homotopy theory

Contents

abstract nonsense

definition.

- An ***initial object*** in a category $\mathcal{C}$ is an object $\varnothing$ such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source $\varnothing$ and target x.
- For $\mathcal{C}$ any category, its ***arrow category*** $\mathrm{Arr}(\mathcal{C})$ is the category such that
  - an object a of $\mathrm{Arr}(C)$ is a morphism $a : a_0 \to a_1$ of $\mathcal{C}$,
  - a morphism $f : a \to b$ of $\mathrm{Arr}(\mathcal{C})$ is a commutative square

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\ a_1 \arrow[r, "f_1"'] & b_1 \end{tikzcd}$$

    in $\mathcal{C}$,

- composition in $\mathrm{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

$$\mathrm{Arr}(\mathcal{C}) := \mathrm{Funct}(I, C) = [I, C] = C^I$$

for I the intervale category $\{0 \to 1\}$.

- An ***equalizer*** is a limit

$$eq \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

over a parallel pair of morphisms f and g. This means that for $f : X \to Y$ and $g : X \to Y$ in a category $\mathcal{C}$, their equalizer, if it exists, is

  - an object $eq(f, g) \in \mathcal{C}$,
  - a morphism $eq(f, g) \to X$
  - such that
    * pulled back to $eq(f, g)$ both morphisms become equal:

$$eq(f, g) \longrightarrow X \xrightarrow{f} Y \quad = \quad [\ eq(f, g) \longrightarrow X \xrightarrow{g} Y$$

    * and $eq(f, g)$ is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions f and g between sets X and Y one forms the set $Y/\sim$ of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the the quotient function $p : Y \to Y/\sim$ satisfies

$$p \circ f = p \circ g.$$

In some category $\mathcal{C}$, the ***coequalizer*** $\mathrm{coeq}(f, g)$ of two parallel morphisms f and g between two objects X and Y, if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{coeq}(f, g) & \end{tikzcd}$$

Equivalently, in a category $\mathcal{C}$ a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Z$$

is called a ***coequalizer*** diagram if

1. $p \circ f = p \circ g$,
2. p is universal for this property: if $q : Y \to W$ is a morphism of $\mathcal{C}$ such that $q \circ f = q \circ g$, then there is a unique morphism $\phi : Z \to W$ such that $\phi \circ p = q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] \arrow[d, "q"] & Z \arrow[dl, dashed, "\phi"] \\ & W & \end{tikzcd}$$

The coequalizer in $\mathcal{C}$ is equivalently an equializer in the opposite category $\mathcal{C}^{op}$.

- A ***pullback*** of the morphisms f and g consists of an object P and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

- A ***pushout*** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\ X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] \arrow[ur, phantom, "\ulcorner", very near end] & P \arrow[dr, dashed, "\phi"] & \\ & & Q \end{tikzcd}$$

remark. Other names for the pushout are ***cofibered product of*** X ***and*** Y (especially in algebraic categories when i_1 and i_2 are monomorphisms), or ***free product of*** X ***and*** Y with Z ***amalgamated sum***, or more simply an ***amalgamation*** or ***amalgarm of*** X ***and*** Y.

remark. If coproducts exist in some category, then the pushout

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X \coprod_Z Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is equivalently the coequalizer

$$X \overset{i_1 \circ f}{\underset{i_2 \circ g}{\rightrightarrows}} X \coprod Y \longrightarrow X \coprod_Z Y$$

of the two morphisms induced by f and g into the coproduct of X with Y.

example (wiki).

- If X, Y and Z are sets and f, g are functions, the pushout of f and g is the disjoint union of X and Y where elements sharing a common preimage in Z are identified, i.e. $P = (X \coprod Y) / \sim$ where $\sim$ is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

  In particular, if X and Y are subsets of some larger set W and Z is their intersection, with f and g the inclusion maps of Z into X and Y, then the pusout can be canonically identified with the union $X \cup Y \subseteq W$.

- The construcion of ***adjunction spaces*** is an example of pushouts in Top. More precisely, if Z is a subspace of Y and $g : Z \to Y$ is the inclusion map, we can glue Y to another space X along Z using an ***attaching map*** $f : Z \to X$. The result is the ***adjunction space*** $X \cup_f Y$ which is just the pushout of f and g. More generally, all identification spaces may be regarded as pushouts in this way. See **??** .

- A ***product*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & X \sqcap Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\ & X & \end{tikzcd}$$

- A ***coproduct*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$,

$i_2 : Y \to X \sqcup Y$ satisfying the following universal property:

$$\begin{tikzcd}
 & Y \arrow[d, "i_2"] \arrow[dr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[rr, bend right, "j_1"'] & X \sqcup Y \arrow[r, dashed, "\phi"] & Q
\end{tikzcd}$$

remark. More generally, for S any set and $F : S \to \mathcal{C}$ a collection of objects in $\mathcal{C}$ indexed by S, their ***coproduct*** is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

- The ***kernel*** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the ***kernel*** $\ker f$ of a morphism $f : A \to B$ is the pullback $\ker(f) \to A$ along f of the unique morphism $0 \to B$

  More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

  > for every object C and every morphism $h : C \to A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \to \ker(f)$ such that $h = p \circ \phi$.

$$\begin{tikzcd}
C \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "h"'] & & \\
 & \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & 0 \arrow[d] \\
 & A \arrow[r, "f"'] & B
\end{tikzcd}$$

- In a category with a terminal object 1, the ***cokernel*** of a morphism $f : A \to B$ is the pushout (arrows h and ϕ apply if terminal object is zero)

$$\operatorname{coker}(f) := 1 \sqcup_A B \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \arrow[ddr, bend left, "h"] & \\ 1 \arrow[r] & \operatorname{coker}(f) \arrow[dr, dashed, "\phi"] & \\ & & C \end{tikzcd}$$

In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\operatorname{coker}(f)$ with the following universal property:

> for every object C and every morphism $h : B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \operatorname{coker}(f) \to C$ such that $h = \phi \circ i$.

- A morphism $f : X \to Y$ is a ***monomorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Z \to X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[rr, bend left, "f \circ g_1"] \arrow[rr, bend right, "f \circ g_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Equivalently, f is a monomorphism if for every Z the hom-functor $\operatorname{Hom}(Z, -)$ takes it to an injective function

$$\operatorname{Hom}(Z, X) \xhookrightarrow{f_*} \operatorname{Hom}(Z, Y).$$

Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A morphism $f : X \to Y$ is a ***epimorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Y \to Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\operatorname{Hom}(-, Z)$ takes it to an injective function

$$\operatorname{Hom}(Y, Z) \xhookrightarrow{f^*} \operatorname{Hom}(X, Z).$$

Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an monomorphism in the opposite category $\mathcal{C}^{op}$.

elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. An ***homotopy*** from f to g is a continuous map

$$H : X \times [0,1] \to Y, \qquad (x,t) \mapsto H(x,t) = H_t(x)$$

) such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. A homotopy of maps $H_t : X \to Y$ is called ***relative to*** $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top, the ***homotopy category*** h-Top, where comoposition of homotopy classes is induced by composition of representing maps. If $f : X \to Y$ represents an isomorphism in h-Top, then f is called a ***homotopy equivalence*** or h-***equivalence***. In explicit termins this means $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$, a ***homotopy inverse of*** f, such that gf and fg are both homotopic to the identity. Spaces X and Y are called ***homotopy equivalent*** or of the same ***homotopy type*** if there exists a homotopy equivalence $X \to Y$. A space is ***contractible*** if it is homotopy equivalent to a point. A map $f : X \to Y$ is ***null homotopic*** if it is homotopic to a constant map.

- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the n-***th homotopy group*** are homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

  Also,

$$\pi_n(X,*) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n/\partial I^n, X]^0$$

  where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \to Y$.

  proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the ***relative homotopy group*** $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n. That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

The elements of such a group are homotopy classes of based maps $D^n \to X$ which carry the boundary S^{n-1} into A. Two maps f, g are called ***homotopic relative to*** A if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0,1] \to X$

such that, for each p in S^{n-1} and t in $[0,1]$, the element $F(p,t)$ is in A. Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

remark. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_{n-1}(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \longrightarrow \cdots \longrightarrow \pi_0(X,x_0)$$

  where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. The map, called the ***boundary map***, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called n-***connected*** if $\pi_i(X, x_0) = 0$ for $i \leqslant n$. Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair (X, A) is n-***connected*** if $\pi(X, A, x_0) = 0$ for $i \leqslant n$.
- Two pointed spaces (X, x_0) and (Y, y_0) are n-***equivalent*** if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i \leqslant n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on ***compactly generated wakly Hausdorff*** spaces.
- X is ***compactly generated*** if, for any subset $C \subset X$, and for all continuous maps $f : K \to X$ from compact Housdorff spaces,

$$\text{if } f^{-1}(C) \text{is closed in } K, \text{ then } C \text{ is closed.}$$

claim (What I picked up from the lecture)**.** If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is k-closed.

From May: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi : X \to Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH, $\mathrm{Hom}(X, Y)$ is a space with the compact-open topology. This is a compactly generated space, $\mathbf{k}(\mathrm{Hom}(X, Y))$.

remark. (Also see wiki on currying)

$$\begin{aligned} \mathrm{Map}(X, Y) &:= \text{ the space of maps } X \to Y. \\ \mathrm{Map}(X \times Y, Z) &\cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ \mathrm{Hom}(X \times Y, Z) &\cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \end{aligned}$$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to $\mathrm{Map}(Y, -)$.

cofibrations

definition.

- A morphism i has the ***left lifting property with respect to a morphism*** p and p has the ***right lifting property with respect to*** i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed, "\phi"] & Y \end{tikzcd}$$

- (wiki) In mathematics, in particular in homotopy theory, a continuous map between topological spaces $i : A \to X$ is a ***cofibration*** if it has the ***homotopy extension property*** with respect to all topological spaces S.

  That is, i is a cofibration if

  - for each topological space S,
  - and for any continuous maps $f, f' : A \to S$
  - and $g : X \to S$ with $g \circ i = f$,
  - for any homotopy $h : A \times I \to S$ from f to f',

there is a continuous map $g' : X \to S$ and a homotopy $h' : X \times I \to S$ from g to g' such that

$$h'(i(a), t) = h(a, t) \qquad \text{for all } a \in A \text{ and } t \in I.$$

- (wiki) Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the ***homotopy extension property*** if, given a homotopy $f_\bullet : A \to Y^I$ and a map $\tilde{f}_0 : X \to Y$ such that

  $$\tilde{f}_0 \circ \iota = f_0$$

  (so $\tilde{f}$ is the lift of $f_0 : A \to Y$) then there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{f}_\bullet : X \to Y^I$ such that$\tilde{}_\bullet \circ \iota = f_\bullet$.

  That is,

  $$\begin{tikzcd} A \arrow[r, "f_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\ X \arrow[r, "\tilde{f}_0"'] \arrow[ur, dashed, "\tilde{f}_\bullet"] & Y \end{tikzcd}$$

  So there's some currying to make usual homotopies $f_\bullet : A \times I \to Y$ look like $f_\bullet : A \to Y^I$. Or, as said in our lectures, !a homotopy $X \times I \to Y$ is the same as a map $X \to \mathrm{Map}(I, Y)$".

- (May) A map $i : A \to X$ is a ***cofibration*** if it satisfies the ***homotopy extension property (HEP)***. This means that if $h \circ i_0 = f \circ i$ in the diagram

  $$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i \times id"] \\ & Y & \\ X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"] \end{tikzcd}$$

  then there exists $\tilde{h}$ that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map $i : A \to X$ between topological spaces is a ***Hurewicz cofibration*** if it satisfies the homotopy extension property.

  Let's say it one more time: for any $g : X \to Y$ and any homotopy $H : A \times I \to Y$ such that

  $$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & A \times I \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

there is $H' : X \times I \to Y$,

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, "g"] & A \times I \arrow[d] \\ X \times I \arrow[r, "H'"'] & Y \end{tikzcd}$$

such that

$$\begin{tikzcd} A \times I \arrow[d] \arrow[dr, "H"] & \\ X \times I \arrow[r, "H'"'] & Y \end{tikzcd}$$

example. $\partial D^n \to D$ is a Huerwicz cofibration. Why?

exercise. Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

definition (Mapping cylinder)**.**

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. make the dashed map unique—up to something maybe). Namely, we can let Y in our original test diagram be the ***mapping cylinder***

  $$Mi \equiv X \cup_i (A \times I)$$

  which is the pushout of i and i_0. Indeed, suppose that we can construct a map r that makes the following diagram commute

  $$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd, "i \times id"] \\ & Mi & \\ X \arrow[ur] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "r"] \end{tikzcd}$$

  By the universal property of the pushouts, given maps f and h in our original test diagram induce a map $Mi \to Y$, and its comoposite with r gives a homotopy $\tilde{h}$ that makes the diagram commute. So just saying that Mi is universal.

- (nLab) Given a continuous map $f : X \to Y$ of topological spaces, one can define its

mapping cylinder as a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \times I \arrow[r, "(\sigma_0)_*(f)"'] & \mathrm{Cyl}(f) \end{tikzcd}$$

in Top, where $I = [0,1]$ and $\sigma : X \to X \times I$ is given by $x \mapsto (x, 0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$\left(X \times I \coprod Y\right) \Big/ \sim$$

where $\sim$ is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

- (wiki) The ***mapping cylinder*** of a function f between topological spaces X and Y is the quotient

  $$M_f = (([0,1] \times X) \amalg Y)/\sim$$

  where $\amalg$ denotes disjoint union, and $\sim$ is the equivalence relation generated by

  $$(0, x) \sim f(x) \text{ for each } x \in X.$$

  That is, the mapping cylinder M_f is obtained by gluing one end of $X \times [0,1]$ to Y via the map f. Notice that the "top" of the cylinder $\{1\} \times X$ is homeomorphic to X, while the "bottom" is the space $f(X) \subset Y$.

  (Dani) So the mapping cylinder is just a deformation of X to $f(X)$.

- (Homework) Let $f : X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f : X \times Y$.

exercise. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\ X \times [0,1] \arrow[r] \arrow[drr, bend right, "(x,t) \mapsto f(x)"'] & M_f \arrow[dr, dashed, "h"] \arrow[ul, phantom, "\ulcorner"] & \\ & & Y \end{tikzcd}$$

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\operatorname{colim} A_i) \times B$ is canonically homeomorphic with $\operatorname{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

1. That h is a deformation retract is consequence of the last diagram. Why is it a homotopy equivalence?
2. Consider the following lifting problem:

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "g"'] & Y^I \arrow[d, "\pi_0"] \\ M_f \arrow[r, "h"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

   Looks nice but why should the dashed arrow exist...?

□

model structures

definition (Riehl). A ***weak factorization system*** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised o two clases of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

1. Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$:

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

2. The maps in $\mathcal{L}$ have the ***left lifting property*** with respect to each map in $\mathcal{R}$ and equivalently the maps in $\mathcal{R}$ have the ***right lifting property*** with respect to each map in $\mathcal{L}$, that is, any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

   admits a diagonal filler as indicated making both triangles commute.

3. The classes $\mathcal{L}$ and $\mathcal{R}$ are each closed under retracts in the arrow category: given a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "t"'] \arrow[rr, bend left, equal] & \bullet \arrow[r] \arrow[d, "s"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r] \arrow[rr, bend right, equal] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

if s is in that class then so is its retract t.

definition (Lecture). A ***model structure*** on a category $\mathcal{A}$ is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called ***weak-equivalences***, ***cofibrations*** and ***fibrations*** with the following properties:

1. Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in $\mathcal{W}$ then all of them are.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system.
   (a) Any morphism in $\mathcal{A}$ can be factored as a morphism in $\mathcal{B}$ followed by a morphism in $\mathcal{D}$.
   (b) Lifts:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \bullet \end{tikzcd}$$

Two interesting model category structures on CGWH.

1. Hurewicz model structure (Strom).
   - Cofibrations:= Huerwicz cofibrations.
   - Fibrations:= maps $E \to B$ such that for all spaces X [Photo1].
   - Weak equivalences:= homotopy equivalences.
2. Quillen model structure.
   - Cofibrations = retracts of relative cell complexes.
   - (Serre) Fibrations = $\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$
   - Weak equivalences: $f : X \to Y$

exercise (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. ***(Coproducts.)*** Sergey: Coproduct of morphisms $A_i \to B_i$ in a category $\mathcal{C}$ is the obvious morphism $\sqcup A_i \to \sqcup B_i$. (Because in this construction morphisms $A_i \to B_i$ are seen as objects of what's called the arrow category of the category $\mathcal{C}$)

Suppose the maps $\ell_i : A_i \to B_i$ are in $\mathcal{L}$. Then their coproduct in the arrow category is the obvious map $\coprod A_i \to \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \to \coprod \ell_i$ such that for any other object $m : A \to B$ in the arrow category and a map $g : \ell \to m$, the following diagram is completed uniquely:

$$\begin{tikzcd} \ell_i \arrow[r, "f_i"] \arrow[rr, bend right, "g"'] & \coprod \ell_i \arrow[r, dashed, "\exists!"] & m \end{tikzcd} \qquad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{tikzcd} A_i \arrow[r, "\ell_i"] \arrow[d, "f_i^1"'] & B_i \arrow[d, "f_i^2"] \\ \coprod A_i \arrow[r, "\coprod \ell_i"] \arrow[d, dashed, "\exists!"'] & \coprod B_i \arrow[d, dashed, "\exists!"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{tikzcd} \coprod A_i \arrow[r] \arrow[d, "\coprod \ell_i"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "\mathcal{L} \ni \ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ B_i \arrow[r] \arrow[urr, dashed] & \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

which in turn means we have unique maps

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

by the universal property of the coproduct $\coprod B_i$.

So, to check that the lower-right triangle commutes, it would be sufficient to show that the map $B_i \to \coprod B_i$ "can be cancelled" since

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}
\quad \text{is already the same as} \quad
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

Likeways, to make sure that the remaining triangle commutes we observe that

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}
\quad \text{is already the same as} \quad
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

Why can we "cancel" the maps $A_i \to \coprod A_i$ and $B_i \to \coprod B_i$? □

remark (Plan). Blakers-Massey excision theorem (relies on technical lema, proof from Tom Dieck's book) $\implies$ Cellular approximation. Also $\implies$ Freudental theorem.

exercise. $X \to M_f \to Y$. Prove $X \to M_f$ is a cofibration.

whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A ***CW complex*** X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called ***vertices***) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along ***attaching maps*** $j : S^n \to X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times$

D^{n+1}) by identifying (j, x) with $j(x)$ for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j (see **??**). Each resulting map $D^{n+1} \to X$ is called a ***cell***. The subspace X^n is called the n-***skeleton*** of X.

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "j"'] & D^{n+1} \arrow[d] \\ X^n \arrow[r] & X^{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

lemma 3 (HELP)**.** content...

Theorem 4 (Whitehead, May)**.** If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 5 (Whitehead, May)**.** An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 6 (Whitehead (4.5), Hatcher)**.** If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12)**.** Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 5. $\square$

lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \to X$$

$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(X', Y)$$

$$\begin{aligned} \mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, B') \text{ natual in } A \implies \\ \mathrm{Hom}(B, B) \cong \mathrm{Hom}(B, B') \& \mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', B') \\ \implies B \cong B'. \end{aligned}$$

- for ($\Longleftarrow$) commutativity of the hypotesis gives us commutativiuty of the rightmost square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an n-equivalence.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 7 (Yoneda)**.**

$$\{\text{Natural transformations } \operatorname{Hom}(-, X) \to F\} \cong F(X)$$

corollary 8. $(\operatorname{Hom}(-, X) \to \operatorname{Hom}(-, Y)) \cong \operatorname{Hom}(X, Y)$.

corollary 9. The correspondence $X \mapsto \operatorname{Hom}(-, X)$ is fully faithful, that is, the correspondence $\operatorname{Hom}(X, X') \to \operatorname{Hom}(\operatorname{Hom}(-, X), \operatorname{Hom}(-, X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. $\square$

lemma 10. Let $E \times_B X$ be the pullback of

$$\begin{tikzcd} & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be such that $E \to B$ is an homotopy fibration and $f : X \to B$ is a homotopy equivalence. Let

$$\begin{tikzcd} E \times_B X \to E \arrow[r, dash, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be the pullback. Then $E \times_B X \to E$ is a homotopy equivalence.

Proof. Let $g : B \to X$ be the homotopy inverse of f.

(Step 1) Construct another pullback

$$\begin{tikzcd}
E \times_B B \arrow[r] \arrow[d, two heads] & X \times_B E \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] \\
B \arrow[r, "g"'] & X \arrow[r, "f"'] & B
\end{tikzcd}$$

(Step 2) Constuct $E \to E \times_B B$.

Consider

$$\begin{tikzcd}
E \arrow[rr, "\mathrm{id}"] \arrow[d] & & E \arrow[d, two heads] \\
E \times I \arrow[r, "f \times \mathrm{id}"'] \arrow[urr, dashed] & B \times I \arrow[r] & B?
\end{tikzcd}$$

And then $E \to E \times_B B \to E \times_B X \to E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{tikzcd}
Z \sqcup Z \arrow[r] \arrow[d, "f_1 \sqcup f_2"'] \arrow[ddr, bend left] & I \times Z \arrow[d] \arrow[dl, dashed] \arrow[dd, bend left] \\
E \times_B X \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r, "\simeq"] & B
\end{tikzcd}$$

∎

corollary 11. $B \xrightarrow{f} B$ is homotopy equivalence and $E \to B$ is a fibration, in

$$\begin{tikzcd}
E \times_B B \arrow[r] \arrow[d] & E \arrow[d] \\
B \arrow[r, "f"'] & B
\end{tikzcd}$$

$E \times_B B \to E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 12. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "g"] & B \arrow[d] \\ X \arrow[r] & X \cup_A B \end{tikzcd}$$

be a pushout with $A \to X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f,g) \to X \cup_A B$ is a homotopy equivalence.

remark.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & M_f \arrow[d] \\ X \arrow[r] & X \cup_A M_f \cong M(f,g) \end{tikzcd}$$

definition.

- The ***homotopy pullback*** of a diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  is

$$\begin{tikzcd} X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y.

- The ***homotopy fiber*** if $f : Y \to Z$ is the pullback of

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ pt \arrow[r] & Z \end{tikzcd}$$

  $F \subset Z^I \times_Z Y \to Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark. The pullback of

$$\begin{tikzcd} & Z^I \times_Z Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is the motopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

lemma 13. If $X \to Z$ is a fibration then for

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r, two heads] & Z \end{tikzcd}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{tikzcd} X \times_Z \arrow[r] \arrow[d, dash, "\simeq"] & Y \arrow[d, "\simeq"] \\ X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r, two heads] \arrow[d] & Z^I \times_Z Y \arrow[d, two heads] \\ X \arrow[r] & Z \end{tikzcd}$$

$\square$

Finally,

$$\begin{tikzcd} \text{hofib } f_1 \arrow[r] \arrow[d] & \text{hofib } f \arrow[r] \arrow[d, two heads] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

and

$$
\begin{tikzcd}
Z \arrow[r] \arrow[dd] & F(f) \arrow[d] \arrow[ddl, dash] \arrow[dd, bend left, two heads] \\
 & X \times_Y Y^I \arrow[d] \\
X \times I \arrow[r] & X
\end{tikzcd}
$$

and an exact sequence

$$\Omega^2 \operatorname{hofib} \to \Omega^2 X \to \Omega^2 Y \to \Omega \operatorname{hofib} f \to \Omega X \to \Omega Y \to \operatorname{hofib} f \to X \xrightarrow{f} Y$$

lemma 14 (Exactness). $\forall z$, $[z \operatorname{hofib} f] \to [Z, X] \to [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \to \pi_0(\Omega^2 Y) \to \pi_0(\Omega \operatorname{hofib} f) \to \pi_0(\Omega X) \to \pi_0(\Omega Y) \to \pi_0(\operatorname{hofib} f) \to \pi_0(X) \to \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewickz fibrations. Let's talk about Serre fibrations. Notice that H. fibration $\implies$ S. fibration. What is the most natural example of a Serre fibration?

proposition 15. Let E be a fiber bundle with fiber F. Then f is a Serre fibration.

Proof. What sdoes it mean to be a Serre fibration? It means that

$$
\begin{tikzcd}
I^n \arrow[r] \arrow[d] & E \arrow[d] \\
I^{n+1} = I^n \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}
$$

So if $\mathcal{U}$ is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U.

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & F \times U \arrow[d, two heads] \\ I^{n+1} \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

has a lift for every little square because

$$\begin{tikzcd} X \arrow[r] \arrow[d] & U \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & pt \end{tikzcd}$$

is always a fibration (think about this) and because pullbacks of fibrations are fibrations:

$$\begin{tikzcd} U \times F \arrow[r, two heads] \arrow[d, two heads] & U \arrow[d, two heads] \\ F \arrow[r, two heads] & pt \end{tikzcd}$$

. Then we may just add up the squares because

$$\begin{tikzcd} D^n \arrow[d] \\ D^n \times I \end{tikzcd}$$

and we're done. □

proposition 16 (Construction of homotopy long exact sequence from relative homotopy long exact sequence)**.** Let $g : E \to B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(E) \arrow[r] & \cdots \\ \cdots \arrow[r] & \pi_n(F) \arrow[r] \arrow[u, "="] & \pi_n(E) \arrow[r] \arrow[u, "="] & \pi_n(E, F) \arrow[r] \arrow[u, dashed, "\cong"] & \pi_{n-1}(F) \arrow[r] \arrow[u, "="] & \pi_{n-1}(E) \arrow[r] \arrow[u, "="] & \cdots \end{tikzcd}$$

example. We have shown that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 17. Let X be a CW-comples, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \varnothing$, so

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

is a pushout (this happens for inclusions, check it?).

If (A, C) is n-connected and (B, C) is m-connected, then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and sujerctive for $i = m + n$.

blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \to Y$ be a map. Consider the pullback

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d, "(q,p)"'] & Y^I \arrow[d, "(ev_0, ev_1)"] \\ X \times Y \arrow[r, "f \times id"'] & Y \times Y \end{tikzcd}$$

where

$$W(f) = \{(x, w) \in X \times Y^I | f(x) = w(0)\},$$
$$q(x, w) = x, \quad p(x, w) = w(1).$$

Since (ev_0, ev_1) is a fibration, the maps (q, p), q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \to W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \to (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \to X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \to *$.
- $I^n \times \{1\} \to A$.

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow \text{hofib} \longrightarrow A \longrightarrow X$$

$$\begin{tikzcd}[column sep=small]
\pi_0(\nearrow) = & \pi_n(A) \arrow[r] & \pi_n(X) \arrow[r] \arrow[dr] & \pi_{n-1}(\mathrm{hofib}) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \\
& & & \pi_n(X, A) \arrow[ur] & &
\end{tikzcd}$$

Theorem 18 (Blakers-Massey 1)**.** Let

$$\begin{tikzcd}
Q \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\
X \arrow[r] & P
\end{tikzcd}$$

be a homotopy pushout, g is m equivalence, f is n-equivalence and $m, n \geqslant 0$. Then $Q \to X \times_P^h$ is $(m+n-1)$-equivalence.

Theorem 19 (Blakers-Massey 2)**.** P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \varnothing$ (***strict pushout***)

$$\begin{tikzcd}
Q \arrow[r, tail] \arrow[d, tail] & Y \arrow[d, tail] \\
X \arrow[r, tail] \arrow[ur, phantom, "\lrcorner", very near end] & X
\end{tikzcd}$$

Then $\pi_i(Y, \mathbb{Q}) \to \pi_i(P, X)$ is epi for $i = m + n$ and iso for $0 \leqslant i < m + n$.

Theorem 20 (Blakers-Massey 3)**.** $P = X \cup Y$, X and Y are open in P, $X \cap Y = Q \neq \varnothing$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a k-***equivalence*** if the induced map on the ith homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W :$ at least p coordinates of x are ¡ the same coordinates of the center of $W\}$

lemma 21. Let W be a cube in $\mathbb{R}^d$ with $\dim W \leqslant d$. If for all faces W' of ∂W, $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \ \mathrm{rel}\, \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

freudenthal theorem (2 april)

definition. The appropriate analogue of the Cartesian product in the category of based spaces is the ***smash product*** $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y.

We also have the ***suspension of pointed spaces***, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X)/(t, x) \sim (0, y) \sim (1, y) \; \forall y \in X.$$

Then we have

$$\mathrm{Hom}_{\mathrm{CGWH}_*}(\Sigma X, \Sigma X) \cong \mathrm{Hom}_{\mathrm{CGWH}_*}(X, \Omega\Sigma X)$$

where $\Sigma X = S^{\wedge} X$ and $\Omega\Sigma X = \mathrm{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\mathrm{Map}(S^1, -)$.

So let X be a space. The identity map $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ then induces a map $X \to \Omega\Sigma X$.

Theorem 22 (Freudenthal)**.** Let X be ℓ-connected space. Then $X \to \Omega\Sigma X$ is a $(2\ell + 1)$-equivalence, that is,

$$\pi_i(X) \to \pi_{i+1}(\Sigma X),$$

$(\ell + 1) + (\ell + 1) - 1 = 2\ell + 1$.

Proof 1. Using pushouts... □

Proof 2. Consider

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

Then we use relative homotopy long exact sequence with (X, CX) to get $\pi_i(CX, X) \cong \pi_{i-i}(X)$, which is zero for $0 \leqslant i \leqslant \ell + 1$. Then use relative homotopy exact sequence for the pair $(\Sigma X, CX)$. then we get that $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$. And then if you use it for $(\Sigma X, X)$ and

But it also turns out that $\pi_i(\Sigma X) = \pi_{i-1}(\Omega\Sigma X)$ because

$$\pi_n(Z) = \mathrm{Hom}_{\text{h-Top},*}(S^n, Z) = \mathrm{Hom}(S^1 \wedge S^{n-1}, Z) = \mathrm{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX, X) \to \pi_i(\Sigma X, CX)$ which is isomorphism for $0 \leqslant i \leqslant 2\ell + 1$ and surjective for $i = 2\ell + 2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \to \pi_{i-1}(\Omega\Sigma X)$ for i as desired. □

corollary 23. If X is ℓ-connected, then ΣX is $(\ell + 1)$-connected for $\ell \geqslant 0$.

$$\begin{array}{cccccc} S^0 & \Sigma S^0 = S^1 & \Sigma^2 S^0 = S^2 & \Sigma^3 S^0 = S^3 & \cdots & \Sigma^n S^0 = S^n \\ -1 & 0 & 1 & 2 & \cdots & (n-1) \end{array}$$

corollary 24. S^n is $(n-1)$-connected.

Back to Hopf fibration:
$$S^1 \hookrightarrow S^3 \to S^2$$
we get
$$0 = \pi_2(S^3) \to \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \to \pi_1(S^3) = 0,$$
so
$$\mathbb{Z} = \pi_2(S^2).$$
Now consider a map $S^n \to S^n$. We get a map $CS^n \to CS^n$ (in general, for $f : X \to Y$ we have $(t, x) \mapsto (t, f(x))$ in the cones). We also have $CS^n \to CS^n/S^n = S^{n+1}$.

Now if we take $\mathrm{id} : S^n \to S^n$ we shall get $\mathrm{id} : S^{n+1} \to S^{n+1}$. Think about this like $\pi_1(S^1) \to \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective because $i = 0$. From Hopf fibration we have $\pi_2(S^2) = \mathbb{Z}$. So we have a surjective map $\mathbb{Z} \to \mathbb{Z}$. So it's an isomorphism and we conclude that id_{S^2} is a generator of $\pi_2(S^2)$.

corollary 25. Since S^n is $(n-1)$-connected, we have
$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$
is isomorphism for $i \leqslant 2(n-1) = 2n-1$ and epimorphism form $i = 2n-1$. (We just shift the indices of theorem 22 by one.)

corollary 26. $\pi_n(S^n) = \mathbb{Z}$ with id_{S^n} as generator.

corollary 27. $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \leqslant n-1$ and epimorphism for $k = n-1$.

So for example
$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$
And in fact they are $\mathbb{Z}/2$. This is what are called the k***th stable homotopy groups of a sphere***. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from
$$S^1 \hookrightarrow S^3 \to S^2$$
we get
$$0 = \pi_i(S^1) \to \pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2) \to \pi_{i-1}(S^2) = 0$$
So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

claim (Serre)**.** $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z}\oplus$finite abelian. And $\pi_i(S^k)$ is finite abelian in all other cases.

another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{tikzcd} S^{n-1} \arrow[r, "(n-1)\text{-equiv}"] \arrow[d, "0\text{-equiv}"'] & D^n \arrow[d] \\ X \arrow[r] & X \cup D^n \end{tikzcd}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is $(n-1)$-equivalence since $S^{n-1} \to D^n$ is an $(n-1)$-equivalence, which is the case since

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

so $\pi_i(D^n, S^{n-1}) = 0$ for $i \leqslant n-1$.

References

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