§1 Lecture 1

We begin with the definition of probability.

Definition 1.1 (**Probability**)**.** A mathematical model about random experiments.

In this class, we fix the notation $\Omega, \mathcal{F}, \mathbb{P}$. We say that Ω is our "sample space" or the set of all possible outcomes. For example, if our experiment was flipping a coin, then $\Omega = \{H, T\}$. It is also important to note that Ω can be countably or uncountably infinite. For example, our experiment might be picking a positive integer in which $\Omega = \mathbb{N}$. Or our experiment might be throwing a dart at a board in which Ω would be uncountably infinite.

We define $\mathcal{F}$ to be the set of all events, mathematically speaking it's a σ algebra. We can define an event to be a subset of Ω. For example, if our experiment is flipping two coins, then an event could be that the first coin is heads - the precise event would then be $\{HH, HT\}$ (notice that both HH and HT are in Ω.

Another interesting example might be where we roll a die as our experiment. Then we could say an event is rolling an *even* number that's greater than 4. The precise event would be $\{4, 6\}$. Notice, however, that we can actually write this event as the intersection of two events. That is

1. Rolling an even number : $\{2, 4, 6\}$.
2. Rolling a number greater than 4 : $\{4, 5, 6\}$.

Then the intersection of these two events is exactly our event of rolling an even number greater than 4.

Remark 1.2. When we say one event A happens, it means we get some outcome $\omega \in \Omega$ s.t. $\omega \in A$. If $\omega \notin A$, then A *doesn't* happen.

Remark 1.3. We can also do set operations on events. That is if A and B are events, then we can say that the event of A or B happening is exactly $A \cup B$.

Definition 1.4. We define the complement of event A to be $A^c = \{\omega \in \Omega : \omega \notin A\}$.

Remark 1.5. If Ω is finite, then one possible choice of $\mathcal{F}$ is $\mathcal{F} = 2^{\Omega}$.

Note that 2^Ω is defined to be the *power set* of Ω, or the "set of all subsets of Ω".

One more thing to note, but not terribly important for this class is that if Ω is infinite, then $\mathcal{F} \neq 2^\Omega$.

Finally, we define $\mathbb{P}$ to be a function from $\mathcal{F}$ to the closed interval $[0, 1]$. That is, $\mathbb{P} : \mathcal{F} \to [0, 1]$. Notice that this means we can only talk about the probability of one event.

If we want to talk about the probability of some outcome ω, then formally we would define the event $\{\omega\}$ and ask what is $\mathbb{P}(\{\omega\})$.

Recall that we said that $\mathcal{F}$ is a "σ algebra". Here we define it formally.

Definition 1.6. Let Ω be a set. We say that $\mathcal{F}$ is a σ algebra of Ω if it satisfies:

1. $\emptyset \in \mathcal{F}$
2. If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{i=1}^\infty A_i \in \mathcal{F}$
3. If $A \in \mathcal{F}$, then $A^c \in F$ (recall that A^c is the *complement* if A)

Now just from these three properties, we can, in fact, prove that if $A_1, A_2, \cdots, A_n \in \mathcal{F}$, then $\bigcap_{i=1}^n A_i \in \mathcal{F}$ as well.