Optics in functional programming
a categorical perspective

2022-01-10

Tony Zorman
TU Dresden

References

C. Pastro, R. Street: Doubles for Monoidal Categories (2008)

B. Milewski: Profunctor Optics: The Categorical View (2017)

M. Riley: Categories of Optics (2018)

Clarke et.al.: Profunctor optics, a categorical update (2020)

B. Clarke, D. Elkins, J. Gibbons, F. Loregian, B. Milewski, E. Pillmore, M. Román

A naïve Definition

For the rest of this talk....

·) $\mathcal{L}$ is a closed cartesian category

$\mathcal{V}$ cocomplete complete closed (symmetric) monoidal,
$\mathcal{L}$ a (monoidal) $\mathcal{V}$-category

A naïve Definition

For the rest of this talk....

·) $\mathcal{L}$ is a closed cartesian category

$\mathcal{V}$ cocomplete complete closed (symmetric) monoidal,
$\mathcal{L}$ a (monoidal) $\mathcal{V}$-category

Def Let $S, A \in \mathcal{L}$. A lens from S to A consists of

$$\mathrm{get}: S \to A, \qquad \mathrm{put}: S \times A \to S.$$

Def A lawful lens is a lens from S to A, such that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\Delta"] \arrow[d, equal] & S \times S \arrow[d, "\mathrm{id} \times \mathrm{get}"] \\ S & S \times A \arrow[l, "\mathrm{put}"] \end{tikzcd} \qquad \begin{tikzcd} S \times A \arrow[r, "\mathrm{put}"] \arrow[dr, "\pi_2"'] & S \arrow[d, "\mathrm{get}"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} S \times A \times A \arrow[r, "\mathrm{put} \times \mathrm{id}"] \arrow[d, "\pi_{13}"'] & S \times A \arrow[d, "\mathrm{put}"] \\ S \times A \arrow[r, "\mathrm{put}"] & A \end{tikzcd}$$

(Co)ends

$\mathrm{Hom} : \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$

Let $P : \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$ be a profunctor

$P : \mathcal{L} \nrightarrow \mathcal{L}$

(Co)ends

$\mathrm{Hom} : \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$

Let $P : \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$ be a **profunctor**

$P : \mathcal{L} \nrightarrow \mathcal{L}$

Def A **wedge** comprises an object $D \in \mathrm{Set}$, as well as a family of maps
$\omega_C : D \longrightarrow P(C, C)$, for all $C \in \mathcal{L}$
such that

$$\begin{tikzcd} D \arrow[r, "\omega_C"] \arrow[d, "\omega_{C'}"'] & P(C,C) \arrow[d, "P(C,f)"] \\ P(C',C') \arrow[r, "P(f,C')"'] & P(C,C') \end{tikzcd}$$

commutes, for all $f : C \longrightarrow C'$.

A **cowedge** $\gamma : P \Rightarrow D$ is the formal dual:
$\gamma_C : P(C,C) \longrightarrow D$
s.t.

$$\begin{tikzcd} P(C',C) \arrow[r, "P(C',f)"] \arrow[d, "P(f,C)"'] & P(C',C') \arrow[d, "\gamma_{C'}"] \\ P(C,C) \arrow[r, "\gamma_C"'] & D \end{tikzcd}$$

commutes, for all $f : C \longrightarrow C'$.

(Co)ends

$\mathrm{Hom}: \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$

Let $P: \mathcal{L}^{op} \times \mathcal{L} \longrightarrow \mathrm{Set}$ be a profunctor

$P: \mathcal{L} \nrightarrow \mathcal{L}$

Def A wedge comprises an object $D \in \mathrm{Set}$, as well as a family of maps
$$\omega_C : D \longrightarrow P(C,C), \quad \text{for all } C \in \mathcal{L}$$
such that
$$\begin{tikzcd} D \arrow[r, "\omega_C"] \arrow[d, "\omega_{C'}"'] & P(C,C) \arrow[d, "P(C,f)"] \\ P(C',C') \arrow[r, "P(f,C')"'] & P(C,C') \end{tikzcd}$$
commutes, for all $f: C \longrightarrow C'$.

A cowedge $\gamma: P \Rightarrow D$ is the formal dual:
$$\gamma_C : P(C,C) \longrightarrow D$$
s.t.
$$\begin{tikzcd} P(C',C) \arrow[r, "P(C',f)"] \arrow[d, "P(f,C)"'] & P(C',C') \arrow[d, "\gamma_{C'}"] \\ P(C,C) \arrow[r, "\gamma_C"'] & D \end{tikzcd}$$
commutes, for all $f: C \longrightarrow C'$.

Def A morphism of wedges between $\omega_1: D \Rightarrow P$ and $\omega_2: E \Rightarrow P$ is a map $f: D \longrightarrow E$ such that for all $C \in \mathcal{L}$ we have
$$\begin{tikzcd} D \arrow[rr, "f"] \arrow[dr, "\omega_{1,C}"'] & & E \arrow[dl, "\omega_{2,C}"] \\ & P(C,C) & \end{tikzcd}$$

Morphism of cowedges between $\gamma_1: P \Rightarrow D$ and $\gamma_2: P \Rightarrow E$:
$$\begin{tikzcd} & P(C,C) \arrow[dl, "\gamma_{1,C}"'] \arrow[dr, "\gamma_{2,C}"] & \\ D \arrow[rr, "f"'] & & E \end{tikzcd}$$

(Co)ends — for real this time

Def An **end** is a terminal wedge.

A **coend** is an initial cowedge:
$\iota : P \Rightarrow C$ s.t.

$\forall_{\gamma : P \Rightarrow D} \; \exists!_{u : C \to D} \; \forall_{f : X \to Y}$

$$\begin{tikzcd}
P(Y,X) \arrow[r, "P(f,X)"] \arrow[d, "P(Y,f)"'] & P(X,X) \arrow[d, "\iota_X"] \arrow[ddr, bend left, "\gamma_X"] & \\
P(Y,Y) \arrow[r, "\iota_Y"] \arrow[drr, bend right, "\gamma_Y"'] & C \arrow[dr, dashed, "u"] & \\
& & D
\end{tikzcd}$$

(Co)ends — for real this time

Def An **end** is a terminal wedge.

Notation The end $\omega : E \Rightarrow P$ will be denoted by

$$\int_{c \in \mathcal{C}} P(c,c).$$

Coends:

$$\int^{c \in \mathcal{C}} P(c,c)$$

A **coend** is an initial cowedge: $\iota : P \Rightarrow C$ s.t. $\forall_{\gamma : P \Rightarrow D}\ \exists!_{u : C \to D}\ \forall_{f : X \to Y}$

$$\begin{tikzcd}
P(Y,X) \arrow[r, "{P(f,X)}"] \arrow[d, "{P(Y,f)}"'] & P(X,X) \arrow[d, "\iota_X"] \arrow[ddr, bend left, "\gamma_X"] & \\
P(Y,Y) \arrow[r, "\iota_Y"] \arrow[drr, bend right, "\gamma_Y"'] & C \arrow[dr, dashed, "u"] & \\
 & & D
\end{tikzcd}$$

Ends: $\forall X.\ P(X,X)$

Coends: $\exists X.\ P(X,X)$

(Co)ends — for real this time

Def An **end** is a terminal wedge.

A **coend** is an initial cowedge: $\iota : P \Rightarrow C$ s.t.

$\forall_{\gamma : P \Rightarrow D} \; \exists!_{u : C \to D} \; \forall_{f : X \to Y}$

$$\begin{tikzcd}
P(Y,X) \arrow[r, "P(f,X)"] \arrow[d, "P(Y,f)"'] & P(X,X) \arrow[d, "\iota_X"] \arrow[ddr, bend left, "\gamma_X"] & \\
P(Y,Y) \arrow[r, "\iota_Y"] \arrow[drr, bend right, "\gamma_Y"'] & C \arrow[dr, dashed, "u"] & \\
 & & D
\end{tikzcd}$$

Notation The end $\omega : E \Rightarrow P$ will be denoted by

$$\int_{c \in \mathcal{C}} P(c,c).$$

Coends:

$$\int^{c \in \mathcal{C}} P(c,c)$$

Ends: $\forall X.\, P(X,X)$
Coends: $\exists X.\, P(X,X)$

Lemma
Let $F, G : \mathcal{C} \to \mathrm{Set}$ be functors. Then:

$$\int_{c \in \mathcal{C}} \mathrm{Set}(Fc, Gc) \cong \mathrm{Nat}(F, G).$$

Proof-idea: For any wedge $\omega : D \Rightarrow \mathrm{Set}(F,G)$ we have

$$\begin{tikzcd}
D \arrow[r, "\omega_c"] \arrow[d, "\omega_{c'}"'] & \mathrm{Set}(Fc, Gc) \arrow[d, "\vartheta(id, Gf)"] \\
\mathrm{Set}(Fc', Gc') \arrow[r, "\vartheta(Ff, id)"'] & \mathrm{Set}(Fc, Gc')
\end{tikzcd}$$

$\Rightarrow \omega_{d,c}$ is natural, for all $d \in D$

$\Rightarrow \{\omega_{d,-} \mid d \in D\} \cong \mathrm{Nat}(F,G)$

$$\Rightarrow \begin{tikzcd}
D \arrow[r, "\omega_c"] \arrow[d, dashed, "\exists! h"'] & \mathrm{Set}(Fc, Gc) \\
\{\omega_{d,-}\} \arrow[r, "\sim"] & \mathrm{Nat}(F,G) \arrow[u, "\eta \mapsto \eta_c"']
\end{tikzcd}$$

which is what we wanted. ∎

Ninja Yoneda

Lemma: For a profunctor $P : \mathcal{C} \nrightarrow \mathcal{C}$ there are isomorphisms

$$\mathrm{Set}\Big(\int^{C\in\mathcal{C}} P(C,C), D\Big) \cong \int_{C\in\mathcal{C}} \mathrm{Set}(P(C,C), D),$$

$$\mathrm{Set}\Big(D, \int_{C\in\mathcal{C}} P(C,C)\Big) \cong \int_{C\in\mathcal{C}} \mathrm{Set}(D, P(C,C)).$$

Proof-idea We have

$$\int_{C\in\mathcal{C}} P(C,C) \cong \lim P^{\tau},$$

where $P^{\tau} : \mathrm{tw}\,\mathcal{C} \longrightarrow \mathrm{Set}$ is a functor and

·) $\mathrm{Ob}(\mathrm{tw}\,\mathcal{C})$ = Arrows $f : C \to C'$ in $\mathcal{C}$

·) an arrow from f to g is a commutative diagram

$$\begin{tikzcd} C \arrow[d, "f"'] & D \arrow[l] \arrow[d, "g"] \\ C' \arrow[r] & D' \end{tikzcd}.$$

■

Ninja Yoneda

Lemma: For a profunctor $P : \mathcal{C} \nrightarrow \mathcal{C}$ there are isomorphisms

$$\mathrm{Set}\left(\int^{C \in \mathcal{C}} P(C,C), D\right) \cong \int_{C \in \mathcal{C}} \mathrm{Set}(P(C,C), D),$$

$$\mathrm{Set}\left(D, \int_{C \in \mathcal{C}} P(C,C)\right) \cong \int_{C \in \mathcal{C}} \mathrm{Set}(D, P(C,C)).$$

Proof-idea We have

$$\int_{C \in \mathcal{C}} P(C,C) \cong \lim P^{\tau},$$

where $P^{\tau} : \mathrm{tw}\,\mathcal{C} \longrightarrow \mathrm{Set}$ is a functor and

·) $\mathrm{Ob}(\mathrm{tw}\,\mathcal{C})$ = Arrows $f : C \to C'$ in $\mathcal{C}$

·) an arrow from f to g is a commutative diagram

$$\begin{tikzcd} C \arrow[d, "f"'] & D \arrow[l] \arrow[d, "g"] \\ C' \arrow[r] & D' \end{tikzcd}.$$

∎

Theorem (Ninja Yoneda)

For any functor $F : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$, we have

$$FX \cong \int^{C \in \mathcal{C}} FC \times \mathcal{C}(X, C)$$

Ninja Yoneda

Lemma: For a profunctor $P : \mathcal{C} \nrightarrow \mathcal{C}$ there are isomorphisms

$$\mathrm{Set}\Big(\int^{c\in\mathcal{C}} P(c,c), D\Big) \cong \int_{c\in\mathcal{C}} \mathrm{Set}(P(c,c), D),$$

$$\mathrm{Set}\Big(D, \int_{c\in\mathcal{C}} P(c,c)\Big) \cong \int_{c\in\mathcal{C}} \mathrm{Set}(D, P(c,c)).$$

Proof-idea We have

$$\int_{c\in\mathcal{C}} P(c,c) \cong \lim P^{\tau},$$

where $P^{\tau} : \mathrm{tw}\,\mathcal{C} \to \mathrm{Set}$ is a functor and

·) $\mathrm{Ob}(\mathrm{tw}\,\mathcal{C})$ = Arrows $f : c \to c'$ in $\mathcal{C}$

·) an arrow from f to g is a commutative diagram

$$\begin{tikzcd} c \arrow[d, "f"'] & d \arrow[l] \arrow[d, "g"] \\ c' \arrow[r] & d' \end{tikzcd}.$$

■

Theorem (Ninja Yoneda)

For any functor $F : \mathcal{C}^{op} \to \mathrm{Set}$, we have

$$FX \cong \int^{c\in\mathcal{C}} Fc \times \mathcal{C}(X,c)$$

Proof For all $Y \in \mathrm{Set}$ we have

$$\mathrm{Set}\Big(\int^{c\in\mathcal{C}} Fc \times \mathcal{C}(X,c), Y\Big)$$

$$\cong \int_{c\in\mathcal{C}} \mathrm{Set}(Fc \times \mathcal{C}(X,c), Y)$$

$$\cong \int_{c\in\mathcal{C}} \mathrm{Set}(\mathcal{C}(X,c), \mathrm{Set}(Fc, Y))$$

$$\cong \mathrm{Nat}(\mathcal{C}(X,-), \mathrm{Set}(F-, Y))$$

$$\cong \mathrm{Set}(FX, Y)$$

■

Example: closure conversion

$$\begin{array}{l} \text{let } x = 42 \\ \quad \text{in } (\lambda y.\ (+\ x\ y)) \end{array} \qquad : \mathbb{N} \to \mathbb{N}$$

Example: closure conversion

$$\begin{aligned}&\text{let } x = 42\\ &\quad \text{in } (\lambda y.\ (+\ x\ y)) \qquad : \mathbb{N} \to \mathbb{N}\end{aligned}$$

$$\begin{aligned}\rightsquigarrow\ &\text{let } x = 42\\ &\quad \text{in } [\{1\ x\}\\ &\qquad (\lambda e\ y.\ (+\ (e\ 1)\ y))] \qquad : \{x : \mathbb{N}\} \times (\{x : \mathbb{N}\} \times \mathbb{N} \to \mathbb{N})\end{aligned}$$

Example: closure conversion

$$\begin{aligned}&\text{let } x = 42\\ &\quad \text{in } (\lambda y.\ (+\ x\ y)) \qquad : \mathbb{N} \to \mathbb{N}\end{aligned}$$

$$\begin{aligned}\rightsquigarrow\ &\text{let } x = 42\\ &\quad \text{in } [\{1\ x\}\\ &\qquad (\lambda e\ y.\ (+\ (e\ 1)\ y))] \qquad : \{x:\mathbb{N}\} \times (\{x:\mathbb{N}\} \times \mathbb{N} \to \mathbb{N})\end{aligned}$$

Idea: use existential types!

$$\exists T.\ T \times (T \times (\mathbb{N} \to \mathbb{N})) \mathrel{\hat{\cong}} \mathbb{N} \to \mathbb{N}$$

$$F\mathcal{X} \mathrel{\hat{=}} \exists T.\ FT \times (\mathcal{X} \to T)$$

$\Rightarrow$ Pick $F := - \times (\mathbb{N} \to \mathbb{N})$ and $\mathcal{X} = 1$.

Optics – the existential encoding

Def / Theorem (Pastro – Street)

There is a functor

$$\mathrm{Optic} : \mathrm{SMCat} \longrightarrow \mathrm{SMCat}$$

that sends a symmetric monoidal category $(\mathcal{U}, \otimes, 1)$ to the category $\mathrm{Optic}^{\mathcal{U}}$ of optics, where an optic from S to A is an element of

$$\mathrm{Optic}^{\mathcal{U}}(S,A) := \int^{M \in \mathcal{U}} \mathcal{U}(S, M \otimes A) \times \mathcal{U}(M \otimes A, S).$$

An optic is a pair of maps

$$l : S \to M \otimes A \qquad r : M \otimes A \to S$$

modulo

$$((f \otimes A) \circ l, r) \sim (l, r \circ (f \otimes A))$$

for some $f : M \to M$.

The composite $(l, r) \circ (l', r')$ is

$$((M \otimes l) \circ l', r' \circ (M \otimes r))$$

Lenses as existential optics

Let $(\mathcal{L}, \times, 1)$ be cartesian closed.
Then we have

$$\mathrm{Lens}(A, S) := \mathcal{L}(S, A) \times \mathcal{L}(S \times A, S).$$

$$\mathrm{get} : S \longrightarrow A$$
$$\mathrm{put} : S \times A \longrightarrow S$$

Lenses as existential optics

$get : S \to A$
$put : S \times A \to S$

Let $(\mathcal{L}, \times, 1)$ be cartesian closed.
Then we have

$$\mathrm{Lens}(A, S) := \mathcal{L}(S, A) \times \mathcal{L}(S \times A, S).$$

A computation yields the desired result:

$$\int^{C \in \mathcal{L}} \mathcal{L}(S, C \times A) \times \mathcal{L}(C \times A, S) \qquad = \mathrm{Optic}^{\mathcal{L}}_{S,A}$$

$$\cong \int^{C \in \mathcal{L}} \mathcal{L}(S, C) \times \mathcal{L}(S, A) \times \mathcal{L}(C \times A, S)$$

$$\cong \mathcal{L}(S, A) \times \mathcal{L}(S \times A, S) \qquad = \mathrm{Lens}(S, A)$$

Property of the product

Ninja Yoneda

The fundamental theorem of optics

There is an equivalence of categories

$$\mathrm{Tamb}_{\mathcal{C}} \simeq [\mathrm{Optic}^{\mathcal{C}}, \mathrm{Set}].$$

The fundamental theorem of optics

There is an equivalence of categories

$$\mathrm{Tamb}_{\mathcal{L}} \simeq [\mathrm{Optic}^{\mathcal{L}}, \mathrm{Set}].$$

The profunctor representation theorem

For all $A, S \in \mathcal{L}$ we have

$$\int_{P \in \mathrm{Tamb}} \mathrm{Set}(P(A,A), P(S,S)) \cong \mathrm{Optic}(A,S)$$

$$\int_{P \in \mathrm{Tamb}} \mathcal{V}(P(A,B), P(S,T)) \cong \mathrm{Optic}((A,B),(S,T))$$