Functional Analysis

github.com/dan-gc/analysis

These are preparation notes for a course on Functional Analysis at IMPA, summer 2024. They are based on Folland, *Real Analysis: Modern Techniques and Their Applications*; Bachman and Narici, *Functional Analysis*; Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.

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1 Measure theory

A measure should surely satisfy:

1. If $E_1, E_2, \ldots$ is a finite or infinite sequence of disjoint sets,
$$\mu(E_1 \cup E_2 \cup \ldots) = \mu(E_1) + \mu(E_2) + \ldots$$
2. If E is congruent to F,
$$\mu(E) = \mu(F)$$
3. If Q is the unit cube,
$$\mu(Q) = 1$$

1.1 σ-algebras

Let X be a nonempty set.

- An ***algebra of sets*** on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements, that is,
  1. If $E_1, \ldots, E_n \in \mathcal{A}$, then $\bigcup_{i=1}^n E_i \in \mathcal{A}$.
  2. If $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$.
- A ***σ-algebra*** is an algebra of sets closed under countable unions.
- The intersection of all σ-algebras containing any subset $\mathcal{E} \subset \mathcal{P}(X)$ is the ***σ-algebra generated by*** $\mathcal{E}$.
- The σ-algebra generated by the open sets of a topological (or metric) space X is the ***Borel algebra*** $\mathcal{B}_X$.
- Let $\{X_\alpha\}_{\alpha \in A}$ is a collection of nonempty sets, $X = \prod_\alpha X_\alpha$ and $\pi_\alpha : X \to X_\alpha$ the coordinate functions. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, the ***product σ-algebra*** on X is the σ-algebra generated by
$$\{\pi_\alpha(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$$
  We denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

  Proposition 1.1. $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}\}$.

  Proposition 1.2. Let $X_1, \ldots, X_n$ be metric spaces and let $X = \prod_i X_i$ be equiped with the product metric. Then $\bigotimes_i \mathcal{B}_i \subset \mathcal{B}_X$. If every X_i is separable equality holds.

  Corollary 1.3. $\mathcal{B}_{\mathbb{R}^n} = \bigotimes_{i=1}^n \mathcal{B}_{\mathbb{R}}$.

1.2 Measures

Let X be a set equipped with a σ-algebra $\mathcal{M}$.

- A ***measure*** on $\mathcal{M}$ (or on $(X, \mathcal{M})$, or simply on X) is a function $\mu : \mathcal{M} \to [0, \infty)$ such that

  1. $\mu(\varnothing)$.
  2. if $\{E_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup_1^\infty\right) = \sum_1^\infty \mu(E_i)$.

- If X is a set and $\mathcal{M} \subset \mathcal{P}(X)$ is a σ-algebra, $(X, \mathcal{M})$ is called a ***measurable space*** and the sets in $\mathcal{M}$ are ***measurable sets***. If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a ***measure space***.

- If $\mu(X) < \infty$ (and hence $\mu(E) < \infty$ for all $E \in \mathcal{M}$), μ is called σ-***finite***. If $X = \bigcup_1^\infty E_j$, with $\mu(E_j) < \infty$, μ is called σ-***finite***. If for every $E \in \mathcal{M}$ with $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $0 < \mu(F) < \infty$, μ is called ***semifinite***.

  Theorem 1.4 (Properties of measure spaces). Let $(X, \mathcal{M}, \mu)$ be a measure space.

  1. **(Monotonicity.)** If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) < \mu(F)$.
  2. **(Subaditivity.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu(E_j)$.
  3. **(Continuity from below.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \subset E_2 \subset \ldots$, then $\mu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.
  4. **(Continuity from above.)** If $\{E_j\}_1^\infty \subset \mathcal{M}$, $E_1 \supset E_2 \supset \ldots$, and $\mu(E_1) < \infty$, then $\mu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$.

- If $E \in \mathcal{M}$ and $\mu(E) = 0$, E es called a ***null set***. If a statement about points in X is true except for points in a null set, we say it is true ***almost everywhere***.

- If $\mu(E) = 0$ and $F \subset E$, then $\mu(F) = 0$ *provided* $F \in \mathcal{M}$. A measure whose domain contains all subsets of null sets is ***complete***. Completeness may help avoid technical difficulties, and it can always be achieved by enlarging the domain of μ:

  Theorem 1.5. Let $(X, \mathcal{M}, \mu)$ be a measure. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$. Then $\overline{\mathcal{M}}$ is a σ-algebra, and there is a unique extension $\bar{\mu}$ of μ to a complete measure on $\overline{\mathcal{M}}$, called the ***completion of*** μ.

- Excercises

  1. If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \ldots, a_n \in [0, \infty)$, then $\sum_1^\infty a_j \mu_j$ is also a measure on $(X, \mathcal{M})$.
  2. $\mu(E \cup F) = \mu(E) + \mu(F) - \mu(E \cap F)$.
  3. A set $E \subset X$ is called ***locally measurable*** if $E \cap A \in \mathcal{M}$ whenever $A \in \mathcal{M}$ and $\mu(A) < \infty$. If $\mathcal{M}$ equals the collection of the saturated sets $\tilde{\mathcal{M}}$, it is called ***saturated***. The saturated measure $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ defined by $\tilde{\mu}(E) = \mu(E)$ for $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwise is called the ***saturation of*** μ.

1.3 Outer measures

This is used to construct measures. The key idea is to approximate the measure of a set by simpler encosing sets, like with the Riemann integral.

- An ***outer measure*** on a nonempty set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ that satisfies

  1. $\mu^*(\varnothing) = 0$,
  2. $\mu^*(A) \leq \mu^*(B)$ if $A \subset B$, and
  3. $\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*(A_j)$.

  Proposition 1.6. Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho\varnothing = 0$. For any $A \subset X$, define

  $$\mu^*(A) = \inf\left\{\sum_1^\infty \mu(E_j) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\} \tag{1.1}$$

  then μ^* is an outer measure.

- A set $A \subset X$ is called ***μ-measurable*** if

  $$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subset X$$

  Which makes sense if we think E is a *well-behaved* set such that $A \subset E$, so that $\mu^*(A) = \mu^*(E) - \mu^*(E \cap A^c)$.

  Theorem 1.7 (Carathèodory). If μ^* is an outer measure on X, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

- If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra, the function $\mu_0 : \mathcal{A} \to [0, \infty]$ is a ***premeasure*** if

  - $\mu_0(\varnothing) = 0$,
  - If $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty \in \mathcal{A}$, then $\mu_0\left(\bigcup_1^\infty A_j\right) = \sum_1^\infty \mu_0(A_j)$.

  Proposition 1.8. If μ_0 is a premeasure on $\mathcal{A}$ and μ^* is defined by eq. (1.1) by taking $\rho = \mu_0$, then

  1. $\mu^*|\mathcal{A} = \mu_0$,
  2. every set in $\mathcal{A}$ is μ^* measureable.

  Theorem 1.9. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, μ_0 a premeasure on $\mathcal{A}$ and $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. There exists a measure μ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is μ_0—namely $\mu = \mu^*|\mathcal{M}$, where μ^* is given by eq. (1.1). (This is a consequence of Carathédory's theorem and the last proposition.)

If ν is another measure on $\mathcal{M}$ that extends μ_0, then $\nu(E) \leq \mu(E)$ for all $E \in \mathcal{M}$ with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then μ is the unique estension of μ_0 to a measure on $\mathcal{M}$.

- A ***Borel measure on*** $\mathbb{R}$ is a measure on $\mathbb{R}$ whose domain is the Borel σ-algebra $\mathcal{B}_\mathbb{R}$.

  Proposition 1.10. Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous. If $(a_j, b_j]$, $j = 1, \dots, n$ are disjoint ***half-open intervals***, so that $0 \leq a_j, b_j \leq \infty$, or (a_j, ∞), or they are empty, define

  $$\mu_0\left(\bigcup_1^n (a_j, b_j]\right) = \sum_1^n [F(b_j) - F(a_j)]$$

  and let $\mu_0(\varnothing) = 0$. Then μ_0 is a premeasure on the algebra $\mathcal{A}$ of finite disjoint unions of half-open intervals.

  Theorem 1.11. If $F : \mathbb{R} \to \mathbb{R}$ is any increasing, right continuous function, there is a unique Borel measure μ_F on $\mathbb{R}$ such that $\mu_F((a, b]) = F(b) - F(a)\ \forall a, b$.

  If G is another such function, $\mu_F = \mu_G$ if and only if $F - G$ is constant. Conversely, given a μ Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets we may define an increasing right continuous function F such that $\mu = \mu_F$.

- By theorem 1.9, there is a complete measure $\bar{\mu}_F$ whose domain includes $\mathcal{B}_\mathbb{R}$. In fact $\bar{\mu}_F$ is the completion of μ_F and its domain is strictly larger than $\mathcal{B}_\mathbb{R}$. This complete measure is called the ***Lebesgue-Stieltjes measure*** and is also denoted by μ_F.

  In the following μ is the Lebesgue-Stieltjes meause associated to some increasing, right-continuous function F, and $\mathcal{M}_\mu$ is the domain of μ.

  Theorem 1.12. If $E \in \mathcal{M}_\mu$, then

  $$\begin{aligned}\mu(E) &= \inf\{\mu(U) : E \subseteq U \text{ and } U \text{ is open}\} \\ &= \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is compact}\}\end{aligned}$$

  Theorem 1.13. If $E \subset \mathbb{R}$, the following are equivalent:

  1. $E \in \mathcal{M}_\mu$.
  2. $E = V \backslash N_1$ where V is a G_δ (countable intersection of open sets) and $\mu(N_1) = 0$.
  3. $E = H \cup N_2$ where H is an F_σ (countable union of closed sets) set and $\mu(N_2) = 0$.

  Proposition 1.14. If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then for every $\varepsilon > 0$, there is a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \varepsilon$.

- The ***Lebesgue measure*** is the Lebesgue-Stieltjes measure of $F(x) = x$. We denote it by m and its domain by $\mathcal{L}$.

Theorem 1.15. Is $E \in \mathcal{L}$, then $E + s \in \mathcal{L}$ and $rE \in \mathcal{L}$ for all $s, r \in \mathbb{R}$. Moreover, $m(E + s) = m(E)$ and $m(rE) = |r|m(E)$.

Proposition 1.16. Let C be the Cantor set. C is compact, nowhere dense and totally disconnected (ie. the only connected subsets of C are single points). C has no isolated points. $m(C) = 0$. $\operatorname{card}(C) = \mathfrak{c}$.

1.4 Measurable functions

Now we construct integrals from simple funtions.

- Recall that a mapping $f : X \to Y$ between two sets induces a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}X$ defined by the inverse image, which preserves unions, intersection and complements, so that if $\mathcal{N}$ is a σ-algebra on Y, then $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X.

  If $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are measureable spaces, a mapping $f : X \to Y$ is called ***measurable*** if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

  Proposition 1.17. A function whose codomain is a product measure space is measurable if precomposing with every projection is measurable.

  Corollary 1.18. A function $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable if and only if $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\mathcal{M}$-measurable.

  Proposition 1.19. If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

  Corollary 1.20. If X and Y are metric (or topological spaces), every continuous function is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.

  Proposition 1.21. If $f, g : X \to \mathbb{C}$ are $\mathcal{M}$-measurable, then so are $f + g$ and fg.

  Proposition 1.22. If $\{f_j\}$ is a sequence of $\overline{\mathbb{R}}$-valued measurable functions on $(X, \mathcal{M})$, then the functions

  $$\sup_j f_j(x) \qquad \limsup_{j\to\infty} f_j(x)$$
  $$\inf_j f_j(x) \qquad \liminf_{j\to\infty} f_j(x)$$

  are measurable. If

  $$f(x) = \lim_{j\to\infty} f_j(x)$$

  exists for every $x \in X$, then f is measurable.

  Corollary 1.23. If $f, g : X \to \overline{\mathbb{R}}$ is measurable, then so are $\max(f, g)$ and $\min(f, g)$.

  If $f : X \to \overline{\mathbb{R}}$ we define de ***positive*** and ***negative*** parts of f as:

  $$f^+(x) = \max(f(x), 0) \qquad\qquad f^-(x) \max(-f(x), 0)$$

  Then $f = f^+ - f^-$, and if f is measurable, so are f^+ and f^- by corollary 1.23.

- Let $(X, \mathcal{M})$ be a measurable space. If $E \subset X$, the ***characteristic or indicator function*** χ_E ***of*** E is

$$\chi(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Iχ_E is measurable if and only if $E \in \mathcal{M}$.

A ***simple function on*** X is a finite linear combination, with complex coefficientes, of characteristic functions of sets in $\mathcal{M}$. (We do not allow simple functions to assume values $\pm\infty$.) Equivalently, $f : X \to \mathbb{C}$ is simple if and only if f is measurable and the range of f is a finite subset of $\mathbb{C}$. The ***standard representation of*** f is

$$f = \sum_{j=1}^{n} z_j \chi_{E_j} \quad \text{where } E_j = f^{-1}(\{z_j\}) \text{ and } \operatorname{range}(f) = \{z_1, \ldots, z_n\}$$

If f and g are simple functions, then so are $f + g$ and fg.

Theorem 1.24. Let $(X, \mathcal{M})$ be a measurable space.

1. If $f : X \to [0, \infty]$ is measurable, there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \le \varphi_1 \le \ldots \le f$, $\varphi_n \to f$ pointwise and $\varphi_n \to f$ uniformly on any set on which f is bounded.
2. If $f : X \to \mathbb{C}$ is measurable, there is a sequence $\{\varphi_n\}$ of simple functions such that $0 \le |\varphi_1| \le \ldots \le |f|$, $\{\varphi_n\} \to f$ pointwise and $\varphi_n \to f$ uniformly on any set on which f is bounded.

Proof.

1. Let $n \in \mathbb{N}$ and $0 \le k \le 2^{2n} - 1$. Set

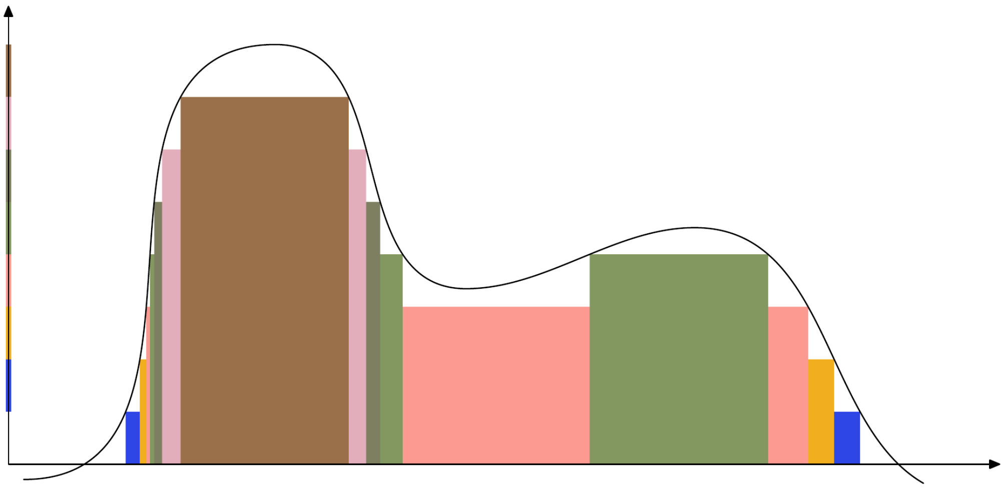
$$E_n^k = f^{-1}((k2^{-n}, (k+1)2^{-1}]) \quad \text{and} \quad F_n = f^{-1}((2^n, \infty]),$$

and finally

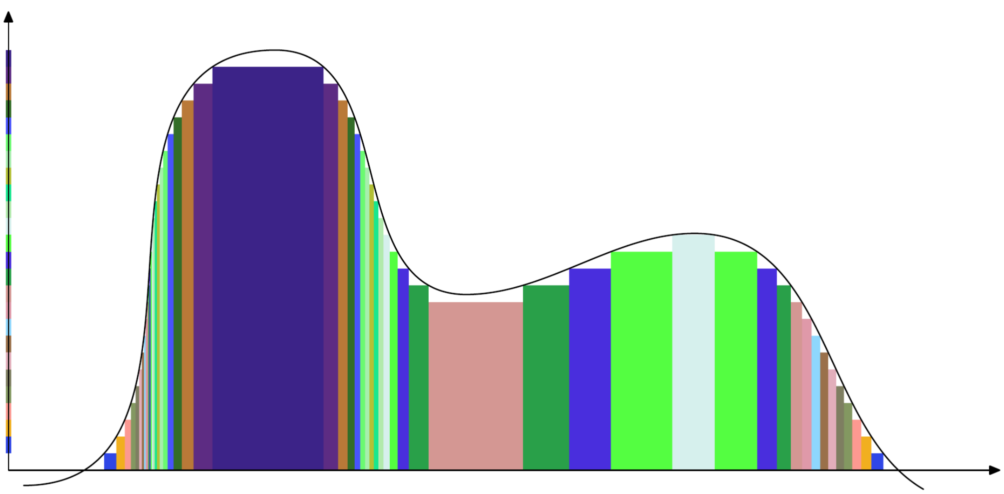
$$\varphi_n = \sum_{k=0}^{2^{2n}-1} k2^{-n} \chi_{E_n^k} + 2^n \chi_{F_n}.$$

Then $\varphi_n \le \varphi_{n+1}$ $\forall n$ and $0 \le f - \varphi_n \le 2^{-n}$ on the set where $f \le 2^n$.

2. If $f = g + ih$ we can apply part a to the positiva and negative parts of g and h.



(a) φ_7.



(b) φ_{24}.

Figure 1: Approximation by simple functions.

Proposition 1.25. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \overline{\mu})$ be its completion. If f is a $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g \overline{\mu}$-almost everywhere.

$\square$

1.5 Riesz and Nagy's Functional Analysis

This section is a short digression to show the approach of Riesz and Nagy, *Functional Analysis*.

Theorem 1.26 (Lebesgue). Every monotonic function $f(x)$ posses a finite derivative at every point with the possible exception of the points x of a set of measure zero, or, as it is often phrased, almost everywhere.

A set of ***measure zero*** is a set of values x which can be covered by a finite number or by a denumerable sequence of intervals whose total length is arbitrarily small.

Functions $f(x)$ continuous or not, for which the sum

$$\sum_{ab} = \sum_{1}^{n} |f(x_k) - f(x_{k-1})|$$

defined in terms of a decomposition of the interval (a, b) into partial intervales (x_{k-1}, x_k) $k = 1, 2, \ldots, n$, does not surpass a finit bound, independant of the particular choice of decomposition, are called ***functions of bounded variation***. The least upper bound is called the ***total variation of*** $f(x)$ ***in the interval*** (a, b).

Theorem 1.27 (Lebesgue). Every function of bounded variation posses a finite derivative almost everywhere.

Theorem 1.28 (Fubini). Let

$$f_1(x) + f_2(x) + \ldots = s(x)$$

be a convergent series all of whose terms are monotonic functions of the same type, defined on the interval $a \leq x \leq b$. Then

$$f_1'(x) + f_2'(x) + \ldots = s'(x)$$

except perhaps on a set of measure zero; that is, term by term differentiation is possible almost everywhere.

Theorem 1.29 (Lebesgue). Almost all points of an arbitrary linear set are density points of that set.

We include the following quote regarding Riemann integrals:

> [p. 23] *(...) we arrive at a necessary condition that $f(x)$ be integrable in the Riemann sense, namely, that the function $f(x)$ be continuous almost everywhere.*

A ***step function*** defined on an interval (a, b) is a function having a constant value c_k in each of a finite number of intervals i_k of finite lenght $|i_k|$ and vanishing outside these intervals. We suppose the integral defined for these functions, as usual, by the sum $\sum c_k |i_k|$.

Lemma 1.30. For every sequence $\{\varphi_n(x)\}$ of step functions which decreases to 0 almost everywhere, the sequene of values of their integrals also tends to zero.

Lemma 1.31. If for an increasing sequence of step functions $\{\varphi_n(x)\}$ the values of their integrals have a common bound, then the sequence $\{\varphi_n(x)\}$ tends almost everywhere to a finite limit.

Denote the class of stepfunctions by C_0 and the class of functions which are limits almost everywhere of the sequences $\{\varphi_n\}$ referred to in lemma 1.31. For $f(x) = \lim \varphi_n(x)$ define

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b \varphi_n(x)dx$$

which does not depend on the particular choice of functions φ_n. In fact, if $\{\psi_n\}$ is increases almost everywhere to a limit function $g(x) \geq f(x)$, we also have

$$\lim_{n\to\infty} \int_a^b \psi_n(x)dx \geq \lim_{n\to\infty} \int_a^b \varphi_n(x)dx$$

so that if $g(x) = f(x)$ almost everywhere is implied.

1.6 Integration

In this section (**in progress**) we continue reading Folland, *Real Analysis: Modern Techniques and Their Applications*.

The ***integral of a simple function*** is . The ***integral of a measurable function*** .

Theorem 1.32 (Monotone convergence). content...

Theorem 1.33 (Dominated convergence). content...

Proposition 1.34 (Aditivity of the integral)**.** content...

- The ***Lebesgue integral*** is the integral we have developed then the measure is the Lebesgue measure.

  Theorem 1.35 (Fubini-Tonelli).

  1. $$\int f d(\mu \times \nu) = \int \left(\int f(x,y) d\nu(y) \right) d\mu(x) = \int \left(\int f(x,y) d\mu(x) \right) d\nu(y)$$

  2.

  Theorem 1.36 (2.44).

  $$\int\int f(x)dx = |\det T| \int f \circ T(x)dx$$

  Theorem 1.37 (2.47, diffeomorphisms). content...

2 Point set topology

2.1 Metric spaces

A ***metric*** on a set X is a function $\rho : X \times X \to [0, \infty)$ such that

1. $\rho(x, x) = 0$ if and only if $x = 0$.
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Two metrics ρ_1 and ρ_2 on a set X are ***equivalent*** if $C\rho_1 \leq \rho_2 \leq C'\rho_2$ for some $C, C' > 0$.

Theorem 2.1. Let (X, d) and (Y, d') be metric spaces. If $f : X \to Y$, the following are equivalent conditions for f to be ***continuous***:

1. $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$
2. $\forall x \in X \forall x_n \to x : f(x_n) \to f(x)$.
3. $\forall F \subseteq Y$ open, $f^{-1}(F)$ is open.
4. $\forall F \subseteq Y$ closed, $f^{-1}(F)$ is closed.

If $f : (X, \rho) \to (Y, \rho')$ is such that

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in X : f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

we say f is ***uniformly continuous***.

Exercise. If (Y, ρ') is complete and $f : A \to Y$ is uniformly continuous on $A \subset X$ and $\overline{A} = X$, then f has a unique continuous extension $g : X \to Y$ which is uniformly continuous on X. Show that this is not true in general if Y is not complete.

If $f : (X, \rho) \to (Y, \rho')$ is a bijective function such that for any $x, y \in X$, $\rho(x, y) = \rho'(f(x), f(y))$ we say f is an ***isometry*** and the two spaces are ***isometric***. A function $f : (X, \rho) \to (X, \rho)$ is a ***contraction*** if there exists a $0 < a < 1$ such that $\rho(f(x), f(y)) \leq a\rho(x, y)$ for any $x, y \in X$. Every contraction is continuous, and if X is complete then any contraction has a unique fixed point.

A sequence $\{x_n\}$ in X ***converges*** to x if $\lim_{n\to\infty} \rho(x_n, x) = 0$. A sequence $\{x_n\}$ in X is called ***Cauchy*** if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$, that is

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n, m > N : \rho(x_n, x_m) < \varepsilon.$$

A subset $E \subseteq X$ is ***complete*** if every Cauchy sequence in E converges to its limit in E. If (X, ρ) and (X^*, ρ^*) are metric spaces and

1. (X, ρ) is isometric to a subspace (X, ρ^*) of (X^*, ρ^*),
2. The closure of X_0 is all of X^* (X_0 is ***everywhere dense*** or simply ***dense***),

we say (X^*, ρ^*) is the ***completion*** of (X, ρ).

Theorem 2.2. Every metric space (X, ρ) has a completion (X^*, ρ^*). If (X^{**}, ρ^{**}) is also a completion of (X, ρ^*), then (X^*, ρ^*) is isometric to (X^{**}, ρ^{**}); that is, the completion of a space is unique up to isometry.

Proof. Consider equivalence classes of Cauchy sequences. □

Proposition 2.3. A closed subset of a metric space is complete, and a complete subset of an arbitrary metric space is closed.

Theorem 2.4. If E is a subset of a metric space (X, ρ), the following are equivalent:

1. E is complete and ***totally bounded*** (it can be covered by finitely many balls of radius ε).
2. **(Bolzano-Wierstrass Property.)** Every sequence in E has a subsequence that converges to a point in E.
3. **(Heine-Borel Property)** If $\{V_\alpha\}_{\alpha \in A}$ is an open cover of E, then there is a finite subset $F \subseteq A$ such that $\{V_\alpha\}_{\alpha \in F}$ covers E.

A set that satsifies any of these condition s is called ***compact***.

Theorem 2.5. If (X, ρ) is a metric space and A is compact, then A is closed and bounded.

If (X, ρ) is a metric space, $A \subseteq X$ is ***relatively compact*** if $\overline{A}$ is compact. If $\varepsilon > 0$, a subset $N \subset X$ is an ***ε-net with respect to*** A if $\forall x \in A \exists n \in N : \rho(x, n) < \varepsilon$. A is ***totally boundad*** if for any $\varepsilon > 0$ there exists a finite ε-net with respect to A.

Theorem 2.6. Let (X, ρ) be a metric space and $A \subseteq X$. If for every sequence of points from A one can select a convergente subsequence, then A is totally bounded.

A set A is ***countably compact*** if every infinite subset of A has a limit point in A. All compact sets are countably compact. A is ***sequentially compact*** if every sequence in A has a subsequence that converges to a point in A. In a metric space, compactness is equivalent to countable and sequential compactness.

Theorem 2.7. Let (X, ρ) be a metric space and $A \subseteq X$.

1. A is relatively compact if and only if a convergent subsequence can be selected from every sequence of points in A. (We do not claim that the limit point is a member of A.)
2. If A is relatively compact, it is also totally bounded.
3. If (X, ρ) is complete and A is totally bounded, then A is relatively compact.
4. If A is compact then A is closed and totally bounded.

2.2 Topological spaces

Let X be a nonempty set. A ***topology*** on X is a family $\mathcal{T}$ of subsets of X that contains $\varnothing$ and X and is closed under arbitrary unions and finite intersections. (That is, if $\{U_\alpha\}_{\alpha\in X} \subset \mathcal{T}$ then $\bigcup_{\alpha\in A} U_\alpha \in \mathcal{T}$ and if $U_1, \ldots, U_n \in \mathcal{T}$ then $\bigcap_{j=1}^n U_j \in \mathcal{T}$.) The pair $(X, \mathcal{T})$ is called a ***topological space***.

The memebers of $\mathcal{T}$ are called ***open sets*** and their complements are called ***closed sets***. If $A \subseteq X$, the union of all open sets contained in A is called the ***interior*** of A, denoted by A^o; so that a point $x \in A^o$ when there is an open set O contained in A such that $x \in O$. The intersection of all closed sets containing A is called the ***closure*** of A. The difference $\bar{A}\backslash A^o = \bar{A} \cap \overline{A^c}$ is called the ***boundary*** of A and is denoted by ∂A.

Following Bachman and Narici, *Functional Analysis*, a point $x \in X$ is an ***adherence point*** of the subset E if every open set containing x conatins a point of E. The set of all adherence points of E is called the ***closure*** of E and it is denoted by $\overline{E}$. It is immediate that $E \subset \overline{E}$.

Proposition 2.8. Let X be a topological space and A and B subsets of X. Then

1. $A \subset B \implies \bar{A} \subset \bar{B}$.
2. $\overline{A \cup B} \implies \bar{A} \cup \bar{B}$,
3. $\bar{A} = \bar{\bar{A}}$.
4. A subset F of X is closed if and only if $F = \overline{F}$.

If X is a topological space and A is a subset of X, a point $x \in X$ is called a ***limit point*** of A if every open set containing x contains a point of A distinct from x. The set of all limit points of A is denoted by A' and is called the ***derived set of*** A. Clearly, $\bar{A} = A \cup A'$, and, in view of the last item in the last proposition, A is closed if and only if $A' \subset A$.

A sequence of points $\{x_n\}$ in a topological space X ***converges*** to the point $x \in X$ if for every open set O containing x there exists an index N (dependending on O) such that $x_n \in O$ for all $n > N$. That is, every open set containing x must contain almost all (but a finite number) of the x_n.

Proposition 2.9. Let X_1, X_2 and X_3 be topologial spaces. Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be mappings.

1. If f and g are continuous, then the composite mapping gf is continuous.
2. f is continuous if and only if for every subset A of X, $f\bar{A} \subset \overline{f(A)}$.
3. Suppose f is a 1:1, onto mapping. Then f is a homeomorphism of X_1 onto X_2 is and only if, for all subsets A of X_1, $f(\bar{A}) = \overline{f(A)}$.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set X, we say $\mathcal{T}_1$ is ***weaker*** (or ***coarser***) and $\mathcal{T}_2$ ***stronger*** (or ***finer***). $E \subseteq X$ is called ***dense*** if $\overline{E} = X$ and ***nowhere dense*** if $\overline{E}$ has empty interior. X is called ***separable*** if it has a countable dense subset.

T_0 If $x \neq y$, there is an open set containing x but not y, or an open set containing y but not x.

T_1 If $x \neq y$, there is an open set containint y but not x. Equivalently, $\{x\}$ is closed for every $x \in X$.

T_2 **(Hausdorff.)** If $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

T_3 **(Regular.)** X is T_1 and for any closed set $A \subset X$ and any $x \in A^c$ there are disjoint open sets U, V with $x \in U$ and $a \subseteq V$.

$\mathrm{T}_{3\frac{1}{2}}$ **(Tychonoff, Completely regular.)** X is T_1 and for each closed $A \subseteq X$ and each $x \notin A$ there exists $f \in C(X, [0,1])$ such that $f(x) = 1$ and $f = 0$ on A.

T_4 **(Normal.)** X is T_1 and for any disjoint closed sets A, B in X there are disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$.

If X is any set and $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in A}$ is a family of maps from X into some topological spaces Y_α, there is a unique weakest topology τ on X that makes all the f_α continuous called the ***weak topology generated by*** $\{f_\alpha\}_{\alpha \in A}$. An example of this topology is the ***product topology*** on $X = \prod_{\alpha \in A} X_\alpha$ with the projections.

Proposition 2.10.

- If X_α is Hausdorff for each $\alpha \in A$ then $X = \prod_{\alpha \in A}$ is Hausdorff.
- If X_α and Y are topological spaces, a function $f : Y \to X = \prod_{\alpha \in A} X_\alpha$ is continuous if and only iff $\pi_\alpha \circ f$ is continuous for each α.
- If X is a topological space, A is a nonempty set and $\{f_n\}$ is a sequence in X^A, then $f_n \to f$ in the product topology if and only if $f_n \to f$ pointwise.

If X is any set and $K = \mathbb{R}$ or $\mathbb{C}$, denote by $B(X, K)$ the ***set of bounded*** K***-valued functions on*** X, $C(X, K)$ the set of ***continuous*** K***-valued functions on*** X, and $BC(X, K)$ the ***set of bounded continuous functions on*** X. If no field is specified we take it to be $\mathbb{C}$.

For $f \in B(X)$ define the ***uniform norm*** of f to be

$$\|f\|_u = \sup\{|f(x)| : x \in X\}$$

Then the function $\rho(f, g) = |Vert f - g|Vert_u$ is a metric on $B(X)$. Convergence in this metric is simply uniform convergence:

$$\{f_n\} \overset{u}{to} f \iff \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N \forall x \in X : |f_n(x) - f(x)| < \varepsilon$$

$B(X)$ is complete with this metric since $\mathbb{C}$ is complete.

Proposition 2.11. If X is a topological space, $BC(X)$ is a closed subspace of $B(X)$ in the uniform metric; in particular $BC(X)$ is complete.

Lemma 2.12 (Urysohn)**.** Let X be a normal space. If A and B are disjoint closed sets in X, there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Theorem 2.13 (Tietze Extension Theorem). Let X be a normal space. If A is a closed subset of X and $f \in C(A, [a, b])$, there exists $F \in C(X, [a, b])$ such that $F|A = f$.

Corollary 2.14. If X is normal, $A \subseteq X$ is closed and $f \in C(A)$, there exists $F \in C(X)$ such that $F|A = f$.

Urysohn's lemma shows that every T_4 space is completely regular ($T_{3\frac{1}{2}}$).

Theorem 2.15 (Dugundji). Sea X un espacio metrizable, $A = \bar{A} \subset X$ y L un espacio vectorial localmente convexo, y $V \subset L$ convexo. Entonces cualquier función $f : A \to V$ admite una extensión F.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook] & V \\ X \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

Además $\operatorname{img} F \subset \operatorname{conv} \operatorname{img} F$.

2.3 Compact spaces

A topological space X is called ***compact*** if whenever $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X there is a finite subset B of A such that $X = \bigcup_{\alpha \in B} U_\alpha$. A subset $Y \subseteq X$ is called ***compact*** if it is compact in the relative topology and ***precompact*** if its closure is compact.

A family $\{F_\alpha\}_{\alpha \in A}$ of subsets of X has the ***finite intersection property*** if $\bigcap_{\alpha \in B} F_\alpha \neq \varnothing$ for all finite $B \subseteq A$.

Proposition 2.16.

- A topological space X is compact if and only if for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$.
- A closed subset of a compact space is compact.
- If K is a compact subset of a Hausdorff space X and $x \notin K$ then there are disjoint open sets U, V such that $x \in U$ and $K \subseteq V$.
- Every compact subset of a Hausdorff space is closed.
- Every compact Hausdorff space is normal.
- If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.
- If X is compact, then $C(X) = BC(X)$.
- If X is compact and Y is Hausdorff, then any continuous bijection $f : X \to Y$ is an homeomorphism.

A topological space X is ***countably compact*** if every countable open cover of X has a finite subcover, and ***sequentially compact*** if every sequence in X has a convergent subsequence. For metric spaces compactness and sequential compactness are the equivalent. There exists no general relation between compactness and sequential compactness.

2.4 Locally Compact Hausdorff spaces

A topological space is called ***locally compact*** if every point has a compact neighbourhood (a set $A \subset X$ such that $x \in A^o$). We call locally compact Hausdorff spaces ***LCH*** for short.

Proposition 2.17. Let X be a LCH space.

- If $U \subseteq X$ is open and $x \in U$, there is a compact neighbourhood K of x such that $K \subset U$.
- If $K \subseteq U \subseteq X$, with K compact and U open, there exists a precompact open V such that $K \subseteq V \subset \overline{V} \subset U$.
- **(Urysohn's Lemma, Locally Compact Version.)** If $K \subset U \subseteq X$, there exists $f \in C(X, [0, 1])$ such that $f = 1$ on K and $f = 0$ outside a compact subset of U.
- Every LCH space is completely regular.
- **(Tietze Extension Theorem, Locally Compact Version)** If $K \subseteq X$ is compact and $f \in C(K)$, there exists $F \in C(X)$ such that $F|K = f$. F may be taken to vanish outside a compact set.

If $f \in C(X)$, the ***support of*** f is the closure of $\{x \in X : f(x) \neq 0\}$ and denote $C_c(X) := \{f \in C(X) : \operatorname{supp} f \text{ is compact}\}$. We say f ***vanishes at infinity*** if for every $\varepsilon > 0$ the set $\{x : |f(x)| \geq \varepsilon\}$ is compact and define $C_0(X) := \{f \in C(X) : f \text{ vanishes at infinity}\}$.

Proposition 2.18. If X is an LCH space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.

If X is a topological space, there are many ways of topologizing $\mathbb{C}^X$. One way is the product topology, that is, the topology of pointwise convergence. Another is the ***topology of uniform convergence***, which is generated by the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in X} |g(x) - f(x)| < n^{-1}\right\} \qquad n \in \mathbb{N}, f \in \mathbb{C}^X.$$

In view of a previous proposition (cite?), we know $C(X)$ is a closed subset of $\mathbb{C}^X$ with the topology of uniform convergence. Another topology is the ***topology of uniform convergence on compact sets***, generated by the sets

$$\left\{g \in \mathbb{C}^X : \sup_{x \in K} |g(x) - f(x)| < n^{-1}\right\} \qquad n \in \mathbb{N}, f \in \mathbb{C}^X, K \subseteq X \text{ compact}.$$

Proposition 2.19. Let X be an LCH space.

- If $E \subseteq X$, then E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.
- $C(X)$ is a closed subspace of $\mathbb{C}^X$ in the topology of uniform convergence on compact sets.

- If $\{U_j\}_{j=1}^n$ is an open cover of a compact subset K of X, then there is a partition of unity on K subordinate to $\{U_j\}_{j=1}^n$ soncisting of compactly supported functions.

Theorem 2.20 (Urysohn Metrization Theorem). Every second countable normal space is metrizable.

2.5 Three compactness theorems

Recall that if $X = \prod_{\alpha \in A} X_\alpha$, an element $x \in X$ is just a mapping from A to $\bigcup_{\alpha \in A} X_\alpha$, with $x(\alpha)$ the αth coordinate of x.

Theorem 2.21. If $\{X_\alpha\}_{\alpha \in A}$ is a family of compact topological spaces, then $X = \prod_{\alpha \in A} X_\alpha$ is compact with the produc topology.

Let X be a topological space and $\mathcal{F} \subseteq C(X)$ a family of complex-valued continuous functions on X. We say $\mathcal{F}$ is ***equicontinuous at*** $x \in X$ if for every $\varepsilon > 0$ there is a neighbourhood U of x such that $|f(x) - f(y)| < \varepsilon$ for all $y \in U$ and all $f \in \mathcal{F}$; and ***equicontinuous*** if it is equicontinuous at every $x \in X$. Also, $\mathcal{F}$ is ***pointwise bounded*** if $\{|f(x)| : f \in \mathcal{F}\}$ is bounded for all $x \in X$.

Theorem 2.22 (Arzelá-Ascoli I). Let X be a compact Housdorff space. If $\mathcal{F}$ is an equicontinuous, pointwise bounded subset of $C(X)$, then $\mathcal{F}$ is totally bounded in the uniform metric, and the closure of $\mathcal{F}$ in $C(X)$ is compact.

Theorem 2.23 (Arzelá-Ascoli II). Let X be a locally compact Housdorff space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, then there exists $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.

2.6 The Stone-Weierstrass Theorem

Recall that the Weierstrass theorem states that any continuous function on a compact interval $[a, b]$ is the uniform limit of polynomials on $[a, b]$. Throughout this subsection, X will denote a compact Hausdorff space, and $C(X)$ is equipped with the uniform metric.

A subset $\mathcal{A}$ of $C(X, \mathbb{R})$ of $C(X)$ is said to ***separate points*** if for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. $\mathcal{A}$ is called an ***algebra*** if it is a real (resp. complex) vector subspace of $C(X, \mathbb{R})$ (resp. $C(X)$) such that $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$. $\mathcal{A}$ is called a ***lattice*** if $\max(f, g)$ and $\min(f, g)$ are in $\mathcal{A}$ whenever $f, g \in \mathcal{A}$. If $\mathcal{A}$ is an algebra or a lattice, so is its closure in the uniform metric.

Theorem 2.24 (Stone-Weierstrass Theorem). Let X be a compact Hausdorff space. If $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ that separates points, then either $A = C(X, \mathbb{R})$ of $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds if and only if $\mathcal{A}$ contains the constant functions.

Corollary 2.25. Supoose $\mathcal{B}$ is a subalgebra of $C(X, \mathbb{R})$ that separates points. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$, then $\mathcal{B}$ is dense in $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$. Otherwise, $\mathcal{B}$ is dense in $C(X, \mathbb{R})$.

The classical Weierstrass approximation theorem is the special case of this corollary where X is the compact subset of $\mathbb{R}^n$ and $\mathscr{B}$ is the algebra of polynomials on $\mathbb{R}^n$ (restricted to X); here $\mathscr{B}$ contains the constant functions, so it is dense in $C(X, \mathbb{R})$.

The Stone-Weirstrass theorem, as stated, is false for complex-valued functions. We may show that $f(z) = \bar{z}$ cannot be approximately uniformly by polynomials on the unit circle.

Theorem 2.26 (Complex Stone-Weirstrass Theorem). Let X be a compact Hausdorff space. If $\mathscr{A}$ is a closed complex subalgebra of $C(X)$ that separates points and is closed under complex conjugation, then either $A = C(X)$ of $\mathscr{A} = \{f \in C(X) : f(x_0) = 0\}$ for some $x_0 \in X$.

Finally, there is a version of the Stone-Weirstrass theorem for noncompact LCH spaces. We state for real functions; the complex analogue is an immediate consequence.

Theorem 2.27 (LCH Stone-Weirstrass Theorem). Let X be a noncompact LCH space. If $\mathscr{A}$ is a closed complex subalgebra of $C_0(X, \mathbb{R})$ that separates points, then either $A = C_0(X, \mathbb{R})$ of $\mathscr{A} = \{f \in C_0(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$.

3 Inner product spaces

Let X be a real or complex vector space.

3.1 Inner products

An ***inner product*** on X is a mapping

$$\langle -, - \rangle : X \times X \to F$$

with the following properties:

(I_1) if $x, y \in X$ then $\langle x, y \rangle = \overline{\langle x, y \rangle}$;

(I_2) if α, β are scalars, $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$;

(I_3) $\langle x, x \rangle \geq 0$ for all $x \in X$ and equal to zero if and only if x is the zero vector. (Since, by I_1, $\langle x, x \rangle$ must be real.)

Examples.

1. Let $X = C[a, b]$ be complex-valued continuous functions on the closed interval $[a, b]$ with pointwise addition and scalar product. As the inner product of any two vectors f and g in this space take

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

2. Let $X = l_2$, the set of all sequences of complex numbers $(a_1, a_2, \ldots)$ with the property that $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. As the inner product of any two vectors $x = (a_i)$ and $y = (b_i)$ in this space take

$$\langle f, g \rangle = \sum_{i=1}^{\infty} a_i \bar{b}_i$$

which converges by the Hölder inequality.

3. Let Y be the closed interval $[a, b]$, S the Lebesgue measurable sets and μ the Lebesgue measure. Then, for the equivalence clasess of square-integrable functions (complex-valued) on $[a, b]$ we can take as the inner product of two clases f and g,

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

where the integral is the Lebesgue integral. This space is denoted by $L_2(a, b)$.

Theorem 3.1 (Cauchy-Schwarz inequality). Let X be an inner product space and let $x, y \in X$. Then

$$|\langle x, y \rangle \leq \|x\| \|y\|$$

with equality holding if and only if x and y are linearly independent.

3.2 Orthogonal projections

Two vectors $x, y \in X$ are ***orthogonal*** if $\langle x, y \rangle = 0$.

Examples.

1. In $L_2(-\pi, \pi)$, the collection (or any subset thereof)

$$x_n = \frac{1}{\sqrt{2\pi}} e^{int}, \qquad n = 0, \pm 1, \dots$$

is an orthonormal set of vectors.

Proof. For any $n \in \mathbb{Z}$,

$$\int_{-\pi}^{\pi} x_n \overline{x}_n dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{int}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|e^{int}\|^2 dt = 1,$$

and if m is another integer,

$$\int_{-\pi}^{\pi} x_n \overline{x}_m dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} \overline{e^{imt}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(n-m)it} dt = \frac{1}{2\pi} \left[\frac{e^{(n-m)it}}{(n-m)i} \right]_{-\pi}^{\pi} = 0.$$

□

2. If we restric out attention to only real-valued functions that are square-integrable on the interval $[-\pi, \pi]$, then the collection (or any subset thereof)

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \cos 2t, \dots$$
$$\frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \sin 2t, \dots$$

is an orthonormal set.

Theorem 3.2. If S is an orthonoromal subset of an inner product space, then it is linearly independent (where linear independence is defined as finite sums).

Theorem 3.3 (Gram-Schmidt process). Let X be an inner product space. If $\{y_1, y_2, \dots\}$ is a linearly independent set of vectors, then there exists an orthonormal set of vectors $\{x_1, x_2, \dots\}$ such that, for any n,

$$\langle y_1, y_2, \dots, y_n \rangle = \langle x_1, x_2, \dots, x_n \rangle$$

where the brackets indicate the subspace spanned by the vectores enclosed.

If S is any subset of X, the ***orthogonal complement of*** S ***in*** X is the linear space $S^\perp := \{x \in X : x \perp s \text{ for all } s \in S\}$.

Theorem 3.4. If M is a finite-dimensional subspace of X, then $X = M \oplus M^\perp$.

3.3 Riesz representation theorem

Theorem 3.5 (Riesz). If X is a finite-dimensional inner product space and f is a linear functional on X, then there exists a unique vector $y \in X$ such that $f(x) = \langle x, y \rangle forall x \in X$.

Proof. Given an orthonormal basis e_i of X, consider $y = \sum_i \overline{f(e_i)} e_i$. □

In Riemannian geometry this is called ***raising an index*** of a 1-form. Indeed, $\omega_p \in \Lambda^1(T_pM)$ is just a linear functional on T_pM, and $(\omega)^\sharp = g^{ij}\omega_j E_i$ at p is just a vector y such that $\omega_p(x) = \langle x, y \rangle$ for all $x \in T_pM$. So the former theorem may also be stated as "$y = f^\sharp$ exists". Recall this is given by viewing the inner product as a nonsingular matrix.

3.4 Adjoint operator

Let $A : X \to X$ be a linear transformation in a finite-dimensional inner product space X. For a given $y \in X$, define the linear functional

$$\begin{aligned} f^y : X &\to F \\ x &\mapsto \langle Ax, y \rangle \end{aligned}$$

which, by the Riesz representation theorem yields a unique $z \in X$ such that

$$f^y(x) = \langle x, z \rangle$$

Then the ***adjoint of*** A is the linear map

$$\begin{aligned} A^* : X &\to X \\ y &\mapsto z \end{aligned}$$

so that $\langle Ax, y \rangle = \langle x, A^* y \rangle$.

Proposition 3.6 (Properties of the adjoint)**.**

1. $(\alpha A)^* = \overline{\alpha} A^*$.
2. $(A + B)^* = A^* + B^*$.
3. $(AB)^* = B^* A^*$.
4. $(A^*)^* = A$.

If $A = A^*$ we say A is ***self-adjoint***, and if $AA^* = A^*A$ we say A is ***normal***.

Theorem 3.7. If A is self-adjoint, its eigenvalues are real. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

Theorem 3.8. If M is an invariant subspace of X under A, then $M^\perp$ is invariant under A^*.

Theorem 3.9. If A is a linear transformation on a finite-dimensional inner product space X, then $\mathrm{range}(A)^\perp = \mathrm{null}(A^*)$.

3.5 Spectral theorem for normal transformations

Theorem 3.10. Let A be a self-adjoint transformation in a finite-dimensional inner product space X. Then there exists an orthonormal basis of X consistinf of eigenvectors of A.

Lemma 3.11. Let A be a normal transformation in a finite-dimensional inner product space X. Then $\|Ax\| = \|A^*x\|$ for all $x \in X$.

Theorem 3.12. Let A be a normal transformation in a complex finite-dimensional inner product space X. Then there exists an orthonormal basis of X consistinf of eigenvectors of A.

Theorem 3.13. If A is a normal transformation on a finite-dimensional inner product space. Eigenvectors associated to distinct eigenvalues of a self-adjoint operator are orthogonal.

Recall that the notation $X = M_1 \oplus \ldots \oplus M_k$ means that X is the ***direct sum*** of the M_i, which means that $X = M_1 + \ldots + M_k$ and $M_i \cap \{M_1 + \ldots \hat{M}_i + \ldots + M_k\} = \{0\}$, (every element in X is expressed as a unique sum of elements in M_i). If $M_i \perp M_j$ for all $i \neq j$, we say this is an ***orthogonal direct sum decomoposition of*** X, and the ***orthogonal projection to*** M_j is just taking the corresponding component of a given element in its decomposition.

Theorem 3.14 (Spectral decomposition theorem for normal transformations). To every normal transformation A on a complex finite-dimensional inner product space there correspond scalar $\lambda_1, \ldots, \lambda_k$, the distinct eigenvalues of A, and orthogonal projections $E_1, \ldots, E_k$ with $k \leq \dim X$, such that

1. E_i is the orthogonal projection on $\mathrm{Null}(A - \lambda_i)$ for $i = 1, \ldots, k$.
2. $E_i \neq 0$ and $E_i E_j = 0$ for $i, j = 1, \ldots, k$.
3. $\sum_{j=1}^{k} E_j = 1$.
4. $\sum_{j=1}^{k} \lambda_j E_j = A$.

If A was self-adjoint, we could weaken the hypotheses to a real inner product space.

3.6 Unitary and orthogonal transformations

Let X be a finite-dimensional inner product space, and $U : X \to X$ a linear transformation with $U^*U = 1$. We say U is ***unitary*** if X is complex and ***orthogonal*** if X is real. The condition $U^*U = 1$ implies that $UU^* = 1$.

Theorem 3.15. Let X be a finite-dimensional inner product space, and $U : X \to X$ a linear transformation. The following statements are equivalent:

1. $U^*U = 1$.
2. $\langle Ux, Uy \rangle = \langle x, y \rangle$.

3. $\|Ux\| = \|x\|$ for all $x \in X$.

Theorem 3.16. If U is a unitary transformation on the finite-dimensional inner product space X, then each of the eigenvalues of U must have an absolute value equal to 1.

To summarize:

Theorem 3.17. Let A be a normal transformation on a complex finite-dimensional inner product space. Then

1. A is self-adjoint is and only if each eigenvalue of A is real.
2. A is unitary if and only if each eigenvalue of A has absolute value equal to 1.

3.7 Normed spaces

Let X be a real or complex vector space. A ***norm*** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda|\|x\|$ for all $x \in X$ and $\lambda \in \mathbb{R}$.
3. **(Triangle inequality.)** $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Every normed space is a metric space with the distance function $\rho(x, y) = \|x - y\|$. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are called ***equivalent*** if there exist $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \qquad \forall x \in X$$

Equivalent norms define the same topology and the same Cauchy sequences.

A normed space that is complete is called a ***Banach space***.

Theorem 3.18. For every normed linear space X there is a complete normed linear space X^* such that X is ***congruent*** (isomorphic and isometric) to a dense subset of X^* and the norm on X^* extends the norm on X.

If $\{x_n\}$ is a sequence in X, the series $\sum_{n=1}^{\infty} x_n$ ***converges to*** x if $\sum_{n=1}^{N} \to x$ as $N \to \infty$, and it is ***absolutely convergent*** is $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

Theorem 3.19. A normed vector space X is complete if and only if every absoultely convergent series in X converges.

Examples.

- If X is a topological space, $B(X)$ and $BC(X)$ are Banach spaces with the uniform norm $\|f\|_u = \sup_{x \in X} |f(x)|$.
- If $(X, \mathcal{M}, \mu)$ is a measure space, $L^1(\mu)$ is a Banach space with the norm $\|f\|_1 = \int |f| d\mu$. (Observe that $\|\cdot\|_1$ is only a seminorm if we do not identify functions that are equal a.e.)

If X and Y are normed vector spaces, $X \times Y$ is a normed vector space with the ***product norm***, $\|(x, y)\| = \max(\|x\|, \|y\|)$. If M is a vector subspace of X, the quotient space X/M consisting of equivalence classes under $x \sim y$ iff $x - y \in M$ is a normed space with the ***quotient norm***, $\|x + M\| = \inf_{y \in M} \|x + y\|$.

A linear map $T : X \to Y$ between two normed vector spaces is ***bounded*** if there exists $C \geq 0$ such that

$$\|Tx\| \leq C\|x\| \qquad \forall x \in X$$

Proposition 3.20. If X and Y are normed vector spaces and $T : X \to Y$ is a linear map, then T is continuous if and only if it is bounded.

Proof. ($\Longrightarrow$) There exists $\delta > 0$ such that $\|x\| \leq \delta$ implies $\|Tx\| \leq 1$. For any nonzero $x \in X$,

$$\|Tx\| = \left\| \frac{\|x\|}{\delta} T\left(\delta \frac{x}{\|x\|}\right) \right\| \leq \frac{1}{\delta}\|x\|.$$

($\Longleftarrow$) If $\|x - y\| < \frac{\varepsilon}{C}$,

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq C\|x - y\| < \varepsilon.$$

□

In fact, if T is bounded it is uniformly continuous and even Lipschitz continuous.

We denote by $L(X, Y)$ the space of bounded linear maps from X to Y, which is a normed vector space with the ***operator norm***

$$\begin{aligned} \|T\| &= \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\| \\ &= \sup\{\|Tx\| : \|x\| = 1\} \\ &= \sup\left\{\frac{\|Tx\|}{\|x\|} : x \neq 0\right\} \\ &= \inf\{C : \|Tx\| \leq C\|x\| \text{ for all } x \in X\} \end{aligned}$$

Proposition 3.21. If Y is complete, so is $L(X, Y)$.

Proof. If $\{T_n\}$ is a Cauchy sequence in $L(X, Y)$, the sequence $\|T_n x\|$ is Cauchy in Y since $\|T_n x - T_m x\| \leq \|T_n - T_m\| \|x\|$. Define pointwise $Tx = \lim_{n \to \infty} T_n x$. □

T is ***invertible*** if it bijective and T^{-1} is bounded. It is called an isometry if $\|Tx\| = \|x\|$ for all $x \in X$. An isometry is injective but not necessarily surjective.

If X is a vector space over $K = \mathbb{R}, \mathbb{C}$, a ***linear functional***. is a linear map from X to K.

Proposition 3.22 (Relationship between real and complex linear functionals)**.** Let X be a vector space over $\mathbb{C}$. If f is a complex linear functional on X, $u := \operatorname{Re} x$ is a real linear functional and $f(x) = u(x) - iu(ix)$. Conversely, if u is a real functional on X, then $f(x) := u(x) - iu(ix)$ is a complex linear functional, and if X is normed, $\|u\| = \|f\|$.

4 Hahn-Banach theorems

4.1 Analytic form of the Hahn-Banach theorem

It is not obvious that there are any nonzero bounded linear functionals on an arbitrary normed vector space. If E is a real vector space, ***sublinear*** or ***Minkoswky functional*** on E is a map $p : E \to \mathbb{R}$ such that

$$p(\lambda x) = \lambda p(x) \qquad \forall x \in X \text{ and } \lambda \geq 0 \tag{4.1}$$

$$p(x + y) \leq p(x) + p(y) \qquad \forall x, y \in X \tag{4.2}$$

Theorem 4.1 (Helly-Hahn-Banach). Let E be a vector space over $\mathbb{R}$ and $p : E \to \mathbb{R}$ a sublinear functional. If $G \subseteq E$ is a linear subspace and $g : G \to \mathbb{R}$ is a linear functional such that

$$g(x) \leq p(x) \quad \forall x \in E,$$

then there exists a linear functional f defined on all of E that extends g, that is, $g(x) = f(x)\ \forall x \in G$ and such that

$$f(x) \leq p(x) \quad \forall x \in E.$$

For a proof first recall that a partial order P is ***inductive*** if every totally ordered subset Q in P has an upper bound, and that

Lemma 4.2 (Zorn)**.** Every nonempty order set that is inductive has a maximal element.

Proof. (Of theorem 4.1) Consider the set

$$P = \left\{ h : D(h) \subseteq E \to \mathbb{R} : \begin{smallmatrix} D(h) \text{ is a linear subspace of } E, \\ h \text{ is linear}, G \subseteq D(h), \\ h \text{ extends } g, \text{ and } h(x) \leq p(x)\ \forall x \in D(h) \end{smallmatrix} \right\}$$

Then P is a partial order with

$$h_1 \leq h_2 \iff D(h_1) \subseteq D(h_2) \text{ and } h_2 \text{ extends } h_1$$

P is nonempty since $g \in P$. To show it is inductive, take $Q \subseteq P$ a partially order subset and write $Q = (h_i)_{i \in I}$. Then define

$$D(h) = \bigcup_{i \in I} D(h_i), \qquad h(x) = h_i(x) \quad \text{if } x \in D(h_i) \text{ for some } i \in I$$

which is an upper bound of Q, so that there is a maximal element f in P by Zorn's Lemma. To finish it suffices to show that $D(f) = E$.

For a contradiction suppose that $D(f) \neq E$ and choose $x_0 \notin D(f)$. We shall construct a function $h \in P$ such that $f < h$. Define $D(h) = D(f) + \mathbb{R}x_0$ and, for every $x \in D(f)$, set

$$h(x + \lambda x_0) = f(x) + t\alpha \quad \forall \lambda \in \mathbb{R}$$

where α is a constant that we choose as follows. We must ensure that

$$h(x + \lambda x_0) = f(x) + \lambda\alpha \leq p(x + tx_0) \quad \forall x \in D(f) \quad \text{and} \quad \forall \lambda \in \mathbb{R}$$

For any $x, y \in D(f)$,

$$\begin{aligned} f(x) + f(y) = f(x+y) \leq p(x+y) &\leq p(x + x_0) + p(y - x_0) \\ \implies f(x) - p(y - x_0) &\leq p(x + x_0) - f(y) \end{aligned}$$

So let α satisfy

$$\sup_{y \in D(f)} \{f(y) - p(y - x_0)\} \leq \alpha \leq \inf_{x \in D(f)} \{p(x + x_0) - f(x)\}$$

If $\lambda = 0$, then $h(x) = f(x) \leq p(x)$. If $\lambda \neq 0$ we must be careful since sublinear functionals only satisfy eq. (4.1) for positve scalars.

If $\lambda > 0$, then

$$\begin{aligned} h(x + \lambda x_0) &= \lambda \cdot h(x/\lambda + x_0) \\ &= \lambda \cdot (f(x/\lambda) + \alpha)) \\ &\leq \lambda \cdot (f(x/\lambda) + p(x/\lambda + x_0) - f(x/\lambda)) \\ &\leq p(x + \lambda x_0) \end{aligned}$$

and if $\lambda = -\mu < 0$,

$$\begin{aligned} h(x + tx_0) &= (-\lambda) \cdot h(-x/\lambda - x_0) \\ &= \mu \cdot (f(x/\mu) - \alpha)) \\ &\leq \mu \cdot (f(x/\mu) - f(x/\mu) + p(x/\mu + x_0)) \\ &\leq p(x - \mu x_0) \\ &= p(x + \lambda x_0). \end{aligned}$$

Then $h \in P$, h extends f and $D(f) \subsetneq D(h)$, which is impossible since f is maximal. $\square$

An example of a sublinear functional is a ***seminorm***, which is a function that satisfies the first two conditions of being norm, but the third. If p is a seminorm,

Theorem 4.3 (Complex Hahn-Banach). Let E be a complex vector space, p a seminorm on E, $G \subseteq E$ a subspace of E and f a complex linear functional on G such that $|g(x)| \leq p(x)$ $\forall x \in G$. Then there exists a complex linear functional f defined on all of E that extends g and $|f(x)| \leq p(x)$.

The space of al continuous linear functionals on E, denoted by $E^* := L(E, K)$, is called the ***dual space*** of E. Since K is complete, X^* is complete with the operator norm. We denote the ***dual norm*** (it remains for me to check wether this is the same as the operator norm) by

$$\|f\|_{E^*} = \sup_{\substack{x \in E \\ \|x\| \leq 1}} |f(x)| = \sup_{\substack{x \in E \\ \|x\| \leq 1}} f(x)$$

Given $f \in E^*$ and $x \in E$ we shall often write $\langle f, x \rangle$ instead of $f(x)$ and call $\langle -, - \rangle$ the ***scalar product for the duality*** E^*, E.

Corollary 4.4. Let $G \subseteq E$ be a linear subspace. If $g : G \to \mathbb{R}$ is a continuos linear functional, then there exists $f \in E^*$ that extends g and such that

$$\|f\|_{E^*} = \sup_{\substack{x \in G \\ \|x\| \leq 1}} = \|g\|_{G^*}$$

Proof. $g(x) \leq p(x) := \|g\|_{G^*}\|x\|$, which is a sublinear functional. □

Corollary 4.5. For every $x_0 \in E$ there exists $f_0 \in E^*$ such that

$$\|f_0\| = \|x_0\| \quad \text{and} \quad \langle f_0, x_0 \rangle = \|x_0\|^2$$

Proof. By corollary 4.4 with $G = \mathbb{R}x_0$ and $g(tx_0) = t\|x_0\|^2$ so that $\|g\|_{G^*} = \|x_0\|$. □

While the element f_0 in corollary 4.5 is in general not unique (excercise), a sufficient condition is that E^* is ***strictly convex***, that is, $\|tx + (1-t)y\| < t \ \forall t \in (0, 1), \forall x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Any Hilbert space and any $L^p(\Omega)$ for $1 < p < \infty$ are.

The (multivalued) ***duality map*** for $x_0 \in E$ is

$$F(x_0) = \left\{ f_0 \in E^* : \|f_0\| = \|x_0\| \text{ and } \langle f, x_0 \rangle = \|x_0\|^2 \right\}$$

Corollary 4.6. For every $x \in E$ we have

$$\|x\| = \sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| = \max_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle|$$

Proof. Assuming $x \neq 0$, if $\|f\| \leq 1$, $\|\langle f, x \rangle\| \leq \|f\|\|x\| \leq \|x\|$, so

$$\sup_{\substack{f \in E^* \\ \|f\| \leq 1}} |\langle f, x \rangle| \leq \|x\|.$$

On the other hand, by corollary 4.5 there is some $f_0 \in E^*$ such that $\|f_0\| = \|x\|$ and $\langle f_0, x \rangle = \|x\|^2$. Set $f_1 = f_0/\|x\|$ so that $\|f_1\| = 1$ and $\langle f_1, x \rangle = \|x\|$. □

4.2 Geometric forms of the Hahn-Banach theorem

An ***affine hyperplane*** is a subset H of E of the form

$$H = \{x \in E : f(x) = \alpha\}$$

where if f is a nonzero linear functional not necesarily continuous and α a is a real constant. We also write $H = [f = \alpha]$.

Proposition 4.7. The hyperplane $H = [f = \alpha]$ is closed if and only if f is continuous.

Proof. It is clear that if f is continuous then H is closed. Conversely... $\square$

Let A and B be two subsets of E. We say that the hyperplane $H = [f = \alpha]$ ***separates*** A and B if

$$f(x) \leq \alpha \quad \forall x \in A \qquad \text{and} \qquad f(x) \geq \alpha \quad \forall x \in B$$

We say H ***strictly separates*** A and B if there is some $\varepsilon > 0$ such that

$$f(x) \leq \alpha - \varepsilon \quad \forall x \in A \qquad \text{and} \qquad f(x) \geq \alpha + \varepsilon \quad \forall x \in B$$

A subset $A \subseteq E$ is ***convex*** if

$$tx + (1-t)y \in A \quad \forall x, y \in A,\ \forall t \in [0,1].$$

Theorem 4.8 (Hahn-Banach, first geometric form). Let $A \subset E$ and $B \subset E$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume that one of them is open. Then there exists a closed hyperplane that separates A and B.

Lemma 4.9. Let $C \subset E$ be an open convex set with $0 \in C$. For every $x \in E$ set

$$p(x) = \inf\{\alpha > 0 : \alpha^{-1}x \in X\}$$

which is called the ***Minkowski functional of*** C or the ***gauge of*** C. Then p is a sublinear functional, that is, satisfies eqs. (4.1) and (4.2), and also

1. there is a constant M such that $0 \leq p(x) \leq M\|x\|\ \forall x \in E$.
2. $C = \{x \in E : p(x) < 1\}$. (In particular, if C is the unit ball centered at 0, then p is just the norm.)

Lemma 4.10. Let $C \subset E$ be a nonempty open convex set and let $x_0 \in E$ with $x_0 \notin C$. Then there exists $f \in E^*$ such that $f(x) < f(x_0)\ \forall x \in C$. In particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C.

Proof. After a translation we may assume that $0 \in C$ and introduce the gauge p of C. For the linear subspace $G = \mathbb{R}x_0$ and the linear functional $g : G \to \mathbb{R}$ defined by $g(t_0) = t$ $(t \in \mathbb{R})$ we have that $g(x) \leq p(x)$ $(\forall x \in G)$. We may thus apply the Helly-Hahn-Banach theorem. $\square$

Proof. (Of theorem 4.8). Set $C = A - B$ so that C is convex (check!). C is also open (since $C = \bigcup_{y \in B}(A - y)$) and $0 \notin C$ (because $A \cap B = \varnothing$). By lemma 4.10 there exists $f \in E^*$ such that

$$f(z) \leq 0 \quad \forall z \in C \qquad \iff \qquad f(x) < f(y) \quad \forall x \in A \,\forall y \in B$$

Fixing a constant α satisfying

$$\sup_{x \in A} f(x) \leq \alpha \leq \inf_{y \in B} f(y)$$

we conclude that the hyperplane $[f = \alpha]$ separates A and B. □

Theorem 4.11 (Hahn-Banach, second geometric form). Let $A \subset E$ and $B \subset E$ be two nonempty convex subsets such that $A \cap B = \varnothing$. Assume that A is closed and B is compact. Then there exists a closed hyperplane that strictly separates A and B.

Proof. Set $C = A - B$, so that C is convex, closed (check!) and $0 \notin C$. Hence there is some $r > 0$ such that $B(0, r) \cap C = \varnothing$. By theorem 4.8 there is a hyperplane that separates $B(0, r)$ and C. Therefore, there is some $f \in E^*$, $f \not\equiv 0$ such that

$$f(x - y) \leq f(rz) \quad \forall x \in A, \quad \forall y \in B \quad \forall z \in B(0, 1)$$

(Incomplete.) □

In infinite-dimensional vector spaces it is in general impossible to separate any two nonempty disjoint convex sets. In finite-dimensional vector spaces, however, it is always possible.

Corollary 4.12. Let $F \subset E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^*$, $f \not\equiv 0$ such that

$$\langle f, x \rangle = 0 \quad \forall x \in F$$

Proof. Let $x_0 \in E$ with $x_0 \notin \overline{E}$. Using theorem 4.11 with $A = \overline{F}$ and $B = \{x_0\}$ we find a closed hyperplane $[f = \alpha]$ that strictly separates $\overline{F}$ and $\{x_0\}$. We thus have

$$\langle f, x \rangle < \alpha \langle f, x_0 \rangle \quad \forall x \in F$$

In particular $\langle f, x \rangle = 0 \,\forall x \in F$ since $\lambda \langle f, x \rangle < \alpha \,\forall \lambda \in \mathbb{R}$. □

By this corollary, we may prove that a linear subspace $F \subset E$ is dense by showing that every continuous linear functional that vanishes on F must vanish in all of E.

4.3 The bidual E^{**}. Orthogonality relations.

Let E be a normed vector space. Recall the norm on its dual space E^* is

$$\|f\|_{E^*} = \sup_{\substack{x \in E \\ \|f\| \leq 1}} |\langle f, x\rangle|.$$

The ***bidual*** space of E is the dual of E^* with norm

$$\|\xi\|_{E^{**}} = \sup_{\substack{f \in E^* \\ Vert f\| \leq 1}} |\langle \xi, f\rangle| \quad \xi \in E^{**}$$

There is a ***canonical injection*** $J : E \to E^{**}$ defined as follows: given $x \in E$, the map $f \mapsto \langle f, x\rangle$ is a continuous linear functional on E^*; thus it is an element of E^{**} which we denote by Jx. We have

$$\langle Jx, f\rangle_{E^{**}, E^*} = \langle f, x\rangle_{E^*, E} \quad \forall x \in x,\ \forall f \in E^{**}.$$

It is clear that J is linear and that J is an isometry, that is, $\|Jx\|_{E^{**}} = \|x\|_E$. Indeed, by corollary 4.6 we have

$$\|x\|_{E^{**}} = \sup_{\substack{f \in E^* \\ Vert f\| \leq 1}} |\langle Jx, f\rangle| = \sup_{\substack{f \in E^* \\ Vert f\| \leq 1}} |\langle f, x\rangle| = \|x\|$$

It may happen that J is not surjective from E onto E^{**}. However, it is convenient to identify E with a subspace of E^{**} using J. If J turns out to be surjective we say E is ***reflexive*** and E^{**} is identified with E. and if $N \subset E^*$ is a linear subspace we set

$$N^{\perp} = \{x \in R : \langle f, x\rangle \ \forall f \in X\}$$

If $M \subset E$ is a linear subspace, we set

$$M^{\perp} = \{f \in E^* : \langle f, x\rangle = 0 \ \forall x \in M\}.$$

Notice that $N^{\perp}$ is a subset of E rather than of E^{**}. Both $M^{\perp}$ and $N^{\perp}$ a closed linear suspaces. We say $M^{\perp}$ (resp. N) is the ***space orthogonoal to*** M ***(resp.*** N***)***.

Proposition 4.13. Let $M \subset E$ be a linear subspace. Then

$$(M^{\perp})^{\perp} = \overline{M}.$$

Let $N \subset E^*$ be a linear subspace. Then

$$(N^{\perp})^{\perp} \supset \overline{N}.$$

5 The Uniform Boundedness Principle and the Closed Graph Theorem

5.1 The Baire Category Theorem

This theorem states that, in a complete metric space, a countable union of closed sets with empty interior has empty interior:

Theorem 5.1 (Baire). Let X be a complete metric space and let $(X_n)_{n\geq 1}$ be a sequence of closed subsets in X. Assume that

$$\operatorname{Int} X_n = \varnothing \text{ for every } n \geq 1$$

Then

$$\operatorname{Int}\left(\bigcup_{n=1}^{\infty} X_n\right) = \varnothing$$

And there's also this other form of the same theorem:

Theorem 5.2 (Baire). Let X be a nonempty complete metric space. Let $(X_n)_{n\geq 1}$ be a sequence of closed subsets such that

$$\bigcup_{n=1}^{\infty} X_n = X.$$

Then there exists some n_0 such that $\operatorname{Int} X_{n_0} \neq \varnothing$.

Remark 5.1. A subset $A \subset X$ has empty interior if and only if its complement A^c is dense in X. Indeed, to see A^c is dense in X notice every point $x \in X$ has a neighbourhood intersecting A^c: since A has empty interior, every open set containing $x \in A$ must intersect A^c. Conversely, if every point $x \in X$ has a neighbourhood intersecting A^c, there cannot be a point in the interior of A.

Proof. (Of the first form.) Set $O_n = X_n^c$ so that O_n is open and dense in X for every $n \geq 1$. To show $\bigcup_{n=1}^{\infty} X_n$ has empty interior, it suffices to show that its complement $G = \bigcap_{n=1}^{\infty} O_n$ is dense in X. Succintly, the procedure is to construct a sequence of a sequence of closed balls inside any given open set ω of X; one ball for every O_n. This produces a Cauchy sequence, whose limit is a point in $\omega \cap G$. □

5.2 The Uniform Boundedness Principle

Let E and F be two normed vector spaces. Recall that the space of continuous (=linear) operators from E into F is denoted by $\mathcal{L}(E, F)$ and is equipped with the norm

$$\|T\|_{\mathcal{L}(E,F)} = \sup_{\substack{x\in E \\ \|x\|\leq 1}} \|Tx\| = \inf_{\substack{x\in E \\ \|Tx\|\leq c\|x\|}} c.$$

Theorem 5.3 (Banach-Steinhaus, uniform boundedness principle). Let E and F be two Banach spaces and let $(T_i)_{i\in I}$ be a family (not necessarily countable) of continuous linear operators from E to F. Assume that

$$\sup_{i\in I} \|T_i x\| < \infty \quad \forall x \in X \tag{5.1}$$

Then

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} < \infty.$$

In other words, there exists a constant c such that

$$\|T_i x\| \le c\|x\| \quad \forall x \in E\, \forall i \in I$$

Remark 5.2. What's remarkable about this theorem is that from *pointwise* (the norm of all the $T_i x$ is bounded at every x) estimates we obtain a *global (uniform)* estimate: there is a unique bound for all the T_i.

Proof. For every $n \ge 1$, let

$$X_n = \{x \in E : \forall i \in I,\ \|T_i\| \le n\}$$

which is closed (why?) and by eq. (5.1) we have

$$\bigcup_{n=1}^{\infty} X_n = E.$$

It follows from the Baire category theorem that there is some $n_0 \ge 1$ such that $\operatorname{Int}(X_0) \ne \varnothing$. Pick a ball $B(x_0, r) \subset X_{n_0}$, so that for every $z \in B(0,1)$ and $i \in I$,

$$\begin{aligned}\|T_i(z)\| &= r^{-1}\|T(x_0 + rz) - T(x_0)\| \\ &\le r^{-1}(n_0 + n_0).\end{aligned}$$

So

$$\sup_{i\in I} \|T_i\|_{\mathcal{L}(E,F)} = \sup_{i \in I}\left(\sup_{z\in B(0,1)} \|T_i z\|_F\right) \le r^{-1}(n_0 + n_0) < \infty.$$

□

Remark 5.3. Recall that the pointwise limit of continuous functions need not be continuous. The former theorem dos not imply that $\|T_n - T\|_{\mathcal{L}(E,F)}$.

Corollary 5.4. Let E and F de two Banach spaces. Let (T_n) be a sequence of continuous linear operators from E to F such that for every $x \in E$, $T_n x$ converges to a limit denoted by Tx. Then

1. $\sup_n \|T_n\|_{\mathcal{L}(E,F)} < \infty$ (uniform boundedness principle),
2. $T \in \mathcal{L}(E,F)$,
3. $\|T\|_{\mathcal{L}(E,F)} \le \liminf_{n\to\infty} \|T_n\|_{\mathcal{L}(E,F)}$. (Contrary, I think, to $\lim_{n\to\infty} \|T_n\| = \|T\|$ if convergence was uniform).

Corollary 5.5. Let G be a Banach space and let B be a subset of G. Assume that

$$\text{for every } f \in G^* \text{ the set } f(B) = \{\langle f, x\rangle : x \in B\} \text{ is bounded} \tag{5.2}$$

then

$$B \text{ is bounded.}$$

Proof. Define a family of linear operators on G^* indexed by B as

$$T_b f = \langle f, b\rangle.$$

By eq. (5.2),

$$\sup_{b\in B} |T_b(f) < \infty \quad \forall f \in E$$

so that by the uniform boundedness theorem there exists a constant c such that

$$|\langle f, b\rangle| \leq c\|f\| \quad \forall f \in G^* \; \forall b \in B$$

and by corollary 4.6,

$$\|b\| = \sup_{\substack{f\in G^* \\ \|f\|\leq 1}} |\langle f, b\rangle \leq c$$

□

Remark 5.4. To prove that a set B is bounded it suffices to *look* at B through the bounded linear functionals.

Corollary 5.6. Let G be a Banach space and let B^* be a subset of G^*. Assume that

$$\text{for every } x \in G \text{ the set } \langle B^*, x\rangle = \{\langle f, x\rangle : f \in B^*\} \text{ is bounded} \tag{5.3}$$

then

$$B^* \text{ is bounded.}$$

Proof. Define for every $b \in B^*$

$$T_b(x) = \langle b, x\rangle \quad \forall x \in G$$

so that there is a constant c such that

$$|\langle b, x\rangle| \leq c\|x\| \quad \forall b \in B^*, \forall x \in G.$$

And this time by definition of operator norm we have

$$\|b\| = \sup_{\substack{x\in G \\ \|x\|\leq 1}} |\langle b, x\rangle| \leq c \quad \forall b \in B^*.$$

□

Remark 5.5. In these last two proofs it has become obvious how the norm of a vector can be described in terms of operators, and, dually, the norm of an operator can be described in terms of points. Recall corollary 4.6 is a consequence of the Hahn-Banach theorem.

5.3 The Open Mapping Theorem and the Closed Graph Theorem

Here are two basic results due to Banach.

Theorem 5.7 (Open mapping theorem). *A surjective continuous linear operator between two Banach spaces is an open map.*

Let E and F be two Banach spaces and T be a linear operator from E into F that is surjective. Then there exists a constant $c > 0$ such that

$$T(B_E(0,1)) \supset B_F(0,c) \tag{5.4}$$

Remark 5.6. First let us make sure that eq. (5.4) implies that T is open. Let U be open in E and fix any point $y_0 \in T(U)$, so that there is some $x_0 \in U$ such that $Tx_0 = y_0$. Choose a ball $x_0 + B(0,r) = B(x_0,r) \subset U$. Then

$$\begin{aligned} & y_0 + T(B(0,r)) \subset T(U) \\ \implies \quad & T(B(0,n)) \subset -y_0 + T(U) \\ \implies \quad & r^{-1}T(B(0,1)) \subset -r^{-1}y_0 + r^{-1}T(U). \end{aligned}$$

And by eq. (5.4) we obtain

$$\begin{aligned} & r^{-1}B_F(0,c) \subset -r^{-1}y_0 + r^{-1}T(U) \\ \implies \quad & B(0,rc) \subset -y_0 + T(U) \\ \implies \quad & B(y_0,rc) \subset T(U). \end{aligned}$$

Corollary 5.8. Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is bijective. Then T^{-1} is also continuous.

Proof of corollary 5.8. Since T is injective, choosing a vector $Tx \in B_F(0,c)$ implies its only preimage x is in $B(0,1)$, so $\|x\| < 1$. By homogeneity,

$$\|x\| \leq \frac{1}{c}\|Tx\|$$

so that T^{-1} is continuous. □

Corollary 5.9. Let E be a vector space provided with two norms $\| \ \|_1$ and $\| \ \|_2$. Assume that E is a Banach space for *both* norms and that there exists a constant $C \geq 0$ such that

$$\|x\|_2 \leq C\|x\|_1 \quad \forall x \in E.$$

Then the two norms are equivalent, that is, there is a constant $c > 0$ such that

$$\|x\|_1 \leq c\|x\|_2 \quad \forall x \in E.$$

(That is, for both inequalities

$$c^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in E)$$

to hold it suffices that the inequality on the right side to holds.)

Proof of corollary 5.9. By corollary 5.8 with $E = (E, \| \|_1)$, $F = (E, \| \|_2)$ and $T = \mathrm{Id}$. (That is, the two norms induce the same topology?) □

Proof of theorem 5.7.

Claim 5.1. Assume that T is a linear surjective operator from E onto F. Then there exists a constant $c > 0$ such that

$$\overline{T(B(0,1))} \supset B(0,2c). \tag{5.5}$$

Proof of claim 5.1. [Uses Baire category theorem.] □

Claim 5.2. Assume that T is a continuous linear operator from E into F that satisfies eq. (5.5). Then

$$T(B(0,1)) \supset B(0,c). \tag{5.6}$$

Proof of claim 5.2. [Uses Cauchy sequences.] (Compare eqs. (5.4) and (5.6).) □

□

Theorem 5.10 (Closed graph theorem). Let E and F be two Banach spaces. Let T be a linear operator from E into F. Assume that the graph of T, denoted by $G(T)$, is closed in $E \times F$. Then T is continuous.

Remark 5.7. The converse is true: the graph of any continuous map is closed.

Proof. Consider, on E the two norms

$$\|x\|_1 = \|x\|_E + \|x\|_F \quad \text{and} \quad \|x\|_2 = \|x\|_E$$

($\| \|_1$ is called the ***graph norm***).

Then prove these two norms are equivalent. (Check details.)

It is easy to check that E is a Banach space with $\| \|_1$. Since it is also a Banach space with $\| \|_2$, it follows from corollary 5.9 that the two norms are equivalent and thus there exists a constant $c > 0$ such that $\|x\|_1 \leq c\|x\|_2$. We conclude that $\|Tx\|_F \leq c\|x\|_E$. □

5.4 An introduction to Unbounded Linear Operators. Definition of the Adjoint