

# Functional Analysis

[github.com/dan-gc/analysis](https://github.com/dan-gc/analysis)

These are preparation notes for a course on Functional Analysis at IMPA, summer 2024. They are based on Folland, *Real Analysis: Modern Techniques and Their Applications*; Bachman and Narici, *Functional Analysis*; Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.

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# 1 Measure theory

A measure should surely satisfy:

1. If  $E_1, E_2, \dots$  is a finite or infinite sequence of disjoint sets,

$$\mu(E_1 \cup E_2 \cup \dots) = \mu(E_1) + \mu(E_2) + \dots$$

2. If  $E$  is congruent to  $F$ ,

$$\mu(E) = \mu(F)$$

3. If  $Q$  is the unit cube,

$$\mu(Q) = 1$$

## 1.1 $\sigma$ -algebras

Let  $X$  be a nonempty set.

- An *algebra of sets* on  $X$  is a nonempty collection  $\mathcal{A}$  of subsets of  $X$  that is closed under finite unions and complements, that is,
  1. If  $E_1, \dots, E_n \in \mathcal{A}$ , then  $\bigcup_{i=1}^n E_i \in \mathcal{A}$ .
  2. If  $E \in \mathcal{A}$ , then  $E^c \in \mathcal{A}$ .
- A  $\sigma$ -*algebra* is an algebra of sets closed under countable unions.
- The intersection of all  $\sigma$ -algebras containing any subset  $\mathcal{E} \subset \mathcal{P}(X)$  is the  $\sigma$ -*algebra generated by*  $\mathcal{E}$ .
- The  $\sigma$ -algebra generated by the open sets of a topological (or metric) space  $X$  is the *Borel algebra*  $\mathcal{B}_X$ .
- Let  $\{X_\alpha\}_{\alpha \in A}$  is a collection of nonempty sets,  $X = \prod_{\alpha} X_\alpha$  and  $\pi_\alpha : X \rightarrow X_\alpha$  the coordinate functions. If  $\mathcal{M}_\alpha$  is a  $\sigma$ -algebra on  $X_\alpha$ , the *product  $\sigma$ -algebra* on  $X$  is the  $\sigma$ -algebra generated by

$$\{\pi_\alpha(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$

We denote this  $\sigma$ -algebra by  $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ .

**Proposition 1.1.**  $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$  is generated by  $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\}$ .

**Proposition 1.2.** Let  $X_1, \dots, X_n$  be metric spaces and let  $X = \prod_i X_i$  be equipped with the product metric. Then  $\bigotimes_i \mathcal{B}_i \subset \mathcal{B}_X$ . If every  $X_i$  is separable equality holds.

**Corollary 1.3.**  $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$ .

## 1.2 Measures

Let  $X$  be a set equipped with a  $\sigma$ -algebra  $\mathcal{M}$ .

- A **measure** on  $\mathcal{M}$  (or on  $(X, \mathcal{M})$ , or simply on  $X$ ) is a function  $\mu : \mathcal{M} \rightarrow [0, \infty)$  such that
  1.  $\mu(\emptyset) = 0$ .
  2. if  $\{E_j\}_1^\infty$  is a sequence of disjoint sets in  $\mathcal{M}$ , then  $\mu(\bigcup_1^\infty E_j) = \sum_1^\infty \mu(E_j)$ .
- If  $X$  is a set and  $\mathcal{M} \subset \mathcal{P}(X)$  is a  $\sigma$ -algebra,  $(X, \mathcal{M})$  is called a **measurable space** and the sets in  $\mathcal{M}$  are **measurable sets**. If  $\mu$  is a measure on  $(X, \mathcal{M})$ , then  $(X, \mathcal{M}, \mu)$  is called a **measure space**.
- If  $\mu(X) < \infty$  (and hence  $\mu(E) < \infty$  for all  $E \in \mathcal{M}$ ),  $\mu$  is called  **$\sigma$ -finite**. If  $X = \bigcup_1^\infty E_j$ , with  $\mu(E_j) < \infty$ ,  $\mu$  is called  **$\sigma$ -finite**. If for every  $E \in \mathcal{M}$  with  $\mu(E) = \infty$  there exists  $F \in \mathcal{M}$  such that  $F \subset E$  and  $0 < \mu(F) < \infty$ ,  $\mu$  is called **semifinite**.

**Theorem 1.4 (Properties of measure spaces).** Let  $(X, \mathcal{M}, \mu)$  be a measure space.

1. **(Monotonicity.)** If  $E, F \in \mathcal{M}$  and  $E \subset F$ , then  $\mu(E) \leq \mu(F)$ .
  2. **(Subadditivity.)** If  $\{E_j\}_1^\infty \subset \mathcal{M}$ , then  $\mu(\bigcup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$ .
  3. **(Continuity from below.)** If  $\{E_j\}_1^\infty \subset \mathcal{M}$ ,  $E_1 \subset E_2 \subset \dots$ , then  $\mu(\bigcup_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ .
  4. **(Continuity from above.)** If  $\{E_j\}_1^\infty \subset \mathcal{M}$ ,  $E_1 \supset E_2 \supset \dots$ , and  $\mu(E_1) < \infty$ , then  $\mu(\bigcap_1^\infty E_j) = \lim_{j \rightarrow \infty} \mu(E_j)$ .
- If  $E \in \mathcal{M}$  and  $\mu(E) = 0$ ,  $E$  is called a **null set**. If a statement about points in  $X$  is true except for points in a null set, we say it is true **almost everywhere**.
  - If  $\mu(E) = 0$  and  $F \subset E$ , then  $\mu(F) = 0$  provided  $F \in \mathcal{M}$ . A measure whose domain contains all subsets of null sets is **complete**. Completeness may help avoid technical difficulties, and it can always be achieved by enlarging the domain of  $\mu$ :

**Theorem 1.5.** Let  $(X, \mathcal{M}, \mu)$  be a measure. Let  $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$  and  $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$ . Then  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra, and there is a unique extension  $\bar{\mu}$  of  $\mu$  to a complete measure on  $\overline{\mathcal{M}}$ , called the **completion of  $\mu$** .

- Exercises
  1. If  $\mu_1, \dots, \mu_n$  are measures on  $(X, \mathcal{M})$  and  $a_1, \dots, a_n \in [0, \infty)$ , then  $\sum_1^n a_j \mu_j$  is also a measure on  $(X, \mathcal{M})$ .
  2.  $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$ .
  3. A set  $E \subset X$  is called **locally measurable** if  $E \cap A \in \mathcal{M}$  whenever  $A \in \mathcal{M}$  and  $\mu(A) < \infty$ . If  $\tilde{\mathcal{M}}$  equals the collection of the saturated sets  $\tilde{\mathcal{M}}$ , it is called **saturated**. The saturated measure  $\tilde{\mu}$  on  $\tilde{\mathcal{M}}$  defined by  $\tilde{\mu}(E) = \mu(E)$  for  $E \in \mathcal{M}$  and  $\tilde{\mu}(E) = \infty$  otherwise is called the **saturation of  $\mu$** .

### 1.3 Outer measures

This is used to construct measures. The key idea is to approximate the measure of a set by simpler enclosing sets, like with the Riemann integral.

- An **outer measure** on a nonempty set  $X$  is a function  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  that satisfies

1.  $\mu^*(\emptyset) = 0$ ,
2.  $\mu^*(A) \leq \mu^*(B)$  if  $A \subset B$ , and
3.  $\mu^*(\bigcup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$ .

**Proposition 1.6.** Let  $\mathcal{E} \subset \mathcal{P}(X)$  and  $\rho : \mathcal{E} \rightarrow [0, \infty]$  be such that  $\emptyset \in \mathcal{E}$ ,  $X \in \mathcal{E}$  and  $\rho\emptyset = 0$ . For any  $A \subset X$ , define

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \rho(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j \right\} \quad (1.1)$$

then  $\mu^*$  is an outer measure.

- A set  $A \subset X$  is called  **$\mu$ -measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X$$

Which makes sense if we think  $E$  is a *well-behaved* set such that  $A \subset E$ , so that  $\mu^*(A) = \mu^*(E) - \mu^*(E \cap A^c)$ .

**Theorem 1.7 (Carathéodory).** If  $\mu^*$  is an outer measure on  $X$ , the collection  $\mathcal{M}$  of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra, and the restriction of  $\mu^*$  to  $\mathcal{M}$  is a complete measure.

- If  $\mathcal{A} \subset \mathcal{P}(X)$  is an algebra, the function  $\mu_0 : \mathcal{A} \rightarrow [0, \infty]$  is a **premeasure** if
  - $\mu_0(\emptyset) = 0$ ,
  - If  $\{A_j\}_1^\infty$  is a sequence of disjoint sets in  $\mathcal{A}$  such that  $\bigcup_1^\infty A_j \in \mathcal{A}$ , then  $\mu_0(\bigcup_1^\infty A_j) = \sum_1^\infty \mu_0(A_j)$ .

**Proposition 1.8.** If  $\mu_0$  is a premeasure on  $\mathcal{A}$  and  $\mu^*$  is defined by eq. (1.1) by taking  $\rho = \mu_0$ , then

1.  $\mu^*|_{\mathcal{A}} = \mu_0$ ,
2. every set in  $\mathcal{A}$  is  $\mu^*$  measurable.

**Theorem 1.9.** Let  $\mathcal{A} \subset \mathcal{P}(X)$  be an algebra,  $\mu_0$  a premeasure on  $\mathcal{A}$  and  $\mathcal{M}$  the  $\sigma$ -algebra generated by  $\mathcal{A}$ . There exists a measure  $\mu$  on  $\mathcal{M}$  whose restriction to  $\mathcal{A}$  is  $\mu_0$ —namely  $\mu = \mu^*|_{\mathcal{M}}$ , where  $\mu^*$  is given by eq. (1.1). (This is a consequence of Carathéodory's theorem and the last proposition.)

If  $\nu$  is another measure on  $\mathcal{M}$  that extends  $\mu_0$ , then  $\nu(E) \leq \mu(E)$  for all  $E \in \mathcal{M}$  with equality when  $\mu(E) < \infty$ . If  $\mu_0$  is  $\sigma$ -finite, then  $\mu$  is the unique extension of  $\mu_0$  to a measure on  $\mathcal{M}$ .

- A **Borel measure on  $\mathbb{R}$**  is a measure on  $\mathbb{R}$  whose domain is the Borel  $\sigma$ -algebra  $\mathcal{B}_{\mathbb{R}}$ .

**Proposition 1.10.** Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be increasing and right continuous. If  $(a_j, b_j]$ ,  $j = 1, \dots, n$  are disjoint **half-open intervals**, so that  $0 \leq a_j, b_j \leq \infty$ , or  $(a_j, \infty)$ , or they are empty, define

$$\mu_0 \left( \bigcup_{j=1}^n (a_j, b_j] \right) = \sum_{j=1}^n [F(b_j) - F(a_j)]$$

and let  $\mu_0(\emptyset) = 0$ . Then  $\mu_0$  is a premeasure on the algebra  $\mathcal{A}$  of finite disjoint unions of half-open intervals.

**Theorem 1.11.** If  $F : \mathbb{R} \rightarrow \mathbb{R}$  is any increasing, right continuous function, there is a unique Borel measure  $\mu_F$  on  $\mathbb{R}$  such that  $\mu_F((a, b]) = F(b) - F(a) \forall a, b$ .

If  $G$  is another such function,  $\mu_F = \mu_G$  if and only if  $F - G$  is constant. Conversely, given a  $\mu$  Borel measure on  $\mathbb{R}$  that is finite on all bounded Borel sets we may define an increasing right continuous function  $F$  such that  $\mu = \mu_F$ .

- By theorem 1.9, there is a complete measure  $\bar{\mu}_F$  whose domain includes  $\mathcal{B}_{\mathbb{R}}$ . In fact  $\bar{\mu}_F$  is the completion of  $\mu_F$  and its domain is strictly larger than  $\mathcal{B}_{\mathbb{R}}$ . This complete measure is called the **Lebesgue-Stieltjes measure** and is also denoted by  $\mu_F$ .

In the following  $\mu$  is the Lebesgue-Stieltjes measure associated to some increasing, right-continuous function  $F$ , and  $\mathcal{M}_{\mu}$  is the domain of  $\mu$ .

**Theorem 1.12.** If  $E \in \mathcal{M}_{\mu}$ , then

$$\begin{aligned} \mu(E) &= \inf \{ \mu(U) : E \subseteq U \text{ and } U \text{ is open} \} \\ &= \sup \{ \mu(K) : K \subseteq E \text{ and } K \text{ is compact} \} \end{aligned}$$

**Theorem 1.13.** If  $E \subset \mathbb{R}$ , the following are equivalent:

1.  $E \in \mathcal{M}_{\mu}$ .
2.  $E = V \setminus N_1$  where  $V$  is a  $G_{\delta}$  (countable intersection of open sets) and  $\mu(N_1) = 0$ .
3.  $E = H \cup N_2$  where  $H$  is an  $F_{\sigma}$  (countable union of closed sets) set and  $\mu(N_2) = 0$ .

**Proposition 1.14.** If  $E \in \mathcal{M}_{\mu}$  and  $\mu(E) < \infty$ , then for every  $\varepsilon > 0$ , there is a set  $A$  that is a finite union of open intervals such that  $\mu(E \triangle A) < \varepsilon$ .

- The **Lebesgue measure** is the Lebesgue-Stieltjes measure of  $F(x) = x$ . We denote it by  $m$  and its domain by  $\mathcal{L}$ .

**Theorem 1.15.** Is  $E \in \mathcal{L}$ , then  $E + s \in \mathcal{L}$  and  $rE \in \mathcal{L}$  for all  $s, r \in \mathbb{R}$ . Moreover,  $m(E + s) = m(E)$  and  $m(rE) = |r|m(E)$ .

**Proposition 1.16.** Let  $C$  be the Cantor set.  $C$  is compact, nowhere dense and totally disconnected (ie. the only connected subsets of  $C$  are single points).  $C$  has no isolated points.  $m(C) = 0$ .  $\text{card}(C) = \mathfrak{c}$ .

## 1.4 Measurable functions

Now we construct integrals from simple functions.

- Recall that a mapping  $f : X \rightarrow Y$  between two sets induces a mapping  $f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$  defined by the inverse image, which preserves unions, intersection and complements, so that if  $\mathcal{N}$  is a  $\sigma$ -algebra on  $Y$ , then  $\{f^{-1}(E) : E \in \mathcal{N}\}$  is a  $\sigma$ -algebra on  $X$ .

If  $(X, \mathcal{M})$  and  $(Y, \mathcal{N})$  are measurable spaces, a mapping  $f : X \rightarrow Y$  is called *measurable* if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{N}$ .

**Proposition 1.17.** A function whose codomain is a product measure space is measurable if precomposing with every projection is measurable.

**Corollary 1.18.** A function  $f : X \rightarrow \mathbb{C}$  is  $\mathcal{M}$ -measurable if and only if  $\text{Re } f$  and  $\text{Im } f$  are  $\mathcal{M}$ -measurable.

**Proposition 1.19.** If  $\mathcal{N}$  is generated by  $\mathcal{E}$ , then  $f : X \rightarrow Y$  is measurable if and only if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ .

**Corollary 1.20.** If  $X$  and  $Y$  are metric (or topological spaces), every continuous function is  $(\mathcal{B}_X, \mathcal{B}_Y)$ -measurable.

**Proposition 1.21.** If  $f, g : X \rightarrow \mathbb{C}$  are  $\mathcal{M}$ -measurable, then so are  $f + g$  and  $fg$ .

**Proposition 1.22.** If  $\{f_j\}$  is a sequence of  $\overline{\mathbb{R}}$ -valued measurable functions on  $(X, \mathcal{M})$ , then the functions

$$\begin{aligned} \sup_j f_j(x) & \quad \limsup_{j \rightarrow \infty} f_j(x) \\ \inf_j f_j(x) & \quad \liminf_{j \rightarrow \infty} f_j(x) \end{aligned}$$

are measurable. If

$$f(x) = \lim_{j \rightarrow \infty} f_j(x)$$

exists for every  $x \in X$ , then  $f$  is measurable.

**Corollary 1.23.** If  $f, g : X \rightarrow \overline{\mathbb{R}}$  is measurable, then so are  $\max(f, g)$  and  $\min(f, g)$ .

If  $f : X \rightarrow \overline{\mathbb{R}}$  we define the *positive* and *negative* parts of  $f$  as:

$$f^+(x) = \max(f(x), 0) \quad f^-(x) = \max(-f(x), 0)$$

Then  $f = f^+ - f^-$ , and if  $f$  is measurable, so are  $f^+$  and  $f^-$  by corollary 1.23.

- Let  $(X, \mathcal{M})$  be a measurable space. If  $E \subset X$ , the **characteristic or indicator function**  $\chi_E$  of  $E$  is

$$\chi(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

$\chi_E$  is measurable if and only if  $E \in \mathcal{M}$ .

A **simple function on  $X$**  is a finite linear combination, with complex coefficients, of characteristic functions of sets in  $\mathcal{M}$ . (We do not allow simple functions to assume values  $\pm\infty$ .) Equivalently,  $f : X \rightarrow \mathbb{C}$  is simple if and only if  $f$  is measurable and the range of  $f$  is a finite subset of  $\mathbb{C}$ . The **standard representation of  $f$**  is

$$f = \sum_{j=1}^n z_j \chi_{E_j} \quad \text{where } E_j = f^{-1}(\{z_j\}) \text{ and } \text{range}(f) = \{z_1, \dots, z_n\}$$

If  $f$  and  $g$  are simple functions, then so are  $f + g$  and  $fg$ .

**Theorem 1.24.** Let  $(X, \mathcal{M})$  be a measurable space.

1. If  $f : X \rightarrow [0, \infty]$  is measurable, there is a sequence  $\{\varphi_n\}$  of simple functions such that  $0 \leq \varphi_1 \leq \dots \leq f$ ,  $\varphi_n \rightarrow f$  pointwise and  $\varphi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.
2. If  $f : X \rightarrow \mathbb{C}$  is measurable, there is a sequence  $\{\varphi_n\}$  of simple functions such that  $0 \leq |\varphi_1| \leq \dots \leq |f|$ ,  $\{\varphi_n\} \rightarrow f$  pointwise and  $\varphi_n \rightarrow f$  uniformly on any set on which  $f$  is bounded.

*Proof.*

1. Let  $n \in \mathbb{N}$  and  $0 \leq k \leq 2^{2n} - 1$ . Set

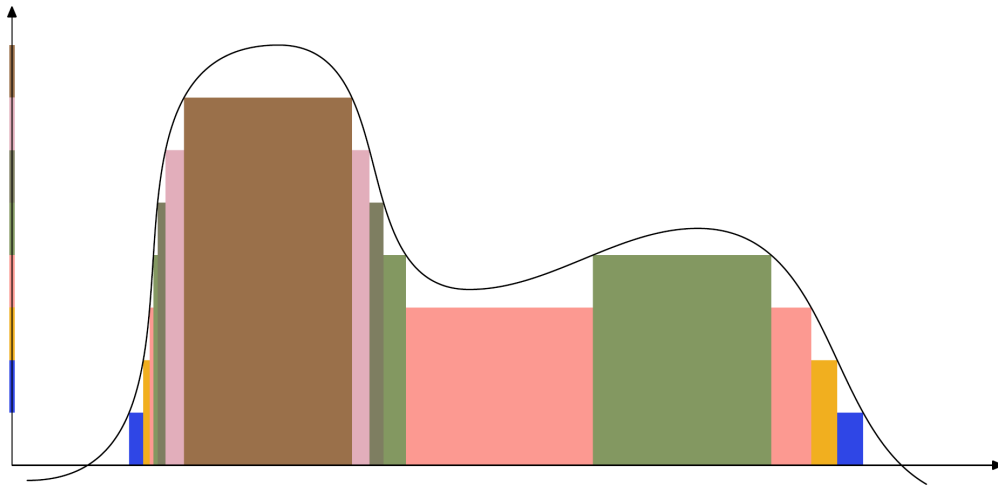
$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-n}]) \quad \text{and} \quad F_n = f^{-1}((2^n, \infty]),$$

and finally

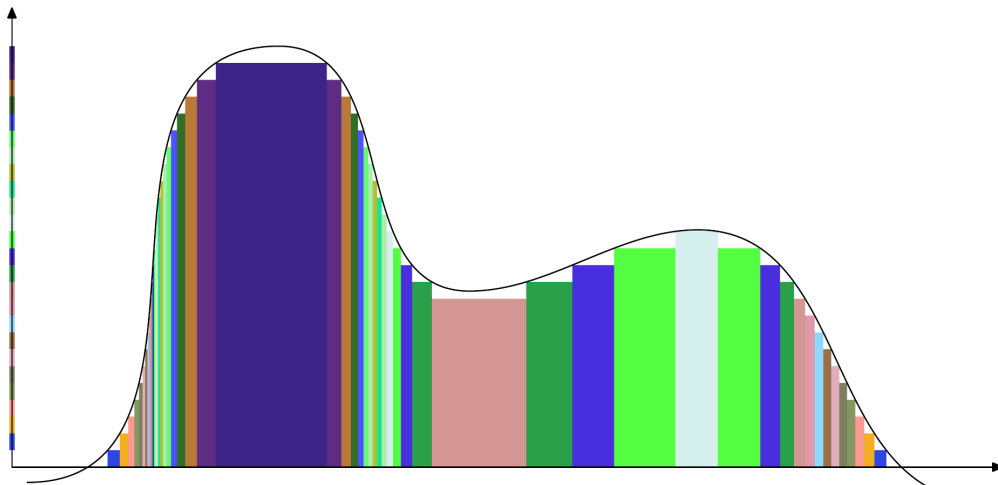
$$\varphi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}.$$

Then  $\varphi_n \leq \varphi_{n+1} \forall n$  and  $0 \leq f - \varphi_n \leq 2^{-n}$  on the set where  $f \leq 2^n$ .

2. If  $f = g + ih$  we can apply part *a* to the positive and negative parts of  $g$  and  $h$ .



(a)  $\varphi_7$ .



(b)  $\varphi_{24}$ .

Figure 1: [Approximation by simple functions.](#)

**Proposition 1.25.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $(X, \overline{\mathcal{M}}, \overline{\mu})$  be its completion. If  $f$  is a  $\overline{\mathcal{M}}$ -measurable function on  $X$ , there is an  $\mathcal{M}$ -measurable function  $g$  such that  $f = g$   $\overline{\mu}$ -almost everywhere.

□



## 1.5 Riesz and Nagy's Functional Analysis

This section is a short digression to show the approach of Riesz and Nagy, *Functional Analysis*.

**Theorem 1.26 (Lebesgue).** Every monotonic function  $f(x)$  possesses a finite derivative at every point with the possible exception of the points  $x$  of a set of measure zero, or, as it is often phrased, almost everywhere.

A set of *measure zero* is a set of values  $x$  which can be covered by a finite number or by a denumerable sequence of intervals whose total length is arbitrarily small.

Functions  $f(x)$  continuous or not, for which the sum

$$\sum_{ab} = \sum_1^n |f(x_k) - f(x_{k-1})|$$

defined in terms of a decomposition of the interval  $(a, b)$  into partial intervals  $(x_{k-1}, x_k)$   $k = 1, 2, \dots, n$ , does not surpass a finite bound, independent of the particular choice of decomposition, are called *functions of bounded variation*. The least upper bound is called the *total variation of  $f(x)$  in the interval  $(a, b)$* .

**Theorem 1.27 (Lebesgue).** Every function of bounded variation possesses a finite derivative almost everywhere.

**Theorem 1.28 (Fubini).** Let

$$f_1(x) + f_2(x) + \dots = s(x)$$

be a convergent series all of whose terms are monotonic functions of the same type, defined on the interval  $a \leq x \leq b$ . Then

$$f'_1(x) + f'_2(x) + \dots = s'(x)$$

except perhaps on a set of measure zero; that is, term by term differentiation is possible almost everywhere.

**Theorem 1.29 (Lebesgue).** Almost all points of an arbitrary linear set are density points of that set.

We include the following quote regarding Riemann integrals:

[p. 23] (...) we arrive at a necessary condition that  $f(x)$  be integrable in the Riemann sense, namely, that the function  $f(x)$  be continuous almost everywhere.

A *step function* defined on an interval  $(a, b)$  is a function having a constant value  $c_k$  in each of a finite number of intervals  $i_k$  of finite length  $|i_k|$  and vanishing outside these intervals. We suppose the integral defined for these functions, as usual, by the sum  $\sum c_k |i_k|$ .

**Lemma 1.30.** For every sequence  $\{\varphi_n(x)\}$  of step functions which decreases to 0 almost everywhere, the sequence of values of their integrals also tends to zero.

**Lemma 1.31.** If for an increasing sequence of step functions  $\{\varphi_n(x)\}$  the values of their integrals have a common bound, then the sequence  $\{\varphi_n(x)\}$  tends almost everywhere to a finite limit.

Denote the class of stepfunctions by  $C_0$  and the class of functions which are limits almost everywhere of the sequences  $\{\varphi_n\}$  referred to in lemma 1.31. For  $f(x) = \lim \varphi_n(x)$  define

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \int_a^b \varphi_n(x)dx$$

which does not depend on the particular choice of functions  $\varphi_n$ . In fact, if  $\{\psi_n\}$  is increases almost everywhere to a limit function  $g(x) \geq f(x)$ , we also have

$$\lim_{n \rightarrow \infty} \int_a^b \psi_n(x)dx \geq \lim_{n \rightarrow \infty} \int_a^b \varphi_n(x)dx$$

so that if  $g(x) = f(x)$  almost everywhere is implied.

## 1.6 Integration

In this section (**in progress**) we continue reading Folland, *Real Analysis: Modern Techniques and Their Applications*.

The *integral of a simple function* is . The *integral of a measurable function* .

[Theorem 1.32 \(Monotone convergence\)](#). content...

[Theorem 1.33 \(Dominated convergence\)](#). content...

[Proposition 1.34 \(Additivity of the integral\)](#). content...

- The *Lebesgue integral* is the integral we have developed then the measure is the Lebesgue measure.

[Theorem 1.35 \(Fubini-Tonelli\)](#).

1.

$$\int f d(\mu \times \nu) = \int \left( \int f(x, y) d\nu(y) \right) d\mu(x) = \int \left( \int f(x, y) d\mu(x) \right) d\nu(y)$$

2.

[Theorem 1.36 \(2.44\)](#).

$$\int \int f(x) dx = |\det T| \int f \circ T(x) dx$$

[Theorem 1.37 \(2.47, diffeomorphisms\)](#). content...

## 2 Point set topology

### 2.1 Metric spaces

A **metric** on a set  $X$  is a function  $\rho : X \times X \rightarrow [0, \infty)$  such that

1.  $\rho(x, x) = 0$  if and only if  $x = 0$ .
2.  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in X$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$  for all  $x, y, z \in X$ .

Two metrics  $\rho_1$  and  $\rho_2$  on a set  $X$  are **equivalent** if  $C\rho_1 \leq \rho_2 \leq C'\rho_1$  for some  $C, C' > 0$ .

**Theorem 2.1.** Let  $(X, d)$  and  $(Y, d')$  be metric spaces. If  $f : X \rightarrow Y$ , the following are equivalent conditions for  $f$  to be **continuous**:

1.  $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$
2.  $\forall x \in X \forall x_n \rightarrow x : f(x_n) \rightarrow f(x)$ .
3.  $\forall F \subseteq Y$  open,  $f^{-1}(F)$  is open.
4.  $\forall F \subseteq Y$  closed,  $f^{-1}(F)$  is closed.

If  $f : (X, \rho) \rightarrow (Y, \rho')$  is such that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

we say  $f$  is **uniformly continuous**.

**Exercise.** If  $(Y, \rho')$  is complete and  $f : A \rightarrow Y$  is uniformly continuous on  $A \subset X$  and  $\overline{A} = X$ , then  $f$  has a unique continuous extension  $g : X \rightarrow Y$  which is uniformly continuous on  $X$ . Show that this is not true in general if  $Y$  is not complete.

If  $f : (X, \rho) \rightarrow (Y, \rho')$  is a bijective function such that for any  $x, y \in X$ ,  $\rho(x, y) = \rho'(f(x), f(y))$  we say  $f$  is an **isometry** and the two spaces are **isometric**. A function  $f : (X, \rho) \rightarrow (X, \rho)$  is a **contraction** if there exists a  $0 < a < 1$  such that  $\rho(f(x), f(y)) \leq a\rho(x, y)$  for any  $x, y \in X$ . Every contraction is continuous, and if  $X$  is complete then any contraction has a unique fixed point.

A sequence  $\{x_n\}$  in  $X$  **converges** to  $x$  if  $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$ . A sequence  $\{x_n\}$  in  $X$  is called **Cauchy** if  $\lim_{n, m \rightarrow \infty} \rho(x_n, x_m) = 0$ , that is

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m > N : \rho(x_n, x_m) < \varepsilon.$$

A subset  $E \subseteq X$  is **complete** if every Cauchy sequence in  $E$  converges to its limit in  $E$ . If  $(X, \rho)$  and  $(X^*, \rho^*)$  are metric spaces and

1.  $(X, \rho)$  is isometric to a subspace  $(X, \rho^*)$  of  $(X^*, \rho^*)$ ,
2. The closure of  $X_0$  is all of  $X^*$  ( $X_0$  is **everywhere dense** or simply **dense**),

we say  $(X^*, \rho^*)$  is the *completion* of  $(X, \rho)$ .

**Theorem 2.2.** Every metric space  $(X, \rho)$  has a completion  $(X^*, \rho^*)$ . If  $(X^{**}, \rho^{**})$  is also a completion of  $(X, \rho)$ , then  $(X^*, \rho^*)$  is isometric to  $(X^{**}, \rho^{**})$ ; that is, the completion of a space is unique up to isometry.

*Proof.* Consider equivalence classes of Cauchy sequences. □

**Proposition 2.3.** A closed subset of a metric space is complete, and a complete subset of an arbitrary metric space is closed.

**Theorem 2.4.** If  $E$  is a subset of a metric space  $(X, \rho)$ , the following are equivalent:

1.  $E$  is complete and **totally bounded** (it can be covered by finitely many balls of radius  $\varepsilon$ ).
2. **(Bolzano-Wierstrass Property.)** Every sequence in  $E$  has a subsequence that converges to a point in  $E$ .
3. **(Heine-Borel Property)** If  $\{V_\alpha\}_{\alpha \in A}$  is an open cover of  $E$ , then there is a finite subset  $F \subseteq A$  such that  $\{V_\alpha\}_{\alpha \in F}$  covers  $E$ .

A set that satisfies any of these conditions is called *compact*.

**Theorem 2.5.** If  $(X, \rho)$  is a metric space and  $A$  is compact, then  $A$  is closed and bounded.

If  $(X, \rho)$  is a metric space,  $A \subseteq X$  is **relatively compact** if  $\overline{A}$  is compact. If  $\varepsilon > 0$ , a subset  $N \subset X$  is an  $\varepsilon$ -*net with respect to*  $A$  if  $\forall x \in A \exists n \in N : \rho(x, n) < \varepsilon$ .  $A$  is **totally bounded** if for any  $\varepsilon > 0$  there exists a finite  $\varepsilon$ -net with respect to  $A$ .

**Theorem 2.6.** Let  $(X, \rho)$  be a metric space and  $A \subseteq X$ . If for every sequence of points from  $A$  one can select a convergent subsequence, then  $A$  is totally bounded.

A set  $A$  is **countably compact** if every infinite subset of  $A$  has a limit point in  $A$ . All compact sets are countably compact.  $A$  is **sequentially compact** if every sequence in  $A$  has a subsequence that converges to a point in  $A$ . In a metric space, compactness is equivalent to countable and sequential compactness.

**Theorem 2.7.** Let  $(X, \rho)$  be a metric space and  $A \subseteq X$ .

1.  $A$  is relatively compact if and only if a convergent subsequence can be selected from every sequence of points in  $A$ . (We do not claim that the limit point is a member of  $A$ .)
2. If  $A$  is relatively compact, it is also totally bounded.
3. If  $(X, \rho)$  is complete and  $A$  is totally bounded, then  $A$  is relatively compact.
4. If  $A$  is compact then  $A$  is closed and totally bounded.

## 2.2 Topological spaces

Let  $X$  be a nonempty set. A **topology** on  $X$  is a family  $\mathcal{T}$  of subsets of  $X$  that contains  $\emptyset$  and  $X$  and is closed under arbitrary unions and finite intersections. (That is, if  $\{U_\alpha\}_{\alpha \in I} \subset \mathcal{T}$  then  $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$  and if  $U_1, \dots, U_n \in \mathcal{T}$  then  $\bigcap_{j=1}^n U_j \in \mathcal{T}$ .) The pair  $(X, \mathcal{T})$  is called a **topological space**.

The members of  $\mathcal{T}$  are called **open sets** and their complements are called **closed sets**. If  $A \subseteq X$ , the union of all open sets contained in  $A$  is called the **interior** of  $A$ , denoted by  $A^\circ$ ; so that a point  $x \in A^\circ$  when there is an open set  $O$  contained in  $A$  such that  $x \in O$ . The intersection of all closed sets containing  $A$  is called the **closure** of  $A$ . The difference  $\bar{A} \setminus A^\circ = \bar{A} \cap \bar{A}^c$  is called the **boundary** of  $A$  and is denoted by  $\partial A$ .

Following Bachman and Narici, *Functional Analysis*, a point  $x \in X$  is an **adherence point** of the subset  $E$  if every open set containing  $x$  contains a point of  $E$ . The set of all adherence points of  $E$  is called the **closure** of  $E$  and it is denoted by  $\bar{E}$ . It is immediate that  $E \subset \bar{E}$ .

**Proposition 2.8.** Let  $X$  be a topological space and  $A$  and  $B$  subsets of  $X$ . Then

1.  $A \subset B \implies \bar{A} \subset \bar{B}$ .
2.  $\overline{A \cup B} \implies \bar{A} \cup \bar{B}$ ,
3.  $\bar{\bar{A}} = \bar{A}$ .
4. A subset  $F$  of  $X$  is closed if and only if  $F = \bar{F}$ .

If  $X$  is a topological space and  $A$  is a subset of  $X$ , a point  $x \in X$  is called a **limit point** of  $A$  if every open set containing  $x$  contains a point of  $A$  distinct from  $x$ . The set of all limit points of  $A$  is denoted by  $A'$  and is called the **derived set** of  $A$ . Clearly,  $\bar{A} = A \cup A'$ , and, in view of the last item in the last proposition,  $A$  is closed if and only if  $A' \subset A$ .

A sequence of points  $\{x_n\}$  in a topological space  $X$  **converges** to the point  $x \in X$  if for every open set  $O$  containing  $x$  there exists an index  $N$  (depending on  $O$ ) such that  $x_n \in O$  for all  $n > N$ . That is, every open set containing  $x$  must contain almost all (but a finite number) of the  $x_n$ .

**Proposition 2.9.** Let  $X_1, X_2$  and  $X_3$  be topological spaces. Let  $f : X_1 \rightarrow X_2$  and  $g : X_2 \rightarrow X_3$  be mappings.

1. If  $f$  and  $g$  are continuous, then the composite mapping  $gf$  is continuous.
2.  $f$  is continuous if and only if for every subset  $A$  of  $X_1$ ,  $f\bar{A} \subset \overline{f(A)}$ .
3. Suppose  $f$  is a 1:1, onto mapping. Then  $f$  is a homeomorphism of  $X_1$  onto  $X_2$  if and only if, for all subsets  $A$  of  $X_1$ ,  $f(\bar{A}) = \overline{f(A)}$ .

If  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are two topologies on a set  $X$ , we say  $\mathcal{T}_1$  is **weaker** (or **coarser**) and  $\mathcal{T}_2$  **stronger** (or **finer**).  $E \subseteq X$  is called **dense** if  $\bar{E} = X$  and **nowhere dense** if  $\bar{E}$  has empty interior.  $X$  is called **separable** if it has a countable dense subset.

- $T_0$  If  $x \neq y$ , there is an open set containing  $x$  but not  $y$ , or an open set containing  $y$  but not  $x$ .
- $T_1$  If  $x \neq y$ , there is an open set containing  $y$  but not  $x$ . Equivalently,  $\{x\}$  is closed for every  $x \in X$ .
- $T_2$  (**Hausdorff.**) If  $x \neq y$  there are disjoint open sets  $U$  and  $V$  such that  $x \in U$  and  $y \in V$ .
- $T_3$  (**Regular.**)  $X$  is  $T_1$  and for any closed set  $A \subset X$  and any  $x \in A^c$  there are disjoint open sets  $U, V$  with  $x \in U$  and  $A \subseteq V$ .
- $T_{3\frac{1}{2}}$  (**Tychonoff, Completely regular.**)  $X$  is  $T_1$  and for each closed  $A \subseteq X$  and each  $x \notin A$  there exists  $f \in C(X, [0, 1])$  such that  $f(x) = 1$  and  $f = 0$  on  $A$ .
- $T_4$  (**Normal.**)  $X$  is  $T_1$  and for any disjoint closed sets  $A, B$  in  $X$  there are disjoint open sets  $U, V$  with  $A \subseteq U$  and  $B \subseteq V$ .

If  $X$  is any set and  $\{f_\alpha : X \rightarrow Y_\alpha\}_{\alpha \in A}$  is a family of maps from  $X$  into some topological spaces  $Y_\alpha$ , there is a unique weakest topology  $\tau$  on  $X$  that makes all the  $f_\alpha$  continuous called the *weak topology generated by*  $\{f_\alpha\}_{\alpha \in A}$ . An example of this topology is the *product topology* on  $X = \prod_{\alpha \in A} X_\alpha$  with the projections.

**Proposition 2.10.**

- If  $X_\alpha$  is Hausdorff for each  $\alpha \in A$  then  $X = \prod_{\alpha \in A} X_\alpha$  is Hausdorff.
- If  $X_\alpha$  and  $Y$  are topological spaces, a function  $f : Y \rightarrow X = \prod_{\alpha \in A} X_\alpha$  is continuous if and only iff  $\pi_\alpha \circ f$  is continuous for each  $\alpha$ .
- If  $X$  is a topological space,  $A$  is a nonempty set and  $\{f_n\}$  is a sequence in  $X^A$ , then  $f_n \rightarrow f$  in the product topology if and only if  $f_n \rightarrow f$  pointwise.

If  $X$  is any set and  $K = \mathbb{R}$  or  $\mathbb{C}$ , denote by  $B(X, K)$  the *set of bounded  $K$ -valued functions on  $X$* ,  $C(X, K)$  the set of *continuous  $K$ -valued functions on  $X$* , and  $BC(X, K)$  the *set of bounded continuous functions on  $X$* . If no field is specified we take it to be  $\mathbb{C}$ .

For  $f \in B(X)$  define the *uniform norm* of  $f$  to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}$$

Then the function  $\rho(f, g) = \|f - g\|_u$  is a metric on  $B(X)$ . Convergence in this metric is simply uniform convergence:

$$\{f_n\} \xrightarrow{u} f \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall x \in X : |f_n(x) - f(x)| < \varepsilon$$

$B(X)$  is complete with this metric since  $\mathbb{C}$  is complete.

**Proposition 2.11.** If  $X$  is a topological space,  $BC(X)$  is a closed subspace of  $B(X)$  in the uniform metric; in particular  $BC(X)$  is complete.

**Lemma 2.12 (Urysohn).** Let  $X$  be a normal space. If  $A$  and  $B$  are disjoint closed sets in  $X$ , there exists  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

**Theorem 2.13 (Tietze Extension Theorem).** Let  $X$  be a normal space. If  $A$  is a closed subset of  $X$  and  $f \in C(A, [a, b])$ , there exists  $F \in C(X, [a, b])$  such that  $F|_A = f$ .

**Corollary 2.14.** If  $X$  is normal,  $A \subseteq X$  is closed and  $f \in C(A)$ , there exists  $F \in C(X)$  such that  $F|_A = f$ .

Urysohn's lemma shows that every  $T_4$  space is completely regular ( $T_{3\frac{1}{2}}$ ).

**Theorem 2.15 (Dugundji).** Sea  $X$  un espacio metrizable,  $A = \bar{A} \subset X$  y  $L$  un espacio vectorial localmente convexo, y  $V \subset L$  convexo. Entonces cualquier función  $f : A \rightarrow V$  admite una extensión  $F$ .

$$\begin{array}{ccc} A & \xrightarrow{f} & V \\ \downarrow & \nearrow F & \\ X & & \end{array}$$

Además  $\text{img } F \subset \text{conv img } f$ .

## 2.3 Compact spaces

A topological space  $X$  is called **compact** if whenever  $\{U_\alpha\}_{\alpha \in A}$  is an open cover of  $X$  there is a finite subset  $B$  of  $A$  such that  $X = \bigcup_{\alpha \in B} U_\alpha$ . A subset  $Y \subseteq X$  is called **compact** if it is compact in the relative topology and **precompact** if its closure is compact.

A family  $\{F_\alpha\}_{\alpha \in A}$  of subsets of  $X$  has the **finite intersection property** if  $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$  for all finite  $B \subseteq A$ .

### Proposition 2.16.

- A topological space  $X$  is compact if and only if for every family  $\{F_\alpha\}_{\alpha \in A}$  of closed sets with the finite intersection property,  $\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$ .
- A closed subset of a compact space is compact.
- If  $K$  is a compact subset of a Hausdorff space  $X$  and  $x \notin K$  then there are disjoint open sets  $U, V$  such that  $x \in U$  and  $K \subseteq V$ .
- Every compact subset of a Hausdorff space is closed.
- Every compact Hausdorff space is normal.
- If  $X$  is compact and  $f : X \rightarrow Y$  is continuous then  $f(X)$  is compact.
- If  $X$  is compact, then  $C(X) = BC(X)$ .
- If  $X$  is compact and  $Y$  is Hausdorff, then any continuous bijection  $f : X \rightarrow Y$  is a homeomorphism.

A topological space  $X$  is **countably compact** if every countable open cover of  $X$  has a finite subcover, and **sequentially compact** if every sequence in  $X$  has a convergent subsequence. For metric spaces compactness and sequential compactness are the equivalent. There exists no general relation between compactness and sequential compactness.

## 2.4 Locally Compact Hausdorff spaces

A topological space is called *locally compact* if every point has a compact neighbourhood (a set  $A \subset X$  such that  $x \in A^\circ$ ). We call locally compact Hausdorff spaces *LCH* for short.

**Proposition 2.17.** Let  $X$  be a LCH space.

- If  $U \subseteq X$  is open and  $x \in U$ , there is a compact neighbourhood  $K$  of  $x$  such that  $K \subset U$ .
- If  $K \subseteq U \subseteq X$ , with  $K$  compact and  $U$  open, there exists a precompact open  $V$  such that  $K \subseteq V \subset \overline{V} \subset U$ .
- **(Urysohn's Lemma, Locally Compact Version.)** If  $K \subset U \subseteq X$ , there exists  $f \in C(X, [0, 1])$  such that  $f = 1$  on  $K$  and  $f = 0$  outside a compact subset of  $U$ .
- Every LCH space is completely regular.
- **(Tietze Extension Theorem, Locally Compact Version)** If  $K \subseteq X$  is compact and  $f \in C(K)$ , there exists  $F \in C(X)$  such that  $F|_K = f$ .  $F$  may be taken to vanish outside a compact set.

If  $f \in C(X)$ , the *support of  $f$*  is the closure of  $\{x \in X : f(x) \neq 0\}$  and denote  $C_c(X) := \{f \in C(X) : \text{supp } f \text{ is compact}\}$ . We say  $f$  *vanishes at infinity* if for every  $\varepsilon > 0$  the set  $\{x : |f(x)| \geq \varepsilon\}$  is compact and define  $C_0(X) := \{f \in C(X) : f \text{ vanishes at infinity}\}$ .

**Proposition 2.18.** If  $X$  is an LCH space,  $C_0(X)$  is the closure of  $C_c(X)$  in the uniform metric.

If  $X$  is a topological space, there are many ways of topologizing  $\mathbb{C}^X$ . One way is the product topology, that is, the topology of pointwise convergence. Another is the *topology of uniform convergence*, which is generated by the sets

$$\left\{ g \in \mathbb{C}^X : \sup_{x \in X} |g(x) - f(x)| < n^{-1} \right\} \quad n \in \mathbb{N}, f \in \mathbb{C}^X.$$

In view of a previous proposition (cite?), we know  $C(X)$  is a closed subset of  $\mathbb{C}^X$  with the topology of uniform convergence. Another topology is the *topology of uniform convergence on compact sets*, generated by the sets

$$\left\{ g \in \mathbb{C}^X : \sup_{x \in K} |g(x) - f(x)| < n^{-1} \right\} \quad n \in \mathbb{N}, f \in \mathbb{C}^X, K \subseteq X \text{ compact}.$$

**Proposition 2.19.** Let  $X$  be an LCH space.

- If  $E \subseteq X$ , then  $E$  is closed if and only if  $E \cap K$  is closed for every compact  $K \subseteq X$ .
- $C(X)$  is a closed subspace of  $\mathbb{C}^X$  in the topology of uniform convergence on compact sets.



- If  $\{U_j\}_{j=1}^n$  is an open cover of a compact subset  $K$  of  $X$ , then there is a partition of unity on  $K$  subordinate to  $\{U_j\}_{j=1}^n$  consisting of compactly supported functions.

**Theorem 2.20 (Urysohn Metrization Theorem).** Every second countable normal space is metrizable.

## 2.5 Three compactness theorems

Recall that if  $X = \prod_{\alpha \in A} X_\alpha$ , an element  $x \in X$  is just a mapping from  $A$  to  $\bigcup_{\alpha \in A} X_\alpha$  with  $x(\alpha)$  the  $\alpha$ th coordinate of  $x$ .

**Theorem 2.21.** If  $\{X_\alpha\}_{\alpha \in A}$  is a family of compact topological spaces, then  $X = \prod_{\alpha \in A} X_\alpha$  is compact with the product topology.

Let  $X$  be a topological space and  $\mathcal{F} \subseteq C(X)$  a family of complex-valued continuous functions on  $X$ . We say  $\mathcal{F}$  is **equicontinuous at**  $x \in X$  if for every  $\varepsilon > 0$  there is a neighbourhood  $U$  of  $x$  such that  $|f(x) - f(y)| < \varepsilon$  for all  $y \in U$  and all  $f \in \mathcal{F}$ ; and **equicontinuous** if it is equicontinuous at every  $x \in X$ . Also,  $\mathcal{F}$  is **pointwise bounded** if  $\{|f(x)| : f \in \mathcal{F}\}$  is bounded for all  $x \in X$ .

**Theorem 2.22 (Arzelà-Ascoli I).** Let  $X$  be a compact Hausdorff space. If  $\mathcal{F}$  is an equicontinuous, pointwise bounded subset of  $C(X)$ , then  $\mathcal{F}$  is totally bounded in the uniform metric, and the closure of  $\mathcal{F}$  in  $C(X)$  is compact.

**Theorem 2.23 (Arzelà-Ascoli II).** Let  $X$  be a locally compact Hausdorff space. If  $\{f_n\}$  is an equicontinuous, pointwise bounded sequence in  $C(X)$ , then there exists  $f \in C(X)$  and a subsequence of  $\{f_n\}$  that converges to  $f$  uniformly on compact sets.

## 2.6 The Stone-Weierstrass Theorem

Recall that the Weierstrass theorem states that any continuous function on a compact interval  $[a, b]$  is the uniform limit of polynomials on  $[a, b]$ . Throughout this subsection,  $X$  will denote a compact Hausdorff space, and  $C(X)$  is equipped with the uniform metric.

A subset  $\mathcal{A}$  of  $C(X, \mathbb{R})$  of  $C(X)$  is said to **separate points** if for every  $x, y \in X$  with  $x \neq y$  there exists  $f \in \mathcal{A}$  such that  $f(x) \neq f(y)$ .  $\mathcal{A}$  is called an **algebra** if it is a real (resp. complex) vector subspace of  $C(X, \mathbb{R})$  (resp.  $C(X)$ ) such that  $fg \in \mathcal{A}$  whenever  $f, g \in \mathcal{A}$ .  $\mathcal{A}$  is called a **lattice** if  $\max(f, g)$  and  $\min(f, g)$  are in  $\mathcal{A}$  whenever  $f, g \in \mathcal{A}$ . If  $\mathcal{A}$  is an algebra or a lattice, so is its closure in the uniform metric.

**Theorem 2.24 (Stone-Weierstrass Theorem).** Let  $X$  be a compact Hausdorff space. If  $\mathcal{A}$  is a closed subalgebra of  $C(X, \mathbb{R})$  that separates points, then either  $\mathcal{A} = C(X, \mathbb{R})$  or  $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$  for some  $x_0 \in X$ . The first alternative holds if and only if  $\mathcal{A}$  contains the constant functions.

**Corollary 2.25.** Suppose  $\mathcal{B}$  is a subalgebra of  $C(X, \mathbb{R})$  that separates points. If there exists  $x_0 \in X$  such that  $f(x_0) = 0$  for all  $f \in \mathcal{B}$ , then  $\mathcal{B}$  is dense in  $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ . Otherwise,  $\mathcal{B}$  is dense in  $C(X, \mathbb{R})$ .

The classical Weierstrass approximation theorem is the special case of this corollary where  $X$  is the compact subset of  $\mathbb{R}^n$  and  $\mathcal{B}$  is the algebra of polynomials on  $\mathbb{R}^n$  (restricted to  $X$ ); here  $\mathcal{B}$  contains the constant functions, so it is dense in  $C(X, \mathbb{R})$ .

The Stone-Weierstrass theorem, as stated, is false for complex-valued functions. We may show that  $f(z) = \bar{z}$  cannot be approximated uniformly by polynomials on the unit circle.

**Theorem 2.26 (Complex Stone-Weierstrass Theorem).** Let  $X$  be a compact Hausdorff space. If  $\mathcal{A}$  is a closed complex subalgebra of  $C(X)$  that separates points and is closed under complex conjugation, then either  $\mathcal{A} = C(X)$  or  $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$  for some  $x_0 \in X$ .

Finally, there is a version of the Stone-Weierstrass theorem for noncompact LCH spaces. We state for real functions; the complex analogue is an immediate consequence.

**Theorem 2.27 (LCH Stone-Weierstrass Theorem).** Let  $X$  be a noncompact LCH space. If  $\mathcal{A}$  is a closed complex subalgebra of  $C_0(X, \mathbb{R})$  that separates points, then either  $\mathcal{A} = C_0(X, \mathbb{R})$  or  $\mathcal{A} = \{f \in C_0(X, \mathbb{R}) : f(x_0) = 0\}$  for some  $x_0 \in X$ .

### 3 Inner product spaces

Let  $X$  be a real or complex vector space.

#### 3.1 Inner products

An *inner product* on  $X$  is a mapping

$$\langle -, - \rangle : X \times X \rightarrow F$$

with the following properties:

- (I<sub>1</sub>) if  $x, y \in X$  then  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ ;
- (I<sub>2</sub>) if  $\alpha, \beta$  are scalars,  $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ ;
- (I<sub>3</sub>)  $\langle x, x \rangle \geq 0$  for all  $x \in X$  and equal to zero if and only if  $x$  is the zero vector. (Since, by I<sub>1</sub>,  $\langle x, x \rangle$  must be real.)

##### Examples.

1. Let  $X = C[a, b]$  be complex-valued continuous functions on the closed interval  $[a, b]$  with pointwise addition and scalar product. As the inner product of any two vectors  $f$  and  $g$  in this space take

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

2. Let  $X = l_2$ , the set of all sequences of complex numbers  $(a_1, a_2, \dots)$  with the property that  $\sum_{i=1}^{\infty} |a_i|^2 < \infty$ . As the inner product of any two vectors  $x = (a_i)$  and  $y = (b_i)$  in this space take

$$\langle f, g \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i$$

which converges by the Hölder inequality.

3. Let  $Y$  be the closed interval  $[a, b]$ ,  $S$  the Lebesgue measurable sets and  $\mu$  the Lebesgue measure. Then, for the equivalence classes of square-integrable functions (complex-valued) on  $[a, b]$  we can take as the inner product of two classes  $f$  and  $g$ ,

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

where the integral is the Lebesgue integral. This space is denoted by  $L_2(a, b)$ .

**Theorem 3.1 (Cauchy-Schwarz inequality).** Let  $X$  be an inner product space and let  $x, y \in X$ . Then

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality holding if and only if  $x$  and  $y$  are linearly independent.

### 3.2 Orthogonal projections

Two vectors  $x, y \in X$  are **orthogonal** if  $\langle x, y \rangle = 0$ .

**Examples.**

1. In  $L_2(-\pi, \pi)$ , the collection (or any subset thereof)

$$x_n = \frac{1}{\sqrt{2\pi}} e^{int}, \quad n = 0, \pm 1, \dots$$

is an orthonormal set of vectors.

*Proof.* For any  $n \in \mathbb{Z}$ ,

$$\int_{-\pi}^{\pi} x_n \overline{x_n} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{int}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|e^{int}\|^2 dt = 1,$$

and if  $m$  is another integer,

$$\int_{-\pi}^{\pi} x_n \overline{x_m} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{imt}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(n-m)it} dt = \frac{1}{2\pi} \left[ \frac{e^{(n-m)it}}{(n-m)i} \right]_{-\pi}^{\pi} = 0.$$

□

2. If we restrict our attention to only real-valued functions that are square-integrable on the interval  $[-\pi, \pi]$ , then the collection (or any subset thereof)

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \cos 2t, \dots \\ & \frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \sin 2t, \dots \end{aligned}$$

is an orthonormal set.

**Theorem 3.2.** If  $S$  is an orthonormal subset of an inner product space, then it is linearly independent (where linear independence is defined as finite sums).

**Theorem 3.3 (Gram-Schmidt process).** Let  $X$  be an inner product space. If  $\{y_1, y_2, \dots\}$  is a linearly independent set of vectors, then there exists an orthonormal set of vectors  $\{x_1, x_2, \dots\}$  such that, for any  $n$ ,

$$\langle y_1, y_2, \dots, y_n \rangle = \langle x_1, x_2, \dots, x_n \rangle$$

where the brackets indicate the subspace spanned by the vectors enclosed.

If  $S$  is any subset of  $X$ , the **orthogonal complement of  $S$  in  $X$**  is the linear space  $S^\perp := \{x \in X : x \perp s \text{ for all } s \in S\}$ .

**Theorem 3.4.** If  $M$  is a finite-dimensional subspace of  $X$ , then  $X = M \oplus M^\perp$ .

### 3.3 Riesz representation theorem

**Theorem 3.5 (Riesz).** If  $X$  is a finite-dimensional inner product space and  $f$  is a linear functional on  $X$ , then there exists a unique vector  $y \in X$  such that  $f(x) = \langle x, y \rangle$  for all  $x \in X$ .

*Proof.* Given an orthonormal basis  $e_i$  of  $X$ , consider  $y = \sum_i \overline{f(e_i)} e_i$ . □

In Riemannian geometry this is called *raising an index* of a 1-form. Indeed,  $\omega_p \in \Lambda^1(T_p M)$  is just a linear functional on  $T_p M$ , and  $(\omega)^{\sharp} = g^{ij} \omega_j E_i$  at  $p$  is just a vector  $y$  such that  $\omega_p(x) = \langle x, y \rangle$  for all  $x \in T_p M$ . So the former theorem may also be stated as “ $y = f^{\sharp}$  exists”. Recall this is given by viewing the inner product as a nonsingular matrix.

### 3.4 Adjoint operator

Let  $A : X \rightarrow X$  be a linear transformation in a finite-dimensional inner product space  $X$ . For a given  $y \in X$ , define the linear functional

$$\begin{aligned} f^y : X &\rightarrow F \\ x &\mapsto \langle Ax, y \rangle \end{aligned}$$

which, by the Riesz representation theorem yields a unique  $z \in X$  such that

$$f^y(x) = \langle x, z \rangle$$

Then the *adjoint of  $A$*  is the linear map

$$\begin{aligned} A^* : X &\rightarrow X \\ y &\mapsto z \end{aligned}$$

so that  $\langle Ax, y \rangle = \langle x, A^*y \rangle$ .

**Proposition 3.6 (Properties of the adjoint).**

1.  $(\alpha A)^* = \overline{\alpha} A^*$ .
2.  $(A + B)^* = A^* + B^*$ .
3.  $(AB)^* = B^* A^*$ .
4.  $(A^*)^* = A$ .

If  $A = A^*$  we say  $A$  is *self-adjoint*, and if  $AA^* = A^*A$  we say  $A$  is *normal*.

**Theorem 3.7.** If  $A$  is self-adjoint, its eigenvalues are real. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

**Theorem 3.8.** If  $M$  is an invariant subspace of  $X$  under  $A$ , then  $M^{\perp}$  is invariant under  $A^*$ .

**Theorem 3.9.** If  $A$  is a linear transformation on a finite-dimensional inner product space  $X$ , then  $\text{range}(A)^{\perp} = \text{null}(A^*)$ .

### 3.5 Spectral theorem for normal transformations

**Theorem 3.10.** Let  $A$  be a self-adjoint transformation in a finite-dimensional inner product space  $X$ . Then there exists an orthonormal basis of  $X$  consisting of eigenvectors of  $A$ .

**Lemma 3.11.** Let  $A$  be a normal transformation in a finite-dimensional inner product space  $X$ . Then  $\|Ax\| = \|A^*x\|$  for all  $x \in X$ .

**Theorem 3.12.** Let  $A$  be a normal transformation in a complex finite-dimensional inner product space  $X$ . Then there exists an orthonormal basis of  $X$  consisting of eigenvectors of  $A$ .

**Theorem 3.13.** If  $A$  is a normal transformation on a finite-dimensional inner product space. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

Recall that the notation  $X = M_1 \oplus \dots \oplus M_k$  means that  $X$  is the **direct sum** of the  $M_i$ , which means that  $X = M_1 + \dots + M_k$  and  $M_i \cap \{M_1 + \dots + M_i + \dots + M_k\} = \{0\}$ , (every element in  $X$  is expressed as a unique sum of elements in  $M_i$ ). If  $M_i \perp M_j$  for all  $i \neq j$ , we say this is an **orthogonal direct sum decomposition of  $X$** , and the **orthogonal projection to  $M_j$**  is just taking the corresponding component of a given element in its decomposition.

**Theorem 3.14 (Spectral decomposition theorem for normal transformations).** To every normal transformation  $A$  on a complex finite-dimensional inner product space there correspond scalar  $\lambda_1, \dots, \lambda_k$ , the distinct eigenvalues of  $A$ , and orthogonal projections  $E_1, \dots, E_k$  with  $k \leq \dim X$ , such that

1.  $E_i$  is the orthogonal projection on  $\text{Null}(A - \lambda_i)$  for  $i = 1, \dots, k$ .
2.  $E_i \neq 0$  and  $E_i E_j = 0$  for  $i, j = 1, \dots, k$ .
3.  $\sum_{j=1}^k E_j = 1$ .
4.  $\sum_{j=1}^k \lambda_j E_j = A$ .

If  $A$  was self-adjoint, we could weaken the hypotheses to a real inner product space.

### 3.6 Unitary and orthogonal transformations

Let  $X$  be a finite-dimensional inner product space, and  $U : X \rightarrow X$  a linear transformation with  $U^*U = 1$ . We say  $U$  is **unitary** if  $X$  is complex and **orthogonal** if  $X$  is real. The condition  $U^*U = 1$  implies that  $UU^* = 1$ .

**Theorem 3.15.** Let  $X$  be a finite-dimensional inner product space, and  $U : X \rightarrow X$  a linear transformation. The following statements are equivalent:

1.  $U^*U = 1$ .
2.  $\langle Ux, Uy \rangle = \langle x, y \rangle$ .

3.  $\|Ux\| = \|x\|$  for all  $x \in X$ .

**Theorem 3.16.** If  $U$  is a unitary transformation on the finite-dimensional inner product space  $X$ , then each of the eigenvalues of  $U$  must have an absolute value equal to 1.

To summarize:

**Theorem 3.17.** Let  $A$  be a normal transformation on a complex finite-dimensional inner product space. Then

1.  $A$  is self-adjoint if and only if each eigenvalue of  $A$  is real.
2.  $A$  is unitary if and only if each eigenvalue of  $A$  has absolute value equal to 1.

### 3.7 Normed spaces

Let  $X$  be a real or complex vector space. A **norm** on  $X$  is a function  $\|\cdot\| : X \rightarrow \mathbb{R}$  such that

1.  $\|x\| \geq 0$  and  $\|x\| = 0$  if and only if  $x = 0$ .
2.  $\|\lambda x\| = |\lambda| \|x\|$  for all  $x \in X$  and  $\lambda \in \mathbb{R}$ .
3. **(Triangle inequality.)**  $\|x + y\| \leq \|x\| + \|y\|$  for all  $x, y \in X$ .

Every normed space is a metric space with the distance function  $\rho(x, y) = \|x - y\|$ . Two norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are called **equivalent** if there exist  $C_1, C_2 > 0$  such that

$$C_1 \|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \quad \forall x \in X$$

Equivalent norms define the same topology and the same Cauchy sequences.

A normed space that is complete is called a **Banach space**.

**Theorem 3.18.** For every normed linear space  $X$  there is a complete normed linear space  $X^*$  such that  $X$  is **congruent** (isomorphic and isometric) to a dense subset of  $X^*$  and the norm on  $X^*$  extends the norm on  $X$ .

If  $\{x_n\}$  is a sequence in  $X$ , the series  $\sum_{n=1}^{\infty} x_n$  **converges to**  $x$  if  $\sum_{n=1}^N x_n \rightarrow x$  as  $N \rightarrow \infty$ , and it is **absolutely convergent** if  $\sum_{n=1}^{\infty} \|x_n\| < \infty$ .

**Theorem 3.19.** A normed vector space  $X$  is complete if and only if every absolutely convergent series in  $X$  converges.

#### Examples.

- If  $X$  is a topological space,  $B(X)$  and  $BC(X)$  are Banach spaces with the uniform norm  $\|f\|_u = \sup_{x \in X} |f(x)|$ .
- If  $(X, \mathcal{M}, \mu)$  is a measure space,  $L^1(\mu)$  is a Banach space with the norm  $\|f\|_1 = \int |f| d\mu$ . (Observe that  $\|\cdot\|_1$  is only a seminorm if we do not identify functions that are equal a.e.)

If  $X$  and  $Y$  are normed vector spaces,  $X \times Y$  is a normed vector space with the **product norm**,  $\|(x, y)\| = \max(\|x\|, \|y\|)$ . If  $M$  is a vector subspace of  $X$ , the quotient space  $X/M$  consisting of equivalence classes under  $x \sim y$  iff  $x - y \in M$  is a normed space with the **quotient norm**,  $\|x + M\| = \inf_{y \in M} \|x + y\|$ .

A linear map  $T : X \rightarrow Y$  between two normed vector spaces is **bounded** if there exists  $C \geq 0$  such that

$$\|Tx\| \leq C\|x\| \quad \forall x \in X$$

**Proposition 3.20.** If  $X$  and  $Y$  are normed vector spaces and  $T : X \rightarrow Y$  is a linear map, then  $T$  is continuous if and only if it is bounded.

*Proof.* ( $\implies$ ) There exists  $\delta > 0$  such that  $\|x\| \leq \delta$  implies  $\|Tx\| \leq 1$ . For any nonzero  $x \in X$ ,

$$\|Tx\| = \left\| \frac{\|x\|}{\delta} T\left(\delta \frac{x}{\|x\|}\right) \right\| \leq \frac{1}{\delta} \|x\|.$$

( $\impliedby$ ) If  $\|x - y\| < \frac{\varepsilon}{C}$ ,

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq C\|x - y\| < \varepsilon.$$

□

In fact, if  $T$  is bounded it is uniformly continuous and even Lipschitz continuous.

We denote by  $L(X, Y)$  the space of bounded linear maps from  $X$  to  $Y$ , which is a normed vector space with the **operator norm**

$$\begin{aligned} \|T\| &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\| \\ &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \end{aligned}$$

**Proposition 3.21.** If  $Y$  is complete, so is  $L(X, Y)$ .

*Proof.* If  $\{T_n\}$  is a Cauchy sequence in  $L(X, Y)$ , the sequence  $\|T_n x\|$  is Cauchy in  $Y$  since  $\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\|$ . Define pointwise  $Tx = \lim_{n \rightarrow \infty} T_n x$ . □

$T$  is **invertible** if it is bijective and  $T^{-1}$  is bounded. It is called an isometry if  $\|Tx\| = \|x\|$  for all  $x \in X$ . An isometry is injective but not necessarily surjective.

If  $X$  is a vector space over  $K = \mathbb{R}, \mathbb{C}$ , a **linear functional** is a linear map from  $X$  to  $K$ .



**Proposition 3.22 (Relationship between real and complex linear functionals).** Let  $X$  be a vector space over  $\mathbb{C}$ . If  $f$  is a complex linear functional on  $X$ ,  $u := \operatorname{Re} x$  is a real linear functional and  $f(x) = u(x) - iu(ix)$ . Conversely, if  $u$  is a real functional on  $X$ , then  $f(x) := u(x) - iu(ix)$  is a complex linear functional, and if  $X$  is normed,  $\|u\| = \|f\|$ .

## 4 Hahn-Banach theorems

### 4.1 Analytic form of the Hahn-Banach theorem

It is not obvious that there are any nonzero bounded linear functionals on an arbitrary normed vector space. If  $E$  is a real vector space, **sublinear** or **Minkoswky functional** on  $E$  is a map  $p : E \rightarrow \mathbb{R}$  such that

$$p(\lambda x) = \lambda p(x) \quad \forall x \in X \text{ and } \lambda \geq 0 \quad (4.1)$$

$$p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X \quad (4.2)$$

**Theorem 4.1 (Helly-Hahn-Banach).** Let  $E$  be a vector space over  $\mathbb{R}$  and  $p : E \rightarrow \mathbb{R}$  a sublinear functional. If  $G \subseteq E$  is a linear subspace and  $g : G \rightarrow \mathbb{R}$  is a linear functional such that

$$g(x) \leq p(x) \quad \forall x \in G,$$

then there exists a linear functional  $f$  defined on all of  $E$  that extends  $g$ , that is,  $g(x) = f(x) \forall x \in G$  and such that

$$f(x) \leq p(x) \quad \forall x \in E.$$

For a proof first recall that a partial order  $P$  is **inductive** if every totally ordered subset  $Q$  in  $P$  has an upper bound, and that

**Lemma 4.2 (Zorn).** Every nonempty order set that is inductive has a maximal element.

*Proof.* (Of theorem 4.1) Consider the set

$$P = \left\{ h : D(h) \subseteq E \rightarrow \mathbb{R} : \begin{array}{l} D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear, } G \subseteq D(h), \\ h \text{ extends } g, \text{ and } h(x) \leq p(x) \forall x \in D(h) \end{array} \right\}$$

Then  $P$  is a partial order with

$$h_1 \leq h_2 \iff D(h_1) \subseteq D(h_2) \text{ and } h_2 \text{ extends } h_1$$

$P$  is nonempty since  $g \in P$ . To show it is inductive, take  $Q \subseteq P$  a partially order subset and write  $Q = (h_i)_{i \in I}$ . Then define

$$D(h) = \bigcup_{i \in I} D(h_i), \quad h(x) = h_i(x) \quad \text{if } x \in D(h_i) \text{ for some } i \in I$$

which is an upper bound of  $Q$ , so that there is a maximal element  $f$  in  $P$  by Zorn's Lemma. To finish it suffices to show that  $D(f) = E$ .

For a contradiction suppose that  $D(f) \neq E$  and choose  $x_0 \notin D(f)$ . We shall construct a function  $h \in P$  such that  $f < h$ . Define  $D(h) = D(f) + \mathbb{R}x_0$  and, for every  $x \in D(f)$ , set

$$h(x + \lambda x_0) = f(x) + t\alpha \quad \forall \lambda \in \mathbb{R}$$

where  $\alpha$  is a constant that we choose as follows. We must ensure that

$$h(x + \lambda x_0) = f(x) + \lambda\alpha \leq p(x + tx_0) \quad \forall x \in D(f) \quad \text{and} \quad \forall \lambda \in \mathbb{R}$$

For any  $x, y \in D(f)$ ,

$$\begin{aligned} f(x) + f(y) &= f(x + y) \leq p(x + y) \leq p(x + x_0) + p(y - x_0) \\ \implies f(x) - p(y - x_0) &\leq p(x + x_0) - f(y) \end{aligned}$$

So let  $\alpha$  satisfy

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}$$

If  $\lambda = 0$ , then  $h(x) = f(x) \leq p(x)$ . If  $\lambda \neq 0$  we must be careful since sublinear functionals only satisfy eq. (4.1) for positive scalars.

If  $\lambda > 0$ , then

$$\begin{aligned} h(x + \lambda x_0) &= \lambda \cdot h(x/\lambda + x_0) \\ &= \lambda \cdot (f(x/\lambda) + \alpha) \\ &\leq \lambda \cdot (f(x/\lambda) + p(x/\lambda + x_0) - f(x/\lambda)) \\ &\leq p(x + \lambda x_0) \end{aligned}$$

and if  $\lambda = -\mu < 0$ ,

$$\begin{aligned} h(x + tx_0) &= (-\lambda) \cdot h(-x/\lambda - x_0) \\ &= \mu \cdot (f(x/\mu) - \alpha) \\ &\leq \mu \cdot (f(x/\mu) - f(x/\mu) + p(x/\mu + x_0)) \\ &\leq p(x - \mu x_0) \\ &= p(x + \lambda x_0). \end{aligned}$$

Then  $h \in P$ ,  $h$  extends  $f$  and  $D(f) \subsetneq D(h)$ , which is impossible since  $f$  is maximal.  $\square$

An example of a sublinear functional is a **seminorm**, which is a function that satisfies the first two conditions of being norm, but the third. If  $p$  is a seminorm,

**Theorem 4.3 (Complex Hahn-Banach).** Let  $E$  be a complex vector space,  $p$  a seminorm on  $E$ ,  $G \subseteq E$  a subspace of  $E$  and  $f$  a complex linear functional on  $G$  such that  $|f(x)| \leq p(x) \quad \forall x \in G$ . Then there exists a complex linear functional  $f$  defined on all of  $E$  that extends  $f$  and  $|f(x)| \leq p(x)$ .

The space of all continuous linear functionals on  $E$ , denoted by  $E^* := L(E, K)$ , is called the **dual space** of  $E$ . Since  $K$  is complete,  $E^*$  is complete with the operator norm. We denote the **dual norm** (it remains for me to check whether this is the same as the operator norm) by

$$\|f\|_{E^*} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} |f(x)| = \sup_{\substack{x \in E \\ \|x\| \leq 1}} f(x)$$

Given  $f \in E^*$  and  $x \in E$  we shall often write  $\langle f, x \rangle$  instead of  $f(x)$  and call  $\langle -, - \rangle$  the **scalar product for the duality**  $E^*, E$ .

**Corollary 4.4.** Let  $G \subseteq E$  be a linear subspace. If  $g : G \rightarrow \mathbb{R}$  is a continuous linear functional, then there exists  $f \in E^*$  that extends  $g$  and such that

$$\|f\|_{E^*} = \sup_{\substack{x \in G \\ \|x\| \leq 1}} = \|g\|_{G^*}$$

*Proof.*  $g(x) \leq p(x) := \|g\|_{G^*} \|x\|$ , which is a sublinear functional. □

**Corollary 4.5.** For every  $x_0 \in E$  there exists  $f_0 \in E^*$  such that

$$\|f_0\| = \|x_0\| \quad \text{and} \quad \langle f_0, x_0 \rangle = \|x_0\|^2$$

*Proof.* By corollary 4.4 with  $G = \mathbb{R}x_0$  and  $g(tx_0) = t\|x_0\|^2$  so that  $\|g\|_{G^*} = \|x_0\|$ . □

While the element  $f_0$  in corollary 4.5 is in general not unique (exercise), a sufficient condition is that  $E^*$  is **strictly convex**, that is,  $\|tx + (1-t)y\| < t\|x\| + (1-t)\|y\|$  for  $t \in (0, 1)$ ,  $x, y \in E$  with  $\|x\| = \|y\| = 1$  and  $x \neq y$ . Any Hilbert space and any  $L^p(\Omega)$  for  $1 < p < \infty$  are.

The (multivalued) **duality map** for  $x_0 \in E$  is

$$F(x_0) = \{f_0 \in E^* : \|f_0\| = \|x_0\| \text{ and } \langle f_0, x_0 \rangle = \|x_0\|^2\}$$

**Corollary 4.6.** For every  $x \in E$  we have

$$\|x\| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle|$$

*Proof.* Assuming  $x \neq 0$ , if  $\|f\| \leq 1$ ,  $|\langle f, x \rangle| \leq \|f\| \|x\| \leq \|x\|$ , so

$$\sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| \leq \|x\|.$$

On the other hand, by corollary 4.5 there is some  $f_0 \in E^*$  such that  $\|f_0\| = \|x\|$  and  $\langle f_0, x \rangle = \|x\|^2$ . Set  $f_1 = f_0/\|x\|$  so that  $\|f_1\| = 1$  and  $\langle f_1, x \rangle = \|x\|$ . □

## 4.2 Geometric forms of the Hahn-Banach theorem

An *affine hyperplane* is a subset  $H$  of  $E$  of the form

$$H = \{x \in E : f(x) = \alpha\}$$

where if  $f$  is a nonzero linear functional not necessarily continuous and  $\alpha$  a real constant. We also write  $H = [f = \alpha]$ .

**Proposition 4.7.** The hyperplane  $H = [f = \alpha]$  is closed if and only if  $f$  is continuous.

*Proof.* It is clear that if  $f$  is continuous then  $H$  is closed. Conversely...  $\square$

Let  $A$  and  $B$  be two subsets of  $E$ . We say that the hyperplane  $H = [f = \alpha]$  *separates*  $A$  and  $B$  if

$$f(x) \leq \alpha \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha \quad \forall x \in B$$

We say  $H$  *strictly separates*  $A$  and  $B$  if there is some  $\varepsilon > 0$  such that

$$f(x) \leq \alpha - \varepsilon \quad \forall x \in A \quad \text{and} \quad f(x) \geq \alpha + \varepsilon \quad \forall x \in B$$

A subset  $A \subseteq E$  is *convex* if

$$tx + (1 - t)y \in A \quad \forall x, y \in A, \forall t \in [0, 1].$$

**Theorem 4.8 (Hahn-Banach, first geometric form).** Let  $A \subset E$  and  $B \subset E$  be two nonempty convex subsets such that  $A \cap B = \emptyset$ . Assume that one of them is open. Then there exists a closed hyperplane that separates  $A$  and  $B$ .

**Lemma 4.9.** Let  $C \subset E$  be an open convex set with  $0 \in C$ . For every  $x \in E$  set

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1}x \in C\}$$

which is called the *Minkowski functional of  $C$*  or the *gauge of  $C$* . Then  $p$  is a sublinear functional, that is, satisfies eqs. (4.1) and (4.2), and also

1. there is a constant  $M$  such that  $0 \leq p(x) \leq M\|x\| \quad \forall x \in E$ .
2.  $C = \{x \in E : p(x) < 1\}$ . (In particular, if  $C$  is the unit ball centered at 0, then  $p$  is just the norm.)

**Lemma 4.10.** Let  $C \subset E$  be a nonempty open convex set and let  $x_0 \in E$  with  $x_0 \notin C$ . Then there exists  $f \in E^*$  such that  $f(x) < f(x_0) \quad \forall x \in C$ . In particular, the hyperplane  $[f = f(x_0)]$  separates  $\{x_0\}$  and  $C$ .

*Proof.* After a translation we may assume that  $0 \in C$  and introduce the gauge  $p$  of  $C$ . For the linear subspace  $G = \mathbb{R}x_0$  and the linear functional  $g : G \rightarrow \mathbb{R}$  defined by  $g(t_0) = t$  ( $t \in \mathbb{R}$ ) we have that  $g(x) \leq p(x) \quad (\forall x \in G)$ . We may thus apply the Helly-Hahn-Banach theorem.  $\square$

*Proof.* (Of theorem 4.8). Set  $C = A - B$  so that  $C$  is convex (check!).  $C$  is also open (since  $C = \bigcup_{y \in B} (A - y)$ ) and  $0 \notin C$  (because  $A \cap B = \emptyset$ ). By lemma 4.10 there exists  $f \in E^*$  such that

$$f(z) \leq 0 \quad \forall z \in C \quad \Longleftrightarrow \quad f(x) < f(y) \quad \forall x \in A \quad \forall y \in B$$

Fixing a constant  $\alpha$  satisfying

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$$

we conclude that the hyperplane  $[f = \alpha]$  separates  $A$  and  $B$ .  $\square$

**Theorem 4.11 (Hahn-Banach, second geometric form).** Let  $A \subset E$  and  $B \subset E$  be two nonempty convex subsets such that  $A \cap B = \emptyset$ . Assume that  $A$  is closed and  $B$  is compact. Then there exists a closed hyperplane that strictly separates  $A$  and  $B$ .

*Proof.* Set  $C = A - B$ , so that  $C$  is convex, closed (check!) and  $0 \notin C$ . Hence there is some  $r > 0$  such that  $B(0, r) \cap C = \emptyset$ . By theorem 4.8 there is a hyperplane that separates  $B(0, r)$  and  $C$ . Therefore, there is some  $f \in E^*$ ,  $f \neq 0$  such that

$$f(x - y) \leq f(rz) \quad \forall x \in A, \quad \forall y \in B \quad \forall z \in B(0, 1)$$

(Incomplete.)  $\square$

In infinite-dimensional vector spaces it is in general impossible to separate any two nonempty disjoint convex sets. In finite-dimensional vector spaces, however, it is always possible.

**Corollary 4.12.** Let  $F \subset E$  be a linear subspace such that  $\overline{F} \neq E$ . Then there exists some  $f \in E^*$ ,  $f \neq 0$  such that

$$\langle f, x \rangle = 0 \quad \forall x \in F$$

*Proof.* Let  $x_0 \in E$  with  $x_0 \notin \overline{F}$ . Using theorem 4.11 with  $A = \overline{F}$  and  $B = \{x_0\}$  we find a closed hyperplane  $[f = \alpha]$  that strictly separates  $\overline{F}$  and  $\{x_0\}$ . We thus have

$$\langle f, x \rangle < \alpha \langle f, x_0 \rangle \quad \forall x \in F$$

In particular  $\langle f, x \rangle = 0 \quad \forall x \in F$  since  $\lambda \langle f, x \rangle < \alpha \quad \forall \lambda \in \mathbb{R}$ .  $\square$

By this corollary, we may prove that a linear subspace  $F \subset E$  is dense by showing that every continuous linear functional that vanishes on  $F$  must vanish in all of  $E$ .

### 4.3 The bidual $E^{**}$ . Orthogonality relations.

Let  $E$  be a normed vector space. Recall the norm on its dual space  $E^*$  is

$$\|f\|_{E^*} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} |\langle f, x \rangle|.$$

The *bidual* space of  $E$  is the dual of  $E^*$  with norm

$$\|\xi\|_{E^{**}} = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle \xi, f \rangle| \quad \xi \in E^{**}$$

There is a *canonical injection*  $J : E \rightarrow E^{**}$  defined as follows: given  $x \in E$ , the map  $f \mapsto \langle f, x \rangle$  is a continuous linear functional on  $E^*$ ; thus it is an element of  $E^{**}$  which we denote by  $Jx$ . We have

$$\langle Jx, f \rangle_{E^{**}, E^*} = \langle f, x \rangle_{E^*, E} \quad \forall x \in E, \forall f \in E^*.$$

It is clear that  $J$  is linear and that  $J$  is an isometry, that is,  $\|Jx\|_{E^{**}} = \|x\|_E$ . Indeed, by corollary 4.6 we have

$$\|x\|_{E^{**}} = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle Jx, f \rangle| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \|x\|_E$$

It may happen that  $J$  is not surjective from  $E$  onto  $E^{**}$ . However, it is convenient to identify  $E$  with a subspace of  $E^{**}$  using  $J$ . If  $J$  turns out to be surjective we say  $E$  is *reflexive* and  $E^{**}$  is identified with  $E$ . and if  $N \subset E^*$  is a linear subspace we set

$$N^\perp = \{x \in E : \langle f, x \rangle = 0 \forall f \in N\}$$

If  $M \subset E$  is a linear subspace, we set

$$M^\perp = \{f \in E^* : \langle f, x \rangle = 0 \forall x \in M\}.$$

Notice that  $N^\perp$  is a subset of  $E$  rather than of  $E^{**}$ . Both  $M^\perp$  and  $N^\perp$  are closed linear subspaces. We say  $M^\perp$  (resp.  $N$ ) is the *space orthogonal to  $M$  (resp.  $N$ )*.

**Proposition 4.13.** Let  $M \subset E$  be a linear subspace. Then

$$(M^\perp)^\perp = \overline{M}.$$

Let  $N \subset E^*$  be a linear subspace. Then

$$(N^\perp)^\perp \supset \overline{N}.$$

## 5 The Baire Category Theorem

This theorem states that, in a complete metric space, a countable union of closed sets with empty interior has empty interior:

**Theorem 5.1 (Baire).** Let  $X$  be a complete metric space and let  $(X_n)_{n \geq 1}$  be a sequence of closed subsets in  $X$ . Assume that

$$\text{Int } X_n = \emptyset \text{ for every } n \geq 1$$

Then

$$\text{Int} \left( \bigcup_{n=1}^{\infty} X_n \right) = \emptyset$$

And there's also this other form of the same theorem:

**Theorem 5.2 (Baire).** Let  $X$  be a nonempty complete metric space. Let  $(X_n)_{n \geq 1}$  be a sequence of closed subsets such that

$$\bigcup_{n=1}^{\infty} X_n = X.$$

Then there exists some  $n_0$  such that  $\text{Int } X_{n_0} \neq \emptyset$ .

**Remark 5.1.** A subset  $A \subset X$  has empty interior if and only if its complement  $A^c$  is dense in  $X$ . Indeed, to see  $A^c$  is dense in  $X$  notice every point  $x \in X$  has a neighbourhood intersecting  $A^c$ : since  $A$  has empty interior, every open set containing  $x \in A$  must intersect  $A^c$ . Conversely, if every point  $x \in X$  has a neighbourhood intersecting  $A^c$ , there cannot be a point in the interior of  $A$ .

*Proof.* (Of the first form.) Set  $O_n = X_n^c$  so that  $O_n$  is open and dense in  $X$  for every  $n \geq 1$ . To show  $\bigcup_{n=1}^{\infty} X_n$  has empty interior, it suffices to show that its complement  $G = \bigcap_{n=1}^{\infty} O_n$  is dense in  $X$ . Succintly, the procedure is to construct a sequence of a sequence of closed balls inside any given open set  $\omega$  of  $X$ ; one ball for every  $O_n$ . This produces a Cauchy sequence, whose limit is a point in  $\omega \cap G$ .  $\square$

## 5.1 The Uniform Boundedness Principle

Let  $E$  and  $F$  be two normed vector spaces. Recall that the space of continuous (=linear) operators from  $E$  into  $F$  is denoted by  $\mathcal{L}(E, F)$  and is equipped with the norm

$$\|T\|_{\mathcal{L}(E, F)} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} \|Tx\| = \inf_{\substack{x \in E \\ \|Tx\| \leq c\|x\|}} c.$$

**Theorem 5.3 (Banach-Steinhaus, uniform boundedness principle).** Let  $E$  and  $F$  be two Banach spaces and let  $(T_i)_{i \in I}$  be a family (not necessarily countable) of continuous linear operators from  $E$  to  $F$ . Assume that

$$\sup_{i \in I} \|T_i x\| < \infty \quad \forall x \in X \tag{5.1}$$

Then

$$\sup_{i \in I} \|T_i\|_{\mathcal{L}(E, F)} < \infty.$$

In other words, there exists a constant  $c$  such that

$$\|T_i x\| \leq c\|x\| \quad \forall x \in E \quad \forall i \in I$$

**Remark 5.2.** What's remarkable about this theorem is that from *pointwise* (the norm of all the  $T_i x$  is bounded at every  $x$ ) estimates we obtain a *global (uniform)* estimate: there is a unique bound for all the  $T_i$ .

*Proof.* For every  $n \geq 1$ , let

$$X_n = \{x \in E : \forall i \in I, \|T_i\| \leq n\}$$

which is closed (why?) and by eq. (5.1) we have

$$\bigcup_{n=1}^{\infty} X_n = E.$$

It follows from the Baire category theorem that there is some  $n_0 \geq 1$  such that  $\text{Int}(X_0) \neq \emptyset$ . Pick a ball  $B(x_0, r) \subset X_{n_0}$ , so that for every  $z \in B(0, 1)$  and  $i \in I$ ,

$$\begin{aligned} \|T_i(z)\| &= r^{-1} \|T(x_0 + rz) - T(x_0)\| \\ &\leq r^{-1}(n_0 + n_0). \end{aligned}$$

So

$$\sup_{i \in I} \|T_i\|_{\mathcal{L}(E, F)} = \sup_{i \in I} \left( \sup_{z \in B(0, 1)} \|T_i z\|_F \right) \leq r^{-1}(n_0 + n_0) < \infty.$$

□