Functional Analysis

github.com/dan-gc/analysis

These are preparation notes for a course on Functional Analysis at IMPA, summer 2024. They are based on Folland, *Real Analysis: Modern Techniques and Their Applications*; Bachman and Narici, *Functional Analysis*; Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.

Contents

1 Measure theory

A measure should surely satisfy:

1. If $E_1, E_2, \ldots$ is a finite or infinite sequence of disjoint sets,
$$\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots$$
2. If E is congruent to F,
$$\mu(E) = \mu(F)$$
3. If Q is the unit cube,
$$\mu(Q) = 1$$

1.1 σ-algebras

Let X be a nonempty set.

- An ***algebra of sets*** on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements, that is,
  1. If $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^n E_i \in \mathcal{A}$.
  2. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.
- A ***σ-algebra*** is an algebra of sets closed under countable unions.
- The intersection of all σ-algebras containing any subset $\mathcal{E} \subset \mathcal{P}(X)$ is the ***σ-algebra generated by*** $\mathcal{E}$.
- The σ-algebra generated by the open sets of a topological (or metric) space X is the ***Borel algebra*** $\mathcal{B}_X$.
- Let $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonempty sets, $X = \prod_\alpha X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ the coordinate functions. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, the ***product σ-algebra*** on X is the σ-algebra generated by
$$\{\pi_\alpha(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$
  We denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

  Proposition 1.1. $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}\}$.

  Proposition 1.2. Let $X_1, \ldots, X_n$ be metric spaces and let $X = \prod_i X_i$ be equiped with the product metric. Then $\bigotimes_i \mathcal{B}_i \subset \mathcal{B}_X$. If every X_i is separable equality holds.

  Corollary 1.3. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$.

1.2 Measures

Let X be a set equipped with a σ-algebra $\mathcal{M}$.

- A ***measure*** on $\mathcal{M}$ (or on $(X, \mathcal{M})$, or simply on X) is a function $\mu : \mathcal{M} \to [0, \infty)$ such that

  1. $\mu(\varnothing)$.
  2. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_1^\infty\right) = \sum_1^\infty \mu(E_i)$.

- If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a ***measurable space*** and the sets in $\mathcal{M}$ are ***measurable sets***. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a ***measure space***.

- If $\mu(X) < \infty$ (and hence $\mu(E) < \infty$ for all $E \in \mathcal{M}$), μ is called ***σ-finite***. If $X = \bigcup_1^\infty E_j$, with $\mu(E_j) < \infty$, μ is called ***σ-finite***. If for every $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$, μ is called ***semifinite***.

  Theorem 1.4 (Properties of measure spaces). Let $(X, \mathcal{M}, \mu)$ be a measure space.

  1. **(Monotonicity.)** If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) < \mu(F)$.
  2. **(Subaditivity.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu(E_j)$.
  3. **(Continuity from below.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \subset E_2 \subset \ldots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.
  4. **(Continuity from above.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \supset E_2 \supset \ldots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

- If $E \in \mathcal{M}$ and $\mu(E) = 0$, E es called a ***null set***. If a statement about points in X is true except for points in a null set, we say it is true ***almost everywhere***.

- If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ *provided* $F \in \mathcal{M}$. A measure whose domain contains all subsets of null sets is ***complete***. Completeness may help avoid technical difficulties, and it can always be achieved by enlarging the domain of μ:

  Theorem 1.5. Let $(X, \mathcal{M}, \mu)$ be a measure. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$, called the ***completion of*** μ.

- Excercises

  1. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^\infty a_j \mu_j$ is also a measure on $(X, \mathcal{M})$.
  2. $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.
  3. A set $E \subset X$ is called ***locally measurable*** if $E \cap A \in \mathcal{M}$ whenever $A \in \mathcal{M}$ and $\mu(A) < \infty$. If $\mathcal{M}$ equals the collection of the saturated sets $\tilde{\mathcal{M}}$, it is called ***saturated***. The saturated measure $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ defined by $\tilde{\mu}(E) = \mu(E)$ for $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise is called the ***saturation of*** μ.

1.3 Outer measures

This is used to construct measures. The key idea is to approximate the measure of a set by simpler encosing sets, like with the Riemann integral.

- An ***outer measure*** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

  1. $\mu^*(\varnothing) = 0$,
  2. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$, and
  3. $\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*(A_j)$.

  Proposition 1.6. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho\varnothing = 0$. For any $A \subset X$, define

  $$\mu^*(A) = \inf\left\{\sum_1^\infty \mu(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\} \tag{1}$$

  then μ^* is an outer measure.

- A set $A \subset X$ is called ***μ-measurable*** if

  $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X$$

  Which makes sense if we think E is a *well-behaved* set such that $A \subset E$, so that $\mu^*(A) = \mu^*(E) - \mu^*(E \cap A^c)$.

  Theorem 1.7 (Carathèodory). If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

- If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, the function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a ***premeasure*** if

  - $\mu_0(\varnothing) = 0$,
  - If $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty \in \mathcal{A}$, then $\mu_0\left(\bigcup_1^\infty A_j\right) = \sum_1^\infty \mu_0(A_j)$.

  Proposition 1.8. If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by eq. (1) by taking $\rho = \mu_0$, then

  1. $\mu^*|\mathcal{A} = \mu_0$,
  2. every set in $\mathcal{A}$ is μ^* measureable.

  Theorem 1.9. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$ and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0—namely $\mu = \mu^*|\mathcal{M}$, where μ^* is given by eq. (1). (This is a consequence of Carathédory's theorem and the last proposition.)

If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \le \mu(E)$ for all $E \in \mathcal{M}$ with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique estension of μ_0 to a measure on $\mathcal{M}$.

- A ***Borel measure on*** $\mathbb{R}$ is a measure on $\mathbb{R}$ whose domain is the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$.

  Proposition 1.10. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$, $j = 1, \dots, n$ are disjoint ***half-open intervals***, so that $0 \le a_j, b_j \le \infty$, or (a_j, ∞), or they are empty, define

  $$\mu_0\left(\bigcup_1^n (a_j, b_j]\right) = \sum_1^n [F(b_j) - F(a_j)]$$

  and let $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$ of finite disjoint unions of half-open intervals.

  Theorem 1.11. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)\ \forall a, b$.

  If G is another such function, $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely, given a μ Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets we may define an increasing right continuous function F such that $\mu = \mu_F$.

- By theorem 1.9, there is a complete measure $\bar{\mu}_F$ whose domain includes $\mathcal{B}_{\mathbb{R}}$. In fact $\bar{\mu}_F$ is the completion of μ_F and its domain is strictly larger than $\mathcal{B}_{\mathbb{R}}$. This complete measure is called the ***Lebesgue-Stieltjes measure*** and is also denoted by μ_F.

  In the following μ is the Lebesgue-Stieltjes meause associated to some increasing, right-continuous function F, and $\mathcal{M}_\mu$ is the domain of μ.

  Theorem 1.12. If $E \in \mathcal{M}_\mu$, then

  $$\begin{aligned} \mu(E) &= \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\} \end{aligned}$$

  Theorem 1.13. If $E \subset \mathbb{R}$, the following are equivalent:

  1. $E \in \mathcal{M}_\mu$.
  2. $E = V \backslash N_1$ where V is a G_δ (countable intersection of open sets) and $\mu(N_1) = 0$.
  3. $E = H \cup N_2$ where H is an F_σ (countable union of closed sets) set and $\mu(N_2) = 0$.

  Proposition 1.14. If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$, there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \varepsilon$.

- The ***Lebesgue measure*** is the Lebesgue-Stieltjes measure of $F(x) = x$. We denote it by m and its domain by $\mathcal{L}$.

Theorem 1.15. Is $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proposition 1.16. Let C be the Cantor set. C is compact, nowhere dense and totally disconnected (ie. the only connected subsets of C are single points). C has no isolated points. $m(C) = 0$. $\operatorname{card}(C) = \mathfrak{c}$.

1.4 Integration

Now we construct integrals from simple funtions.

- **(Measurable maps.)** Recall that a mapping $f : X \to Y$ between two sets induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}X$ defined by the inverse image, which preserves unions, intersection and complements, so that if $\mathcal{N}$ is a σ-algebra on Y, then $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X.

  If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measureable spaces, a mapping $f : X \to Y$ is called ***measurable*** if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

  Proposition 1.17. A function whose codomain is a product measure space is measurable if precomposing with every projection is measurable.

  Corollary 1.18. A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{M}$-measurable.

  Proposition 1.19. If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

  Corollary 1.20. If X and Y are metric (or topological spaces), every continuous function is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

  Proposition 1.21. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f + g$ and fg.

  Proposition 1.22. If $\{f_j\}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

  $$\sup_j f_j(x) \qquad \limsup_{j\to\infty} f_j(x)$$
  $$\inf_j f_j(x) \qquad \liminf_{j\to\infty} f_j(x)$$

  are measurable. If

  $$f(x) = \lim_{j\to\infty} f_j(x)$$

  exists for every $x \in X$, then f is measurable.

  Corollary 1.23. If $f, g : X \to \overline{\mathbb{R}}$ is measurable, then so are $\max(f, g)$ and $\min(f, g)$.

  If $f : X \to \overline{\mathbb{R}}$ we define de ***positive*** and ***negative*** parts of f as:

  $$f^+(x) = \max(f(x), 0) \qquad\qquad f^-(x) \max(-f(x), 0)$$

  Then $f = f^+ - f^-$, and if f is measurable, so are f^+ and f^- by corollary 1.23.

- Let $(X, \mathcal{M})$ be a measurable space. If $E \subset X$, the ***characteristic or indicator function*** χ_E ***of*** E is

$$\chi(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

  A ***simple function*** is . The ***integral of a simple function*** is . The ***integral of a measurable function*** .

  Theorem 1.24 (Monotone convergence). content...

  Theorem 1.25 (Dominated convergence). content...

  Proposition 1.26 (Aditivity of the integral)**.** content...

- The ***Lebesgue integral*** is the integral we have developed then the measure is the Lebesgue measure.

  Theorem 1.27 (Fubini-Tonelli).

  1.

$$\int f d(\mu \times \nu) = \int \left(\int f(x,y) d\nu(y) \right) d\mu(x) = \int \left(\int f(x,y) d\mu(x) \right) d\nu(y)$$

  2.

  Theorem 1.28 (2.44).

$$\int \int f(x) dx = |\det T| \int f \circ T(x) dx$$

  Theorem 1.29 (2.47, diffeomorphisms). content...

2 Point set topology

2.1 Metric spaces

A ***metric*** on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

1. $\rho(x, x) = 0$ if and only if $x = 0$.
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Two metrics ρ_1 and ρ_2 on a set X are ***equivalent*** if $C\rho_1 \leq \rho_2 \leq C'\rho_2$ for some $C, C' > 0$.

Theorem 2.1. Let (X, d) and (Y, d') be metric spaces. If $f : X \to Y$, the following are equivalent conditions for f to be ***continuous***:

1. $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$
2. $\forall x \in X \forall x_n \to x : f(x_n) \to f(x)$.
3. $\forall F \subseteq Y$ open, $f^{-1}(F)$ is open.
4. $\forall F \subseteq Y$ closed, $f^{-1}(F)$ is closed.

If $f : (X, \rho) \to (Y, \rho')$ is such that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

we say f is ***uniformly continuous***.

Exercise. If (Y, ρ') is complete and $f : A \to Y$ is uniformly continuous on $A \subset X$ and $\overline{A} = X$, then f has a unique continuous extension $g : X \to Y$ which is uniformly continuous on X. Show that this is not true in general if Y is not complete.

If $f : (X, \rho) \to (Y, \rho')$ is a bijective function such that for any $x, y \in X$, $\rho(x, y) = \rho'(f(x), f(y))$ we say f is an ***isometry*** and the two spaces are ***isometric***. A function $f : (X, \rho) \to (X, \rho)$ is a ***contraction*** if there exists a $0 < a < 1$ such that $\rho(f(x), f(y)) \leq a\rho(x, y)$ for any $x, y \in X$. Every contraction is continuous, and if X is complete then any contraction has a unique fixed point.

A sequence $\{x_n\}$ in X ***converges*** to x if $\lim_{n\to\infty} \rho(x_n, x) = 0$. A sequence $\{x_n\}$ in X is called ***Cauchy*** if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$, that is

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m > N : \rho(x_n, x_m) < \varepsilon.$$

A subset $E \subseteq X$ is ***complete*** if every Cauchy sequence in E converges to its limit in E. If (X, ρ) and (X^*, ρ^*) are metric spaces and

1. (X, ρ) is isometric to a subspace (X, ρ^*) of (X^*, ρ^*),
2. The closure of X_0 is all of X^* (X_0 is ***everywhere dense*** or simply ***dense***),

we say (X^*, ρ^*) is the ***completion*** of (X, ρ).

Theorem 2.2. Every metric space (X, ρ) has a completion (X^*, ρ^*). If (X^{**}, ρ^{**}) is also a completion of (X, ρ^*), then (X^*, ρ^*) is isometric to (X^{**}, ρ^{**}); that is, the completion of a space is unique up to isometry.

Proof. Consider equivalence classes of Cauchy sequences. □

Proposition 2.3. A closed subset of a metric space is complete, and a complete subset of an arbitrary metric space is closed.

Theorem 2.4. If E is a subset of a metric space (X, ρ), the following are equivalent:

1. E is complete and ***totally bounded*** (it can be covered by finitely many balls of radius ε).
2. **(Bolzano-Wierstrass Property.)** Every sequence in E has a subsequence that converges to a point in E.
3. **(Heine-Borel Property)** If $\{V_\alpha\}_{\alpha \in A}$ is an open cover of E, then there is a finite subset $F \subseteq A$ such that $\{V_\alpha\}_{\alpha \in F}$ covers E.

A set that satsifies any of these condition s is called ***compact***.

Theorem 2.5. If (X, ρ) is a metric space and A is compact, then A is closed and bounded.

If (X, ρ) is a metric space, $A \subseteq X$ is ***relatively compact*** if $\overline{A}$ is compact. If $\varepsilon > 0$, a subset $N \subset X$ is an ***ε-net with respect to*** A if $\forall x \in A \exists n \in N : \rho(x, n) < \varepsilon$. A is ***totally boundad*** if for any $\varepsilon > 0$ there exists a finite ε-net with respect to A.

Theorem 2.6. Let (X, ρ) be a metric space and $A \subseteq X$. If for every sequence of points from A one can select a convergente subsequence, then A is totally bounded.

A set A is ***countably compact*** if every infinite subset of A has a limit point in A. All compact sets are countably compact. A is ***sequentially compact*** if every sequence in A has a subsequence that converges to a point in A. In a metric space, compactness is equivalent to countable and sequential compactness.

Theorem 2.7. Let (X, ρ) be a metric space and $A \subseteq X$.

1. A is relatively compact if and only if a convergent subsequence can be selected from every sequence of points in A. (We do not claim that the limit point is a member of A.)
2. If A is relatively compact, it is also totally bounded.
3. If (X, ρ) is complete and A is totally bounded, then A is relatively compact.
4. If A is compact then A is closed and totally bounded.

2.2 Topological spaces

If τ_1 and τ_2 are two topologies on a set X, we say τ_1 is ***weaker*** (or ***coarser***) and τ_2 ***stronger*** (or ***finer***). $E \subseteq X$ is called ***dense*** if $\overline{E} = X$ and ***nowhere dense*** if $\overline{E}$ has empty interior. X is called ***separable*** if it has a countable dense subset.

T_0 If $x \neq y$, there is an open set containing x but not y, or an open set containing y but not x.

T_1 If $x \neq y$, there is an open set containint y but not x. Equivalently, $\{x\}$ is closed for every $x \in X$.

T_2 **(Hausdorff.)** If $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

T_3 **(Regular.)** X is T_1 and for any closed set $A \subset X$ and any $x \in A^c$ there are disjoint open sets U, V with $x \in U$ and $a \subseteq V$.

$\mathrm{T}_{3\frac{1}{2}}$ **(Tychonoff, Completely regular.)** X is T_1 and for each closed $A \subseteq X$ and each $x \notin A$ there exists $f \in C(X, [0,1])$ such that $f(x) = 1$ and $f = 0$ on A.

T_4 **(Normal.)** X is T_1 and for any disjoint closed sets A, B in X there are disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$.

If X is any set and $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ is a family of maps from X into some topological spaces Y_α, there is a unique weakest topology τ on X that makes all the f_α continuous called the ***weak topology generated by*** $\{f_\alpha\}_{\alpha \in A}$. An example of this topology is the ***product topology*** on $X = \prod_{\alpha \in A} X_\alpha$ with the projections.

Proposition 2.8.

- If X_α is Hausdorff for each $\alpha \in A$ then $X = \prod_{\alpha \in A}$ is Hausdorff.
- If X_α and Y are topological spaces, a function $f : Y \to X = \prod_{\alpha \in A} X_\alpha$ is continuous if and only iff $\pi_\alpha \circ f$ is continuous for each α.
- If X is a topological space, A is a nonempty set and $\{f_n\}$ is a sequence in X^A, then $f_n \to f$ in the product topology if and only if $f_n \to f$ pointwise.

If X is any set and $K = \mathbb{R}$ or $\mathbb{C}$, denote by $B(X, K)$ the ***set of bounded*** K***-valued functions on*** X, $C(X, K)$ the set of ***continuous*** K***-valued functions on*** X, and $BC(X, K)$ the ***set of bounded continuous functions on*** X. If no field is specified we take it to be $\mathbb{C}$.

For $f \in B(X)$ define the ***uniform norm*** of f to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}$$

Then the function $\rho(f, g) = |Vertf - g|Vert_u$ is a metric on $B(X)$. Convergence in this metric is simply uniform convergence:

$$\{f_n\}\overset{u}{to}f \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall x \in X : |f_n(x) - f(x)| < \varepsilon$$

$B(X)$ is complete with this metric since $\mathbb{C}$ is complete.

Proposition 2.9. If X is a topological space, $BC(X)$ is a closed subspace of $B(X)$ in the uniform metric; in particular $BC(X)$ is complete.

Lemma 2.10 (Urysohn). Let X be a normal space. If A and B are disjoint closed sets in X, there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Theorem 2.11 (Tietze Extension Theorem). Let X be a normal space. If A is a closed subset of X and $f \in C(A, [a,b])$, there exists $F \in C(X, [a,b])$ such that $F|A = f$.

Corollary 2.12. If X is normal, $A \subseteq X$ is closed and $f \in C(A)$, there exists $F \in C(X)$ such that $F|A = f$.

Urysohn's lemma shows that every T_4 space is completely regular ($\mathrm{T}_{3\frac{1}{2}}$).

Theorem 2.13 (Dugundji). Sea X un espacio metrizable, $A = \bar{A} \subset X$ y L un espacio vectorial localmente convexo, y $V \subset L$ convexo. Entonces cualquier función $f : A \to V$ admite una extensión F.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & V \\ X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Además $\operatorname{img} F \subset \operatorname{conv} \operatorname{img} F$.

2.3 Compact spaces

A topological space X is called ***compact*** if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X there is a finite subset B of A such that $X = \bigcup_{\alpha \in B} U_\alpha$. A subset $Y \subseteq X$ is called ***compact*** if it is compact in the relative topology and ***precompact*** if its closure is compact.

A family $\{F_\alpha\}_{\alpha \in A}$ of subsets of X has the ***finite intersection property*** if $\bigcap_{\alpha \in B} F_\alpha \neq \varnothing$ for all finite $B \subseteq A$.

Proposition 2.14.

- A topological space X is compact if and only if for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$.
- A closed subset of a compact space is compact.
- If K is a compact subset of a Hausdorff space X and $x \notin K$ then there are disjoint open sets U, V such that $x \in U$ and $K \subseteq V$.
- Every compact subset of a Hausdorff space is closed.
- Every compact Hausdorff space is normal.
- If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.
- If X is compact, then $C(X) = BC(X)$.
- If X is compact and Y is Hausdorff, then any continuous bijection $f : X \to Y$ is an homeomorphism.

A topological space X is ***countably compact*** if every countable open cover of X has a finite subcover, and ***sequentially compact*** if every sequence in X has a convergent subsequence. For metric spaces compactness and sequential compactness are the equivalent. There exists no general relation between compactness and sequential compactness.

2.4 Locally Compact Hausdorff spaces

A topological space is called ***locally compact*** if every point has a compact neighbourhood (a set $A \subset X$ such that $x \in A^o$). We call locally compact Hausdorff spaces ***LCH*** for short.

Proposition 2.15. Let X be a LCH space.

- If $U \subseteq X$ is open and $x \in U$, there is a compact neighbourhood K of x such that $K \subset U$.
- If $K \subseteq U \subseteq X$, with K compact and U open, there exists a precompact open V such that $K \subseteq V \subset \overline{V} \subset U$.
- **(Urysohn's Lemma, Locally Compact Version.)** If $K \subset U \subseteq X$, there exists $f \in C(X, [0,1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.
- Every LCH space is completely regular.
- **(Tietze Extension Theorem, Locally Compact Version)** If $K \subseteq X$ is compact and $f \in C(K)$, there exists $F \in C(X)$ such that $F|K = f$. F may be taken to vanish outside a compact set.

If $f \in C(X)$, the ***support of*** f is the closure of $\{x \in X : f(x) \neq 0\}$ and denote $C_c(X) := \{f \in C(X) : \operatorname{supp} f \text{ is compact}\}$. We say f ***vanishes at infinity*** if for every $\varepsilon > 0$ the set $\{x : |f(x)| \geq \varepsilon\}$ is compact and define $C_0(X) := \{f \in C(X) : f \text{ vanishes at infinity}\}$.

Proposition 2.16. If X is an LCH space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.

If X is a topological space, there are many ways of topologizing $\mathbb{C}^X$. One way is the product topology, that is, the topology of pointwise convergence. Another is the ***topology of uniform convergence***, which is generated by the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in X} |g(x) - f(x)| < n^{-1}\right\} \qquad n \in \mathbb{N}, f \in \mathbb{C}^X.$$

In view of a previous proposition (cite?), we know $C(X)$ is a closed subset of $\mathbb{C}^X$ with the topology of uniform convergence. Another topology is the ***topology of uniform convergence on compact sets***, generated by the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in K} |g(x) - f(x)| < n^{-1}\right\} \qquad n \in \mathbb{N}, f \in \mathbb{C}^X, K \subseteq X \text{ compact}.$$

Proposition 2.17. Let X be an LCH space.

- If $E \subseteq X$, then E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.
- $C(X)$ is a closed subspace of $\mathbb{C}^X$ in the topology of uniform convergence on compact sets.
- If $\{U_j\}_{j=1}^n$ is an open cover of a compact subset K of X, then there is a partition of unity on K subordinate to $\{U_j\}_{j=1}^n$ soncisting of compactly supported functions.

Theorem 2.18 (Urysohn Metrization Theorem). Every second countable normal space is metrizable.

2.5 Three compactness theorems

Recall that if $X = \prod_{\alpha \in A} X_\alpha$, an element $x \in X$ is just a mapping from A to $\bigcup_{\alpha \in A} X_\alpha$, with $x(\alpha)$ the αth coordinate of x.

Theorem 2.19. If $\{X_\alpha\}_{\alpha \in A}$ is a family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact with the produc topology.

Let X be a topological space and $\mathcal{F} \subseteq C(X)$ a family of complex-valued continuous functions on X. We say $\mathcal{F}$ is ***equicontinuous at*** $x \in X$ if for every $\varepsilon > 0$ there is a neighbourhood U of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and all $f \in \mathcal{F}$; and ***equicontinuous*** if it is equicontinuous at every $x \in X$. Also, $\mathcal{F}$ is ***pointwise bounded*** if $\{|f(x)| : f \in \mathcal{F}\}$ is bounded for all $x \in X$.

Theorem 2.20 (Arzelá-Ascoli I). Let X be a compact Housdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.

Theorem 2.21 (Arzelá-Ascoli II). Let X be a locally compact Housdorff space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, then there exists $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.

2.6 The Stone-Weierstrass Theorem

Recall that the Weierstrass theorem states that any continuous function on a compact interval $[a, b]$ is the uniform limit of polynomials on $[a, b]$. Throughout this subsection, X will denote a compact Hausdorff space, and $C(X)$ is equipped with the uniform metric.

A subset $\mathcal{A}$ of $C(X, \mathbb{R})$ of $C(X)$ is said to ***separate points*** if for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. $\mathcal{A}$ is called an ***algebra*** if it is a real (resp. complex) vector subspace of $C(X, \mathbb{R})$ (resp. $C(X)$) such that $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$. $\mathcal{A}$ is called a ***lattice*** if $\max(f, g)$ and $\min(f, g)$ are in $\mathcal{A}$ whenever $f, g \in \mathcal{A}$. If $\mathcal{A}$ is an algebra or a lattice, so is its closure in the uniform metric.

Theorem 2.22 (Stone-Weierstrass Theorem). Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ that separates points, then either $A = C(X, \mathbb{R})$ of $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds if and only if $\mathcal{A}$ contains the constant functions.

Corollary 2.23. Supoose $\mathscr{B}$ is a subalgebra of $C(X, \mathbb{R})$ that separates points. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathscr{B}$, then $\mathscr{B}$ is dense in $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$. Otherwise, $\mathscr{B}$ is dense in $C(X, \mathbb{R})$.

The classical Weierstrass approximation theorem is the special case of this corollary where X is the compact subset of $\mathbb{R}^n$ and $\mathscr{B}$ is the algebra of polynomials on $\mathbb{R}^n$ (restricted to X); here $\mathscr{B}$ contains the constant functions, so it is dense in $C(X, \mathbb{R})$.

The Stone-Weirstrass theorem, as stated, is false for complex-valued functions. We may show that $f(z) = \bar{z}$ cannot be approximately uniformly by polynomials on the unit circle.

Theorem 2.24 (Complex Stone-Weirstrass Theorem). Let X be a compact Hausdorff space. If $\mathscr{A}$ is a closed complex subalgebra of $C(X)$ that separates points and is closed under complex conjugation, then either $A = C(X)$ of $\mathscr{A} = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$.

Finally, there is a version of the Stone-Weirstrass theorem for noncompact LCH spaces. We state for real functions; the complex analogue is an immediate consequence.

Theorem 2.25 (LCH Stone-Weirstrass Theorem). Let X be a noncompact LCH space. If $\mathscr{A}$ is a closed complex subalgebra of $C_0(X, \mathbb{R})$ that separates points, then either $A = C_0(X, \mathbb{R})$ of $\mathscr{A} = \{f \in C_0(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$.

3 Inner product spaces

Let X be a real or complex vector space.

3.1 Inner products

An ***inner product*** on X is a mapping

$$\langle -, - \rangle : X \times X \to F$$

with the following properties:

(I_1) if $x, y \in X$ then $\langle x, y \rangle = \overline{\langle x, y \rangle}$;

(I_2) if α, β are scalars, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

(I_3) $\langle x, x \rangle \geq 0$ for all $x \in X$ and equal to zero if and only if x is the zero vector. (Since, by I_1, $\langle x, x \rangle$ must be real.)

Examples.

1. Let $X = C[a, b]$ be complex-valued continuous functions on the closed interval $[a, b]$ with pointwise addition and scalar product. As the inner product of any two vectors f and g in this space take

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

2. Let $X = l_2$, the set of all sequences of complex numbers $(a_1, a_2, \ldots)$ with the property that $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. As the inner product of any two vectors $x = (a_i)$ and $y = (b_i)$ in this space take

$$\langle f, g \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i$$

which converges by the Hölder inequality.

3. Let Y be the closed interval $[a, b]$, S the Lebesgue measurable sets and μ the Lebesgue measure. Then, for the equivalence clasess of square-integrable functions (complex-valued) on $[a, b]$ we can take as the inner product of two clases f and g,

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

where the integral is the Lebesgue integral. This space is denoted by $L_2(a, b)$.

Theorem 3.1 (Cauchy-Schwarz inequality). Let X be an inner product space and let $x, y \in X$. Then

$$|\langle x, y \rangle \leq \|x\| \|y\|$$

with equality holding if and only if x and y are linearly independent.

3.2 Orthogonal projections

Two vectors $x, y \in X$ are ***orthogonal*** if $\langle x, y\rangle = 0$.

Examples.

1. In $L_2(-\pi, \pi)$, the collection (or any subset thereof)

$$x_n = \frac{1}{\sqrt{2\pi}} e^{int}, \qquad n = 0, \pm 1, \ldots$$

is an orthonormal set of vectors.

Proof. For any $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} x_n \overline{x}_n dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{int}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|e^{int}\|^2 dt = 1,$$

and if m is another integer,

$$\int_{-\pi}^{\pi} x_n \overline{x}_m dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{imt}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(n-m)it} dt = \frac{1}{2\pi} \left[\frac{e^{(n-m)it}}{(n-m)i} \right]_{-\pi}^{\pi} = 0.$$

□

2. If we restric out attention to only real-valued functions that are square-integrable on the interval $[-\pi, \pi]$, then the collection (or any subset thereof)

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \cos 2t, \ldots$$
$$\frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \sin 2t, \ldots$$

is an orthonormal set.

Theorem 3.2. If S is an orthonoromal subset of an inner product space, then it is linearly independent (where linear independence is defined as finite sums).

Theorem 3.3 (Gram-Schmidt process). Let X be an inner product space. If $\{y_1, y_2, \ldots\}$ is a linearly independent set of vectors, then there exists an orthonormal set of vectors $\{x_1, x_2, \ldots\}$ such that, for any n,

$$\langle y_1, y_2, \ldots, y_n\rangle = \langle x_1, x_2, \ldots, x_n\rangle$$

where the brackets indicate the subspace spanned by the vectores enclosed.

If S is any subset of X, the ***orthogonal complement of*** S ***in*** X is the linear space $S^\perp := \{x \in X : x \perp s \text{ for all } s \in S\}$.

Theorem 3.4. If M is a finite-dimensional subspace of X, then $X = M \oplus M^\perp$.

3.3 Riesz representation theorem

Theorem 3.5 (Riesz). If X is a finite-dimensional inner product space and f is a linear functional on X, then there exists a unique vector $y \in X$ such that $f(x) = \langle x, y \rangle forall x \in X$.

Proof. Given an orthonormal basis e_i of X, consider $y = \sum_i \overline{f(e_i)} e_i$. □

In Riemannian geometry this is called ***raising an index*** of a 1-form. Indeed, $\omega_p \in \Lambda^1(T_pM)$ is just a linear functional on T_pM, and $(\omega)^\sharp = g^{ij}\omega_j E_i$ at p is just a vector y such that $\omega_p(x) = \langle x, y \rangle$ for all $x \in T_pM$. So the former theorem may also be stated as "$y = f^\sharp$ exists". Recall this is given by viewing the inner product as a nonsingular matrix.

3.4 Adjoint operator

Let $A : X \to X$ be a linear transformation in a finite-dimensional inner product space X. For a given $y \in X$, define the linear functional

$$\begin{aligned} f^y : X &\to F \\ x &\mapsto \langle Ax, y \rangle \end{aligned}$$

which, by the Riesz representation theorem yields a unique $z \in X$ such that

$$f^y(x) = \langle x, z \rangle$$

Then the ***adjoint of*** A is the linear map

$$\begin{aligned} A^* : X &\to X \\ y &\mapsto z \end{aligned}$$

so that $\langle Ax, y \rangle = \langle x, A^* y \rangle$.

Proposition 3.6 (Properties of the adjoint)**.**

1. $(\alpha A)^* = \overline{\alpha} A^*$.
2. $(A + B)^* = A^* + B^*$.
3. $(AB)^* = B^* A^*$.
4. $(A^*)^* = A$.

If $A = A^*$ we say A is ***self-adjoint***, and if $AA^* = A^*A$ we say A is ***normal***.

Theorem 3.7. If A is self-adjoint, its eigenvalues are real. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

Theorem 3.8. If M is an invariant subspace of X under A, then $M^\perp$ is invariant under A^*.

Theorem 3.9. If A is a linear transformation on a finite-dimensional inner product space X, then $\mathrm{range}(A)^\perp = \mathrm{null}(A^*)$.

3.5 Spectral theorem for normal transformations

Theorem 3.10. Let A be a self-adjoint transformation in a finite-dimensional inner product space X. Then there exists an orthonormal basis of X consistinf of eigenvectors of A.

Lemma 3.11. Let A be a normal transformation in a finite-dimensional inner product space X. Then $\|Ax\| = \|A^*x\|$ for all $x \in X$.

Theorem 3.12. Let A be a normal transformation in a complex finite-dimensional inner product space X. Then there exists an orthonormal basis of X consistinf of eigenvectors of A.

Theorem 3.13. If A is a normal transformation on a finite-dimensional inner product space. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

Recall that the notation $X = M_1 \oplus \ldots \oplus M_k$ means that X is the ***direct sum*** of the M_i, which means that $X = M_1 + \ldots + M_k$ and $M_i \cap \{M_1 + \ldots \hat{M}_i + \ldots + M_k\} = \{0\}$, (every element in X is expressed as a unique sum of elements in M_i). If $M_i \perp M_j$ for all $i \neq j$, we say this is an ***orthogonal direct sum decomoposition of*** X, and the ***orthogonal projection to*** M_j is just taking the corresponding component of a given element in its decomposition.

Theorem 3.14 (Spectral decomposition theorem for normal transformations). To every normal transformation A on a complex finite-dimensional inner product space there correspond scalar $\lambda_1, \ldots, \lambda_k$, the distinct eigenvalues of A, and orthogonal projections $E_1, \ldots, E_k$ with $k \leq \dim X$, such that

1. E_i is the orthogonal projection on $\mathrm{Null}(A - \lambda_i)$ for $i = 1, \ldots, k$.
2. $E_i \neq 0$ and $E_i E_j = 0$ for $i, j = 1, \ldots, k$.
3. $\sum_{j=1}^{k} E_j = 1$.
4. $\sum_{j=1}^{k} \lambda_j E_j = A$.

If A was self-adjoint, we could weaken the hypotheses to a real inner product space.

3.6 Unitary and orthogonal transformations

Let X be a finite-dimensional inner product space, and $U : X \to X$ a linear transformation with $U^*U = 1$. We say U is ***unitary*** if X is complex and ***orthogonal*** if X is real. The condition $U^*U = 1$ implies that $UU^* = 1$.

Theorem 3.15. Let X be a finite-dimensional inner product space, and $U : X \to X$ a linear transformation. The following statements are equivalent:

1. $U^*U = 1$.
2. $\langle Ux, Uy \rangle = \langle x, y \rangle$.

3. $\|Ux\| = \|x\|$ for all $x \in X$.

Theorem 3.16. If U is a unitary transformation on the finite-dimensional inner product space X, then each of the eigenvalues of U must have an absolute value equal to 1.

To summarize:

Theorem 3.17. Let A be a normal transformation on a complex finite-dimensional inner product space. Then

1. A is self-adjoint is and only if each eigenvalue of A is real.
2. A is unitary if and only if each eigenvalue of A has absolute value equal to 1.

4 Normed spaces

Let X be a real or complex vector space. A ***norm*** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.
3. **(Triangle inequality.)** $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Every normed space is a metric space with the distance function $\rho(x, y) = \|x - y\|$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called ***equivalent*** if there exist $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \qquad \forall x \in X$$

Equivalent norms define the same topology and the same Cauchy sequences.

A normed space that is complete is called a ***Banach space***.

Theorem 4.1. For every normed linear space X there is a complete normed linear space X^* such that X is ***congruent*** (isomorphic and isometric) to a dense subset of X^* and the norm on X^* extends the norm on X.

If $\{x_n\}$ is a sequence in X, the series $\sum_{n=1}^{\infty} x_n$ ***converges to*** x if $\sum_{n=1}^{N} \to x$ as $N \to \infty$, and it is ***absolutely convergent*** is $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem 4.2. A normed vector space X is complete if and only if every absoultely convergent series in X converges.

Examples.

- If X is a topological space, $B(X)$ and $BC(X)$ are Banach spaces with the uniform norm $\|f\|_u = \sup_{x \in X} |f(x)|$.
- If $(X, \mathcal{M}, \mu)$ is a measure space, $L^1(\mu)$ is a Banach space with the norm $\|f\|_1 = \int |f| d\mu$. (Observe that $\|\cdot\|_1$ is only a seminorm if we do not identify functions that are equal a.e.)

If X and Y are normed vector spaces, $X \times Y$ is a normed vector space with the ***product norm***, $\|(x,y)\| = \max(\|x\|, \|y\|)$. If M is a vector subspace of X, the quotient space X/M consisting of equivalence classes under $x \sim y$ iff $x - y \in M$ is a normed space with the ***quotient norm***, $\|x + M\| = \inf_{y \in M} \|x + y\|$.

A linear map $T : X \to Y$ between two normed vector spaces is ***bounded*** if there exists $C \geq 0$ such that

$$\|Tx\| \leq C\|x\| \qquad \forall x \in X$$

Proposition 4.3. If X and Y are normed vector spaces and $T : X \to Y$ is a linear map, then T is continuous if and only if it is bounded.

Proof. ($\Longrightarrow$) There exists $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$. For any nonzero $x \in X$,

$$\|Tx\| = \left\| \frac{\|x\|}{\delta} T\left(\delta \frac{x}{\|x\|}\right) \right\| \leq \frac{1}{\delta}\|x\|.$$

($\Longleftarrow$) If $\|x - y\| < \frac{\varepsilon}{C}$,

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq C\|x - y\| < \varepsilon.$$

$\square$

In fact, if T is bounded it is uniformly continuous and even Lipschitz continuous.

We denote by $L(X,Y)$ the space of bounded linear maps from X to Y, which is a normed vector space with the ***operator norm***

$$\begin{aligned}\|T\| &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|tx\|}{\|x\|} : x \neq 0\right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x \in X\}\end{aligned}$$

Proposition 4.4. If Y is complete, so is $L(X,Y)$.

Proof. If $\{T_n\}$ is a Cauchy sequence in $L(X,Y)$, the sequence $\|T_n x\|$ is Cauchy in Y since $\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\|$. Define pointwise $Tx = \lim_{n\to\infty} T_n x$. $\square$

T is ***invertible*** if it bijective and T^{-1} is bounded. It is called an isometry if $\|Tx\| = \|x\|$ for all $x \in X$. An isometry is injective but not necessarily surjective.

If X is a vector space over $K = \mathbb{R}, \mathbb{C}$, a ***linear functional***. is a linear map from X to K. The space $X^* := L(X,K)$ is called the ***dual space*** of X. Since K is complete, X^* is complete with the operator norm.

Proposition 4.5 (Relationship between real and complex linear functionals)**.** Let X be a vector space over $\mathbb{C}$. If f is a complex linear functional on X, $u := \operatorname{Re} x$ is a real linear functional and $f(x) = u(x) - iu(ix)$. Conversely, if u is a real functional on X, then $f(x) := u(x) - iu(ix)$ is a complex linear functional, and if X is normed, $\|u\| = \|f\|$.

It is not obvious that there are any nonzero bounded linear functionals on an arbitrary normed vector space. If X is a real vector space, ***sublinear*** or ***Minkoswky functional*** on X is a map $p : X \to \mathbb{R}$ such that

$$p(\lambda x) = \lambda p(x) \qquad \forall x \in X \text{ and } \lambda \geq 0 \tag{2}$$

$$p(x + y) \leq p(x) + p(y) \qquad \forall x, y \in X \tag{3}$$

Theorem 4.6 (Helly-Hahn-Banach). Let E be a vector space over $\mathbb{R}$ and $p : E \to \mathbb{R}$ a sublinear functional. If $G \subseteq E$ is a linear subspace and $g : G \to \mathbb{R}$ is a linear functional such that

$$g(x) \leq p(x) \quad \forall x \in E,$$

then there exists a linear functional f defined on all of E that extends g, that is, $g(x) = f(x)\ \forall x \in G$ and such that

$$f(x) \leq p(x) \quad \forall x \in E.$$

For a proof first recall that a partial order P is ***inductive*** if every totally ordered subset Q in P has an upper bound, and that

Lemma 4.7 (Zorn)**.** Every nonempty order set that is inductive has a maximal element.

Proof. (Of theorem 4.6) Consider the set

$$P = \left\{ h : D(h) \subseteq E \to \mathbb{R} : \substack{D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear}, G \subseteq D(h), \\ h \text{ extends } g, \text{ and } h(x) \leq p(x)\ \forall x \in D(h)} \right\}$$

Then P is a partial order with

$$h_1 \leq h_2 \iff D(h_1) \subseteq D(h_2) \text{ and } h_2 \text{ extends } h_1$$

P is nonempty since $g \in P$. To show it is inductive, take $Q \subseteq P$ a partially order subset and write $Q = (h_i)_{i \in I}$. Then define

$$D(h) = \bigcup_{i \in I} D(h_i), \qquad h(x) = h_i(x) \quad \text{if } x \in D(h_i) \text{ for some } i \in I$$

which is an upper bound of Q, so that there is a maximal element f in P by Zorn's Lemma. To finish it suffices to show that $D(f) = E$.

For a contradiction suppose that $D(f) \neq E$ and choose $x_0 \notin D(f)$. We shall construct a function $h \in P$ such that $f < h$. Define $D(h) = D(f) + \mathbb{R}x_0$ and, for every $x \in D(f)$, set

$$h(x + \lambda x_0) = f(x) + t\alpha \quad \forall \lambda \in \mathbb{R}$$

where α is a constant that we choose as follows. We must ensure that

$$h(x + \lambda x_0) = f(x) + \lambda\alpha \leq p(x + tx_0) \quad \forall x \in D(f) \quad \text{and} \quad \forall \lambda \in \mathbb{R}$$

For any $x, y \in D(f)$,

$$\begin{aligned} f(x) + f(y) = f(x + y) \leq p(x + y) \leq p(x + x_0) + p(y - x_0) \\ \implies f(x) - p(y - x_0) \leq p(x + x_0) - f(y) \end{aligned}$$

So let α satisfy

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}$$

If $\lambda = 0$, then $h(x) = f(x) \leq p(x)$. If $\lambda \neq 0$ we must be careful since sublinear functionals only satisfy eq. (2) for positve scalars.

If $\lambda > 0$, then

$$\begin{aligned} h(x + \lambda x_0) &= \lambda \cdot h(x/\lambda + x_0) \\ &= \lambda \cdot (f(x/\lambda) + \alpha)) \\ &\leq \lambda \cdot (f(x/\lambda) + p(x/\lambda + x_0) - f(x/\lambda)) \\ &\leq p(x + \lambda x_0) \end{aligned}$$

and if $\lambda = -\mu < 0$,

$$\begin{aligned} h(x + tx_0) &= (-\lambda) \cdot h(-x/\lambda - x_0) \\ &= \mu \cdot (f(x/\mu) - \alpha)) \\ &\leq \mu \cdot (f(x/\mu) - f(x/\mu) + p(x/\mu + x_0)) \\ &\leq p(x - \mu x_0) \\ &= p(x + \lambda x_0). \end{aligned}$$

Then h extends f and $D(f) \subseteq D(h)$ but f is maximal. $\square$

5 Exercises

Exercise 5.1. Let $A : (C[-1,1], \|\cdot\|_\infty) \to \mathbb{R}$ be defined by

$$Ax = x(0)$$

Show A is linear, bounded and find its norm.

Solution.

1. $A(x + \lambda y) = (x + \lambda y)(0) = x(0) + \lambda y(0) = Ax + \lambda Ay$.
2. $|Ax| = |x(0)| \leq \|x\|_\infty$, so 1 is a bound.
3. The bound is attained with $\|1\|_\infty = 1$, so $\|A\|$ cannot be lower than 1.

□

Exercise 5.2. Let $A : (C[0,1], \|\cdot\|_\infty) \to \mathbb{R}$ be defined by

$$Ax = \int_a^b x(t)\varphi(t)dt$$

where $\varphi \in C[a,b]$ is a fixed function. Show that $\|A\| = \int_a^b |\varphi(t)|dt = \|\varphi(t)\|_1$.

Solution. See here. First we show A is bounded:

$$\begin{aligned}
|Ax| &= \left|\int_a^b x(t)\varphi(t)dt\right| \\
&\leq \int_a^b |x(t)||\varphi(t)|dt \\
&\leq \|x\|_\infty \int_a^b |\varphi(t)|dt \\
&= \|x\|_\infty \|\varphi(t)\|_1.
\end{aligned}$$

Which proves that $\|A\| \leq \|\varphi\|_1$. To show the reverse inequality, define the sequence

$$x_n(t) := \frac{\varphi(t)}{|\varphi(t)| + \frac{1}{n}}$$

so that

$$\|x_n\|_\infty = \frac{\|\varphi\|_\infty}{\|\varphi\|_\infty + \frac{1}{n}} \longrightarrow 1$$

And also

$$Ax_n = \int_a^b \frac{\varphi(t)^2}{|\varphi(t)| + \frac{1}{n}}dt \longrightarrow \int_a^b |\varphi(t)|dt$$

since

And since $Ax_n = |Ax_n| \leq \|x_n\|_\infty \|A\|$, we have

$$\|A\| \geq \frac{Ax_n}{\|x_n\|_\infty} \longrightarrow \frac{\int_a^b |\varphi(t)| dt}{1} = \|\varphi\|_1,$$

so $\|A\| \geq \|\varphi\|_1$ and we are finished.

$\square$

Exercise 5.3. Sejam $(C([0,1]), \|\cdot\|_\infty)$ um espaço vetorial normado com $\|f\|_\infty = \max_{x\in[0,1]} |f(x)|$ e $T : (C([0,1]) \to (C([0,1])$ dado por

$$(Tf)(x) = x \int_0^x f(y) dy$$

1. Mostre que T é linear, limitado e calcule $\|T\|$.
2. Mostre que $T^{-1} : \mathcal{R}(T) \to C([0,1])$ existe mais não é limitado.

Solução.

1. T é linear, pois $T(f + \lambda g) = x \int_0^1 (f(y) + \lambda g(y)) dy = x \int_0^1 f(y) dy + \lambda x \int_0^1 g(y) dy$.

   T é limitado, pois

   $$\begin{aligned} Tf(x) = x \int_0^1 f(y) dy \leq x \int_0^x \|f\|_\infty dy = x\left((x-0)\|f\|_\infty\right) = \|f\|_\infty x^2 \\ \implies \|Tf\|_\infty \leq \|f\|_\infty \|x^2\|_\infty = \|f\|_\infty \end{aligned}$$

   Assim que 1 é uma cota. Para ver que de fato $\|T\| = 1$, basta considerar a função constante $f \equiv 1$, caso em que a cota é alcançada: $\|Tf\|_\infty = \|x^2\|_\infty = 1 = \|f\|_\infty$.

2.

$\square$