Duality in Monoidal Categories

A tale of how details are important sometimes

Based on `arXiv:2301.03545` with Sebastian Halbig.

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The Goal: establishing connections

$$\begin{array}{c}\text{Closed monoidal}\\ |\\ \text{Grothendieck–Verdier}\\ |\\ \text{Tensor representable}^{+}\\ |\\ \text{Rigid}\end{array}$$

Notions of duality

Closed monoidal categories

Definition

A (strict) monoidal category $(\mathcal{C}, \otimes, 1)$ is called *(left) closed*, if for every $x \in \mathcal{C}$ there exists an adjunction

$$- \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : [x, -]$$

with unit $\eta^x \colon - \Longrightarrow [x, - \otimes x]$ and counit $\varepsilon^x \colon [x, -] \otimes x \Longrightarrow -$.

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$(\mathrm{Cat}, \times)$,

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(Co)evaluation morphisms in closed categories

Every object x in a closed category $\mathcal{C}$ is equipped with natural *(co)evaluation* morphisms

$$\mathrm{coev}_x \coloneqq \eta_1^x \colon 1 \longrightarrow [x, x], \qquad \mathrm{ev}_x \coloneqq \varepsilon_1^x \colon [x, 1] \otimes x \longrightarrow 1$$

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$$\begin{tikzcd} x \arrow[r, "\eta_1^x \otimes x"] \arrow[dr, "\mathrm{id}_x"'] & {[x,x] \otimes x} \arrow[d, "\varepsilon_x^x"] \\ & x \end{tikzcd}$$

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We need a coherent isomorphism $\phi_x \colon [x, x] \longrightarrow x \otimes [x, 1]$.

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Tensor representability

Every rigid category yields adjunctions

$$- \otimes x : \mathcal{C} \rightleftarrows \mathcal{C} : - \otimes {}^{\vee}x.$$

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Starting with an adjunction

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A monoidal category is *tensor representable*, if for every $x \in \mathcal{C}$ there is an adjunction as in Equation (1).

Tensor representability in context

Relationships among different types of duality

$$\begin{tikzcd}
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Tensor representable categories are Grothendieck–Verdier

Theorem ([HZ23, Theorem 3.2])

If for every $x \in \mathcal{C}$, there exist adjunctions

$$- \otimes Lx \dashv - \otimes x \dashv - \otimes Rx,$$

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Proof.

The assumption implies that L is a quasi-inverse of R; for example,

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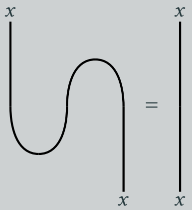
Is tensor representability and rigidity the same thing?

The problem in string diagrams

The rigid case



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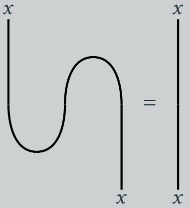

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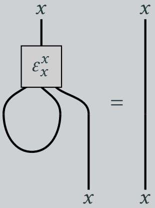

$\varepsilon_x^x \circ (\eta_1^x \otimes x) = \mathrm{id}_x$

A negative result

Theorem ([HZ23, Theorems 2.5 and 4.15])

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$$(M \star N)x := \int^{a,b} \Bbbk\mathsf{Sp}_G(a \otimes b, x) \otimes_k Ma \otimes_k Nb,$$

$$[M, N]x := \int_{a,b} \mathsf{vect}_\Bbbk(\Bbbk\mathsf{Sp}_G(x \otimes a, b), \mathsf{vect}_\Bbbk(Ma, Nb)) \cong \mathsf{mky}(M(x^* \otimes -), N).$$

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5. By [Bou05, Lemma 2.2]: mky is rigid iff all short exact sequences split.

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The category mky *is tensor representable, but not rigid.*

Proof idea.

1. In $\mathsf{vect}_{\Bbbk}$: $\int_y \int^x \cong \int^x \int_y$.
2. By [Day06]: $\int^x \cong \int_y$.
3. By [PS07, Section 9]: mky is *-autonomous.
4. Calculation: it is even tensor representable.
5. By [Bou05, Lemma 2.2]: mky is rigid iff all short exact sequences split.
6. If the order of G divides the characteristic of $\Bbbk$, then mky is not rigid.

$\square$

Thank you for your attention!

arXiv:2301.03545

[Bou05] Serge Bouc. “**The Burnside dimension of projective Mackey functors**”. English. In: *RIMS Kôkyûroku* 1440 (2005), pp. 107–120. ISSN: 1880-2818.

[Day06] Brian J. Day. “**Compact convolution**”. In: *arXiv e-prints* (2006). arXiv: `math/0605463 [math.CT]`.

[HZ23] Sebastian Halbig and Tony Zorman. “**Duality in Monoidal Categories**”. In: *arXiv e-prints* (2023). arXiv: `2301.03545 [math.CT]`.

[PS07] Elango Panchadcharam and Ross Street. “**Mackey functors on compact closed categories**”. English. In: *J. Homotopy Relat. Struct.* 2.2 (2007), pp. 261–293. ISSN: 2193-8407.

Spans

Definition

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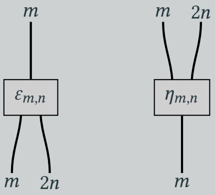
$$\begin{tikzcd} & s \arrow[dl, "l"'] \arrow[dr, "r"] & \\ a & & b \end{tikzcd}$$

The explicit construction

Define a category $\mathcal{D}$ whose objects are natural numbers, with morphisms generated by compositions and tensor products of the identities, as well as



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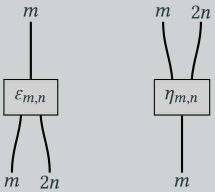

subject to appropriate naturality conditions. Define a *monoidal localisation* $\mathcal{C}$ of $\mathcal{D}$, in which adjunction properties for ε and η hold.

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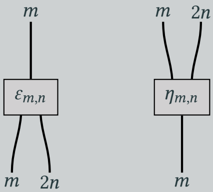

subject to appropriate naturality conditions. Define a *monoidal localisation* $\mathcal{C}$ of $\mathcal{D}$, in which adjunction properties for ε and η hold. By studying strong monoidal functors $\mathcal{C} \longrightarrow \mathsf{vect}_{\Bbbk}$, we find that the snake equation and the identity must lie in different equivalence classes.