Duality in Monoidal Categories

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A symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is called **closed**, if there exists a tensor-hom adjunction

$$-\otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : [x, -] \qquad \text{for all } x \in \mathcal{C}.$$

For any $x \in \mathcal{C}$, we write $x^* := [x, 1]$ and define

$$\varphi_x := x \otimes x^* \xrightarrow{\eta} [x, x \otimes x^* \otimes x] \xrightarrow{[x, x\otimes\varepsilon]} [x, x],$$

using the unit and counit of the above adjunction.

The category $\mathcal{C}$ is **rigid** if ϕ_x is invertible for all objects $x \in \mathcal{C}$. This implies the existence of an ambidextrous adjunction

$$-\otimes x^* \dashv -\otimes x \dashv -\otimes x^*.$$

To investigate whether rigidity can be characterised by these adjunctions, we first promote the previous observations to a definition.

The internal hom $[-,-]\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ is said to be **tensor representable** if for all $x \in \mathcal{C}$ we have

$$-\otimes x^* \dashv -\otimes x \dashv -\otimes x^*.$$

In this case, there is a natural isomorphism

$$\zeta_{x,y}\colon [x, y] \longrightarrow y \otimes x^*.$$

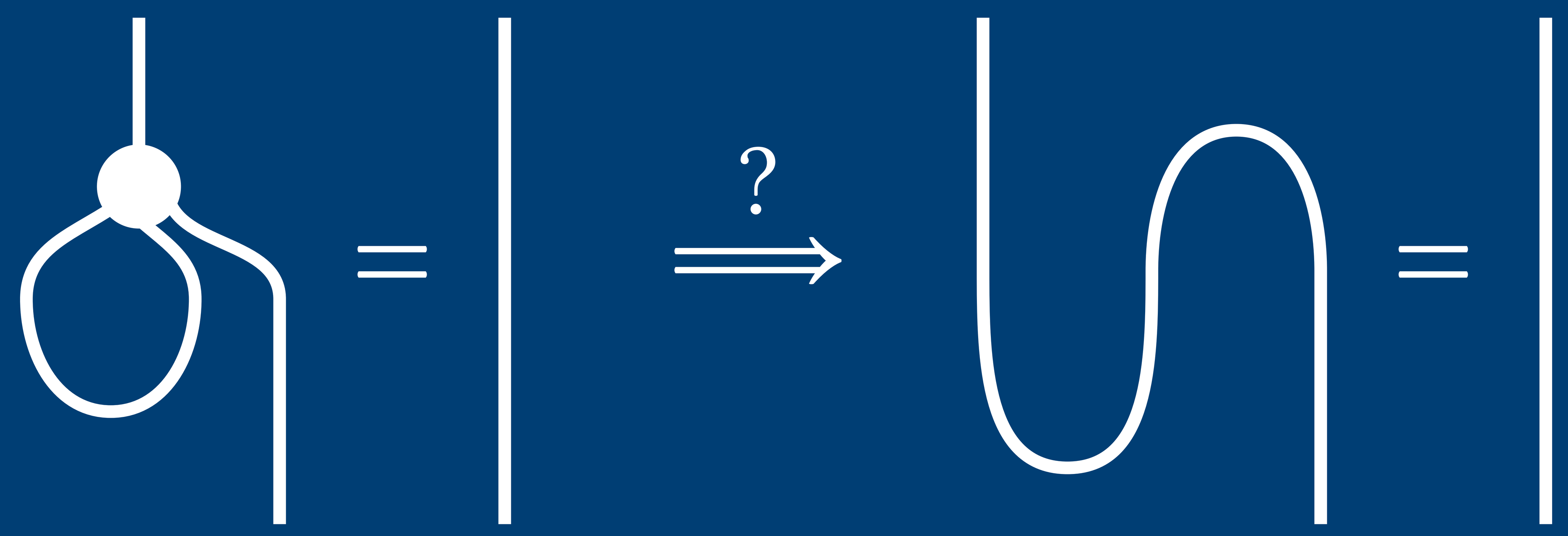
All of this motivates the question:

Given a closed monoidal category, does the tensor representability of its internal hom imply rigidity?

In many (algebraic) contexts the answer is: *yes*; given a commutative ring R and an R-module M, the following are equivalent:

1. M admits a dual in the rigid sense.
2. $-\otimes_R M$ is left adjoint to $-\otimes_R \mathrm{Hom}_R(M, R)$.
3. M is finitely generated and projective.

Do adjunctions characterise rigidity?



There exists a **strict** ordering

Closed

|

$\ast$-autonomous/Grothendieck–Verdier

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Tensor representable

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Rigid

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Fix a finite group G and a field $\Bbbk$. The category of finite-dimensional $\Bbbk$-valued **Mackey functors** of G is defined as

$$[\mathsf{Sp}_{G\text{-set}}, \mathsf{vect}_{\Bbbk}]_+ =: \mathsf{mky}_G;$$

that is, additive functors from spans of G-sets into finite-dimensional $\Bbbk$-vector spaces.

Day convolution endows mky_G with the structure of a closed symmetric monoidal category.

An illuminating characterisation of Mackey functors is due to Peter Webb:

> *A Mackey functor is an algebraic structure possessing operations which behave like the induction, restriction and conjugation mappings in group representation theory.*

In line with this description, examples of Mackey functors of G include representations of G, the Grothendieck group of $\mathrm{Rep}_{\Bbbk}(G)$, as well as many homological and K-theoretic constructions.

Theorem (HZ). The internal hom of mky_G is tensor representable. It is rigid if and only if the characteristic of $\Bbbk$ does not divide the order of G.

Although tensor representability does not imply rigidity, it is part of a well-studied duality concept.

A **$\ast$-autonomous** or **Grothendieck–Verdier** category is a closed monoidal category $\mathcal{C}$ endowed with a fixed object $d \in \mathcal{C}$, the **dualising object**, such that there is an anti-equivalence

$$D := [-, d]\colon \mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}.$$

Theorem (HZ). Every tensor representable category is $\ast$-autonomous, with the monoidal unit as the dualising object.