

# algebraic topology exercises

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## Homework 1

### 0 Preliminaries

In the category of sets there is a bijection  $\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, \text{Hom}(Y, Z))$  that depends naturally on  $X$ ,  $Y$  and  $Z$ . The notions related to this bijection are “Cartesian closed category”, “currying” and “internal Hom”.

**Definition.** A category  $\mathcal{C}$  is *Cartesian closed* if:

1.  $\mathcal{C}$  has all finite products (Caveat: some require that  $\mathcal{C}$  has all finite limits)
2. For any object  $Y$  the functor  $- \times Y$  has a right adjoint, which we will denote by  $\text{Map}(Y, -)$  or by  $-^Y$ .

**Remark.** By section 3 [here](#), the second property above implies that we get a functor  $\text{Map}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ , and moreover we get natural isomorphisms  $\text{Hom}(X, \text{Map}(Y, Z)) \cong \text{Hom}(X \times Y, Z)$  and  $\text{Map}(X, \text{Map}(Y, Z)) \cong \text{Map}(X \times Y, Z)$ .

**Lemma (Yoneda, [wiki](#)).** Let  $F$  be a functor from a locally small category  $\mathcal{C}$  to  $\text{Set}$ . Then for each object  $X$  of  $\mathcal{C}$ , the natural transformations  $\text{Nat}(\text{Hom}(X, -), F)$  are in one-to-one correspondence with the elements of  $F(X)$ , that is

$$\text{Nat}(\text{Hom}(X, -), F) \cong F(X)$$

Moreover, this isomorphism is natural in  $X$  and  $F$  when both sides are regarded as functors from  $\mathcal{C} \times \text{Set}^{\mathcal{C}}$  to  $\text{Set}$ . ( $\text{Set}^{\mathcal{C}}$  denotes the category of functors from  $\mathcal{C}$  to  $\text{Set}$ .)

There is a contravariant version of Yoneda lemma asserting that if  $F$  is a contravariant functor from  $\mathcal{C}$  to  $\mathbf{Set}$ ,

$$\mathbf{Nat}(\mathbf{Hom}(-, X), F) \cong F(X).$$

**Corollary.**  $\mathbf{Nat}(\mathbf{Hom}(-, X), \mathbf{Hom}(-, Y)) \cong \mathbf{Hom}(X, Y)$ .

**Remark.** The correspondence  $X \mapsto \mathbf{Hom}(-, X)$  is fully faithful, that is, the correspondence  $\mathbf{Hom}(X, X') \rightarrow \mathbf{Nat}(\mathbf{Hom}(-, X), \mathbf{Hom}(-, X'))$  is injective and bijective.

**Exercise (a).** Let  $\mathcal{C}$  be any category. Show that if for some objects  $X$  and  $X'$  we have  $\mathbf{Hom}(X, Y) \cong \mathbf{Hom}(X', Y)$  for all objects  $Y$ , with isomorphisms being natural in  $Y$ , then  $X \cong X'$ . Dually, if  $\mathbf{Hom}(Y, X) \cong \mathbf{Hom}(Y, X')$  naturally in  $Y$ , then also  $X \cong X'$ .

*Solution.* The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude  $X \cong X'$ . The dual statement follows from the analogue formulation of Yoneda lemma.  $\square$

**Exercise.** Let  $\mathcal{C}$  be a Cartesian closed category and  $\mathbf{pt}$  be the terminal object. Show that for any object  $X$  we have  $X \cong \mathbf{Map}(\mathbf{pt}, X)$ .

*Solution.* Using item (a) with  $X$  and  $X' = \mathbf{Map}(\mathbf{pt}, X)$ , it suffices to show that

$$\mathbf{Hom}(Y, X) \cong \mathbf{Hom}(Y, \mathbf{Map}(\mathbf{pt}, X))$$

for all objects  $Y$  and isomorphisms natural in  $Y$ .

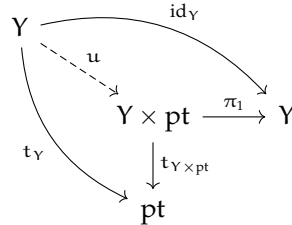
Since  $\mathcal{C}$  is Cartesian closed, we have isomorphisms **natural** in  $Y$

$$\mathbf{Hom}(Y, \mathbf{Map}(\mathbf{pt}, X)) \cong \mathbf{Hom}(Y \times \mathbf{pt}, X) \cong \mathbf{Hom}(Y, X)$$

since  $\mathbf{pt}$  is a terminal object. Indeed:

**Claim.** In a Cartesian closed category  $\mathcal{C}$  with terminal object  $\mathbf{pt}$ , we have that  $Y \times \mathbf{pt} \cong Y$  for any object  $Y$ .

*Proof of claim.* (**From StackExchange**) The universal property of the product  $Y \times \mathbf{pt}$  shows that the maps  $\mathbf{id}_Y$  and  $t_Y : Y \rightarrow \mathbf{pt}$  must factor through some  $u : Y \rightarrow Y \times \mathbf{pt}$ , making  $\pi_1 \circ u = \mathbf{id}_Y$ .

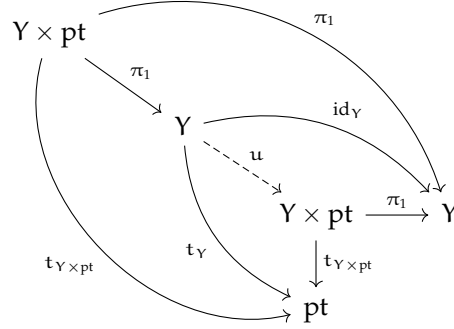


It is also true that  $u \circ \pi_1 = \mathbf{id}_{Y \times \mathbf{pt}}$ , since

- $\pi_1 \circ u \circ \pi_1 = \mathbf{id}_Y \circ \pi_1 = \pi_1$  and

- $t_{Y \times pt} \circ u \circ \pi_1 = t_{Y \times pt}$

so by uniqueness of the universal property we get that  $u \circ \pi_1 = \text{id}_{Y \times pt}$ .



□

□

## 1 Based spaces and smash product

[In progress...]

## 2 Mapping cylinders and Hurewicz cofibrations

**Definition** ([wikipedia](#)). Let  $X$  be a topological space and let  $A \subset X$ . We say that the pair  $(X, A)$  has the *homotopy extension property* if for any space  $Y$ , any homotopy  $g_\bullet : A \rightarrow Y^I$  and any map  $\tilde{g}_0 : X \rightarrow Y$  such that  $\tilde{g}_0 \circ \iota = g_0$ , there exists an *extension* of  $f_\bullet$  to a homotopy  $\tilde{g}_\bullet : X \rightarrow Y^I$  such that  $\tilde{g}_\bullet \circ \iota = g_\bullet$ .

$$\begin{array}{ccc} A & \xrightarrow{g_\bullet} & Y^I \\ \downarrow \iota & \nearrow \tilde{g}_\bullet & \downarrow \pi_0 \\ X & \xrightarrow{\tilde{g}_0} & Y \end{array}$$

A **Hurewicz cofibration** is a map  $\iota : A \rightarrow X$  satisfying the homotopy extension property.

**Exercise (a).** Prove that an inclusion  $f : A \rightarrow X$  is a Hurewicz cofibration if and only if  $A \times I \cup X \times \{0\}$  is a retract of  $X \times I$ .

**Remark.** A little late I noticed the comment on Telegram that we may assume  $A$  to be a closed subspace. Maybe I wouldn't have tried the solution following [Miller](#) if I had knew this earlier, hehe— still it was nice to see two different solutions.

*Solution following [Hatcher](#).* (  $\implies$  ) According to the former definition, choose  $Y = (X \times \{0\}) \cup (A \times I)$ . The inclusion  $A \times I \hookrightarrow Y$  is an homotopy  $g_\bullet$  from  $A$  to  $Y$ . Also, the inclusion

$X \times \{0\} \hookrightarrow Y$  is an extension  $\tilde{g}_0$ . Then there exists an extension  $\tilde{g}_\bullet$  of the whole homotopy, which is just a map from  $X \times I$  to  $Y$ . We have thus produced a retraction:

$$\begin{array}{ccc} (X \times \{0\}) \cup (A \times I) & \xrightarrow{\text{id}} & (X \times \{0\}) \cup (A \times I) = Y \\ \downarrow & \nearrow & \\ X \times I & & \end{array}$$

( $\Leftarrow$ ) Now suppose that  $(X \times \{0\}) \cup (A \times I)$  is a retract of  $X \times I$ . Let  $Y$  be any space,  $g_\bullet : A \rightarrow Y$  an homotopy and  $\tilde{g}_0$  a map such that  $\tilde{g}_0 = g_0 \circ f$ .

The homotopy  $g_\bullet$  along with  $\tilde{g}_0$  yield a map  $\varphi : (A \times I) \cup (X \times \{0\}) \rightarrow Y \cup (X \times \{0\})$ . The key observation is that if  $A$  is closed in  $X$ , then this map is continuous by the [gluing lemma](#). Then we simply compose the given retraction  $r$  with this map to obtain the homotopy extension:

$$\begin{array}{ccc} A \times I & \xrightarrow{g_\bullet} & Y \\ \downarrow & \nearrow \varphi & \\ X \times I & \xrightarrow{r} & (X \times \{0\}) \cup (A \times I) \end{array}$$

A complicated argument in Hatcher's appendix shows that such a function is continuous even without the assumption that  $A$  is closed.  $\square$

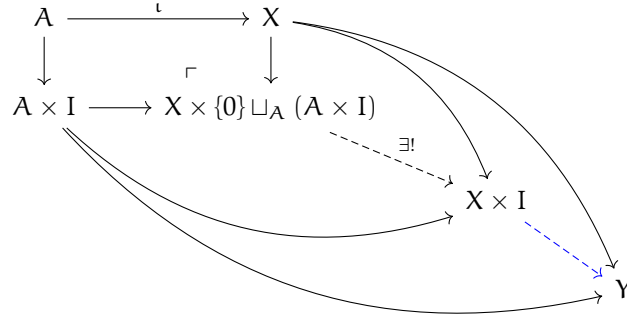
*Solution following Miller.* The homotopy extension property may be defined as a map  $\iota : A \rightarrow X$  such that for any solid-arrow diagram as below, a dotted blue arrow exists making the whole diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & X \times I \\ & \searrow & \downarrow \\ & & Y \end{array}$$

(A dotted blue arrow from  $A \times I$  to  $Y$  completes the commutative diagram.)

Now consider the pushout corresponding to  $\iota$  and the inclusion  $A \rightarrow A \times I$ . By the universal property of the pushout, the former diagram must factor by the pushout, and

we get the following diagram:



The implication  $(\implies)$  of our exercise again follows by choosing  $Y = (X \times \{0\}) \cup (A \times I)$ . For the implication  $(\impliedby)$  **it appears that we have the same problem as before**: we need to construct the blue dashed arrow from the rest of the diagram (using that the black dashed arrow has a left inverse), but it seems that the natural thing to do is defining this function from the two pieces just like before, and we must make sure it is continuous.  $\square$

**Definition.** Let  $f : X \rightarrow Y$  be a map. Let  $M_f = X \times [0, 1] \cup_f Y$  be the *mapping cylinder of  $f$* , i.e. the pushout of  $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$  and of  $f : X \rightarrow Y$ . Let  $g : X \rightarrow M_f$  be the map  $X \xrightarrow{\cong} X \times \{1\} \rightarrow M_f$ . Let  $h : M_f \rightarrow Y$  be the map that is induced by  $X \times [0, 1] \rightarrow Y : (x, t) \mapsto f(x)$  and  $\text{id}_Y : Y \rightarrow Y$ . Observe that  $f$  is the composition of  $g$  and  $h$ .

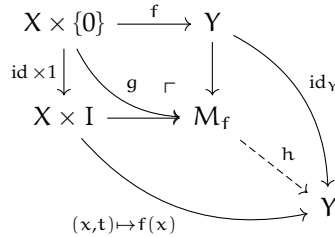
**Remark.** In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e.  $(\text{colim } A_i) \times B$  is canonically homeomorphic with  $\text{colim}(A_i \times B)$ .

### Exercise.

- Show that  $h$  is a deformation retract, and in particular is a homotopy equivalence.
- Show that  $g : X \rightarrow M_f$  is a cofibration. You may use exercise (a), but the direct proof might be simpler.

*Solution.*

- We have that



We must show that there is a homotopy between the identity map on  $M_f$  and a retraction from  $M_f$  to  $Y$ . So we want  $h : M_f \times I \rightarrow M_f$  such that

$$h(-, 0) = \text{id}_{M_f}, \quad \text{img } h(-, 1) \subset Y \quad \text{and} \quad h(-, 1)|_Y = \text{id}_Y$$

Since  $M_f$  is a pullback, we can see it as a colimit, that is

$$M_f = \text{colim}(X \times I \leftarrow X \rightarrow Y)$$

and, since colimits commute with products in CGWH, we get

$$M_f \times I = \text{colim}(X \times I \times I \leftarrow X \times I \rightarrow Y \times I)$$

that is,

$$\begin{array}{ccc} X \times \{0\} \times I & \longrightarrow & Y \times I \\ \downarrow & \lrcorner & \downarrow ? \\ X \times I \times I & \xrightarrow{?} & M_f \times I \end{array} \quad \begin{array}{c} \searrow (y,s) \mapsto y \\ \downarrow \\ \searrow (x,t,s) \mapsto f(x) \end{array} \quad \begin{array}{c} \\ \\ M_f \end{array}$$

[I certainly got stuck in concluding...]

- c. [Also in progress...] Consider the following lifting problem:

$$\begin{array}{ccc} X & \xrightarrow{H} & Z^I \\ g \downarrow & \nearrow & \downarrow \pi_0 \\ M_f & \xrightarrow{h} & Z \end{array}$$

□

### 3 Path spaces and fibrations

#### Exercise.

- a. Show that  $\text{Map}(I, Y)$  deformation retracts on  $\text{Map}(\text{pt}, Y)$ . Most likely you'll have to find a correct map  $I \times I \rightarrow I$ . Also show that  $\text{Map}(I, Y) \rightarrow \text{Map}(\text{pt}, Y)$  is a Hurewicz fibration. The key map will be of the form  $I \times I \rightarrow I \times I$ .

*Solution.*

- a. ( **$\text{Map}(I, Y) \rightarrow \text{Map}(\text{pt}, Y)$  is a Hurewicz fibration.**) Let  $A$  be any space. We must show that for any homotopy  $H$  and lift  $h_0$  there exists an homotopy  $\tilde{H}$  as in the following diagram:

$$\begin{array}{ccc} A \times \{0\} & \xrightarrow{h_0} & \text{Map}(I, Y) \\ \downarrow & \nearrow \tilde{H} & \downarrow p \\ A \times I & \xrightarrow{H} & \text{Map}(0, Y) \end{array}$$

From the isomorphism  $\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$  we may rewrite the problem as

$$\begin{array}{ccc}
 (A \times \{0\}) \times I & & \\
 & \searrow H_0 & \\
 (A \times I) \times I & \xrightarrow{\tilde{H}} & Y \\
 & \nearrow H & \\
 (A \times I) \times \{0\} & & 
 \end{array}$$

So we define the dashed arrow by

$$(a, s, t) \mapsto \begin{cases} H_0(a, 0, s - t) & \text{when } s - t \geq 0 \\ H(a, t - s, 0) & \text{when } s - t \leq 0 \end{cases}$$

so that when  $s = t$  the functions coincide, when  $s = 0$  we get  $H$  and when  $t = 0$  we get  $H_0$ .

( $\text{Map}(I, Y)$  **deformation retracts on**  $\text{Map}(\text{pt}, Y)$ .) We must show there is a homotopy

$$h : \text{Map}(I, Y) \times I \longrightarrow \text{Map}(I, Y)$$

such that

$$h(-, 0) = \text{id}_{\text{Map}(I, Y)}, \quad h(-, 1) \subset \text{Map}(\text{pt}, Y)$$

$$\text{and} \quad h(-, 1)|_{\text{Map}(\text{pt}, Y)} = \text{id}_{\text{Map}(\text{pt}, Y)}.$$

Consider the map

$$\begin{aligned}
 I \times I &\rightarrow I \\
 (s, t) &\mapsto s - st
 \end{aligned}$$

Our deformation retract may be written like

$$h : \text{Map}(I, Y) \times I \longrightarrow \text{Map}(I, Y)$$

$$(f(s), t) \longmapsto f(s - st)$$

Then for  $t = 0$  we have the identity on  $\text{Map}(I, Y)$ , and when  $t = 1$  we have  $\text{ev}_0$ .

□

Let  $f : X \rightarrow Y$  be a map. Let  $E_f$  be the pullback of  $f : X \rightarrow Y$  and of  $\text{ev}_0 : \text{Map}(I, Y) \rightarrow Y$ . Let  $h : X \rightarrow E_f$  be the map that sends  $x$  to  $(x, \text{const}(f(x)))$ , where  $\text{const}(f(x)) : I \rightarrow Y$  is the constant path at  $f(x)$ . Let  $g : E_f \rightarrow Y$  be the composition of projection map  $E_f \rightarrow \text{Map}(I, Y)$  with  $\text{ev}_1 : \text{Map}(I, Y) \rightarrow Y$ .

## Homework 1.5

### 1 Exercise on model categories

**Exercise (3.1.8 from Riehl).** Verify that the class of morphisms  $\mathcal{L}$  characterized by the left lifting property against a fixed class of morphisms  $\mathcal{R}$  is closed under coproducts, closed under retracts, and contains the isomorphisms.

*Solution. (Coproducts.)* Suppose the maps  $\ell_i : A_i \rightarrow B_i$  are in  $\mathcal{L}$ . Then their coproduct in the arrow category is the obvious map  $\coprod A_i \rightarrow \coprod B_i$ .

Explicitly, their coproduct is an arrow  $\coprod \ell_i$  and a collection of maps  $f_i : \ell_i \rightarrow \coprod \ell_i$  such that for any other object  $m : A \rightarrow B$  in the arrow category and a map  $g : \ell \rightarrow m$ , the following diagram is completed uniquely:

$$\begin{array}{ccc} \ell_i & \xrightarrow{f_i} & \coprod \ell_i \xrightarrow{\exists!} m \\ & \searrow g & \uparrow \\ & & \end{array} \quad \forall i$$

So we conclude that the source of  $\coprod \ell_i$  is  $\coprod A_i$  and its target  $\coprod B_i$ . Indeed, we really looking at

$$\begin{array}{ccc} A_i & \xrightarrow{\ell_i} & B_i \\ f_i^1 \downarrow & & \downarrow f_i^2 \\ \coprod A_i & \xrightarrow{\coprod \ell_i} & \coprod B_i \\ \exists! \downarrow & & \downarrow \exists! \\ A & \xrightarrow{m} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism  $r \in \mathcal{R}$ :

$$\begin{array}{ccc} \coprod A_i & \longrightarrow & \bullet \\ \coprod \ell_i \downarrow & & \downarrow r \in \mathcal{R} \\ \coprod B_i & \longrightarrow & \bullet \end{array}$$

Since  $\ell_i \in \mathcal{L}$ , we have maps

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$



which in turn means we have a unique map

$$\begin{array}{ccccc}
 A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\
 \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\
 B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet
 \end{array}$$

by the universal property of the coproduct  $\coprod B_i$ .

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts  $\coprod A_i$  and  $\coprod B_i$  since, **in general**,

$$\text{Hom}\left(\coprod X_i, Y\right) \cong \prod \text{Hom}(X_i, Y).$$

More explicitly, we now that the red paths in the following diagrams are the same:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\
 \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\
 B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet
 \end{array} & \text{and} & \begin{array}{ccccc}
 A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\
 \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\
 B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet
 \end{array}
 \end{array}$$

and also

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\
 \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\
 B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet
 \end{array} & \text{and} & \begin{array}{ccccc}
 A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\
 \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\
 B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet
 \end{array}
 \end{array}$$

so the conclusion follows from the former comment.

**(Closed under retracts.)** Let us at least state what a retract of a morphism  $g$  should be in the arrow category. Recall that a retract is just

$$\begin{array}{ccccc}
 X & \longrightarrow & Y & \longrightarrow & X \\
 & & \searrow & \nearrow & \\
 & & \text{id}_X & & 
 \end{array}$$

So in the arrow category we get

$$\begin{array}{ccccc}
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
 & & \searrow & \nearrow & \\
 & & \text{id} & & 
 \end{array}$$

□

## 2 Hatcher's exercise on Whitehead's theorem

**Theorem 1** (Whitehead, May). If  $X$  is a CW complex and  $e : Y \rightarrow Z$  is an  $n$ -equivalence, then  $e_* : [X, Y] \rightarrow [X, Z]$  is a bijection if  $\dim X < n$  and surjection if  $\dim X = n$ .

**Theorem 2** (Whitehead, May). An  $n$ -equivalence between CW complexes of dimension less than  $n$  is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

**Theorem 3** (Whitehead (4.5), Hatcher). If a map  $f : X \rightarrow Y$  between connected CW complexes induces isomorphisms  $f_* : \pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then  $f$  is a homotopy equivalence. In case  $f$  is the inclusion of a subcomplex  $X \hookrightarrow Y$ , the conclusion is stronger:  $X$  is a deformation retract of  $Y$ .

**Exercise** (Hatcher 4.1.12). Show that an  $n$ -connected,  $n$ -dimensional CW complex is contractible.

*Solution.* Just recall that  $n$ -connectedness means that  $\pi_i(X) = 0$  for all  $i \leq n$ , which means that  $X$  is contractible by theorem 2.  $\square$

## Homework 2

**Exercise** (4.1.3). For an  $H$ -space  $(X, x_0)$  with multiplication  $\mu : X \times X \rightarrow X$ , show that the group operation in  $\pi_n(X, x_0)$  can also be defined by the rule  $(f + g)(x) = \mu(f(x), g(x))$ .

**Exercise** (4.1.19). Consider the equivalence relation  $\simeq_w$  generated by weak homotopy equivalence:  $X \simeq_w Y$  if there are spaces  $X = X_1, X_2, \dots, X_n = Y$  with weak homotopy equivalences  $X_i \rightarrow X_{i+1}$  or  $X_i \leftarrow X_{i+1}$  for each  $i$ . Show that  $X \simeq_w Y$  iff  $X$  and  $Y$  have a common CW approximation.

*Solution.* ( $\Leftarrow$ ) Suppose  $Z$  is a common CW approximation of  $X$  and  $Y$ , that is,  $Z$  is a CW complex and there are weak homotopy equivalences  $Z \rightarrow X$  and  $Z \rightarrow Y$ . Then the sequence of spaces  $X = X_1, Z = X_2$  and  $Y = X_3$  shows that  $X \simeq_w Y$ .

( $\Rightarrow$ ) Suppose  $Z$  is a CW approximation of  $X$  and let's show it can be made (somehow) into a CW approximation of  $Y$ . There is a weak homotopy equivalence  $Z \rightarrow X$ , and also a weak homotopy equivalence either  $X = X_1 \rightarrow X_2$  or  $X = X_1 \leftarrow X_2$ . I wonder if this implies that the composition  $Z \rightarrow X = X_1 \rightarrow X_2$  is also a weak homotopy equivalence  $\square$

**Exercise** (4.2.1). Use homotopy groups to show that there is no retraction  $\mathbb{R}P^n \rightarrow \mathbb{R}P^k$  for  $n > k > 0$ .

*Solution (in progress...)* Suppose there is a retraction

$$\begin{array}{ccccc} \mathbb{R}P^k & \hookrightarrow & \mathbb{R}P^n & \longrightarrow & \mathbb{R}P^k \\ & & \searrow & \nearrow & \\ & & \text{id} & & \end{array}$$

it induces isomorphisms

$$\begin{array}{ccccc} \pi_i(\mathbb{R}P^k) & \longrightarrow & \pi_i(\mathbb{R}P^n) & \longrightarrow & \pi_i(\mathbb{R}P^k) \\ & & \searrow & \nearrow & \\ & & \cong & & \end{array}$$

If we had that this is a Serre fibration, we would get that

content...

□

**Exercise (4.2.13).** Show that a map between connected  $n$ -dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on  $\pi_i$  for  $i \leq n$ . [Pass to universal covers and use homology.]

*Solution.* Let  $X$  and  $Y$  be  $n$ -dimensional CW complexes and  $f : X \rightarrow Y$  such that  $f_* : \pi_i(X) \rightarrow \pi_i(Y)$  is an isomorphism for  $i \leq n$ . Let's try to use Hurewicz theorem, which states that a map between simply connected CW complexes is a homotopy equivalence if it induces isomorphisms on all homology groups.

Consider the universal covers  $\tilde{X}$  and  $\tilde{Y}$ , which are simply connected and also [have CW structures](#). By prop. 4.1, the cover projections induce isomorphisms in the homotopy groups for all  $i \geq 2$ . By [StackExchange](#) there is a unique lift  $\tilde{f}$  to the universal covers making the diagram on the left commute, and by functoriality the diagram on the right also commutes.

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array} \quad \begin{array}{ccc} \pi_i(\tilde{X}) & \xrightarrow{\tilde{f}_*} & \pi_i(\tilde{Y}) \\ p_* \downarrow \cong & & \cong \downarrow q_* \\ \pi_i(X) & \xrightarrow{f_*} & \pi_i(Y) \end{array} \quad i \geq 2$$

We conclude that  $\tilde{f}$  is a **weak homotopy equivalence**, and by prop. 4.21 it induces isomorphisms on homology groups. Finally, by Hurewicz theorem (coro. 4.33) it is an homotopy equivalence and so is  $f$ . □

**Exercise.** There is a fiber sequence  $U(n) \hookrightarrow U(n+1) \rightarrow U(n+1)/U(n) \cong S^{2n+1}$ . Use this to show that  $\pi_k(U(n)) \rightarrow \pi_k(U(n+1))$  is isomorphism for  $n > k/2$ . Compute  $\pi_k(U(n))$  for  $n \geq 2$  and  $k = 1, 2, 3$ . In fact, if  $k$  is even then  $\pi_k(U(N)) = 0$  and if  $k$  is odd then  $\pi_k(U(N)) = \mathbb{Z}$ , where again  $N > k/2$ . These equalities are known as Bott periodicity.

*Solution.* The required isomorphisms  $\pi_k(U(n)) \rightarrow \pi_k(U(n+1))$  follow simply from the fact that  $S^{2n+1}$  is  $2n+1$ -connected: in the long homotopy sequence of the fiber bundle we have

$$\pi_{k+1}(S^{2n+1}) \longrightarrow \pi_k(U(n)) \longrightarrow \pi_k(U(n+1)) \longrightarrow \pi_k(S^{2n+1})$$

so when  $2n + 1 > k + 1 \iff n > k/2$  the homotopy groups of the spheres vanish and we have an isomorphism.

The group  $\pi_1(U(n))$  is isomorphic to  $\mathbb{Z}$ . This follows from the fact that  $U(1)$  is homeomorphic to a circle and by induction using the former isomorphism  $\pi_1(U(n)) \cong \pi_1(U(n+1))$ . We also have  $\pi_2(U(1)) = 0$ , so that again by induction we get  $\pi_2(U(n)) = 0$ . Finally, a similar argument shows  $\pi_3(U(n)) = 0$ .  $\square$

**Exercise (4.2.32).** Show that if  $S^k \rightarrow S^m \rightarrow S^n$  is a fiber bundle, then  $k = n - 1$  and  $m = 2n - 1$ . [Look at the long exact sequence of homotopy groups.]

*Solution (in progress...)* The sequence of homotopy groups is

$$\dots \rightarrow \pi_i(S^k) \rightarrow \pi_i(S^m) \rightarrow \pi_i(S^n) \rightarrow \pi_{i-1}(S^k) \rightarrow \pi_{i-1}(S^m) \rightarrow \dots$$

so that if  $i \leq k$  we have  $\pi_i(S^m) = \pi_i(S^n)$ ,  $\square$

## cohomology ring of $\mathbb{CP}^n$

We know that cohomology of  $\mathbb{CP}^n$  is ...

So that

$$\begin{aligned} H^\bullet(\mathbb{CP}^n) &= H^0(\mathbb{CP}^2) \oplus H^1(\mathbb{CP}^2) \oplus \dots \oplus H^{2n}(\mathbb{CP}^n) \\ &= \mathbb{Z} \oplus \mathbb{Z}a_1 \oplus \dots \oplus \mathbb{Z}a_n \end{aligned}$$

We wish to show that

$$a_1 = \alpha, a_2 = \alpha^2 \dots a_n = \alpha^n$$

for the generator of the cohomology group of degree  $2\alpha$ . To show that  $H^\bullet(\mathbb{CP}^n, \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(\alpha^{n+1})$  with  $\deg \alpha = 2$  we use induction.

By Poincaré duality on  $\mathbb{CP}^2$

$$\begin{aligned} H^2(\mathbb{CP}^2) \times H^2(\mathbb{CP}^2) &\rightarrow H^4(\mathbb{CP}^2) \\ (\alpha, \alpha) &\mapsto \alpha \smile \alpha = \alpha^2 \end{aligned}$$

must map a generator to a generator, so the case of  $\mathbb{CP}^2$  is done.

Now the case of  $\mathbb{CP}^3$  is as follows:

$$\begin{aligned} H^2(\mathbb{CP}^3) \times H^4(\mathbb{CP}^3) &\rightarrow H^6(\mathbb{CP}^3) \\ (\alpha, \alpha^2) &\mapsto \alpha \smile \alpha^2 = \alpha^3 \end{aligned}$$

Notice we have used the isomorphism  $H^i(\mathbb{CP}^3) \approx H^i(\mathbb{CP}^2)$  when choosing generators of each of the groups.

So we have the ring morphism

$$\begin{aligned}\varphi : \mathbb{Z}[\alpha] &\rightarrow H^\bullet(\mathbb{CP}^2) = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \dots \oplus \mathbb{Z}\alpha^n \\ \sum \lambda_i \alpha^i &\mapsto \bigoplus \lambda_i \alpha^i\end{aligned}$$

with

$$\ker \varphi = \{0\}$$

## References

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