoardo Salati
Zbigniew Wojciechowski Tony Zorman

15–16th July 2023

Abstract. These are notes from a seminar on Mackey functors held at TU Dresden, in collaboration with the University of Marburg. They cover several equivalent definitions of Mackey functors, and—along the way—introduce representation theory of finite groups, relative homological algebra, and some monoidal category theory.

Contents

1 Introduction

> A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation mappings in group representation theory. Operations such as these appear in quite a variety of diverse contexts — for example group cohomology, the algebraic K-theory of group rings, and algebraic number theory — and it is their widespread occurrence which motivates the study of such operations in abstract.
>
> Peter Webb, *A guide to Mackey functors*, [Web00]

This seminar is supposed to be a modest introduction to a more ambitious scheme. The latter might be summarised under the slogan: "Investigate algebraic structures up to homotopy using operads." In particular, we are interested in formalising $(\infty, 1)$-Grothendieck–Verdier categories and express additional structures such as ribbon twists by group actions. In doing so, we follow the program laid out by Boyarchenko and Drinfeld in *A duality formalism in the spirit of Grothendieck and Verdier*, [BD13]. A section, contributed by Lurie, to the aforementioned article is especially helpful as it states profound conjectures about the shape of these theories in the $(\infty, 1)$-categorical setting. (Spectral) Mackey functors enter this story as their non-homotopical analogues from a ribbon Grothendieck–Verdier category. Thus, spectral Mackey functors are good test candidates for formulating and checking these tools in the $(\infty, 1)$-categorical world.

The seminar consists of two main parts:

1. concrete and abstract Mackey functors, as well as
2. coend calculus and Day convolution.

Acknowledgements

This document was LaTeX'd by S.H., M.M., E.S., and T.Z. We thank the other—both active and passive—participants of the seminar for sharing their notes with us, as well as for many interesting discussions, suggestions, and insightful questions.

2 Representation theoretic preliminaries (Manuel)

THIS TALK WILL INTRODUCE much of the representation theoretic language that is needed when one wants to study classical Mackey functors. In particular, we will see what *restriction*, *induction*, and *conjugation* look like, and conclude with *Mackey decomposition*—Theorem 2.17—modelling the interaction between these operations. In particular, we are interested in studying the "original" Mackey functor: the assignment $H \leq G \longrightarrow M(H)$, where

$$M(H) = \operatorname{Rep} H = \mathbb{k}[H]\text{-mod} = Z[\mathbb{k}[H]].$$

Throughout, let $\mathbb{k}$ denote a field; we usually think of the complex numbers. Suppose that G is a finite group. I will follow [Web00], [TW95], [Ste12], [Bum13], and [Ser98].

There are many equivalent ways of describing representations of G:

1. Representations $\rightsquigarrow \rho\colon G \longrightarrow GL(V)$ group morphism $\rightsquigarrow \operatorname{Rep}_G$.[1]
2. Modules $\rightsquigarrow \mathbb{k}[G] \otimes V \longrightarrow V$ associative and unital $\rightsquigarrow \mathbb{k}[G]$-mod.[2]
3. Class functions $\rightsquigarrow f\colon G \longrightarrow \mathbb{k}$ conjugation invariant $\rightsquigarrow Z(\mathbb{k}[G])$.
4. Corepresentations of the dual group algebra $(\mathbb{k}G)^*$.

[1] More precisely, take the Grothendieck completion and extend scalars to $\mathbb{k}$; i.e., consider $R(G) \otimes \mathbb{k}$. See Section 3.1.1 for a more thorough introduction to the Grothendieck group G_0.

[2] Again, this is to be read as $G_0(\mathbb{k}[G]\text{-Mod}) \otimes \mathbb{k}$.

2.1 Representations, characters, and class functions

Definition 2.1. The *group algebra* $\mathbb{k}[G]$ is the vector space of $\mathbb{k}$-linear combinations of elements of G, with the algebra structure extended linearly from the group structure of G.

Equivalently, it is the same (for finite G) as $\mathbb{k}^G$, the Euclidean vector space of all linear functionals $G \longrightarrow \mathbb{k}$ (via $g \leftrightarrow \delta_g$). It admits, as a $\mathbb{k}$-basis, the functions $\{\,\delta_g \mid g \in G\,\}$ given by

$$\delta_g(x) = \begin{cases} 1, & \text{if } g = x, \\ 0, & \text{else.} \end{cases}$$

The algebra structure in $\mathbb{k}^G$ is determined by:

$$\begin{aligned} \mu\colon \mathbb{k}[G] \otimes \mathbb{k}[G] &\longrightarrow \mathbb{k}[G], \\ \delta_g \otimes \delta_h &\longmapsto \delta_g * \delta_h = \delta_{gh}, \end{aligned} \qquad \begin{aligned} \eta\colon \mathbb{k} &\longrightarrow \mathbb{k}[G] \\ 1 &\longmapsto \delta_e, \end{aligned}$$

where the convolution is defined by $(f * g)(x) \coloneqq \sum_{y\in G} f(xy^{-1})g(y)$.

Definition 2.2. A (finite-dimensional) *representation* of G is a group homomorphism $\rho\colon G \longrightarrow GL(V)$, where V is a (finite-dimensional) vector space. Equivalently, ρ defines a $\mathbb{k}[G]$-module structure on V.

A (nonzero) representation (ρ, V) is *irreducible* if the only G-invariant subspaces of V are 0 and V itself. In the language of modules, V is a simple $\mathbb{k}[G]$-module. For instance, nonzero 1-dimensional representations are irreducible, since if $V \cong \mathbb{k}$ with an action of $\mathbb{k}[G]$, then any proper submodule of $\mathbb{k}$ is 0. By induction, one can prove that any representation V decomposes as a direct sum of irreducible

Fact. Given a representation (ρ, V), its *character* $\chi_\rho \colon G \longrightarrow \mathtt{k}$ is given by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g)).$$

An irreducible character is the character of a irreducible representation. Any representation is uniquely determined by its character.

If χ_ρ, χ_θ are two characters, one can define

$$\langle \chi_\rho\, , \, \chi_\theta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g) \chi_\theta(g^{-1}).$$

This defines a bilinear form on linear combinations of characters.

The set of irreducible characters forms orthonormal system, meaning that $\langle \chi_\rho, \chi_\rho \rangle = 1$ and $\langle \chi_\rho, \chi_\theta \rangle = 0$ if ρ and θ are non-isomophic. In fact, an irreducibility criterion is $\langle \chi, \chi \rangle = 1$.

Definition 2.3. A linear map $f \colon G \longrightarrow \mathtt{k}$ is called a *class function*, if it is invariant in conjugacy classes. That is, for all $g, x \in G$,

$$f(xgx^{-1}) = f(g).$$

We denote the subspace (!) of class functions by $Z(\mathtt{k}[G])$.

This notation is consistent with the fact that $Z(\mathtt{k}[G])$ is the center of $\mathtt{k}^G$.

Lemma 2.4. *If $f \in Z(\mathtt{k}[G])$ and $\delta_x \in \mathtt{k}[G]$ is in the dual basis, then*

$$(f * \delta_x)(g) = f(x^{-1}g) = f(gx^{-1}) = (\delta_x * f)(g)$$

for all $g \in G$.

A character of a representation is a class function. Indeed one has that

$$\mathrm{tr}(\pi(xgx^{-1})) = \mathrm{tr}(\pi(gx{-}1)\pi(x)) = \mathrm{tr}(\pi(g)).$$

Fact. The irreducible characters form a basis of $Z(\mathtt{k}[H])$. That is, any class function is a linear combination of irreducible characters. This is the analogue of the statement that any representation is a linear combination of irreducible representations.

In fact, the irreducibly characters form an orthonormal basis with respect to $\langle\, , \rangle$, meaning that $\langle \chi_\rho, \chi_\rho \rangle = 1$ and $\langle \chi_\rho, \chi_\theta \rangle = 0$ if ρ and θ are non-isomophic.

Another way of interpreting the above bilinear form is as the dimension of a hom-space. If (V, ρ) and (W, θ) are representations of G, then define

$$\langle V, W \rangle_G \coloneqq \dim \mathrm{Hom}_{\mathtt{k}G}(V, W).$$

This defines a bilinear form

$$\langle\, , \rangle_G \colon \mathtt{k}\,\mathrm{Rep}_G \otimes \mathtt{k}\,\mathrm{Rep}_G \longrightarrow \mathtt{k}.$$

Notice that, by Schur's lemma[3] if V and W are irreducible, then

$$\langle V, W\rangle_G = \begin{cases} 1, & \text{if } V \cong W \text{ as } \mathtt{k}[G]\text{-modules},\\ 0, & \text{otherwise.}\end{cases}$$

Hence, the bilinear form on representations agrees with the bilinear form on characters for irreducible reps/characters. Notice that $\chi_{\rho+\theta} = \chi_\rho + \chi_\theta$ and therefore, extending bilinearly, shows that:

$$\langle V, W\rangle_G = \langle \chi_\rho, \chi_\theta\rangle$$

If $\chi_1, \cdots, \chi_p$ are all the irreducible characters corresponding to irreducible representations $\rho_1, \cdots, \rho_p$ of G then $p = \dim_{\mathtt{k}} |Z(\mathtt{k}[G])|$ and any representation ρ is of the form

$$\rho = m_1\rho_1 \oplus \cdots \oplus m_p\rho_p$$

where each $m_i = \langle \rho, \rho_1\rangle$.

In particular, every irreducible represention is contained in the regular representation with multiplicity equal to its degree (aka, $\mathtt{k}$-linear dimension). Since

$$\mathtt{k}[G] = \mathtt{k}[G] \otimes_{\mathtt{k}} \mathtt{k} = \operatorname{Ind}_{\{e\}}^G \mathtt{k}_{\text{triv}},$$

Frobenius reciprocity tells us that for any V_i irreducible

$$\dim \operatorname{Hom}_{\mathtt{k}[G]}(\mathtt{k}[G], V_i) = \dim_{\mathtt{k}} \operatorname{Hom}_{\mathtt{k}}(\mathtt{k}, \operatorname{Res}_{\{e\}^G} V_i) = \dim_{\mathtt{k}} V_i =: \deg V_i.$$

Since the regular representation ρ has as character $\chi_\rho(1) = |G|$, $\chi_\rho(g) = 0$ for $g \neq 0$ and in general $\chi_\theta(1) = \deg\theta$, we have

$$|G|\chi_{r_G}(1) = d_1\chi_1(1) + \cdots \chi_p(1) = d_1^2 + \cdots + d_p^2.$$

2.2 Restriction, induction, and conjugation in terms of class functions

Having warmed up, we would now like to study—from the perspective of class functions—the three operations of interest. For the rest of this section, let $H \leq G$ be a subgroup.

Definition 2.5. Given $f\colon G \longrightarrow \mathtt{k}$, define the *restriction of f from G to H* by

$$\operatorname{Res}_H^G f := f|_H.$$

If $f \in Z(\mathtt{k}[G])$, then we have $\operatorname{Res}_H^G f \in Z(\mathtt{k}[H])$, and so the resulting map $\operatorname{Res}_H^G\colon Z(\mathtt{k}[G]) \longrightarrow Z(\mathtt{k}[H])$ is linear.

Lemma 2.6 (Properties of restriction)**.** *For subgroups $J \leq K \leq H$, we have that*

$$\operatorname{Res}_H^H = \operatorname{Id}_{Z(\mathtt{k}[H])} \qquad \textit{and} \qquad \operatorname{Res}_J^K \operatorname{Res}_K^H = \operatorname{Res}_J^H.$$

We know define the induction operation, that is, a way to obtain a class function of G out of a class function of H. First, some notation:

Definition 2.7. For $f \in \mathtt{k}[H]$, define $\dot{f} \in \mathtt{k}[G]$ by

$$\dot{f}(g) = \begin{cases} f(g), & \text{if } g \in H,\\ 0, & \text{else.}\end{cases}$$

[3] Schur's lemma states that any linear map between finite-dimensional irreducible representations, which also commutes with the action of the group, is either zero, or an isomorphism. A quick proof: the kernel and image of a $\mathtt{k}G$-linear morphism f are submodules, so if $f \neq 0$, then $\ker f = 0$ and $\operatorname{im} f = W$, therefore f is an isomorphism. Now let λ be an eigenvalue of f and consider $f - \lambda$, which has nontrivial kernel. By the same argument, it must be zero, therefore f is a scalar multiple of identity and $\operatorname{Hom}(V, W)$ is 1-dimensional.

It is clear that this assignment is linear.

Definition 2.8 (Induction for class functions)**.** Given $f \in Z(\mathtt{k}[H])$, define the *induction of f from H to G* by

$$\begin{aligned} \operatorname{Ind}_H^G f\colon\ G &\longrightarrow \mathtt{k} \\ g &\longmapsto \frac{1}{|H|}\sum_{x\in G}\dot{f}(x^{-1}gx). \end{aligned}$$

Then $\operatorname{Ind}_H^G f \in Z(\mathtt{k}[G])$ and the map $\operatorname{Ind}_H^G\colon\ Z(\mathtt{k}[H]) \longrightarrow Z(\mathtt{k}[G])$ is linear.

We expect $\deg \operatorname{Ind}_H^G \rho = [G : H] \cdot \deg \rho$ from the formula of induction for modules. Indeed this can be verified directly.

Definition 2.9. Let M be a $\mathtt{k}[H]$-module. Then,

$$\operatorname{Ind}_H^G M = \mathtt{k}[G] \otimes_{\mathtt{k}[H]} M$$

as a $\mathtt{k}[G]$-module.

Definition 2.10 (Induction for representations)**.** Let $\rho\colon\ H \longrightarrow GL_d(\mathtt{k})$ be a representation of V and let $\{t_1, \cdots, t_m\}$ be a complete set of representatives of G/H. Linearly extend ρ to G via

$$\dot{\rho}(g) = \begin{cases} \rho(g), & g \in H, \\ 0, & g \notin H. \end{cases}$$

Let $\rho^G(g)$ be the block matrix with $m\times m$ blocks of size $d\times d$—more precisely, $[\rho^G(g)]_{i,j} = \dot{\rho}(t_i^{-1}gt_j)$. Then the *induction of ρ from H to G* is defined as

$$\operatorname{Ind}_H^G \rho \coloneqq \rho^G\colon\ G \longrightarrow GL_{md}(\mathtt{k}).$$

Lemma 2.11 (Properties of induction)**.** *For subgroups $J \le K \le H$, we have*

$$\operatorname{Ind}_H^H = \operatorname{Id}_{Z(\mathtt{k}[H])} \qquad \textit{and} \qquad \operatorname{Ind}_K^H \operatorname{Ind}_J^K = \operatorname{Ind}_J^H.$$

Theorem 2.12 (Frobenius Reciprocity)**.** *Induction and restriction are adjoint with respect to $\langle\ ,\ \rangle$; that is, for*

$$\operatorname{Res}_H^G\colon\ Z(\mathtt{k}[G]) \longrightarrow Z(\mathtt{k}[H]) \quad \textit{and} \quad \operatorname{Ind}_H^G\colon\ Z(\mathtt{k}[H]) \longrightarrow Z(\mathtt{k}[G])$$

we have

$$\langle \operatorname{Ind}_H^G a, b\rangle = \langle a, \operatorname{Res}_H^G b\rangle.$$

Proof. We show the formula on characters of representations, which is enough by linearity:

$$\begin{aligned} \langle \operatorname{Ind}_H^G a, b\rangle &= \frac{1}{|G|}\sum_{g\in G}(\operatorname{Ind}_H^G a)(g)b(g^{-1}) \\ &= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{x\in G}\dot{a}(x^{-1}gx)b(g^{-1}). \end{aligned}$$

In order for $\dot{a}$ not to vanish, we must have $x^{-1}gx \in H$, which is equivalent to $g = xhx^{-1}$ for some $h \in H$. As such we can reindex the sum above as

$$\begin{aligned} \frac{1}{|G||H|}\sum_{x\in G}\sum_{h\in H} a(h)b(xh^{-1}x^{-1}) &= \frac{1}{|G||H|}\sum_{x\in G}\sum_{h\in H} a(h)b(h^{-1}) \\ &= \frac{1}{|G|}\sum_{x\in G}\langle a, \operatorname{Res}^G_H b\rangle_H \\ &= \langle a, \operatorname{Res}^G_H b\rangle_H. \end{aligned}$$

□

Theorem 2.13 (Frobenius Reciprocity for modules). *Let $M \in \mathtt{k}[H]$-mod and $N \in \mathtt{k}[G]$-mod. Then*

$$\operatorname{Hom}_{\mathtt{k}[H]}(M, \operatorname{Res}^G_H N) \cong \operatorname{Hom}_{\mathtt{k}[G]}(\operatorname{Ind}^G_H M, N).$$

Proof. Since $\operatorname{Ind}^G_H M = \mathtt{k}[G] \otimes_{\mathtt{k}[H]} N$, we can think in terms of the tensor-hom adjunction:

$$\operatorname{Hom}_{\mathtt{k}[H]}(M, \operatorname{Hom}_{\mathtt{k}[G]}(\mathtt{k}[G], N)) \cong \operatorname{Hom}_{\mathtt{k}[G]}(\mathtt{k}[G] \otimes_{\mathtt{k}[H]} M, N)$$

and $\operatorname{Hom}_{\mathtt{k}[G]}(\mathtt{k}[G], N) \cong N, \phi \longmapsto \phi(1)$ as a left $\mathtt{k}[H]$-module, with the action coming from restriction to $\mathtt{k}[H]$. □

Definition 2.14 (Conjugation). Given $f \in Z(\mathtt{k}[H])$ and an element $g \in G$, define the *conjugation of f by g* by

$$\begin{aligned} c_{H,g} f\colon\ gHg{-}1 &\longrightarrow \mathtt{k} \\ ghg^{-1} &\longmapsto f(h). \end{aligned}$$

Here, for a subgroup $H \leq G$ and an element $g \in G$, the notation gHg^{-1} denotes the conjugacy class of H by g by; i.e.,

$$gHg^{-1} := \{\, ghg^{-1} \mid h \in H \,\}.$$

Then $c_{H,g} f \in Z(\mathtt{k}[gHg^{-1}])$ and the map $c_{H,g}\colon Z(\mathtt{k}[H]) \longrightarrow Z(\mathtt{k}[gHg^{-1}])$ is a linear isomorphism.

Lemma 2.15. *The following compatibility conditions hold for conjugation, restriction, and induction:*

1. $c_{H,h} = \operatorname{Id}_{Z(\mathtt{k}[H])}$, *for $h \in H$.*
2. $c_{yHy-1,x}c_{H,y} = c_{H,xy}$, *for $x, y \in G$.*
3. $c_{K,g}\operatorname{Res}^H_K = \operatorname{Res}^{gHg^{-1}}_{gKg^{-1}} c_{H,g}$, *for $K \leq H, g \in G$.*
4. $c_{H,g}\operatorname{Ind}^H_K = \operatorname{Ind}^{gHg^{-1}H}_{gKg^{-1}} c_{K,g}$, *for $K \leq H, g \in G$.*

Besides the properties encoded by Lemmas 2.6, 2.11, and 2.15, the most crucial property satisfied by the restriction, induction, and conjugation operations is given by *Mackey's Theorem*, which we will state in Theorem 2.17.

Note that, given subgroups $H, K \leq G$, the Cartesian product $H \times K$ acts on G by

$$(h, k) \cdot g = hgk^{-1},$$

for $h \in H$, $g \in G$, and $k \in K$.

Definition 2.16. Let $H, K \leq G$ be subgroups. The *double coset of $g \in G$* is its $(H \times K)$-orbit

$$HgK := \{ hgk \mid h \in H, k \in K \}.$$

That is, the double cosets of H and K in G are the left cosets of $H \times K$ in G. We write

$$H \backslash G / K$$

for the set of double cosets of H and K in G.

Theorem 2.17 (Mackey Decomposition)**.** *We have the following direct sum decomposition of operators:*

$$\mathrm{Res}^G_H \mathrm{Ind}^G_K = \bigoplus_{s \in H \backslash G / K} \mathrm{Ind}^H_{H \cap xKx^{-1}} \mathrm{Res}^{xKx^{-1}}_{H \cap xKx^{-1}} c_{K,x},$$

where the left-hand side corresponds to the bottom path, and the right-hand side corresponds to the top path in the following diagram:

$$\begin{tikzcd} M(K) \arrow[r, "c_{K,x}"] \arrow[d, "\mathrm{Ind}^G_K"] & M(xKx^{-1}) \arrow[r, "\mathrm{Res}^{xKx^{-1}}_{H \cap xKx^{-1}}"] & M(H \cap xKx^{-1}) \arrow[d, "\mathrm{Ind}^H_{H \cap xKx^{-1}}"] \\ M(G) \arrow[rr, "\mathrm{Res}^G_H"] & & M(H) \end{tikzcd}$$

Proof. Let S be a complete set of representatives of $H\backslash G/K$, i.e., we have $\Bbbk G = \oplus_{s \in S} \Bbbk(HsK)$ as $(\Bbbk H, \Bbbk[K])$-bimodules. Let M be a $\Bbbk[K]$-module; then

$$\mathrm{Ind}^G_K M = \Bbbk[G] \otimes_{\Bbbk[K]} M = \bigoplus_{s \in S} \Bbbk(HsK) \otimes_{\Bbbk[K]} M.$$

Claim. For each $s \in S$, let $K_s := H \cap sKs^{-1} \leq H$ and let M^s denote the module with underlying vector space M and action of sKs^{-1} given by $(sks^{-1}) \triangleleft m := k \triangleleft m$, $k \in K, m \in M$. There is an isomorphism

$$\begin{aligned} \varphi_s \colon \ & \Bbbk[H] \otimes_{\Bbbk[K_s]} M^s|_{K_s} \longrightarrow \Bbbk(HsK) \otimes_{\Bbbk[K]} M, \\ & h \otimes m \longmapsto hs \otimes m \end{aligned}$$

of $\Bbbk H$-modules.

This is well defined by the following argument. Let $h \in H$, $m \in M$ and $x \in K_s$. Then, in particular, $x = sks^{-1}$ for some $k \in K$. On the one hand, we have that $hx \otimes m$ is mapped to $hxs \otimes m = hsk \otimes m = hs \otimes km$. On the other hand, $hx \otimes m = h \otimes s^{-1}xsm = h \otimes km$ is again mapped to $hs \otimes km$. It is also clearly surjective

If $\{ y_1, \cdots y_l \}$ is a complete set of representatives of K_s cosets of H, then

$$\Bbbk[H] = \bigoplus_{i=1}^{l} y_i \Bbbk[K_s]$$

$\{ y_1 s, \cdots y_l s \}$ is a basis of $\Bbbk(HsK)$ as a right $\Bbbk[K]$-module, since

$$\Bbbk(HsK) = \bigoplus_{i=1}^{l} \Bbbk(y_i)K_s sK = \bigoplus_{i=1}^{l} \Bbbk(y_i s)K$$

because $K_s sK = sK$. In particular $\{ y_1 s, \cdots, y_l s \}$ is a basis $\Bbbk(HsK)$ as a right $\Bbbk[K]$-module and this shows that the map φ_s is surjective.

We now compare dimensions

$$\begin{aligned}\dim_{\Bbbk} \Bbbk(HsK) \otimes_{\Bbbk[K]} M &= l \cdot \dim_{\Bbbk} M = [H : K_s] \cdot \dim_{\Bbbk} M \\ &= \dim_{\Bbbk} \Bbbk[H] \otimes_{\Bbbk[K_s]} M^s|_{K_s},\end{aligned}$$

which implies that φ_s is an isomorphism. □

A consequence of Mackey decomposition, in the case $H = K$, is that it allows to prove the following irreducibility criterion for induced representations. First, some notation. For $g \in G$, let $H_g \coloneqq H \cap gHg^{-1}$ and if $\rho\colon H \longrightarrow GL(W)$ is a representation, let $\rho^g\colon gHg^{-1} \longrightarrow GL(W)$ be the representation defined via $\rho^g(x) = \rho(gxg^{-1})$.

Theorem 2.18. *Let $H \leq G$ be a subgroup and $\rho\colon H \longrightarrow GL(W)$ a representation of H. Then, $\operatorname{Ind}_H^G \rho$ is irreducible if and only if*

1. *ρ is irreducible, and*
2. *for all $g \in G/H$, the two representations $\operatorname{Res}^H_{H_g}(\rho)$ and $\operatorname{Res}^{gHg^{-1}}_{H_g}(\rho^g)$ of H_g are disjoint*[4].

[4] Two representations are *disjoint* if they have no irreducible components in common, i.e.,, if they have orthogonal characters.

3 Mackey functors and the Mackey algebra (Sebastian)

3.1 Two definitions of Mackey functors

3.1.1 A closer look at the Grothendieck group

Let us assume for a second that we intend to study the category $\operatorname{Rep}_{\Bbbk}(G)$ of finite-dimensional representations of G. Equivalently, this means considering the category $\Bbbk G\text{-}\mathrm{Mod}_{\mathrm{fg}}$ of finitely-generated modules over the group-algebra of G. By the *Krull–Remak–Schmidt theorem*, any object in $\Bbbk G\text{-}\mathrm{Mod}_{\mathrm{fg}}$ is a finite direct sum of indecomposables and this decomposition is unique up to permutation of its factors. Categorically, indecomposablity of an object M means that any splitting short exact sequence

Throughout the talk $\Bbbk$ is a field, R is a commutative ring with unit 1, and G a finite group.

$$0 \longrightarrow A \xrightarrow{\alpha} M \xrightarrow{\beta} B \longrightarrow 0$$

has either $\alpha\colon A \longrightarrow M$ or $\beta\colon M \longrightarrow B$ an isomorphism (and the complementary morphism being zero).

In order to understand the underlying combinatorics better, we want to associate to this category any invariant that describes the behaviour of the composition factors—the *Grothendieck group* $G_0(\Bbbk G)$ of $\Bbbk G\text{-}\mathrm{Mod}_{\mathrm{fg}}$.

We construct $G_0(\Bbbk G)$ in two steps. First we take the free abelian group $F_{\Bbbk G}$ generated by the set $\{ [X] \mid X \in \Bbbk G\text{-}\mathrm{Mod}_{\mathrm{fg}} \}$ of isomorphism classes of objects in $\Bbbk G\text{-}\mathrm{Mod}_{\mathrm{fg}}$. The Grothendieck group $G_0(\Bbbk G)$ is the quotient of $F_{\Bbbk G}$ by the relations

$[X] + [Z] = [Y]$

$\Longleftrightarrow$

There is a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$.

In order to understand $G_0(\mathtt{k}G)$, we are now faced with the problem of determining the indecomposable $\mathtt{k}G$-modules. Unless we have specific information about the group G (and the field $\mathtt{k}$) this might be a challenging task, necessitating the need of relating $\mathtt{k}G\text{-Mod}_{\mathrm{fg}}$ with other, possibly better behaved, categories. Given subgroups $K \leq H \leq G$ we obtain a pair of adjoint functors

$$I_K^H\colon\ \mathtt{k}K\text{-Mod}_{\mathrm{fg}} \rightleftarrows \mathtt{k}H\text{-Mod}_{\mathrm{fg}}\ :R_K^H$$

given by inducing and restricting representations of H. Explicity, these functors are

$$\begin{aligned} I_K^H &\coloneqq \mathtt{k}H \otimes_{\mathtt{k}K} -\colon\ \mathtt{k}K\text{-Mod}_{\mathrm{fg}} \longrightarrow \mathtt{k}H\text{-Mod}_{\mathrm{fg}}, \\ R_K^H &\coloneqq \mathrm{Forg}\colon\ \mathtt{k}H\text{-Mod}_{\mathrm{fg}} \longrightarrow \mathtt{k}K\text{-Mod}_{\mathrm{fg}}, \end{aligned}$$

admitting for all $M \in \mathtt{k}K\text{-Mod}$ and $N \in \mathtt{k}H\text{-Mod}$ the natural isomorphism

$$\phi\colon\ \mathrm{Hom}_{\mathtt{k}H}(\mathtt{k}H \otimes_{\mathtt{k}K} M, N) \longrightarrow \mathrm{Hom}_{\mathtt{k}K}(M,N), \quad \phi(f)(m) = f(1 \otimes_{\mathtt{k}K} m).$$

Notice that $\mathtt{k}H$ is isomorphic as a right $\mathtt{k}K$ module to the direct sum $\oplus_{x\in[H/K]}\mathtt{k}K_x$. Here $[H/K]$ is a complete set of representatives of the cosets H/K and $\mathtt{k}K_x$ is the free $\mathtt{k}K$ module with vector space basis $\{\, x\lambda \mid \lambda \in K \,\}$. From this argument, it follows that besides the forgetful functor R_K^H, the induction I_K^H is also exact. Subsequently, we obtain group homomorphisms

$$R_K^H\colon\ G_0(\mathtt{k}H) \longrightarrow G_0(\mathtt{k}K), \qquad\qquad I_K^H\colon\ G_0(\mathtt{k}K) \longrightarrow G_0(\mathtt{k}H).$$

Assume G admits a subgroup K and an element $g \in G$ such that $K \neq gKg^{-1}$. Suppose further K and gKg^{-1} are contained in a supergroup $H \leq G$. Using induction and restriction, we can non-trivially transfer representations of K to its conjugate gKg^{-1}. In order to analyse the underling combinatorics of this map, we need the conjugation operation $c_g^K\colon\ G_0(\mathtt{k}K) \longrightarrow G_0(\mathtt{k}gKg^{-1})$, parametrised by an element $g \in G$ as well as a subgroup $K \leq G$ and in correspondence with the equivalence of categories

$$c_g^K\colon\ \mathtt{k}K\text{-Mod}_{\mathrm{fg}} \longrightarrow \mathtt{k}gKg^{-1}\text{-Mod}_{\mathrm{fg}}.$$

In short, restriction, induction and conjugation between representations of groups have as "shadows" homomorphisms between the corresponding Grothendieck groups. Many of the relations satisfied by these maps can be deduced immediately from the functorial nature of their categorical counterparts. A trivial yet instructive example is that restricting, inducing, or conjugating from a group to itself results in identity maps between the corresponding Grothendieck groups.[5] The arguably most unexpected identity of these operations and eponymous for Mackey functors is the *Mackey identity*

$$R_L^H I_K^H(M) = \mathtt{k}H \otimes_{\mathtt{k}K} M \cong \bigoplus_{x\in[L\backslash H/K]} I_{L\cap xKx^{-1}}^L c_x^{x^{-1}Lx\cap K} R_{x^{-1}Lx\cap K}^K(M),$$

for all $M \in \mathtt{k}K\text{-Mod}$. As the proof is quite instructive for the kinds of arguments we will encounter when studying Mackey functors, we will recall it here.

[5] Note in particular that only the restriction functor is the identity in this example.

Proof. First, we observe that $\mathtt{k}H$ is a free $\mathtt{k}L$ module, which is isomorphic to $\bigoplus_{x[L\backslash H]} \mathtt{k}L_x$. Here $[L\backslash H]$ denotes a complete set of representatives of the right cosets $L\backslash H$. If K were the trivial group, this would suggest a direct sum decomposition of $\mathtt{k}H \otimes_{\mathtt{k}K} M$ into submodules indexed by the right cosets $L\backslash H$. However, for non-trivial $\mathtt{k}K$ we have to take its action into account and simultaneously decompose $\mathtt{k}H$ as a left $\mathtt{k}L$ and right $\mathtt{k}K$ module. In other words, we want to consider $\mathtt{k}H \cong \bigoplus_{x\in[L\backslash H/K]} \mathtt{k}LxK$ were x ranges over a complete set of representatives of the double coset $L\backslash H/K$.

For each summand $\mathtt{k}LxK$ and $\mathtt{k}K$ module M there is a surjective $\mathtt{k}L$-linear map

$$\pi\colon\ \mathtt{k}L \otimes_{\mathtt{k}} M \longrightarrow \mathtt{k}LxK \otimes_{\mathtt{k}K} M, \qquad l \otimes_{\mathtt{k}} m \longmapsto lx \otimes_{\mathtt{k}K} m$$

To conclude the proof, we are now faced with the task of determining the kernel of π. Hereto we introduce the embedding $\iota\colon\ \mathtt{k}L \otimes_{\mathtt{k}} M \longrightarrow \mathtt{k}LxK \otimes_{\mathtt{k}} M$, $\iota(l \otimes_{\mathtt{k}} m) = lx \otimes_{\mathtt{k}} m$, leading to a commutative triangle in the category of modules over $\mathtt{k}L$:

$$\begin{tikzcd} \mathtt{k}LxK \otimes_{\mathtt{k}} M \arrow[rr, "\mathrm{can}"] & & \mathtt{k}LxK \otimes_{\mathtt{k}K} M \\ & \mathtt{k}L \otimes_{\mathtt{k}} M \arrow[ul, "\iota"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

We have $\iota(\ker \pi) = \operatorname{im} \iota \cap \ker(\mathrm{can})$. Using that $\{lxk \mid l \in L, k \in K\}$ is a basis of $\mathtt{k}LxK$ we get $\ker(\mathrm{can}) = \operatorname{span}\{lxk \otimes_{\mathtt{k}} m - lx \otimes_{\mathtt{k}} k \triangleright m \mid l \in L, k \in K \text{ and } m \in M\}$. Thus

$$\iota(\ker(\pi)) = \left\{ \sum_{l\in L, k\in K, m\in M} \lambda_{lkm} lxk \otimes_{\mathtt{k}} m - lx \otimes_{\mathtt{k}} k \triangleright m \;\middle|\; \begin{array}{l} \lambda_{lkm} = 0 \text{ almost always, and} \\ \lambda_{lkm} = 0 \text{ if } lxk \notin Lx \end{array} \right\}.$$

For any $l \in L$ and $k \in K$ we have $lxk \in Lx$ if and only if $k \in x^{-1}Lx$ or, equivalently, $xkx^{-1} \in L \cap xKx^{-1}$. Therefore we get

$$\ker(\pi) = \operatorname{span}_{\mathtt{k}}(lk \otimes_{\mathtt{k}} m - l \otimes_{\mathtt{k}} x^{-1}kx \triangleright m \mid l \in L, k \in L \cap xKx^{-1}, m \in M).$$

In other words, the kernel of π coincides with the kernel of the canonical projection $\mathtt{k}L \otimes_k M \longrightarrow \mathtt{k}L \otimes_{\mathtt{k}L\cap xKx^{-1}} M$. Putting all arguments together, we get an isomorphism of $\mathtt{k}L$-modules

$$\mathtt{k}LxK \otimes_{\mathtt{k}K} M \cong \mathtt{k}L \otimes_{\mathtt{k}L\cap xKx^{-1}} M = I^L_{\mathtt{k}L\cap xKx^{-1}} c_x^{x^{-1}Lx\cap K} R^K_{x^{-1}Lx\cap K}(M).$$

This concludes the proof. □

3.1.2 Green's definition of Mackey functors

Mackey functors abstract the induction, restriction, and conjugation operations introduced on the level of Grothendieck groups. This becomes most transparent in the initial definition given by Green.

Definition 3.1 (Green). An *R-valued Mackey functor over G* is a map

$$M\colon\ \{H \leq G\} \longrightarrow R\text{-Mod}, \qquad H \longmapsto M(H)$$

endowed with R-linear *induction*, *restriction*, and *conjugation* morphisms

$$I_K^H\colon\ M(K) \longrightarrow M(H),\quad R_K^H\colon\ M(H) \longrightarrow M(K),\quad c_g^H\colon\ M(H) \longrightarrow M(gHg^{-1}),$$

which are parametrised by subgroups $K \leq H \leq G$ and elements $g \in G$. These are required to satisfy for all $L \leq K \leq H \leq G$ and $f, g \in G$, $h \in H$ the following list of axioms:

1. $I_H^H = R_H^H = c_h^H = \mathrm{id}_{M(H)}$,
2. $R_L^K R_K^H = R_L^H$, $I_K^H I_L^K = I_L^H$ and $c_g^{fHf^{-1}} c_f^H = c_{gf}^H$,
3. $R_{gKg^{-1}}^{gHg^{-1}} c_g^H = c_g^K R_K^H$ and $I_{gKg^{-1}}^{gHg^{-1}} c_g^K = c_g^H I_K^H$, and
4. $R_L^H I_K^H = \sum_{x \in [L\backslash H/K]} I_{L\cap xKx^{-1}}^L c_x^{x^{-1}Lx\cap K} R_{x^{-1}Lx\cap K}^K$.[6]

A *morphism of Mackey functors* is a collection of R-linear maps

$$\phi_H\colon\ M(H) \longrightarrow N(H), \qquad\qquad H \leq M,$$

which commute with the respective induction, restriction, and conjugation operations.

[6] Again, $[L\backslash H/K]$ denotes a set of representatives of the double coset $L\backslash H/K$.

We have already seen that the collection of Grothendieck groups associated to the (sub-)groups of a finite group G are an example of a Mackey functor.

Besides the question of examples of Mackey functors—we will shortly see that there is an abundance of these—we want to investigate their structure from an algebraic perspective. In particular, we set as our goal to understand the projective Mackey functors. Hereto, it is helpful to have different equivalent definitions at hand.

3.1.3 G-sets and Dress' definition of Mackey functors

The next characterisation we want to investigate is based on the notion of G-sets. In order to contextualise and motivate it, let us state some useful facts about G-sets first. The starting point is the orbit-stabiliser theorem.

Theorem 3.2. *Let X be a transitive G-set. For any $x \in X$ there exists a G-equivariant bijection*

$$G/G_x \longrightarrow X, \qquad [g] \longmapsto g \triangleright x. \tag{3.1}$$

Furthermore, for any $h \in G$ the groups hG_xh^{-1} and $G_{h\triangleright x}$ coincide.

There are two "observations" of this theorem we want to emphasise. First, transitive G-sets correspond, by choosing a base point, to subgroups of G. Second, changing the base point amounts to appropriately conjugating the stabiliser groups.

Lemma 3.3. *Suppose $K, H \leq G$ are subgroups and $\phi\colon G/K \longrightarrow G/H$ is a morphism of G-sets. Fix $g \in G$ such that $\phi(K) = gH$. The morphism of G-sets $\psi\colon G \longrightarrow G$, $h \longmapsto hg$ lets the following square with canonical projections as vertical morphisms commute:*

$$\begin{tikzcd} G \arrow[r, "\psi"] \arrow[d] & G \arrow[d] \\ G/K \arrow[r, "\phi"] & G/H \end{tikzcd} \tag{3.2}$$

Furthermore, K is a subgroup of gHg^{-1}.

Proof. Given $x \in G$ we compute

$$\phi(xK) = \phi(x \triangleright 1K) = x \triangleright \phi(K) = xgH = \psi(x)H.$$

Let $k \in K$. We have $kgH = \psi(k)H = \phi(kK) = gH$ and therefore $g^{-1}kg \in H$ or, equivalently, $k \in gHg^{-1}$. □

Notice that the above lemma suggests the existence of two "special" morphisms of G-sets. The first is given by changing the base-point of the given G-set; i.e., conjugating the stabiliser group. The second is projecting onto a 'smaller' orbit. More precisely, given two subgroups $K \leq H \leq G$ and an element $g \in G$, we set

$$\begin{aligned} \pi_K^H &\colon G/K \longrightarrow G/H, && xK \longmapsto xH \\ c_g^H &\colon G/H \longrightarrow G/gHg^{-1}, && xH \longmapsto xg^{-1}(gHg^{-1}). \end{aligned}$$

Theorem 3.4. *Let K, H be subgroups of G and let $\phi\colon G/K \longrightarrow G/H$ be a morphism of G-sets. For any $g \in G$ such that $\phi(K) = gH$, we have*

$$\phi = \pi^H_{g^{-1}Kg} c^K_{g^{-1}} = c^{gHg^{-1}}_{g^{-1}} \pi_K^{gHg^{-1}}. \tag{3.3}$$

Furthermore, for any $g, q \in G$ we have $\pi^H_{g^{-1}Kg} c^K_{g^{-1}} = \pi^H_{q^{-1}Kq} c^K_{q^{-1}}$ if and only if $gH = qH$.

Proof. For any $x \in G$ and $q \in G$, we have

$$\pi^H_{q^{-1}Kq} c^K_{q^{-1}}(xK) = \pi^H_{q^{-1}Kq}(xq(q^{-1}Kq)) = xqH$$

and

$$c^{q^{-1}Hq}_{q^{-1}} \pi_K^{qHq^{-1}}(xK) = c^{q^{-1}Hq}_{q^{-1}}(x(qHq^{-1})) = xqH.$$

Thus $\phi = \pi^H_{g^{-1}Kg} c^K_{g^{-1}} = c^{gHg^{-1}}_{g^{-1}} \pi_K^{gHg^{-1}}$ and $\pi^H_{g^{-1}Kg} c^K_{g^{-1}} = \pi^H_{q^{-1}Kq} c^K_{q^{-1}}$ if and only if $gH = qH$. □

The above computations already give a slight hint at the fact that we might be able to model the induction and conjugation morphisms of Mackey functors using G-sets but what really makes us anticipate an equivalent formulation of Mackey functor based on the category G-Set is the appearance of the induction-restriction formula in certain pullback diagrams.

Theorem 3.5. *Suppose we are given three subgroups $L, K \leq H$ of G. Then the G-set*

$$\Sigma := \coprod_{x \in [L \backslash H / K]} G / L \cap xKx^{-1}$$

endowed with the maps

$$\coprod_{x \in [L \backslash H / K]} \pi^{L}_{L \cap xKx^{-1}} \colon \Sigma \longrightarrow G/L, \qquad \coprod_{x \in [L \backslash H / K]} \pi^{K}_{x^{-1}Lx \cap K} c^{L \cap xKx^{-1}}_{x^{-1}} \colon \Sigma \longrightarrow G/K$$

is a pullback of the diagram

$$\begin{tikzcd} \Sigma \arrow[r] \arrow[d] & G/K \arrow[d, "\pi^H_K"'] \\ G/L \arrow[r, "\pi^H_L"] & G/H \end{tikzcd}$$

Proof. We proceed in two steps. First we consider the product of H-sets $H/L \times H/K$. It decomposes into a disjoint union of transitive H-sets which we can identify, using the orbit-stabiliser theorem with certain subgroups of H. By definition, we have that $H/L \times H/K = \{(aL, bK) \mid a, b \in H\}$ implying that any orbit contains a point (L, xK). Its stabiliser group is given by

$$\begin{aligned} \{h \in H \mid (hL, hxK) = (L, xK)\} &= \{h \in H \mid h \in L \text{ and } x^{-1}hx \in K\} \\ &= L \cap xKx^{-1}. \end{aligned}$$

It remains to count the number of choices we have for $x \in H$ to implement different orbits. Suppose $h, x, y \in H$ are such that $(hL, hxK) = (L, yK)$. This is equivalent to $h \in L$ and $hxk = y$ for some $k \in K$. In other words (L, xK) and (L, yK) are contained in the same orbit under the H action if and only if x and y are representants of the same class of the double centraliser $L \backslash H / K$.

The product $H/L \times H/K$ comes equipped with projections

$$\mathrm{pr}_L \colon H/L \times H/K \longrightarrow H/L \qquad \text{and} \qquad \mathrm{pr}_K \colon H/L \times H/K \longrightarrow H/K.$$

In light of our decomposition of the above product of H-sets into its orbits these correspond to

$$\begin{aligned} \mathrm{pr}_L &= \coprod_{x \in [L \backslash H / K]} \pi^{L}_{L \cap xKx^{-1}} \colon \coprod_{x \in [L \backslash H / K]} H / L \cap xKx^{-1} \longrightarrow H/L, \\ \mathrm{pr}_K &= \coprod_{x \in [L \backslash H / K]} \pi^{K}_{x^{-1}Lx \cap K} c^{L \cap xKx^{-1}}_{x^{-1}} \colon \coprod_{x \in [L \backslash H / K]} H / L \cap xKx^{-1} \longrightarrow H/K. \end{aligned}$$

The forgetful functor H-Set $\longrightarrow$ G-Set has a left-adjoint

$$G \times_H - \colon H\text{-}\mathsf{Set} \longrightarrow G\text{-}\mathsf{Set}, \qquad S \longmapsto G \times_H M,$$

with $G \times_H S$ being given by equivalence classes of $G \times S$ under the relation $(gh, s) \sim (g, hs)$ for all $g \in G$, $h \in H$ and $s \in S$. This functor does in general not preserve all limits.[7] Instead, as the second step of our proof, we will show that the previously considered products of H-sets are mapped to pushouts of G-sets.

In doing so, we use that for any subgroup $I \leq H$ we have an isomorphism of G-sets

$$\phi_{H/I}\colon\ G \times_H {}^{H}\!/_{I} \longrightarrow {}^{G}\!/_{I}, \qquad\qquad \phi_{H/I}(g, hI) = ghI.$$

A direct computation shows that with respect to the previous identification we have

$$G \times_H \pi_I^J = \pi_I^J\colon\ {}^{G}\!/_{I} \longrightarrow {}^{G}\!/_{J}, \qquad G \times_H c_h^I = c_h^I\colon\ {}^{G}\!/_{I} \longrightarrow {}^{G}\!/_{hIh^{-1}},$$

for $I \leq J \leq H, h \in H$.

Now consider the pullback Σ of

$$G/L \xrightarrow{\ \pi_L^H\ } G/H \xleftarrow{\ \pi_K^H\ } G/K.$$

It is given by

$$\Sigma = \left\{(aL, bK) \in {}^{G}\!/_{L} \times {}^{G}\!/_{K} \mid \pi_L^H(aL) = \pi_K^H(bK)\right\} = G \times_H \left({}^{H}\!/_{L} \times {}^{H}\!/_{K}\right).$$

Moreover, we have

$$G \times_H \mathrm{pr}_L = \mathrm{pr}_L\colon\ \Sigma \longrightarrow {}^{G}\!/_{L} \qquad \text{and} \qquad G \times_H \mathrm{pr}_K = \mathrm{pr}_K\colon\ \Sigma \longrightarrow {}^{G}\!/_{K}.$$

The claim now follows from the fact that $G \times_H -$ commutes with disjoint unions (colimits) and our previously derived decomposition of the product $H/L \times H/K$ of H-sets into its orbits. □

[7] For example the terminal object is not mapped to the terminal object unless $G = H$.

Having discussed G-sets in quite some detail, we will now state Dress' definition of Mackey functors.

Definition 3.6 (Dress). Let G-Set be the category of finite G-sets. An *R-valued Mackey functor M over G* is a pair of functors $M = (M_*, M^*)$ with

$$M_*\colon\ G\text{-}\mathsf{Set} \longrightarrow R\text{-Mod} \qquad\qquad M^*\colon\ G\text{-}\mathsf{Set}^{\mathrm{op}} \longrightarrow R\text{-Mod}$$

such that

1. $M_*(X) = M^*(X)$ for all objects $X \in G$-Set;
2. for every pullback-diagram

$$\begin{tikzcd} & X \times_U Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & U & \end{tikzcd}$$

in G-Set we have $M^*(f)M_*(g) = M_*(p_X)M^*(p_Y)$; and

3. the canonical inclusions $\iota_{1,2}\colon X_1, X_2 \longrightarrow X_1 \sqcup X_2$ of two G-sets X_1 and X_2 into their disjoint union induce isomorphisms

$$M_*(\iota_1) + M_*(\iota_2)\colon\ M_*(X_1) \oplus M_*(X_2) \longrightarrow M(X_1 \sqcup X_2),$$
$$M^*(\iota_1) + M^*(\iota_2)\colon\ M(X_1 \sqcup X_2) \longrightarrow M_*(X_1) \oplus M_*(X_2).$$

A natural transformation of two Mackey functors M, N is a collection of R-linear maps $(\phi_X\colon\ M(X) \longrightarrow N(X))_{X\in G\text{-}\mathsf{Set}}$ that is natural with respect to the co- and contravariant parts of M and N, simultaneously.

A slight reformulation of the above definition is due to Linder. Let Sp_G be the category of spans of G-sets. Its morphisms form a free abelian monoid. By extending the morphism spaces to finite R-linear combinations of such spans we obtain the category Sp_G^R.

Definition 3.7 (Lindner)**.** The category Mky of *Mackey functors* of G with values in modules over R is the category of R-linear functors

$$[\mathsf{Sp}_G^R, R\text{-Mod}]^{\mathrm{lin}}.$$

We will study this definition in detail in Section 6.

Exercise 3.8. Let G be a group with subgroups K and L.

1. Prove that for any $x, y \in G$ either $KxL = KyL$ or $KxL \cap KyL = 0$.

2. Suppose K is normal. Show $K\backslash G/L = G/KL$.

3. Show that

This is Burnside's Lemma.

$$K\backslash G/L = \frac{1}{|K||L|} \sum_{(h,k)\in H\times K} \#\{\, g \in G \mid hgk = g \,\}.$$

4. Consider the group $G = \mathrm{GL}_2(\mathbb{R})$ and let $K = L$ be the subgroup of upper triangular matrices. Determine all double cosets $K\backslash G/L$.

3.1.4 Equivalence of the constructions

In order to prove that the two definitions of Mackey functors stated so far are equivalent, we will, for a brief moment, speak of *Green* and *Dress Mackey functors.* For any Mackey functor $(M_*, M^*)\colon\ G\text{-}\mathsf{Set} \longrightarrow R\text{-Mod}$ in the sense of Dress we define for all $H \le G$ the R-Module $M(H) = M_*(G/H)$. We can now define for any $K \le H \le G$ and $g \in G$ the prototypical operations of a Mackey functor à la Green:

$$I_K^H = M_*(\pi_H^K)\colon\ M(K) \longrightarrow M(H), \qquad R_K^H = M_*(\pi_H^K)\colon\ M(H) \longrightarrow M(K),$$
$$c_g^H = M_*(c_g^H) = M^*(c_{g^{-1}}^{gHg^{-1}})\colon\ M(H) \longrightarrow M(gHg^{-1}).$$

Note that the two equivalent ways to define the conjugation operation is a first application of Axiom (2).

Theorem 3.9. *There is a functor*

$$\mathfrak{F}\colon \mathsf{Mky}_{\mathrm{Dress}} \longrightarrow \mathsf{Mky}_{\mathrm{Green}}$$

from the category of Dress Mackey functors to Green Mackey functors given for all $H \leq G$ by

$$\mathfrak{F}(M_*, M^*)(H) = M\left({}^{G}\!/\!_{H}\right), \qquad \mathfrak{F}(\phi)_H = \phi_{G/H}. \tag{3.4}$$

Proof. Suppose $(M_*, M^*)\colon G\text{-}\mathsf{Set} \longrightarrow R\text{-Mod}$ is a Dress-Mackey functor. In order to show that the family of R-Modules $\{M(H) \mid H \leq G\}$ with the previously stated induction, restriction, and conjugation operations obey the axioms of the original definition of Mackey functors, we prove their validity one-by-one. Note that Axioms (1) and (2) follow from the functoriality of M_* and M^*. Axiom (3) follows from Equation (3.3). Finally, suppose $L \leq K \leq H$ to be subgroups of G. The additivity and pullback property of Dress Mackey functors imply—in combination with Theorem 3.5—Axiom (4).

Suppose $\phi\colon (M_*, M^*) \longrightarrow (N_*, N^*)$ is a natural transformation of Dress-Mackey functors. For any $g \in G$ and $K \leq H \leq G$ we have

$$\phi_{G/H} M_*(\pi^H_K) = N_*(\pi^H_K)\phi_{G/K}, \qquad \phi_{G/K} M^*(\pi^H_K) = N^*(\pi^H_K)\phi_{G/H},$$
$$\phi_{G/gHg^{-1}} M_*(c^H_g) = N_*(c^H_g)\phi_{G/H}.$$

Therefore $\mathfrak{F}\colon \mathsf{Mky}_{Dress} \longrightarrow \mathsf{Mky}_{Green}$ is a well-defined functor. □

In order to associate to a Green Mackey functor a Dress Mackey functor we have to solve the following problem.

Suppose X is a transitive G-set and $x \in X$. Given a Mackey functor in the sense of Green $M\colon \{H \leq G\} \longrightarrow R\text{-Mod}$, we would like to associate to $X \cong G/G_x$ the R-module $M(G_x)$. How can we make this construction base-point independent?

One solution is to first consider the direct sum $\oplus_{x\in X} M(G_x)$ and then pass to a quotient where the various choices of base-points are identified using the action of G on X.

For any $x \in X$ and $m \in M(G_x)$ we write m_x for the image of m under the canonical embedding $\iota_x\colon M(G_x) \longrightarrow \oplus_{x\in X} M(G_x)$. Let $B(X)$ be the submodule of $\oplus_{x\in X} M(G_x)$ spanned by

$$\{m_x - (c^{G_x}_g(m))_{g\triangleright x} \mid x \in X, g \in G, xm \in M(G_x)\}.$$

We write

$$M(X) = \bigoplus_{x\in X} M(G_x) \Big/ B(X).$$

While this construction seems to be depended on certain choices, $M(X)$ can also be characterised by its universal property.

Lemma 3.10. *Let M be a Mackey functor in the sense of Green, X a transitive G-set. For any $x \in X$, R-module P and R-linear map $f\colon M(G_x) \longrightarrow P$ there exists a unique R-linear map $\xi\colon M(X) \longrightarrow P$ such that $\xi([m_x]) = f(m)$ for all $m \in M(G_x)$.*

Proof. Suppose we are given a map $f\colon M(G_x) \longrightarrow P$. For any $g \in G$ we consider the composition

$$f_{g \triangleright x} = M(G_{g \triangleright x}) \xrightarrow{c_{g^{-1}}^{gG_xg^{-1}}} M(G_x) \xrightarrow{f} P.$$

The collection of these maps induces $\oplus_{g \in [G/G_x]} f_{g \triangleright x}\colon \oplus_{x \in X} M(G_x) \longrightarrow P$. The kernel of this map contains the subspace $B(X)$, implying the existence of the map $\xi\colon M(X) \longrightarrow P$.

Since any vector in $M(X)$ has a representant whose entries in any coordinate different from x are zero, ξ is uniquely determined by f. □

Theorem 3.11. *There is a functor $\mathfrak{G}\colon \mathsf{Mky}_{\mathfrak{Green}} \longrightarrow \mathsf{Mky}_{\mathfrak{Dress}}$ specified by*

$$\mathfrak{G}(M) := (M_*, M^*)\colon G\text{-}\mathsf{Set} \longrightarrow R\text{-Mod},$$

with

$$M_*({}^{G}\!/\!_{H}) = M(H) = M^*({}^{G}\!/\!_{H}), \quad M_*(\pi_K^H) = M(I_K^H), \quad M_*(c_g^H) = M(c_g^H),$$
$$M^*(\pi_K^H) = M(R_K^H), \quad \text{and} \quad M^*(c_g^H) = M(c_{g^{-1}}^{gHg^{-1}}).$$

Proof. Let X be a transitive G-set. We define $M_*(X) = M(X) = M^*(X)$. Now suppose $f\colon X \longrightarrow Y$ is a morphism of transitive G-sets. Unless X is empty (and we have $M_*(f) = M^*(f) = 0$) we can fix a point $x \in X$ with stabiliser group K and a point $y \in f(X)$ whose stabiliser group we denote by H. The unique factorisation of morphisms of transitive G-sets stated in Theorem 3.4 allows us to uniquely decompose $f = \pi_{gKg^{-1}}^H c_g^K$. Due to Lemma 3.10 there are unique arrows $M_*(f)\colon M(X) \longrightarrow M(Y)$ and $M^*(f)\colon M(Y) \longrightarrow M(X)$ induced by the compositions $M(I_{gKg^{-1}}^H)M(c_g^K)$ and $M(c_{g^{-1}}^{gKg^{-1}})M(R_{gKg^{-1}}^H)$. This extends to a pair $(M_*, M^*)\colon G\text{-}\mathsf{Set} \longrightarrow R\text{-Mod}$ of functors satisfying Axioms (1) and (3) of Dress' definition.

Now suppose we are given the pullback Σ of $X \xrightarrow{f} U \xleftarrow{g} Y$. If $f(X) \cap g(Y) = \varnothing$, then the pullback is the emtpy set and $M^*(f)M_*(g) = 0$.

Otherwise we can assume without loss of generality that X and Y are transitive and that there are $x \in X$ and $y \in Y$ with $f(x) = g(y)$. We write $L = G_x$, $K = G_y$ and $H = G_{f(x)} = G_{g(y)}$. Then Σ is the pullback of

$$G/L \xrightarrow{\pi_L^H} G/H \xleftarrow{\pi_K^H} G/K\ .$$

By Theorem 3.5 the Mackey formula of the Green Mackey functor M implies Axiom 2.

Lastly, the decomposition of maps of G-sets into projections and conjugations, combined with the relations of morphisms of Mackey functors in the sense of Green imply that $\mathcal{G}$ maps arrows between Green Mackey functors to bivariant natural transformations. □

A direct computation now shows that $\mathfrak{F}$ and $\mathfrak{G}$ are quasi-inverse to each other.

Theorem 3.12. *The categories of Green and Dress Mackey functors are equivalent.*

3.2 The Mackey algebra

A REPRESENTATION of a quiver[8] Γ is an assignment of an R-module to any vertex of Γ and an R-linear map for every edge with sources and targets given by the modules assigned to the source and target vertex, respectively. The similarity between this construction and Green's definition of a Mackey functor suggest that we can think of them as quiver representations.

[8] A *quiver* is a directed graph in the most general sense, meaning we allow things such as loops and multiple edges between the same two vertices.

Definition 3.13. The *Mackey quiver* Γ_G of a group G is the (labelled quiver) given by the set of vertices $\{H \le G\}$ and for all $K \le H \le G$ and $g \in G$ edges $K \xrightarrow{I_K^H} H$, $H \xrightarrow{R_K^H} K$ and $H \xrightarrow{c_g^H} gHg^{-1}$.

The *Mackey algebra* μ_G^R is the quotient of the path algebra of Γ_G by the two-sided ideal generated by Axioms (2)–(4) of Definition 3.1 and I_H^H, R_H^H, as well as c_h^H, are identified with the path of length zero starting at H for all $H \le G$ and $h \in H$.

Before stating how modules over the Mackey algebra relate to Mackey functors, let us remark that pointwise addition and scalar multiplication of Mackey functors turns Mky_G into an R-linear category.

Theorem 3.14. *The category Mky_G of Mackey functors of G is equivalent as an R-linear category to the category μ_G^R-*Mod *of modules over the Mackey algebra.*

Proof. This statement follows from a direct computation. □

In order to show that the Mackey algebra is free and of finite rank, a second definition is needed. Recall that Mackey functors could be understood as linear functors on spans of G-sets.

Inspired by Lindner's definition of Mackey functors, see Definition 3.7, we consider the R-module

$$\mu_G^{R'} := \bigoplus_{H,K \le G} \mathsf{Sp}^R_{G\text{-}\mathsf{Set}}\left(G/H, G/K\right).$$

It admits a natural algebra structure with the multiplication of two homogeneous elements ϕ, ψ being either $\phi \circ \psi$ if they are composable or 0 otherwise. Again, its modules are equivalent to the Mackey functors of G. As a finite direct sum of free R-module of finite rank, itself is free and of finite rank.

Theorem 3.15. *The algebras μ_G^R and $\mu_G^{R'}$ are isomorphic; a basis of μ_G^R is given by elements*

$$\left\{ I^K_{xLx^{-1}} c_x^L R_L^H \,\middle|\, x \in \left[K\backslash G/H\right], L \le H \cap x^{-1}Kx \text{ up to conjugation with } H \cap x^{-1}Hx \right\}$$

Since we did not yet discuss Lindner's definition in detail, we will not give a proof of the claim here. Instead we refer the reader to Section 2 of [TW95]

For any subgroup $H \le G$ we write v_H for the vertex of Γ_G labelled by H. Given that v_H is an idempotent, we obtain projective Mackey functors indexed by subgroups $H \le G$.

Theorem 3.16. *Every subgroup $H \le G$ induces a projective Mackey functor $\mu_G^{\Bbbk} v_H$.*

4 Relative homological algebra (Edoardo)

4.1 The Dress construction

THIS TALK is mainly based on [Dre73], and later also on [EJ00]. Let $\mathcal{A}$ be a small category with finite products.[9] Denote by $T_{\mathcal{A}}$ the empty product—i.e., the terminal object.

[9] This is also often called a *Cartesian category*.

If $\mathcal{B}$ is any abelian category, recall that the functor category $[\mathcal{A}, \mathcal{B}]$ is still abelian; see [ML98, p. 199].

Proposition 4.1. *For all $M^*\colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$ and $M\colon \mathcal{A} \longrightarrow \mathcal{B}$, there exist functors*

$$P^*\colon \mathcal{A}^{\mathrm{op}} \longrightarrow [\mathcal{A}^{\mathrm{op}}, \mathcal{B}] \qquad\qquad P\colon \mathcal{A} \longrightarrow [\mathcal{A}, \mathcal{B}]$$
$$x \longmapsto M^*_x, \qquad\qquad x \longmapsto M_{*,x},$$

where

$$M^*_x \coloneqq M^*(- \times x), \qquad\qquad M_{*,x} \coloneqq M_*(x \times -).$$

On morphisms, these functors do the obvious thing; e.g., for $f\colon x \longrightarrow y$ in $\mathcal{A}$, we have $P^* f \coloneqq M^*(f \times \mathrm{id})$.

We will often denote $M_{*,x}$ by just M_x.

Remark 4.2. Note that, for a contravariant functor $M^* : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$, one obtains natural transformations

$$\pi^{*,x}\colon M^* \Longrightarrow M^*_x,$$

given by precomposition with the respective projections π, coming from the Cartesian structure. More explicitly, we have

$$\pi^{*,x}_y\colon M^* y \xrightarrow{M^*\pi_y} M^*(y \times x)$$

on components.

Likewise, there exists a natural transformation $\pi^x_*\colon M_* \Longrightarrow M_x$ for all functors $M\colon \mathcal{A} \longrightarrow \mathcal{B}$ and objects $x \in \mathcal{A}$.

Definition 4.3. The *Dress construction* is the functor

$$\mathfrak{D}\colon \mathcal{A} \times [\mathcal{A}, \mathcal{B}] \longrightarrow [\mathcal{A}, \mathcal{B}], \qquad\qquad (x, M) \longmapsto M_x.$$

If we apply this to Lindner's category of Mackey functors, see Definition 6.24, it turns out that the Dress construction is a strong monoidal $\Bbbk$-linear functor [PS07, Proposition 8.1].[10]

[10] Even more, this extends to the centre of $\mathcal{A}$! According to [PS07, Remark 8.5], there is in fact a strong monoidal $\Bbbk$-linear functor

$$\mathfrak{D}\colon \mathsf{Z}\mathcal{A} \otimes \mathsf{Mky} \longrightarrow \mathsf{Mky}.$$

4.1.1 Relative projectiveness

Definition 4.4. The sequence $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{A}$ is *Dress split in y*, if there exist $r\colon y \longrightarrow x$ and $s\colon z \longrightarrow y$, such that

$$f \circ r + s \circ g = \mathrm{id}_y.$$

Lemma 4.5 (Relationship with split exact sequences)**.**

1. *If* $x \xrightarrow{f} y \xrightarrow{g} z$ *is Dress split, then the sequence*

$$0 \longrightarrow \operatorname{im} f \xrightarrow{f_0} y \xrightarrow{\hat{g}} \operatorname{im} g \longrightarrow 0$$

is split exact[11].

2. *If* $0 \longrightarrow x \xrightarrow{f} y \xrightarrow{g} z \longrightarrow 0$ *is split exact, then* $x \xrightarrow{f} y \xrightarrow{g} z$ *is Dress split in* y.

[11] That is, equivalently, either f_0 or $\hat{g}$ has a split.

Definition 4.6. Let $x\mathcal{A}$, and $M^*, N^*, P^* \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$. The sequence

$$M^* \xrightarrow{f} N^* \xrightarrow{g} P^*$$

is called *x-split at N^**, if $M^*_x \longrightarrow N^*_x \longrightarrow P^*_x$ is split exact at N^*_x.

Lemma 4.7.

1. *The sequences* $0^* \longrightarrow M^* \xrightarrow{\pi^*} M^*_x$ *and* $M_x \longrightarrow M \longrightarrow 0$ *are x-split.*

2. *If* $\mathcal{A}(y, x) \neq \varnothing$*, then a sequence that is x-split is also y-split.*

Proposition 4.8. *Let* $M^* \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$ *and* $x \in \mathcal{A}$*. Then the following are equivalent:*

1. $0^* \longrightarrow M^* \xrightarrow{\pi^*} M^*_x$ *splits*

2. *There exists a functor* $\mathfrak{N} \colon (\mathcal{A}/x)^{\mathrm{op}} \longrightarrow \mathcal{B}$*, such that* M^* *is a direct summand in the composition:*

$$\mathcal{A}^{\mathrm{op}} \longrightarrow \left(\mathcal{A}\!\big/\!x\right)^{\mathrm{op}} \xrightarrow{\mathfrak{N}} \mathcal{B}.$$

Above, $\mathcal{A}/x$ *is the* slice category[12].

3. *For every diagram*

$$\begin{tikzcd} 0^* \arrow[r] & N^* \arrow[r] \arrow[d] & P^* \\ & M^* & \end{tikzcd}$$

with an x-split line, there exists a $\psi \colon P^* \Longrightarrow M^*$ *making it commute.*

4. *Any x-split sequence* $0^* \longrightarrow M^* \longrightarrow N^*$ *splits.*

[12] Objects in the slice category are arrows $f \colon x \longrightarrow a \times b$, and morphisms are morphisms between the respective sources, making the obvious triangle commute.

Definition 4.9. A presheaf M^* satisfying any of the equivalent properties of Proposition 4.8 is called *x-injective.*

Dually, one arrives at the notion of *x-projectivity.*

Proposition 4.10 (Amitsur)**.** *For any* $x \in \mathcal{A}$*, consider the simplicial complex* $\mathrm{Am}(x)$*, where all of the arrows are induced from the projections:*

$$x \longleftarrow\!\!\!\!\longleftarrow x \times x \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; x \times x \times x \qquad \dots$$

Then for any $M^* \colon \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{B}$*, one obtains a complex of x-injective functors*

$$\mathrm{Am}(x, M) \colon \; 0 \longrightarrow M_x \xrightarrow{\partial^1} M_{x^2} \xrightarrow{\partial^2} M_{x^3} \longrightarrow \dots,$$

with $\partial^n \coloneqq \sum_{i=1}^{n} (-1)^n \cdot M\pi^n_i$*. With the augmentation* $M \longrightarrow M_x$*, we have an x-split complex, hence*

$$H^i_x M = {\ker \partial^{i+1}}\big/{\operatorname{im} \partial^i}.$$

4.2 Relative Homological Algebra

Throughout this section, let R be a commutative ring, and R-Mod the category of R-modules.[13] Furthermore, suppose that $\mathfrak{F} \subseteq R\text{-Mod}$ is a class of R-modules closed under isomorphisms.

[13] One may well replace the category of R-modules in this talk with an appropriate category C having similar properties.

Definition 4.11. Let $M \in R\text{-Mod}$. An *$\mathfrak{F}$-cover of M* comprises an object $C \in \mathfrak{F}$, and a morphism $\varphi \in \mathrm{Hom}_{R\text{-Mod}}(C, M)$, such that the following conditions are satisfied:

1. For any $C' \in R\text{-Mod}$ and $\psi \in \mathrm{Hom}_{R\text{-Mod}}(C', M)$, there exists an arrow from C' to C, such that

$$\begin{tikzcd} C' \arrow[d, dashed] \arrow[dr, "\psi"] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

commutes.[14]

2. The diagram

$$\begin{tikzcd} C \arrow[d, dashed] \arrow[dr, "\varphi"] & \\ C \arrow[r, "\varphi"'] & M \end{tikzcd}$$

may only be filled by an automorphism of C.

A morphism such that only Condition 1 is satisfied is called an *$\mathfrak{F}$-precover*.

If $\mathfrak{F}$ and M are clear from the context, we may also just say that φ is a *(pre)cover*.

[14] Note that the morphism is not require to be unique!

Proposition 4.12. *Let M be an R-module.*

1. *If an $\mathfrak{F}$-cover of M exists, then it is unique up to isomorphism.*
2. *If the class $\mathfrak{F}$ contains R as a module, then every $\mathfrak{F}$-precover is surjective.*

Proof.

1. Let $\phi\colon C \longrightarrow M$ and $\phi'\colon C' \longrightarrow M$ be covers and consider the diagram

$$\begin{tikzcd} C \arrow[d, dashed, "\alpha"'] \arrow[dr, "\phi"] & \\ C' \arrow[r, "\phi'"] \arrow[d, dashed, "\beta"'] & M. \\ C \arrow[ur, "\phi"'] & \end{tikzcd}$$

By Condition 2 of Definition 4.11, the composition $\beta \circ \alpha$ is an automorphism of C; by reasoning in a similar way we also get that $\alpha \circ \beta$ is an automorphism of C', hence both ϕ and ϕ' are isomorphisms.

2. For every $m \in M$ we may always define the morphism

$$f_m \colon R \longrightarrow M, \qquad f_m(1_R) := m.$$

As its image, this has the cyclic R submodule Rm of M. There must exist $\hat{f}_m$ such that $\phi \circ \hat{f}_m = f_m$, and thus $m = f_m(1_R) \in \operatorname{im} \phi$.

□

Example 4.13. If $\mathfrak{F} := \mathsf{Proj}$ is the class of projective modules, an $\mathfrak{F}$-precover of M—also called a *projective precover* in this case—is a surjective homomorphism from a projective module $\phi \colon P_1 \longrightarrow M$, as the following diagram shows.

$$\begin{tikzcd} P_1 \arrow[dr, two heads, "\phi"] & \\ P_2 \arrow[u, dashed] \arrow[r] & M \end{tikzcd}$$

We say that the class $\mathfrak{F}$ is a *(pre)covering* if every R-module admits an $\mathfrak{F}$-(pre)cover.

Proposition 4.14. *Let M be an R-module. If an $\mathfrak{F}$-cover of M exists, then it is a direct summand of any $\mathfrak{F}$-precover of M.*

Proof. Let $\varphi \colon C \longrightarrow M$ be an $\mathfrak{F}$-cover, and $f \colon C' \longrightarrow M$ an $\mathfrak{F}$-precover. The the following diagram commutes, by the properties of covers and precovers:

$$\begin{tikzcd} C \arrow[d, dashed, "\alpha"'] \arrow[dr, "\varphi"] & \\ C' \arrow[r, "f"] \arrow[d, dashed, "\beta"'] & M \\ C \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Since φ is a cover, it just be that $\beta \circ \alpha$ is an automorphism of C.

Defining $\gamma := \beta \circ \alpha$, we have that $\mathrm{id}_C = \gamma \circ \gamma^{-1} = \beta \circ \alpha \circ \gamma^{-1}$, and thus $\alpha \circ \gamma^{-1}$ splits β, so we are done. □

One way of thinking about this result is as a generalization of the well known fact that every projective module is a direct summand of a free module. If M admits a projective cover $\phi \colon P \longrightarrow M$, then among all the projective precovers of M we may choose a free one $\psi \colon F \longrightarrow M$ and P will be a direct summand of F.

Dual to the notion of covers we have that of envelopes.

Definition 4.15. Let M be an R-module. A morphism $\phi \colon M \longrightarrow C$ is called an $\mathfrak{F}$-*envelope* of M if it satisfies the following:

1. For any morphism $f \colon M \longrightarrow C'$ with $C' \in \mathfrak{F}$, there exists a arrow $\hat{f} \colon C \longrightarrow C'$ making the following diagram commute:

As before, the morphism $\hat{f}$ need not be unique.

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "f"'] & C \arrow[d, dashed, "\hat{f}"] \\ & C' \end{tikzcd}$$

2. If $f = \phi$, then $\hat{f}$ is an automorphisms of C.

If $\phi\colon\ M \longrightarrow C$ satisfies only Condition 1, then it is called an $\mathfrak{F}$-*preenvelope*.

Just like for covers, any $\mathfrak{F}$-envelope—if it exists—is unique up to isomorphism, and a direct summand of any $\mathfrak{F}$-preenvelope.

4.2.1 From classical to relative homological algebra

In classical homological algebra one gives the following (dual) definitions.

Definition 4.16. Let M be an R-module; a *projective resolution* of M is an exact sequence

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,$$

where each P_i is a projective module.

Dually, an *injective resolution* of M is an exact sequence

$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \ldots,$$

where each I_j is an injective module.

These definitions clearly make sense for any abelian category[15] C, not only for R-modules. However, the hypothesis of being abelian is not sufficient to guarantee the existence of such resolutions: one needs C to have *enough* projectives and injectives.

[15] An *abelian category* is a category C, such that all hom-sets $C(x, y)$ have a canonical abelian group structure, C has biproducts, every morphism has a kernel and cokernel, every mono is a kernel, and every epi is a cokernel.
Note that abelian categories are always finitely complete and cocomplete.

Definition 4.17. The abelian category C is said to have *enough projectives* if for every object $M \in C$ there exists an epimorphism $P \twoheadrightarrow M$ with P projective.

Dually, C has *enough injectives*, if for each $M \in C$ there exists a monomorphism $M \hookrightarrow I$ with I injective.

For C to have enough projectives, we may equivalently require the existence of a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0,$$

for all objects M, with P projective.

Dually, for C to have enough injectives, we would need

$$0 \longrightarrow M \longrightarrow I \longrightarrow Q \longrightarrow 0$$

with I injective.

For a category C the problem of "having enough projectives (injectives)" in classical homological algebra corresponds to the existence of precovers (preenvelopes) in relative homological algebra. Making this correspondence precise goes beyond the scope of these notes, henceforth we will only sketch the main ideas; the interested reader may find this topic explained in Chapter 7 of [EJ00].

Generalising the notions of projective and injective objects to the relative context requires the following technical definition.

Definition 4.18. Let $\mathfrak{F}$ and $\mathfrak{G}$ be classes of objects of the abelian category $\mathcal{C}$ with enough projectives and injectives. Denote by $\mathfrak{F}^{\perp}$ the class of objects $N \in \mathcal{C}$, such that $\mathrm{Ext}^1(M, N) = 0$ for all $M \in \mathfrak{F}$. Similarly, denote by ${}^{\perp}\mathfrak{F}$ the class of objects $L \in \mathcal{C}$, such that $\mathrm{Ext}^1(L, M) = 0$ for all $M \in \mathfrak{F}$.

The pair $(\mathfrak{F}, \mathfrak{G})$ is a *cotorsion theory for* $\mathcal{C}$ if $\mathfrak{F}^{\perp} = \mathfrak{G}$ and ${}^{\perp}\mathfrak{G} = \mathfrak{F}$.

Example 4.19. Setting $\mathfrak{M} \coloneqq \mathrm{Ob}\, R\text{-Mod}$, Proj to be the class of projective R-modules, and Inf to be the class of injective R-modules, examples of cotorsion theories are the pairs $(\mathfrak{M}, \mathsf{Inj})$ and $(\mathsf{Proj}, \mathfrak{M})$.

Definition 4.18 allows us to express the property of having enough projectives and/or injectives in the relative setting.

Definition 4.20. A cotorsion theory $(\mathfrak{F}, \mathfrak{G})$ has *enough injectives* if for every object $M \in \mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow G \longrightarrow F \longrightarrow 0$$

with $G \in \mathcal{G}$ and $F \in \mathfrak{F}$.

Dually, $(\mathfrak{F}, \mathfrak{G})$ has *enough projectives* if for every object $M \in \mathcal{C}$ there exists a short exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

with $G \in \mathcal{G}$ and $F \in \mathfrak{F}$.

Theorem 4.21 ([EJ00])**.** *Let $(\mathfrak{F}, \mathcal{G})$ be a cotorsion theory for R-Mod, and suppose that $\mathfrak{F}$ is closed under well-ordered inductive limits*[16]*. If $(\mathfrak{F}, \mathfrak{G})$ has enough injectives and projectives, then every R-module has an $\mathfrak{F}$-cover and a $\mathfrak{G}$-envelope.*

[16] A different name for well-ordered inductive limit is *filtered colimit.*

Going back to classical homological algebra, let us momentarily restrict our attention to projective resolutions. Hence, let $P_{\bullet} \longrightarrow M \longrightarrow 0$ be a projective resolution of R-Mod over M, and let $T\colon\ R\text{-Mod} \longrightarrow \mathcal{C}$ be a right exact functor[17] to the abelian category $\mathcal{C}$; by applying T to the resolution wo obtain the complex $TP_{\bullet} \longrightarrow TM$

$$\ldots \longrightarrow TP_2 \longrightarrow TP_1 \longrightarrow TP_0 \longrightarrow TM \longrightarrow 0.$$

[17] A functor is called *right exact,* if it preserves finite colimits. In the abelian case, this is equivalent to turning exact sequences into right exact sequences.

We define the *left derived functors* L_iT by

$$L_iT(M) \coloneqq H_i(TP_{\bullet}).$$

Right derived functors $R^iT(M)$ are dually defined for left exact functors T and injective resolutions. The intuition here is that derived functors measure loss of exactness of the functor T.

Considering the relative case, let us now consider an additive functor $T\colon\ \mathcal{C}^{op} \times \mathcal{D} \longrightarrow \mathcal{E}$, where $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ are all abelian categories. Jenda and Enochs give the following definition, see [EJ00].

Definition 4.22. Let $\mathfrak{F}$ be a class of objects of $\mathcal{C}$. The complex

$$\dots \longrightarrow D_2 \longrightarrow D_1 \longrightarrow D_0 \longrightarrow D^0 \longrightarrow D^1 \longrightarrow \dots$$

in $\mathcal{D}$ is $T(\mathfrak{F}, -)$*-exact* if for every $F \in \mathfrak{F}$ the complex

$$\dots \longrightarrow T(F, D_1) \longrightarrow T(F, D_0) \longrightarrow T(F, D^0) \longrightarrow T(F, D^1) \longrightarrow \dots$$

is exact.

Similarly, if $\mathfrak{G}$ is a class of objects of $\mathcal{D}$, then the complex

$$\dots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots$$

is $T(-, \mathfrak{G})$*-exact* if for every $G \in \mathfrak{G}$ the complex

$$\dots \longrightarrow T(C_1, G) \longrightarrow T(C_0, G) \longrightarrow T(C^0, G) \longrightarrow T(C^1, G) \dots$$

is exact.

Definition 4.23. Let $\mathfrak{F}$ be a class of objects of $\mathcal{C}$ and suppose $M \in \mathcal{C}$. A *left* $\mathfrak{F}$*-resolution* of M is a $\mathcal{C}(\mathfrak{F}, -) := \mathrm{Hom}_{\mathcal{C}}(\mathfrak{F}, -)$-exact complex

$$\dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with all $F_i \in \mathfrak{F}$.

A *right* $\mathfrak{F}$*-resolution* of M is a $\mathcal{C}(-, \mathfrak{F})$-exact complex

$$0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \dots$$

with all $F_i \in \mathfrak{F}$.

Remark 4.24. An object $P \in \mathcal{C}$ is projective if and only if the functor $\mathcal{C}(P, -)$ is exact. Similarly, $I \in \mathcal{C}$ is injective if and only if the functor $\mathcal{C}(-, I)$ is exact. This shows that the relative definitions truly specialize the classical case.

Definition 4.25. Let $\mathcal{C}$ be an abelian category, $\mathfrak{F}$ a precovering class of $\mathcal{C}$, $\mathfrak{G}$ a preenveloping class of $\mathcal{C}$, and $T \colon \mathcal{C} \longrightarrow \mathcal{D}$ an additive functor. For $M \in \mathcal{C}$, suppose that $F_\bullet \longrightarrow M \longrightarrow 0$ is a left $\mathfrak{F}$-resolution of M. The *left* $\mathfrak{F}$*-derived functor for* T is given by

$$L_i T(M) := H_i(TF_\bullet).$$

Similarly, for $T \colon \mathcal{C}^{op} \longrightarrow \mathcal{D}$ additive, and $0 \longrightarrow M \longrightarrow G^\bullet$ a right $\mathfrak{G}$-resolution of M, the *right* $\mathfrak{G}$*-derived functors for* T are defined by

$$R^i T(M) := H^i(TG^\bullet).$$

Remark 4.26. Unlike in the classical case, L_n and R^n *do not* measure the loss of exactness of T, but rather the loss of exactness of T “relative to the hom functor”.

Proposition 4.27. *Let* $\mathcal{C} := [\mathcal{A}^{\mathrm{op}}, \mathcal{B}]$*, with* $\mathcal{B}$ *abelian. Then the class* $\mathfrak{F}_x$ *of* x*-injective functors is a preenveloping. Further, for an object* $M \in \mathcal{C}$*, the sequence* $0 \longrightarrow M \longrightarrow M_x \longrightarrow M_{x^2} \longrightarrow \dots$ *is a right* $\mathfrak{F}_x$*-resolution.*

5 Coends (Ivan)

COENDS PLAY AN IMPORTANT ROLE whenever one wants to talk about, for example, monoidal structures on functor categories. As we will see in Definition 5.19, one such canonical tensor product—*Day convolution*—is best phrased in this language.

While coends can be defined in any appropriately *enriched category*[18], we will specialise this for very concrete categories, like the category of sets, or the category of (finite-dimensional) $\Bbbk$-vector spaces. For a more general account see [Kel05, Chapter 3].

[18] A bit more on the concept of enrichment will be said in Section 6.2.

This talks follows [Lor21], [Ric20], and [ML98].

5.1 Abstract definition

Let $P, Q\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be functors for categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 5.1. A *dinatural transformation* $\alpha\colon P \Longrightarrow Q$ comprises a family of morphisms

$$\alpha := \{\, \alpha_c\colon P(c,c) \longrightarrow Q(c,c) \,\}_{c \in \mathcal{C}},$$

such that for all arrows $f\colon c \longrightarrow d$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
 & P(c,c) \arrow[r, "\alpha_c"] & Q(c,c) \arrow[dr, "{Q(c,f)}"] & \\
P(d,c) \arrow[ur, "{P(f,c)}"] \arrow[dr, "{P(d,f)}"'] & & & Q(c,d) \\
 & P(d,d) \arrow[r, "\alpha_d"'] & Q(d,d) \arrow[ur, "{Q(f,d)}"'] &
\end{tikzcd}$$

Example 5.2 (Evaluation)**.** Let $\Bbbk$ be a commutative ring. If the category $\mathcal{C}$ is modules over $\Bbbk$ (or its group algebra), or even over a finite-dimensional Hopf algebra over $\Bbbk$, we can define

$$\begin{aligned} P\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (x,y) &\longmapsto {}^{\vee}x \otimes y \\ (f,g) &\longmapsto {}^{\vee}f \otimes g \end{aligned} \qquad\qquad \begin{aligned} Q\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (x,y) &\longmapsto \Bbbk \\ (f,g) &\longmapsto \mathrm{id}_{\Bbbk}. \end{aligned}$$

This turns the family

$$\alpha := \{\, \alpha_x\colon {}^{\vee}x \otimes x \xrightarrow{\mathrm{ev}_x} \Bbbk \,\}_{x \in \mathcal{C}}$$

of evaluation morphisms into a dinatural transformation.

For ends and coends, we are interested in a very particular kind of dinatural transformation: one where either P or Q is a constant functor.

Definition 5.3. A *wedge* for $P\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ comprises an object $d \in \mathcal{D}$, and a dinatural transformation $\omega\colon \Delta_d \Longrightarrow P$, where Δ_d is the constant functor on d.[19]

Similarly, a *cowedge* for P is a dinatural transformation $P \Longrightarrow d$.

[19] Often, one identifies d with Δ_d, and just writes $\omega\colon d \Longrightarrow P$ for a wedge.

In fact, there is a category Wed_P: objects are wedges for P, and a morphism between $\omega\colon d \Longrightarrow\!\!\diamond\, P$ and $\omega'\colon d' \Longrightarrow\!\!\diamond\, P$ is a morphism $f\colon d \longrightarrow d'$, such that the triangle

$$\begin{tikzcd} d \arrow[rr, "f"] \arrow[dr, "\omega"'] & & d' \arrow[dl, "\omega'"] \\ & P(c,c) & \end{tikzcd}$$

commutes for all $c \in C$.

Dually, there is also a category Cwd_P.

Definition 5.4. The *end* of $P\colon C^{\mathrm{op}} \times C \longrightarrow \mathcal{D}$ is a terminal wedge, and the *coend* is an initial cowedge for P. In particular, ends and coends are unique up to unique isomorphism.

Following [ML98], we use the notation

$$\int^{c\in C} P(c,c), \qquad \text{and} \qquad \int_{c\in C} P(c,c)$$

for coends and ends, respectively.

Unpacking these definitions, and end is equipped with arrows

$$\left\{ \omega_c\colon \int_{x\in C} P(x,x) \longrightarrow P(c,c) \right\}_{c\in C}$$

such that for all $f\colon c \longrightarrow d$, the diagram

$$\begin{tikzcd} \int_{x\in C} P(x,x) \arrow[r, "\omega_c"] \arrow[d, "\omega_d"'] & P(c,c) \arrow[d, "P(c,f)"] \\ P(d,d) \arrow[r, "P(f,d)"'] & P(c,d) \end{tikzcd}$$

commutes, and we have the following universal property:

> For all other wedges (e, ω'), for which we also have
>
> $$P(c,f) \circ \omega'_c = P(f,d) \circ \omega'_d,$$
>
> there exists a unique morphism $g\colon e \longrightarrow \int_{x\in C} P(x,x)$, such that ω' factors through the end via g.

In diagrams, the following must commute:

$$\begin{tikzcd} e \arrow[dr, dashed, "\exists! g"] \arrow[drr, bend left, "\omega'_c"] \arrow[ddr, bend right, "\omega'_d"'] & & \\ & \int_x P(x,x) \arrow[r, "\omega_c"] \arrow[d, "\omega_d"'] & P(c,c) \arrow[d, "P(c,f)"] \\ & P(d,d) \arrow[r, "P(f,d)"'] & P(c,d) \end{tikzcd}$$

Exercise 5.5. Unpack the definition of a coend.

The "integral notation" for ends and coends is—in part—justified by results like the following.

Theorem 5.6 (Fubini–Tonelli for ends). *Suppose that we have a functor*

$$P\colon\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{E}.$$

If any of the following three ends exist, then they all exist and are isomorphic:

$$\int_{c\in\mathcal{C}} \int_{d\in\mathcal{D}} P(c,c,d,d), \qquad \int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} P(c,c,d,d), \qquad \int_{(c,d)\in\mathcal{C}\times\mathcal{D}} P(c,c,d,d).$$

Remark 5.7. While we have defined (co)ends with the help of dinatural transformation, which is enough for our purposes here, once one dips into enriched category theory, the slightly less general concept of an *extranatural transformation* may be better suited for the task.

An extranatural transformation is a collection of arrows between functors of the form

$$\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{E},$$

satisfying appropriate conditions. Quoting from [Lor21, p. 9]:

> We should prefer extranaturality for a variety of reasons:
>
> - it is less general (see 1.1.12), but it still makes co/ends available;
> - it gives rise to a fairly intuitive graphical calculus (see 1.1.10); moreover, it behaves better under composition (see Exercise 1.4);
> - extranaturality is the correct notion in the enriched setting (see 4.3.7 and the caveat right after).

5.2 (Co)ends as limits

Perhaps unsurprisingly, (co)ends can be expressed as certain (co)limits. This has important consequences for their existence, and so we will go through one construction in detail.

Definition 5.8. The *limit* of a functor $F\colon\ \mathcal{D} \longrightarrow \mathcal{C}$ comprises an object $\lim_{\mathcal{D}} F$ in $\mathcal{C}$, as well as a natural transformation $\tau\colon\ \lim_{\mathcal{D}} F \Longrightarrow F$ with the following universal property: if (l, τ') is another such pair, there exists a unique morphism

$$\xi\colon\ l \longrightarrow \lim_{\mathcal{D}} F,$$

such that $\tau' = \tau \circ \xi$.

While not immediately obvious, ends and coends may be realised as limits in a relevant category.

Definition 5.9. Let $\mathcal{C}$ be a category. The *twisted arrow category* $\mathsf{Tw}_{\mathcal{C}}$ of $\mathcal{C}$ is defined as follows:

- Objects of $\mathsf{Tw}_{\mathcal{C}}$ are morphisms in $\mathcal{C}$.

- A morphism between $f\colon c \longrightarrow d$ and $g\colon x \longrightarrow y$ is a pair
$$(k\colon y \longrightarrow c, l\colon d \longrightarrow y),$$
such that
$$\begin{tikzcd} c & x \arrow[l, "k"'] \arrow[d, "g"] \\ d \arrow[u, leftarrow, "f"] \arrow[r, "l"'] & y \end{tikzcd}$$
commutes.

- Composition is done by composing commutative diagrams:
$$(m, n) \circ (k, l) \coloneqq (k \circ m, n \circ l);$$
i.e.,
$$\begin{tikzcd} c \arrow[d, "f"'] & x \arrow[l, "k"'] \arrow[d, "g"] & a \arrow[l, "m"'] \arrow[d, "h"] \\ d \arrow[r, "l"'] & y & b \arrow[l, "n"] \end{tikzcd}$$

- The identity arrow for $f\colon c \longrightarrow d$ is $(\mathrm{id}_c, \mathrm{id}_d)$.

Exercise 5.10. Prove that
$$\begin{aligned} \chi\colon \mathsf{Tw}_C &\longrightarrow C^{\mathrm{op}} \times C \\ c \xrightarrow{f} d &\longmapsto (c, d) \\ (k, l) &\longmapsto (k^{\mathrm{op}}, l) \end{aligned}$$
defines a functor.

Note that the above functor lifts to one on the functor categories
$$[C^{\mathrm{op}} \times C, \mathcal{D}] \longrightarrow [\mathsf{Tw}_C, \mathcal{D}], \qquad (C^{\mathrm{op}} \times C \xrightarrow{P} \mathcal{D}) \longmapsto (\mathsf{Tw}_C \xrightarrow{\bar{P}} \mathcal{D}),$$
with $\bar{P}(c \xrightarrow{f} d) \coloneqq P(c, d)$. In fact, bifunctoriality of P corresponds exactly to functoriality of $\bar{P}$. There is even more to this story, as the next proposition tells us.

Proposition 5.11. *Let $P\colon C^{\mathrm{op}} \times C \longrightarrow \mathcal{D}$ be a functor. Then the end*
$$(e, \tau\colon e \Longrightarrow\!\!\diamond\, P \circ \chi)$$
is isomorphic to $\lim_{\mathsf{Tw}_C}(P \circ \chi)$.

Note that $\tau\colon e \Longrightarrow\!\!\diamond\, P \circ \chi$ is defined by setting
$$\tau_f \coloneqq P(a, f) \circ \omega_a \overset{\text{dinat}}{=} P(f, b) \circ \omega_b, \qquad \text{for } f\colon a \longrightarrow b \text{ in } \mathsf{Tw}_C.$$

Proof. Naturality of τ boils down to showing that for all arrows (k, l) between $f\colon a \longrightarrow b$ and $g\colon c \longrightarrow d$ in Tw_C, the diagram

$$\begin{tikzcd} e \arrow[r, equal] \arrow[d, "\tau_f"'] & e \arrow[d, "\tau_g"] \\ P \circ \chi(f) \arrow[r, "P\circ\chi(k,l)"'] & P \circ \chi(g) \end{tikzcd}$$

commutes. Recall that for (k, l) we—by definition—have $l \circ f \circ k = g$. To show that $P \circ \chi(k, l) \circ \tau_f = \tau_g$, consider the following diagram:

$$\begin{tikzcd} & P(a,a) \arrow[r, "P(a,f)"] \arrow[d, "P(k,a)"] \arrow[dr, phantom, "(ii)"] & P(a,b) \arrow[d, "P(k,b)"'] \arrow[dd, bend left, "P\circ\chi(k,l)"] \\ e \arrow[ur, "\omega_a"] \arrow[dr, "\omega_c"'] \arrow[r, phantom, "(i)"] & P(c,a) \arrow[r, "P(c,f)"] \arrow[dr, phantom, "(iii)"] & P(c,b) \arrow[d, "P(c,l)"'] \arrow[r, phantom, "(iv)"] & {} \\ & P(c,c) \arrow[u, "P(c,k)"'] \arrow[r, "P(c,g)"'] & P(c,d) \end{tikzcd}$$

We have that (i) commutes by the dinaturality of ω, the squares (ii) and (iii) are functoriality of P, and (iv) commutes by definition of $P \circ \chi(k, l)$.

The universality of left as an exercise. □

Exercise 5.12. Write down (and prove) a dual version of this statement for coends.

Taken together, Proposition 5.11 and Exercise 5.12 give existence results for ends and coends: if C is small, and $\mathcal{D}$ is (co)complete, the (co)end of $P\colon C^{\mathrm{op}} \times C \longrightarrow \mathcal{D}$ always exists!

Remark 5.13. There is also a more direct way to write down coends as coequalisers:

$$\int^{c\in C} P(c,c) \cong \operatorname{coeq}\Big(\coprod_{f\colon c\longrightarrow d} P(d,c) \overset{P^*}{\underset{P_*}{\rightrightarrows}} \coprod_{c} P(c,c) \Big),$$

where the arrows P_* and P^* are induced by

$$P(d, f)\colon P(d,c) \longrightarrow P(d,d), \qquad \text{and} \qquad P(f, c)\colon P(d,c) \longrightarrow P(c,c),$$

respectively.

Likewise, it is possible to write down ends as equalisers:

$$\int_{c\in C} P(c,c) \cong \operatorname{eq}\Big(\prod_{f\colon c\longrightarrow d} P(d,c) \overset{P_*}{\underset{P^*}{\leftleftarrows}} \prod_{c} P(c,c) \Big).$$

Example 5.14 (Recovering the tensor product of modules)**.** Let R be a ring (with unit), and suppose that A is a left, and B is a right R-module.

The delooping $\mathbf{B}R$ of R is an Ab-enriched category with one object $\star$, such that $\mathbf{B}R(\star, \star) = R$. Multiplication is recovered by composition, and addition by the abelian group structure on the hom-set.

In this setting, B becomes a functor

$$B\colon \mathbf{B}R^{\mathrm{op}} \longrightarrow \mathsf{Ab}, \quad \star \longmapsto B, \quad (\star \xrightarrow{r} \star) \longmapsto r^* := (b \longmapsto br).$$

Equivalently, A can be seen as a functor $\mathbf{B}R \longrightarrow \mathsf{Ab}$.

Defining a functor

$$P\colon \mathbf{B}R^{\mathrm{op}} \times \mathbf{B}R \longrightarrow \mathsf{Ab}, \qquad (\star,\star) \longmapsto B \otimes_{\mathbb{Z}} A, \qquad (r,s) \longmapsto r^* \otimes s_*,$$

one can see that

$$\int^{\star} P(\star,\star) \cong B \otimes_R A.$$

Example 5.15 (Elmendorf's reconstruction)**.** Let G be a topological group, and suppose that X is a G-space[20], meaning the action $G \times X \longrightarrow X$ is continuous.

[20] By *space* we always mean a *nice enough* topological space; e.g., a compactly generated (weak) Hausdorff space. We may still denote the category of such spaces by Top.

Define the category Orb_G as follows:

- objects are subgroups of G, and
- a morphism between H and K is a G-equivariant map $G/H \longrightarrow G/K$.

We in particular obtain functors

$$\begin{aligned} X^{(-)}\colon \mathsf{Orb}_G &\longrightarrow \mathsf{Top} \\ H \le G &\longmapsto X^H \end{aligned} \qquad\qquad \begin{aligned} {}^{G}\!/_{(-)}\colon \mathsf{Orb}_G &\longrightarrow \mathsf{Top} \\ H \le G &\longmapsto {}^{G}\!/_{H}, \end{aligned}$$

with which we can calculate

$$X \cong \int^{H \in \mathsf{Orb}_G} X^H \times {}^{G}\!/_{H}.$$

Exercise 5.16. Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} :U$ be an adjunction, and $G\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{E}$ a functor. Show that there is an isomorphism

$$\int^{c \in \mathcal{C}} G(Fc, c) \cong \int^{d \in \mathcal{D}} G(d, Ud).$$

Exercise 5.17 (Ends and coends are functorial)**.** Let $P, Q\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be two functors, and suppose that $\eta\colon P \Longrightarrow Q$ is a natural transformation.

1. Show that there is a unique morphism

$$\int_c \eta_c\colon \int_c P(c,c) \longrightarrow \int_c Q(c,c),$$

such that for all objects $x \in \mathcal{C}$, the diagram

$$\begin{tikzcd} P(x,x) \arrow[r, "\omega_c"] \arrow[dr, "\omega'_c"'] & \int_c P(c,c) \arrow[d, "\int_c \eta_c"] \\ & \int_c Q(c,c) \end{tikzcd}$$

commutes.

2. Prove that the above construction is functorial.
3. Formulate the equivalent statement for coends.

5.3 Day convolution

We are now ready to define the tensor product on Mackey functors. Suppose that $\mathcal{C}$ is a small symmetric monoidal $\Bbbk$-linear category.

Definition 5.18. Let $F, G\colon \mathcal{C} \longrightarrow \mathsf{Vect}_{\Bbbk}$ be $\Bbbk$-linear functors. The *external tensor product* of F and G is defined by

$$\begin{aligned} F \boxtimes G\colon\ \mathcal{C} \times \mathcal{C} &\longrightarrow \mathsf{Vect}_{\Bbbk} \\ (c, c') &\longmapsto Fc \otimes_{\Bbbk} Gc' \\ (f, f') &\longmapsto Ff \otimes_{\Bbbk} Gf'. \end{aligned}$$

Definition 5.19. Let $F, G\colon \mathcal{C} \longrightarrow \mathsf{Vect}_{\Bbbk}$ be two $\Bbbk$-linear functors. The *Day convolution* of F and G, denoted by $F \star G$, is a left Kan extension of $F \boxtimes G$ along $\otimes\colon\ \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$:

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, "F \boxtimes G"] \arrow[d, "\otimes"'] & \mathsf{Vect}_{\Bbbk} \\ \mathcal{C} \arrow[ur, dashed, "F \star G"'] & \end{tikzcd}$$

More concretely, the Day convolution can—like all Kan extensions—be written as a coend:

$$\begin{aligned} (F \star G)(a) &\overset{\text{def}}{=} \int^{c,c'} \mathcal{C}(c \otimes c', a) \otimes_{\Bbbk} Fc \otimes_{\Bbbk} Gc' \\ &\cong \operatorname{coeq}\Big(\coprod_{\substack{f\colon c \longrightarrow c' \\ g\colon x \longrightarrow x'}} \mathcal{C}(c' \otimes x', a) \otimes_{\Bbbk} Fc \otimes_{\Bbbk} Gx \rightrightarrows \coprod_{c,x} \mathcal{C}(c \otimes x, a) \otimes_{\Bbbk} Fc \otimes_{\Bbbk} Gx \Big). \end{aligned}$$

Since the last object is a coequaliser in the category of $\Bbbk$-vector spaces, one can explicitly write it as

$$\coprod_{c,x} \mathcal{C}(c \otimes x, a) \otimes_{\Bbbk} Fc \otimes_{\Bbbk} Gx \Big/ T,$$

where T is a submodule generated by the differences

$$(h \circ (f \otimes g)) \otimes \alpha \otimes \beta \quad - \quad h \otimes Ff\alpha \otimes Gg\beta,$$

for $h \in \mathcal{C}(c' \otimes x', a)$, $\alpha \in Fc$, $\beta \in Gx$.

Theorem 5.20. *The category* $([\mathcal{C}, \mathsf{Vect}_{\Bbbk}], \star, \mathcal{C}(1, -))$ *is monoidal.*

We will return to this theorem in Section 6.1, after we have introduced the tools to prove it.

Exercise 5.21 (Tensor product over a small category)**.** Let $\Bbbk$ be a commutative ring with unit, and suppose that $\mathcal{C}$ is a small category. For functors $F\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \Bbbk\text{-Mod}$ and $G\colon \mathcal{C} \longrightarrow \Bbbk\text{-Mod}$, one can define a *tensor product over* $\mathcal{C}$:

$$F \otimes_{\mathcal{C}} G := \bigoplus_{c \in \mathcal{C}} Fc \otimes_{\Bbbk} Gc \Big/ M,$$

where M is the $\Bbbk$-submodule generated by

$$Ffx \otimes_{\Bbbk} y \quad - \quad x \otimes_{\Bbbk} Gfy,$$

for $x \in Fc'$, $y \in Gc$, and $f\colon c \longrightarrow c'$. Interpret this construction as a coend of an appropriate functor.

Write down the Day convolution as such a tensor product.

Exercise 5.22. The geometric realisation of a simplicial set X is defined to be the space defined by

$$|X| := \coprod_{n\geq 0} X_n \times \Delta_n \big/ \sim,$$

where Δ_n is the geometric n-simplex, and the equivalence relation $\sim$ is generated by

$$(x, (\Delta_n\phi)y) \sim ((X\phi)x, y),$$

for any $x \in X_n$, $y \in \Delta_m$, and non-decreasing morphism $\phi\colon [m] \longrightarrow [n]$. Interpret this construction as a coend of an appropriate functor.

6 Abstract Mackey functors (Tony)

So far, all of the definitions of Mackey functor that we have seen had—from the categorical point of view—a few drawbacks: the assignment from Definition 3.1 was not functorial, and Definition 3.6, while featuring two functors, split everything up between a covariant and a contravariant one.[21] However, there was also Definition 3.7, featuring just a single functor as data, with all of the necessary conditions "tucked away" in a suitable category. The goal of this talk is to understand this definition—and the resulting category of Mackey functors—in detail. I will follow [Kel05], [Lin76], [Lor21], [Bén67], [PS07], [Day70], and [Gar22].

[21] In particular, this made morphisms between Mackey functors quite awkward to define.

Before we get to the heart of things, however, we need to lay some additional groundwork.

6.1 Day convolution redux

In the following, unless otherwise stated, let C always be a small $\Bbbk$-linear category. The prime example to think of is $C := \Bbbk G\text{-}\mathsf{Set}$.

Theorem 6.1 (coYoneda/ninja Yoneda). *Suppose that $F\colon C \longrightarrow \mathsf{Vect}_\Bbbk$ is a $\Bbbk$-linear functor. Then*

$$Fx \cong \int^c C(c,x) \otimes_\Bbbk Fc \cong \int_c \mathsf{Vect}_\Bbbk(C(x,c), Fc). \tag{6.1}$$

For the proof of the theorem, we need the following important lemma.

Lemma 6.2. *For two $\Bbbk$-linear functors $F, G\colon C \longrightarrow \mathsf{Vect}_\Bbbk$, we have*

$$\mathrm{Nat}(F,G) \cong \int_c C(Fc, Gc).$$

Proof. For any wedge $\omega\colon V \Longrightarrow\!\!\diamond\ \mathsf{Vect}_\Bbbk(F,G)$, we have the commutative diagram

$$\begin{tikzcd}
V \arrow[r, "\omega_c"] \arrow[d, "\omega_{c'}"'] & \mathsf{Vect}_\Bbbk(Fc, Gc) \arrow[d, "\mathsf{Vect}_\Bbbk(Fc, Gf)"] \\
\mathsf{Vect}_\Bbbk(Fc', Gc') \arrow[r, "\mathsf{Vect}_\Bbbk(Ff, Gc')"'] & \mathsf{Vect}_\Bbbk(Fc, Gc')
\end{tikzcd}$$

Fixing a $v \in V$, this is exactly the naturality condition for the resulting $\omega_{d,-}$. Thus, the following diagram can be filled uniquely

$$\begin{tikzcd} D \arrow[r, "\omega_c"] \arrow[dr, dotted, "d \mapsto \omega_{d,-}"'] & \mathsf{Vect}_{\Bbbk}(Fc, Gc) \\ & \mathrm{Nat}(F, G) \arrow[u, "\eta \mapsto \eta_c"'] \end{tikzcd}$$

This is precisely the universal property of an end. □

Proof of Theorem 6.1. The proof of the theorem is an application of the Yoneda lemma;[22] we calculate:

$$\begin{aligned} \mathsf{Vect}_{\Bbbk}\Big(\int^c C(c,-) \otimes_{\Bbbk} Fc, y\Big) &\cong \int_c \mathsf{Vect}_{\Bbbk}(C(c,-) \otimes_{\Bbbk} Fc, y) \\ &\cong \int_c \mathsf{Vect}_{\Bbbk}(C(c,-), \mathsf{Vect}_{\Bbbk}(Fc, y)) \\ &\cong \mathrm{Nat}(C(-,x), \mathsf{Vect}_{\Bbbk}(F, y)) \\ &\cong \mathsf{Vect}_{\Bbbk}(F, y). \end{aligned}$$

□

[22] More precisely, we use that the Yoneda embedding

$$よ\colon C \longrightarrow [C^{\mathrm{op}}, \mathsf{Vect}_{\Bbbk}]$$

is fully faithful, and thus

$$C(a,-) \cong C(b,-) \iff a \cong b.$$

Exercise 6.3. Formulate a version of Theorem 6.1 for presheaves.

Proof-sketch of Theorem 5.20. In order to show that the Day convolution is a tensor product, we need to specify coherent isomorphisms

$$\alpha\colon F \star (G \star H) \xrightarrow{\sim} (F \star G) \star H, \quad \rho\colon F \star \mathbb{I} \xrightarrow{\sim} F, \quad \lambda\colon \mathbb{I} \star F \xrightarrow{\sim} F,$$

where $\mathbb{I} := C(1,-)$. For simplicity, let us assume that C is strict monoidal.[23]

Define the associator α by

$$\begin{aligned} \alpha &:= (F \star (G \star H))x \\ &\overset{\text{def}}{=} \int^{a,b} C(a \otimes b, x) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} \int^{c,d} C(c \otimes d, b) \otimes_{\Bbbk} Gc \otimes_{\Bbbk} Hd \\ &\cong \int^{a,b,c,d} C(a \otimes b, x) \otimes_{\Bbbk} C(c \otimes d, b) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gc \otimes_{\Bbbk} Hd \\ &\overset{6.1}{\cong} \int^{a,c,d} C(a \otimes c \otimes d, x) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gc \otimes_{\Bbbk} Hd \\ &\cong \int^{a,c,d,e} C(e \otimes d, x) \otimes_{\Bbbk} C(a \otimes c, e) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} Gc \otimes_{\Bbbk} Hd \\ &\cong \int^{d,e} C(e \otimes d, x) \otimes_{\Bbbk} \Big(\int^{a,c} C(a \otimes c, e) \otimes_{\Bbbk} Fa \otimes Gc\Big) \otimes Hd \\ &\overset{\text{def}}{=} ((F \star G) \star H)x, \end{aligned}$$

and the unitors ρ and λ by

$$\begin{aligned} \rho &:= F \star \mathbb{I} \overset{\text{def}}{=} \int^{a,b} C(a \otimes b, -) \otimes_{\Bbbk} Fa \otimes_{\Bbbk} C(1, b) \cong \int^a C(a,-) \otimes_{\Bbbk} Fa \cong F, \\ \lambda &:= \mathbb{I} \star F \overset{\text{def}}{=} \int^{a,b} C(a \otimes b, -) \otimes_{\Bbbk} C(1, a) \otimes_{\Bbbk} Fb \cong \int^b C(b,-) \otimes_{\Bbbk} Fb \cong F. \end{aligned}$$

[23] It is well-known that this is not a restriction, as every monoidal category is monoidally equivalent to a strict one; see [Mac63].

Let よ denote the first isomorphism in Equation (6.1). The diagrams get very big, so in the following we only give a sketch[24] of the coherence diagram for the triangle

$$\begin{tikzcd}
F \star (\mathbb{I} \star G) \arrow[r, "\alpha_{F,\mathbb{I},G}"] \arrow[dr, "F \star \rho_G"'] & (F \star \mathbb{I}) \star G \arrow[d, "\lambda_F \star G"] \\
& F \star G
\end{tikzcd}$$

Further, for the sake of saving space, let us abbreviate $\mathcal{C}(a,b)$ to $\mathcal{C}^a_b$, $\otimes$ (in $\mathcal{C}$) to juxtaposition, and $\otimes_{\Bbbk}$ to $\otimes$.

$$\begin{tikzcd}[column sep=small]
& F \star \int^{a,b} \mathcal{C}^{ab}_{-} \otimes \mathcal{C}^1_a \otimes Gb \arrow[r, "\cong"] \arrow[dl, "F\star \text{よ}_a"'] & \int^{c,d,a,b} \mathcal{C}^{cd}_{-} \otimes Fc \otimes \mathcal{C}^{ab}_d \otimes \mathcal{C}^1_a \otimes Gb \arrow[dl, "F\star \text{よ}_a;\, c=a, d=c, b=b"'] \arrow[d, "F\star \text{よ}_b"] & \\
F \star \int^{b} \mathcal{C}^{b}_{-} \otimes Gb \arrow[r, shift left, "\cong"] \arrow[dddr, "F\star \text{よ}_b"'] & \int^{a,c,b} \mathcal{C}^{ac}_{-} \otimes Fa \otimes \mathcal{C}^b_c \otimes Gb \arrow[l, shift left, "\cong"] \arrow[ddd, "\text{よ}_c"] & \int^{a,c,d} \mathcal{C}^{acd}_{-} \otimes Fa \otimes \mathcal{C}^1_c \otimes Gd \arrow[r, "\text{よ}_e^{-1}"] \arrow[d, "\text{よ}_c"'] & \int^{a,c,d,e} \mathcal{C}^{ed}_{-} \otimes \mathcal{C}^{ac}_e \otimes Fa \otimes \mathcal{C}^1_c \otimes Gd \arrow[ddl, "\text{よ}_c"'] \arrow[ddd, "\cong"] \\
& & \int^{a,d} \mathcal{C}^{ad}_{-} \otimes Fa \otimes Gd \arrow[ddl, equal] \arrow[d, shift left, "\text{よ}_e^{-1}"] & \\
& & \int^{a,d,e} \mathcal{C}^{ed}_{-} \otimes \mathcal{C}^a_e \otimes Fa \otimes Gd \arrow[u, shift left, "\text{よ}_e"] \arrow[d, "\cong"] & \\
& F \star G & \left(\int^a \mathcal{C}^a_e \otimes Fa\right) \star G \arrow[l, "\text{よ}_a \star G"'] & \left(\int^{a,c} \mathcal{C}^{ac}_e \otimes Fa \otimes \mathcal{C}^1_c\right) \star G \arrow[l, "\text{よ}_c \star G"']
\end{tikzcd}$$

The pentagon for α also uses the same techniques, although it is much more work. See [Day70] for a more rigorous proof of the coherence relations, in the setting of promonoidal categories. □

[24] That is, we are very liberal with the different kinds of natural isomorphisms involved when transforming coends, and only pay attention to よ, as it is the most non-trivial operation.

6.2 Enriched Category Theory

While we mainly need the language of $\Bbbk$-linear categories for this seminar, enriched categories crop up all the time in mathematics, so it is still good to at least know the basic definitions.

Let $(\mathcal{V}, \otimes, 1)$ be a symmetric closed monoidal complete and cocomplete category.

Definition 6.4. A *$\mathcal{V}$-category $\mathcal{C}$* comprises data:

- A collection $\operatorname{Ob}\mathcal{C}$ of objects.
- For all $x, y \in \mathcal{C}$[25] a *hom-object* $\mathcal{C}(x,y) \in \mathcal{V}$.
- For all $x, y, z \in \mathcal{C}$, a *composition map*

  $$\circ_{xyz} \colon \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z)$$

  in $\mathcal{V}$.

[25] In the interest of brevity, we will often abuse notation and write $x \in \mathcal{C}$ instead of $x \in \operatorname{Ob}\mathcal{C}$.

- For all $x \in \mathcal{C}$, an *identity map* $1_x \colon 1 \longrightarrow \mathcal{C}(x,x)$.

This data needs to satisfy the following coherence conditions:

$$\begin{tikzcd}
\mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \otimes \mathcal{C}(w,x) \arrow[r, "\circ_{xyz}\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\circ_{wyz}"'] & \mathcal{C}(x,z) \otimes \mathcal{C}(w,x) \arrow[d, "\circ_{wxz}"] \\
\mathcal{C}(y,z) \otimes \mathcal{C}(w,y) \arrow[r, "\circ_{wyz}"'] & \mathcal{C}(w,z)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{C}(y,y) \otimes \mathcal{C}(x,y) \arrow[dr, "\circ_{xyy}"'] & \mathcal{C}(x,y) \arrow[l, "1_y\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes 1_x"] \arrow[d, equal] & \mathcal{C}(x,y) \otimes \mathcal{C}(x,x) \arrow[dl, "\circ_{xxy}"] \\
& \mathcal{C}(x,y) &
\end{tikzcd}$$

Example 6.5. Various examples of categories from everyday life can be formulated as being enriched over a well-behaved base.

- A category enriched in Set is just an ordinary category.
- A Cat-category is a (strict) 2-category.
- An Ab-enriched category is a preadditive category.
- Important for our purposes, a $\mathsf{Vect}_{\Bbbk}$-category is a $\Bbbk$-linear category.

Exercise 6.6. Let $\mathcal{V} \coloneqq \mathsf{Set}^{\rightarrow}$ be the functor category from the category $[1]$ to Set.[26] What is a category enriched in $\mathcal{V}$?

[26] That is, $[1]$ is the poset $\{0,1\}$, seen as a category with two objects and one non-trivial morphism:

$$\begin{tikzcd}
0 \arrow[loop above, "\mathrm{id}_0"] \arrow[r] & 1 \arrow[loop above, "\mathrm{id}_1"]
\end{tikzcd}$$

Definition 6.7. Given a $\mathcal{V}$-category $\mathcal{C}$, its *underlying category* is the (ordinary) category $\mathcal{C}_0$ with the same objects, and

$$\mathcal{C}_0(x,y) \coloneqq \mathcal{C}(1, \mathcal{C}(x,y)).$$

Definition 6.8. A *$\mathcal{V}$-functor* $F \colon \mathcal{C} \longrightarrow \mathcal{V}$ between $\mathcal{V}$ categories $\mathcal{C}$ and $\mathcal{D}$ comprises:

- an object mapping $\operatorname{Ob}\mathcal{C} \longrightarrow \operatorname{Ob}\mathcal{D}$,
- for all $x, y \in \mathcal{C}$, a map
$$F_{xy} \colon \mathcal{C}(x,y) \longrightarrow \mathcal{D}(Fx, Fy)$$
in $\mathcal{V}$,

such that the following *functoriality* properties are satisfied:

$$\begin{tikzcd}
\mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \arrow[r, "F_{yz}\otimes F_{xy}"] \arrow[d, "\circ_{xyz}"'] & \mathcal{D}(Fy,Fz) \otimes \mathcal{D}(Fx,Fy) \arrow[d, "\circ_{Fx,Fy,Fz}"] \\
\mathcal{C}(x,z) \arrow[r, "F_{xz}"'] & \mathcal{D}(Fx,Fz)
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[d, "1_x"'] \arrow[dr, "1_{Fx}"] & \\
\mathcal{C}(x,x) \arrow[r, "F_{xx}"'] & \mathcal{C}(Fx,Fx)
\end{tikzcd}$$

Continuing Example 6.5, all of the naturally associated functors to these categories—e.g., $\Bbbk$-linear functors between $\Bbbk$-linear categories—are examples of $\mathcal{V}$-functors. One has to be a bit careful in the case of 2-categories, as there are several coherent notions of functor between such categories. The very strictest notion, that of a *2-functor*, is what corresponds to a Cat-enriched functor.

Definition 6.9. Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ be $\mathcal{V}$-functors. A *$\mathcal{V}$-natural transformation* $\eta\colon F \Longrightarrow G$ between F and G comprises, for all $x \in \mathcal{C}$, maps[27]

$$\eta_x\colon 1 \longrightarrow \mathcal{D}(Fx, Gx)$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r, "G_{xy}\otimes\eta_x"] \arrow[d, "\eta_y\otimes F_{xy}"'] & \mathcal{D}(Gx,Gy)\otimes\mathcal{D}(Fx,Gx) \arrow[d, "\circ_{Fx,Gx,Gy}"] \\ \mathcal{D}(Fy,Gy)\otimes\mathcal{D}(Fx,Fy) \arrow[r, "\circ_{Fx,Fy,Gy}"'] & \mathcal{D}(Fx,Gy) \end{tikzcd}$$

[27] i.e., elements of the underlying hom-set:

$$\eta_x \in \mathcal{D}_0(Fx, Gx).$$

Exercise 6.10. Suppose that $\mathcal{V}$ and $\mathcal{W}$ are symmetric closed monoidal bicomplete categories Prove that any monoidal functor

$$F\colon \mathcal{V} \longrightarrow \mathcal{W}$$

induces a 2-functor

$$F_*\colon \mathcal{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{W}\text{-}\mathsf{Cat}$$

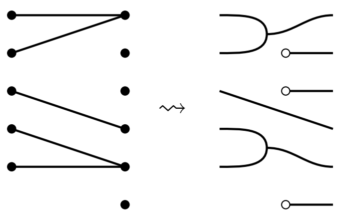
by setting $\operatorname{Ob} F_*\mathcal{C} \coloneqq \operatorname{Ob}\mathcal{C}$, and $F_*\mathcal{C}(x,y) \coloneqq F\mathcal{C}(x,y)$.

This functor is also called a *change of enrichment*.

Example 6.11. Let $\mathcal{C}$ be a category, and suppose that T is a monad on $\mathcal{C}$. We can enrich $\mathcal{C}$ in sSet—the category of simplicial sets—as follows:

$$\begin{aligned} \underline{\mathcal{C}}(x,y)\colon \Delta^{\mathrm{op}} &\longrightarrow \mathsf{Set} \\ [n] &\longmapsto \mathcal{C}(T^n x, y). \end{aligned}$$

Functoriality uses the monad structure on T, and is best explained in string diagrams:



The composition, *a priori* given by Day convolution, simplifies to

$$\begin{aligned} \underline{\mathcal{C}}(y,z)[m] \times \underline{\mathcal{C}}(x,y)[n] &\longrightarrow \underline{\mathcal{C}}(x,z)[n+m] \\ (f,g) &\longmapsto T^{m+n}x \xrightarrow{T^m g} T^m y \xrightarrow{f} z. \end{aligned}$$

If S is a monad on a category $\mathcal{D}$, one can think about sSet-enriched functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$. Indeed, these are morphisms of simplicial sets

$$\alpha_{xy}\colon \underline{\mathcal{C}}(x,y) \longrightarrow \underline{\mathcal{D}}(Fx, Fy);$$

more explicitly,

$$\mathcal{C}(T^n x, y) \longrightarrow \mathcal{D}(FT^n x, Fy) \longrightarrow \mathcal{D}(S^n Fx, Fy).$$

This means that such a morphism consists of a natural transformation

$$\sigma\colon FT \Longrightarrow SF,$$

satisfying exactly the axioms of a *monad morphism* in the sense of [Str72].

Perhaps surprisingly, analogues of Theorems 5.6 and 6.1 also hold in this setting, though (co)ends now have to be defined by means of *weighted (co)limits*. We will not get into this here, but see [Kel05, Sections 2.4 and 3.10].

6.3 Spans

Definition 6.12. Let C be any category, and let $a, b \in C$. A *span from a to b* is a triple (s, f, g), where $f\colon\ s \longrightarrow a$ and $g\colon\ s \longrightarrow b$ are morphisms in C.

Spans are often depicted graphically:

$$\begin{tikzcd} & s \arrow[dl] \arrow[dr] & \\ a & & b \end{tikzcd}$$

Definition 6.13. Let (s, f, g) and t, f', g' be spans from a to b. A *morphism of spans* is an arrow $\xi\colon\ s \longrightarrow t$ in C, such that

$$\begin{tikzcd} & s \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\xi"] & \\ a & & b \\ & t \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

commutes.

Spans equip a nice enough category with a 2-dimensional structure. However, the notion of a Cat-enriched (i.e., 2-)category is not weak enough to capture this notion.

Definition 6.14. A *bicategory* $\mathbb{B}$ comprises:

- A collection $\operatorname{Ob}\mathbb{B}$ of objects.
- For all $x, y \in \mathbb{B}$, a hom-category $\mathbb{B}(x, y)$.
- For all $x, y, z \in \mathbb{B}$, a horizontal composition functor

$$\otimes_y\colon\ \mathbb{B}(y, z) \times \mathbb{B}(x, y) \longrightarrow \mathbb{B}(x, z).$$

- For all $x \in \mathbb{B}$, an identity 1-cell $1_x \in \mathbb{B}(x, x)$
- For all $x \xrightarrow{P} y \xrightarrow{N} z \xrightarrow{M} w$, a natural isomorphism

$$\alpha_{MNP}\colon\ (M \otimes_z N) \otimes_y P \longrightarrow M \otimes_z (N \otimes_y P)$$

  in $\mathbb{B}(x, w)$.
- For all $x \xrightarrow{M} y$, natural isomorphisms

$$\lambda_M\colon\ 1_y \otimes_y M \longrightarrow M, \qquad\qquad \rho_M M \longrightarrow M \otimes_x 1_x.$$

This data has to satisfy the following axioms:

$$
\begin{tikzcd}
(M \otimes_z (N \otimes_y P)) \otimes_x R \arrow[rr, "\alpha_{M,N\otimes_y P,R}"] & & M \otimes_z ((N \otimes_y P) \otimes_x R) \arrow[d, "M\otimes\alpha_{NPR}"] \\
((M \otimes_z N) \otimes_y P) \otimes_x R \arrow[u, "\alpha_{MNP}\otimes R"] \arrow[dr, "\alpha_{M\otimes_z N,P,R}"'] & & M \otimes_z (N \otimes_y (P \otimes_x R)) \\
& (M \otimes_z N) \otimes_y (P \otimes_x R) \arrow[ur, "\alpha_{M,N,P\otimes_x R}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
(M \otimes_x 1_x) \otimes_x N \arrow[rr, "\alpha_{M,1_x,N}"] & & M \otimes_x (1_x \otimes_x N) \arrow[d, "M\otimes\lambda_N"] \\
M \otimes_x N \arrow[u, "\rho_M\otimes N"] \arrow[rr, equal] & & M \otimes_x N
\end{tikzcd}
$$

The above definition should make you think of a monoidal category. Indeed, one can view bicategories as a kind of "many-object monoidal category"; for any object x in a bicategory $\mathbb{B}$, the hom-category $(\mathbb{B}(x,x), \otimes_x, 1_x)$ is monoidal.

The following examples are taken from [Lac10], which also contains lots of examples of 2-categories.

Example 6.15. There is a bicategory $\mathbb{B}\mathsf{im}$ whose objects are rings, a morphism from R to S is an R-S-bimodule, and 2-cells are module morphisms. Composition is the tensor product of modules over the respective ring.

Whenever possible, we notationally indicate a 2- or bicategorical structure by making the first letter blackboard bold. For example, Cat would be the 1-category of categories and functors, and $\mathbb{C}\mathsf{at}$ denotes the 2-category of categories, functors, and natural transformations.

In fact, "bimodules over a structure" turns out to be a great source of examples of bicategories. As we have seen in Example 5.14, modules may equivalently be seen as functors. Thus, there is a bicategory $\mathbb{P}\mathsf{rof}$, build as follows:

- Objects are categories.
- A morphism $\mathcal{C} \rightsquigarrow \mathcal{D}$ is a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$.
- A 2-cell is a natural transformation.
- If $P\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$ and $Q\colon \mathcal{E}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathsf{Set}$ are 1-cells, their composition is given by the following coend:

$$(Q \circ P)(e,c) := \int^{d\in\mathcal{D}} P(d,c) \times Q(e,d).$$

- The hom-functors $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathsf{Set}$ are the identities.

This construction also still works if everything is enriched over a (nice enough) category $\mathcal{V}$.

Another special case of this is the bicategory $\mathbb{M}\mathsf{at}_{\mathcal{V}}$ of *matrices*, over a monoidal category $(\mathcal{V}, \otimes, 1)$ with coproducts, such that $\otimes$ preserves them in each variable:

- Objects are finite sets.
- A 1-cell between I and J is a functor $I \times J \longrightarrow \mathcal{V}$, where I and J are seen as discrete categories.

- 2-cells are natural transformations.
- The composition of two 1-cells $I \xrightarrow{M} J \xrightarrow{N} K$ is given by matrix multiplication:
$$N \otimes_J M := \coprod_{j \in J} N_{jk} \otimes M_{ij}.$$
- Identities are matrices $\mathbb{I}\colon I \longrightarrow I$, such that $\mathbb{I}_{ij} = 1$ if $i = j$, and 0 otherwise. We will return to this example in Example 6.18.

Proposition 6.16. *If C has pullbacks, then there is a bicategory $\mathbb{S}\mathsf{p}_C$ of spans in C:*

- *0-cells of $\mathbb{S}\mathsf{p}_C$ are objects of C.*
- *A 1-cell between a and b is a span from a to b.*
- *2-cells are morphisms of spans.*
- *The composition of two spans is given by a pullback:*

$$\begin{tikzcd}
 & & s \times_b t \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\vee" very near start] & & \\
 & s \arrow[dl] \arrow[dr] & & t \arrow[dl] \arrow[dr] & \\
a & & b & & c
\end{tikzcd}$$

Definition 6.17. The *classifying category* of spans of C is the 1-category obtained from $\mathbb{S}\mathsf{p}_C$ by taking arrows to be isomorphism classes of spans.

This category will often also be denoted by $\mathbb{S}\mathsf{p}_C$.

Example 6.18 (Spans are matrices)**.** Given any span $(N, a, b)\colon X \longrightarrow Y$ in the category FinSet of finite sets, define N_{xy} as the cardinality of the fibre of x and y; more precisely:

$$N_{xy} := \#\{\, n \in N \mid a(n) = x, b(n) = y \,\}.$$

This turns N into a matrix, with matrix multiplication given by composition of spans. Indeed, given any two finite sets A and B, the pullback along $f\colon A \longrightarrow C$ and $g\colon B \longrightarrow B$ is given by

$$A \times_C B = \{\, (a, b) \in A \times B \mid f(a) = g(b) \,\}.$$

For two spans $(N, a, b)\colon X \longrightarrow Y$ and $(M, c, d)\colon Y \longrightarrow Z$ we get:

$$\begin{aligned}
(N \times_Y M) &= \#\{\, (n, m) \in N \times M \mid a(n) = i, d(m) = j \,\} \\
&= \#\{\, (n, m) \in N \times M \mid a(n) = i, d(m) = j, b(m) = c(n) \,\} \\
&= \#\Big(\bigcup_{y \in Y} \{\, n \in N \mid a(n) = i, b(m) = y \,\} \times \{\, m \in M \mid c(m) = y, d(m) = j \,\} \Big) \\
&= \sum_{k \in Y} M_{ik} N_{kj}.
\end{aligned}$$

In fact, there is an biequivalence $\mathbb{S}\mathsf{p}_{\mathsf{FinSet}} \simeq \mathbb{M}\mathsf{at}_{\mathsf{FinSet}}$.

Remark 6.19. Given $a, b \in \mathsf{Sp}_C$, we have a canonical isomorphism

$$\mathsf{Sp}_C(a, b) \cong \left[{}^{C}\!/_{a \times b}\right],$$

where, as in Proposition 4.8, $C/a \times b$ denotes the slice category over $a \times b$, and we additionally have to take isomorphism classes in it.

Proposition 6.20. *If C has products, then they become a semi-cartesian monoidal structure on Sp_C.*

Proof. The tensor product is defined by

$$\begin{aligned} \times \colon \mathsf{Sp}_C \times \mathsf{Sp}_C &\longrightarrow \mathsf{Sp}_C \\ (s, t) &\longmapsto s \times t \\ ((s, f, g), (t, f', g')) &\longmapsto (s \times t, f \times t', g \times g'). \end{aligned}$$

This structure is semi-cartesian by virtue of the empty product $\{\varnothing\}$ being a terminal object: for every $x \in C$, we have a span

$$\begin{tikzcd} & x \arrow[dl, "\mathrm{id}_x"'] \arrow[dr, "!"] & \\ x & & \{\varnothing\} \end{tikzcd}$$

□

In fact, $\mathsf{Sp}_C(a, b) \cong \mathsf{Sp}_C(b, a)$, and the category is even rigid via the identification

$$\begin{tikzcd} & s \arrow[dl] \arrow[dr] & & & s \arrow[dl] \arrow[d] \arrow[dr] & & & s \arrow[dl] \arrow[dr] & \\ a \times b & \cong & c & a & b & c & a & \cong & b \times c \end{tikzcd}$$

Proposition 6.21. *If the canonical functor*

$$\begin{aligned} {}^{C}\!/_{a} \times {}^{C}\!/_{b} &\longrightarrow {}^{C}\!/_{a + b} \\ (f, g) &\longmapsto f + g \end{aligned}$$

is an equivalence,[28] *then Sp_C has coproducts.*

28 This is the case for $C = G$-Set, which is the prime example you should be thinking of here.

Proof. We have that

$$\begin{aligned} \mathsf{Sp}_C(a + b, c) &\cong \left[{}^{C}\!/_{(a + b) \times c}\right] \\ &\cong \left[{}^{C}\!/_{(a \times c) + (b \times c)}\right] \\ &\cong \left[{}^{C}\!/_{a \times c}\right] \times \left[{}^{C}\!/_{b \times c}\right] \\ &\cong \mathsf{Sp}_C(a, c) \times \mathsf{Sp}_C(b, c). \end{aligned}$$

Adding two spans is straightforward:

$$\begin{tikzcd} & s \arrow[dl, "s_a"'] \arrow[dr, "s_b"] & & & t \arrow[dl, "t_a"'] \arrow[dr, "t_b"] & & & & s + t \arrow[dl, "s_a + t_b"'] \arrow[ddl, "\langle s_a, t_a \rangle"] \arrow[ddr, "\langle s_b, t_b \rangle"'] \arrow[dr, "s_b + t_b"] & \\ a & + & b & a & = & b & & a + a \arrow[d, "\langle \mathrm{id}_a, \mathrm{id}_a \rangle"'] & & b + b \arrow[d, "\langle \mathrm{id}_b, \mathrm{id}_b \rangle"] \\ & & & & & & & a & & a \end{tikzcd}$$

□

All in all, Sp_C is enriched in the category of commutative monoids. More than that is to be said about relevant functor categories.

Lemma 6.22. *The category* $[\mathsf{Sp}_C, \mathsf{Vect}_\Bbbk]$ *is a* $\mathcal{V}$*-category.*

Proof. We have to show that the hom-sets can be endowed with the structure of a $\Bbbk$-module, such that composition becomes $\Bbbk$-linear.

To that end, fix two functors $F, G\colon \mathsf{Sp}_C \longrightarrow \mathsf{Vect}_\Bbbk$. The $\Bbbk$-module structure on the natural transformations between F and G is inherited by the $\Bbbk$-module structure of $\mathrm{Hom}_\Bbbk(Fx, Gx)$, for any object $x \in \mathsf{Sp}_C$. That is, for any pair of natural transformations $\alpha, \beta\colon F \Longrightarrow G$ and scalar $\lambda \in \Bbbk$, we have

$$(\lambda\alpha + \beta)_x\colon Fx \longrightarrow Gx, \qquad v \longmapsto \lambda(\alpha_x v) + \beta_x v.$$

In order to show that this yields a natural transformation, we consider a morphism $f \in \mathsf{Sp}_C(x, y)$ and compute

$$\begin{aligned}((\lambda\alpha + \beta)_y Ff)(v) &= \lambda(\alpha_y(Ffv)) + \beta_y(Ffv) = \lambda(Gf\alpha_x v) + (Gf\beta_x)v \\ &= (Gf(\lambda\alpha_x v + \beta_x))v.\end{aligned}$$

The claim follows, as the composition of $\Bbbk$-linear maps is itself $\Bbbk$-linear. □

In fact, linearisation is an extremely nice operation for functor categories.

Proposition 6.23. *For any small category* C*, there exists an isomorphism of monoidal categories*

$$[C, \mathsf{Vect}_\Bbbk]_{\mathrm{conv}} \cong [\Bbbk C, \mathsf{Vect}_\Bbbk]^{\mathrm{lin}}_{\mathrm{conv}}$$

between functors from C *to* $\mathsf{Vect}_\Bbbk$*, and* $\Bbbk$*-linear functors from* $\Bbbk C$ *to* $\mathsf{Vect}_\Bbbk$*.*

Proof. We define a functor $A\colon [C, \mathsf{Vect}_\Bbbk]_{\mathrm{conv}} \longrightarrow [\Bbbk C, \mathsf{Vect}_\Bbbk]^{\mathrm{lin}}_{\mathrm{conv}}$; on objects it is given by mapping any functor F to its linearisation $\Bbbk F$. The functor $\Bbbk F\colon C \longrightarrow \mathsf{Vect}_\Bbbk$ agrees with F on objects, and linearly extends F on hom-spaces; that is, $\Bbbk Ff = Ff$ for all $f \in C(x, y)$. By definition, $\Bbbk F$ is $\Bbbk$-linear.

On morphisms, A is the identity, which implies its faithfulness. To see that A is full, suppose that $\Bbbk\alpha\colon \Bbbk F \Longrightarrow \Bbbk G$ is a natural transformation. Pointwise restricting α to the canonical bases yields a corresponding natural transformation $\alpha\colon F \Longrightarrow G$ such that $A\alpha = \Bbbk\alpha$.

The proof is concluded by showing that A is essentially surjective. Again, this follows by considering any $\Bbbk$-linear functor $F\colon \Bbbk C \longrightarrow \mathsf{Vect}_\Bbbk$ and restricting it to the induced bases on the morphism spaces. This gives rise to a functor $F'\colon C \longrightarrow \mathsf{Vect}_\Bbbk$ such that $AF' = F$. □

6.4 The Lindner definition

The following definition is due to Lindner [Lin76]. Let G be a finite group.

Definition 6.24. A *Mackey functor* is a coproduct preserving functor from $\mathsf{Sp}_{G\text{-}\mathsf{Set}}$ to $\mathsf{Vect}_\Bbbk$.

Note that, in view of Lemma 6.22, one could also define Mackey functors as $\Bbbk$-linear functors from $\Bbbk\mathsf{Sp}_C$ to $\mathsf{Vect}_\Bbbk$, which is what we have already seen in Definition 3.7.

Definition 6.25. A Mackey functor M is *finite-dimensional*, if every Mx is a finite-dimensional vector space, for all $x \in \mathsf{Sp}_{G\text{-}\mathsf{Set}}$.

Theorem 6.26. *The category of Mackey functors is closed monoidal.*

The monoidal structure is given by Day convolution

$$\begin{aligned}(F \star G)(x) &:= \int^{a,b} \Bbbk\mathsf{Sp}_C(a \otimes b, x) \otimes_{\Bbbk} Ma \otimes_{\Bbbk} Nb \\ &\cong \int^{b} M(x \otimes b^*) \otimes_{\Bbbk} Nb,\end{aligned}$$

and the closed structure is

$$\begin{aligned}[M, N]x &:= \int_{a,b} \mathsf{Vect}_{\Bbbk}(\Bbbk\mathsf{Sp}_G(x \otimes a, b), \mathsf{Vect}_{\Bbbk}(Ma, Nb)) \\ &\cong \mathsf{Mky}(M(x^* \otimes -), N).\end{aligned}$$

Theorem 6.27 ([PS07, Theorem 9.2])**.** *The category of finite-dimensional Mackey functors is $*$-autonomous.*

References

[BD13] Mitya Boyarchenko and Vladimir Drinfeld. “A duality formalism in the spirit of Grothendieck and Verdier”. English. In: *Quantum Topol.* 4.4 (2013), pp. 447–489. ISSN: 1663-487X. DOI: 10.4171/QT/45.

[Bén67] Jean Bénabou. *Introduction to bicategories.* English. Bénabou, Jean et al., Reports of the Midwest Category Seminar. Lect. Notes Math. 47, 1-77 (1967). 1967. DOI: 10.1007/BFb0074299.

[Bum13] Daniel Bump. *Lie groups.* English. 2nd ed. Vol. 225. Grad. Texts Math. New York, NY: Springer, 2013. ISBN: 978-1-4614-8023-5; 978-1-4614-8024-2. DOI: 10.1007/978-1-4614-8024-2.

[Day70] Brian Day. *On closed categories of functors.* English. Rep. Midwest Category Semin. 4, Lect. Notes Math. 137, 1-38 (1970). 1970. DOI: 10.1007/BFb0060438.

[Dre73] Andreas W. M. Dress. “Contributions to the theory of induced representations”. In: *“Classical” Algebraic K-Theory, and Connections with Arithmetic.* Ed. by H. Bass. Berlin, Heidelberg: Springer Berlin Heidelberg, 1973, pp. 181–240. ISBN: 978-3-540-37770-2.

[EJ00] Edgar E. Enochs and Overtoun M. G. Jenda. *Relative homological algebra.* English. Vol. 30. De Gruyter Expo. Math. Berlin: Walter de Gruyter, 2000. ISBN: 3-11-016633-X.

[Gar22] Richard Garner. *Bicategories.* Lecture course at the *Research School on Bicategories, Categorification and Quantum Theory.* 2022.

[Kel05] G. M. Kelly. “Basic concepts of enriched category theory”. English. In: *Repr. Theory Appl. Categ.* 2005.10 (2005), pp. 1–136.

[Lac10] Stephen Lack. “A 2-categories companion”. English. In: *Towards higher categories.* Berlin: Springer, 2010, pp. 105–191. ISBN: 978-1-4419-1523-8; 978-1-4419-1524-5. DOI: 10.1007/978-1-4419-1524-5_4.

[Lin76] Harald Lindner. “A remark on Mackey-functors”. English. In: *Manuscr. Math.* 18 (1976), pp. 273–278. ISSN: 0025-2611. DOI: 10.1007/BF01245921.

[Lor21] Fosco Loregian. *(Co)end calculus.* Vol. 468. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2021, p. 308. ISBN: 978-1-108-74612-0.

[Mac63] S. Mac Lane. “Natural associativity and commutativity”. In: *Rice University Studies* 49 (1963), pp. 28–46.

[ML98] Saunders Mac Lane. *Categories for the working mathematician.* Second. Vol. 5. Graduate Texts in Mathematics. Springer-Verlag, New York, 1998, pp. xii+314. ISBN: 0-387-98403-8.

[PS07] E. Panchadcharam and R. Street. "Mackey functors on compact closed categories". English. In: *J. Homotopy Relat. Struct.* 2.2 (2007), pp. 261–293. ISSN: 2193-8407.

[Ric20] Birgit Richter. *From categories to homotopy theory.* English. Vol. 188. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2020. ISBN: 978-1-108-47962-2; 978-1-108-85589-1. DOI: 10.1017/9781108855891.

[Ser98] Jean-Pierre Serre. *Représentations linéaires des groupes finis. 5ième éd., corr. et augm. de nouveaux exercices.* French. 5ième éd., corr. et augm. de nouveaux exercices. Paris: Hermann, 1998. ISBN: 2-7056-6352-5.

[Ste12] Benjamin Steinberg. *Representation theory of finite groups. An introductory approach.* English. Universitext. Berlin: Springer, 2012. ISBN: 978-1-4614-0775-1; 978-1-4614-0776-8. DOI: 10.1007/978-1-4614-0776-8.

[Str72] Ross Street. "The formal theory of monads". In: *Journal of Pure and Applied Algebra* 2.2 (1972), pp. 149–168. ISSN: 0022-4049. DOI: 10.1016/0022-4049(72)90019-9.

[TW95] Jacques Thévenaz and Peter Webb. "The structure of Mackey functors". English. In: *Trans. Am. Math. Soc.* 347.6 (1995), pp. 1865–1961. ISSN: 0002-9947. DOI: 10.2307/2154915.

[Web00] Peter Webb. "A guide to Mackey functors". English. In: *Handbook of algebra. Volume 2.* Amsterdam: North-Holland, 2000, pp. 805–836. ISBN: 0-444-50396-X.