

homotopy theory

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abstract nonsense

definition.

- An *initial object* in a category \mathcal{C} is an object \emptyset such that for any object $x \in \mathcal{C}$ there is a unique morphism $\emptyset \rightarrow x$ with source \emptyset and target x .
- For \mathcal{C} any category, its *arrow category* $\text{Arr}(\mathcal{C})$ is the category such that
 - an object a of $\text{Arr}(\mathcal{C})$ is a morphism $a : a_0 \rightarrow a_1$ of \mathcal{C} ,
 - a morphism $f : a \rightarrow b$ of $\text{Arr}(\mathcal{C})$ is a commutative square

$$\begin{array}{ccc} a_0 & \xrightarrow{f_0} & b_0 \\ a \downarrow & & \downarrow b \\ a_1 & \xrightarrow{f_1} & b_1 \end{array}$$

in \mathcal{C} ,

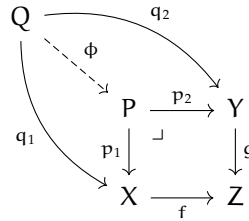
- composition in $\text{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

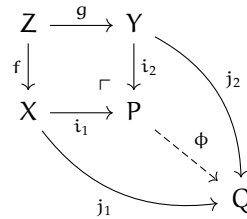
$$\text{Arr}(\mathcal{C}) := \text{Funct}(I, \mathcal{C}) = [I, \mathcal{C}] = \mathcal{C}^I$$

for I the interval category $\{0 \rightarrow 1\}$.

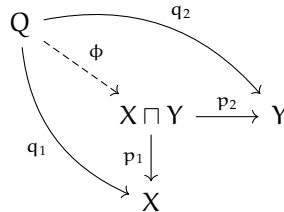
- A **pullback** of the morphisms f and g consists of an object P and two morphisms $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ satisfying the following universal property:



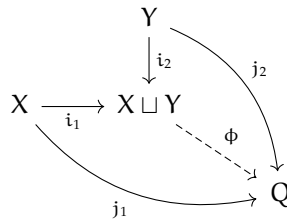
- A **pushout** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \rightarrow X$ and $i_2 : P \rightarrow Y$ satisfying the following universal property:



- A **product** of X and Y is an object $X \sqcap Y$ and a pair of morphisms $p_1 : X \sqcap Y \rightarrow X$, $p_2 : X \sqcap Y \rightarrow Y$ satisfying the following universal property:



- A **coproduct** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \rightarrow X \sqcup Y$, $i_2 : Y \rightarrow X \sqcup Y$ satisfying the following universal property:



remark. More generally, for S any set and $F : S \rightarrow \mathcal{C}$ a collection of objects in \mathcal{C} indexed by S , their *coproduct* is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

$$F(s) \rightarrow \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

- A morphism i has the *left lifting property with respect to a morphism* p and p has the *right lifting property with respect to* i if for each morphisms f and g , if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \nearrow \phi & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

- The *kernel* of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the *kernel* $\ker f$ of a morphism $f : A \rightarrow B$ is the pullback $\ker(f) \rightarrow A$ along f of the unique morphism $0 \rightarrow B$

More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

for every object C and every morphism $h : C \rightarrow A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \rightarrow \ker(f)$ such that $h = p \circ \phi$.

$$\begin{array}{ccccc} C & & \xrightarrow{\phi} & \ker(f) & \longrightarrow & 0 \\ & \searrow h & & \downarrow & & \downarrow \\ & & A & \xrightarrow{f} & B \end{array}$$

- In a category with a terminal object 1 , the *cokernel* of a morphism $f : A \rightarrow B$ is the

pushout (arrows h and ϕ apply if terminal object is zero)

$$\text{coker}(f) := 1 \sqcup_A B$$

In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\text{coker}(f)$ with the following universal property:

for every object C and every morphism $h : B \rightarrow C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \text{coker}(f) \rightarrow C$ such that $h = \phi \circ i$.

- A morphism $f : X \rightarrow Y$ is a **monomorphism** if for every object Z and every pair of morphisms $g_1, g_2 : Z \rightarrow X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

Equivalently, f is a monomorphism if for every Z the hom-functor $\text{Hom}(Z, -)$ takes it to an injective function

$$\text{Hom}(Z, X) \xleftarrow{f_*} \text{Hom}(Z, Y).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an epimorphism in the opposite category \mathcal{C}^{op} .

- A morphism $f : X \rightarrow Y$ is a **epimorphism** if for every object Z and every pair of morphisms $g_1, g_2 : Y \rightarrow Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\text{Hom}(-, Z)$ takes it to an injective function

$$\text{Hom}(Y, Z) \xleftarrow{f^*} \text{Hom}(X, Z).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an monomorphism in the opposite category \mathcal{C}^{op} .

elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \rightarrow Y$ continuous maps. An *homotopy* from f to g is a continuous map

$$H : X \times [0, 1] \rightarrow Y, \quad (x, t) \mapsto H(x, t) = H_t(x)$$

such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation \simeq is an equivalence relation on the set of continuous maps $X \rightarrow Y$. A homotopy of maps $H_t : X \rightarrow Y$ is called *relative to* $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top , the *homotopy category* h-Top , where composition of homotopy classes is induced by composition of representing maps. If $f : X \rightarrow Y$ represents an isomorphism in h-Top , then f is called a *homotopy equivalence* or *h-equivalence*. In explicit terms this means $f : X \rightarrow Y$ is a homotopy equivalence if there exists $g : Y \rightarrow X$, a *homotopy inverse of* f , such that gf and fg are both homotopic to the identity. Spaces X and Y are called *homotopy equivalent* or of the same *homotopy type* if there exists a homotopy equivalence $X \rightarrow Y$. A space is *contractible* if it is homotopy equivalent to a point. A map $f : X \rightarrow Y$ is *null homotopic* if it is homotopic to a constant map.
- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the *n-th homotopy group* are homotopy classes of maps $(S^n, s_0) \rightarrow (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \rightarrow (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

Also,

$$\pi_n(X, *) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n / \partial I^n, X]^0$$

where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \rightarrow Y$.

proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the *relative homotopy group* $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n . That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

The elements of such a group are homotopy classes of based maps $D^n \rightarrow X$ which carry the boundary S^{n-1} into A . Two maps f, g are called *homotopic relative to* A if they are homotopic by a basepoint-preserving homotopy $F : D^n \times [0, 1] \rightarrow X$ such that, for each p in S^{n-1} and t in $[0, 1]$, the element $F(p, t)$ is in A . Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

remark. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(A, x_0) \xrightarrow{\partial} \pi_{n-1}(X, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0)$$

where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ to I^{n-1} , or by restricting maps $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$. The map, called the **boundary map**, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called **n -connected** if $\pi_i(X, x_0) = 0$ for $i \leq n$. Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair (X, A) is **n -connected** if $\pi_i(X, A, x_0) = 0$ for $i \leq n$.
- Two pointed spaces (X, x_0) and (Y, y_0) are **n -equivalent** if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i \leq n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on **compactly generated weakly Hausdorff** spaces.
- X is **compactly generated** if, for any subset $C \subset X$, and for all continuous maps $f : K \rightarrow X$ from compact Hausdorff spaces,

if $f^{-1}(C)$ is closed in K , then C is closed.

claim (What I picked up from the lecture). If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is **k -closed**.

From May, *A Concise Course in Algebraic Topology*: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi : X \rightarrow Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta_Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a “closed equivalence relation” is compactly generated.

Several other propositions follow in May, *A Concise Course in Algebraic Topology*. Now some other notes from the lectures:

In CGWH, $\text{Hom}(X, Y)$ is a space with the compact-open topology. This is a compactly generated space, $\mathbf{k}(\text{Hom}(X, Y))$.

$\text{Map}(X, Y) :=$ the space of maps $X \rightarrow Y$.

$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$

$\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, \text{Map}(Y, Z))$

In the last line, product is product in CGWH, not in Top .

The functor $- \times Y$ is left adjoint to $\text{Map}(Y, -)$.

cofibrations

- A *homotopy* $X \times I \rightarrow Y$ is the same as a map $X \rightarrow \text{Map}(I, Y)$.
- A map $A \rightarrow X$ is a *Hurewicz cofibration* for any $g : X \rightarrow Y$ and any homotopy $H : A \times I \rightarrow Y$ such that

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & A \times I \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

there is $H : X \times I \rightarrow Y$,

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & A \times I \\ \downarrow g & & \downarrow \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

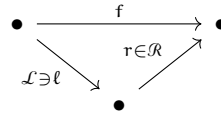
$$\begin{array}{ccc} A \times I & & \\ \downarrow & \searrow H & \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

example. $\partial D^n \rightarrow D^n$ is a Hurewicz cofibration. Why?

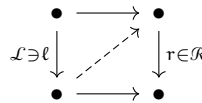
model structures

definition (Riehl, *Homotopical categories: from model categories to $(\infty, 1)$ -categories*). A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{M} is comprised of two classes of morphisms \mathcal{L} and \mathcal{R} so that

1. Every morphism in \mathcal{M} may be factored as a morphism in \mathcal{L} followed by a morphism in \mathcal{R} :

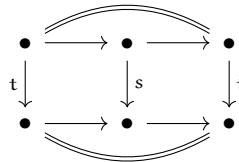


2. The maps in \mathcal{L} have the *left lifting property* with respect to each map in \mathcal{R} and equivalently the maps in \mathcal{R} have the *right lifting property* with respect to each map in \mathcal{L} , that is, any commutative square



admits a diagonal filler as indicated making both triangles commute.

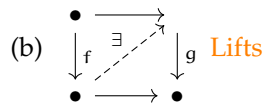
3. The classes \mathcal{L} and \mathcal{R} are each closed under retracts in the arrow category: given a commutative diagram



if s is in that class then so is its retract t .

definition (Lecture). A *model structure* on a category \mathcal{A} is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called *weak-equivalences*, *cofibrations* and *fibrations* with the following properties:

1. Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in \mathcal{W} then all of them are.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system.
 - (a) Any morphism in \mathcal{A} can be factored as a morphism in \mathcal{B} followed by a morphism in \mathcal{D} .



Two interesting model category structures on CGWH.

1. Hurewicz model structure (Strom).
 - Cofibrations:= Hurewicz cofibrations.
 - Fibrations:= maps $E \rightarrow B$ such that for all spaces X [Photo1].

- Weak equivalences:= homotopy equivalences.

2. Quillen model structure.

- Cofibrations = retracts of relative cell complexes.

• (Serre) Fibrations =

$$\begin{array}{ccc} D^n & \longrightarrow & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ D^n \times I & \longrightarrow & B \end{array}$$

- Weak equivalences: $f : X \rightarrow Y$

exercise (3.1.8 from Riehl's "Homotopical categories: ..."). Verify that the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. (Coproducts.) Sergey: Coproduct of morphisms $A_i \rightarrow B_i$ in a category \mathcal{C} is the obvious morphism $\coprod A_i \rightarrow \coprod B_i$. (Because in this construction morphisms $A_i \rightarrow B_i$ are seen as objects of what's called the arrow category of the category \mathcal{C})

Suppose the maps $\ell_i : A_i \rightarrow B_i$ are in \mathcal{L} . Then their coproduct in the arrow category is the obvious map $\coprod A_i \rightarrow \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \rightarrow \coprod \ell_i$ such that for any other object $m : A \rightarrow B$ in the arrow category and a map $g : \ell \rightarrow m$, the following diagram is completed uniquely:

$$\begin{array}{ccccc} \ell_i & \xrightarrow{f_i} & \coprod \ell_i & \xrightarrow{\exists!} & m \\ & \searrow & \downarrow g & \nearrow & \\ & & A & \xrightarrow{m} & B \end{array} \quad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{array}{ccc} A_i & \xrightarrow{\ell_i} & B_i \\ f_i^1 \downarrow & & \downarrow f_i^2 \\ \coprod A_i & \xrightarrow{\coprod \ell_i} & \coprod B_i \\ \exists! \downarrow & & \downarrow \exists! \\ A & \xrightarrow{m} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{array}{ccc} \coprod A_i & \longrightarrow & \bullet \\ \downarrow \coprod \ell_i & & \downarrow r \in \mathcal{R} \\ \coprod B_i & \longrightarrow & \bullet \end{array}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

which in turn means we have unique maps

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

by the universal property of the coproduct $\coprod B_i$.

So, to check that the lower-right triangle commutes, it would be sufficient to show that the map $B_i \rightarrow \coprod B_i$ "can be cancelled" since

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} \quad \text{is already the same as} \quad \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

Likeways, to make sure that the remaining triangle commutes we observe that

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} \quad \text{is already the same as} \quad \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

□

remark (Plan). Blakers-Massey excision theorem (relies on technical lemma, proof from Tom Dieck's book) \implies Cellular approximation. Also \implies Freudenthal theorem.

exercise. $X \rightarrow M_f \rightarrow Y$. Prove $X \rightarrow M_f$ is a cofibration.

Whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A CW complex X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called vertices) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along **attaching maps** $j : S^n \rightarrow X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying (j, x) with $j(x)$ for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j . Each resulting map $D^{n+1} \rightarrow X$ is called a *cell*. The subspace X^n is called the *n-skeleton* of X .

$$\begin{array}{ccc} S^n & \xhookrightarrow{i} & D^{n+1} \\ j \downarrow & \lrcorner & \downarrow \\ X^n & \longrightarrow & X^{n+1} \end{array}$$

lemma 3 (HELP). content...

Theorem 4 (Whitehead, May). If X is a CW complex and $e : Y \rightarrow Z$ is an n -equivalence, then $e_* : [X, Y] \rightarrow [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 5 (Whitehead, May). An n -equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 6 (4.5, Hatcher). If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ for all n , then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y .

exercise (Hatcher 4.1.12). Show that an n -connected, n -dimensional CW complex is contractible.

Solution. Just recall that n -connectedness means that $\pi_i(X) = 0$ for all $i \leq n$, which means that X is contractible by ??.

lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \rightarrow X$$

$$\text{Hom}(X, Y) \mapsto \text{Hom}(X', Y)$$

$$\begin{aligned} \text{Hom}(A, B) \cong \text{Hom}(A, B') \text{ natural in } A &\implies \\ \text{Hom}(B, B) \cong \text{Hom}(B, B') \&\text{Hom}(B', B) \cong \text{Hom}(B', B') \\ &\implies B \cong B'. \end{aligned}$$

- for (\Leftarrow) commutativity of the hypothesis gives us commutativity of the right-most square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- Some good concepts are pushouts, coproducts, direct limits.
- What we did? Prove the bijection between the homotopic sets given an n -equivalence.
- Defined smash.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 7 (Yoneda).

$$\{\text{Natural transformations } \text{Hom}(-, X) \rightarrow F\} \cong F(X)$$

corollary 8. $(\text{Hom}(-, X) \rightarrow \text{Hom}(-, Y)) \cong \text{Hom}(X, Y)$.

corollary 9. The correspondence $X \mapsto \text{Hom}(-, X)$ is fully faithful, that is, the correspondence $\text{Hom}(X, X') \rightarrow \text{Hom}(\text{Hom}(-, X), \text{Hom}(-, X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. \square

lemma 10. Let $E \times_B X$ be the pullback of

$$\begin{array}{ccc} & E & \\ & \downarrow & \\ X & \xrightarrow{\simeq} & B \end{array}$$

be such that $E \rightarrow B$ is an homotopy fibration and $f : X \rightarrow B$ is a homotopy equivalence. Let

$$\begin{array}{ccccc} E \times_B X & \rightarrow & E & \xrightarrow{\simeq} & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\simeq} & B & & \end{array}$$

be the pullback. Then $E \times_B X \rightarrow E$ is a homotopy equivalence.

Proof. Let $g : B \rightarrow X$ be the homotopy inverse of f .

(Step 1) Construct another pullback

$$\begin{array}{ccccc} E \times_B B & \longrightarrow & X \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{g} & X & \xrightarrow{f} & B \end{array}$$

(Step 2) Construct $E \rightarrow E \times_B B$.

Consider

$$\begin{array}{ccccc} E & \xrightarrow{\text{id}} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ E \times I & \xrightarrow{f \times \text{id}} & B \times I & \longrightarrow & B? \end{array}$$

And then $E \rightarrow E \times_B B \rightarrow E \times_B X \rightarrow E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{array}{ccccc} Z \sqcup Z & \longrightarrow & I \times Z \\ \downarrow f_1 \sqcup f_2 & \nearrow \text{dashed} & \downarrow & \searrow \text{curved} & \\ E \times_B X & \longrightarrow & E & & \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{\simeq} & B & & \end{array}$$

□

corollary 11. $B \xrightarrow{f} B$ is homotopy equivalence and $E \rightarrow B$ is a fibration, in

$$\begin{array}{ccc} E \times_B B & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B \end{array}$$

$E \times_B B \rightarrow E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 12. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \\ X & \longrightarrow & X \cup_A B \end{array}$$

be a pushout with $A \rightarrow X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f, g) \rightarrow X \cup_A B$ is a homotopy equivalence.

remark.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ X & & \end{array} \quad \begin{array}{ccc} A & \hookrightarrow & M_f \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_A M_f \cong M(f, g) \end{array}$$

definition.

- The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is

$$\begin{array}{ccc} X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y .

- The *homotopy fiber* if $f : Y \rightarrow Z$ is the pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ \text{pt} & \longrightarrow & Z \end{array}$$

$F \subset Z^I \times_Z Y \rightarrow Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark. The pullback of

$$\begin{array}{ccc} & & Z^I \times_Z Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is the motopy pullback of

$$\begin{array}{ccc} & Y & \\ & \downarrow & \\ X & \longrightarrow & Z \end{array}$$

lemma 13. If $X \rightarrow Z$ is a fibration then for

$$\begin{array}{ccc} & Y & \\ & \downarrow & \\ X & \longrightarrow & Z \end{array}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & Y \\ \downarrow \simeq & & \downarrow \simeq \\ X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Z^I \times_Z Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

□

Finally,

$$\begin{array}{ccccc} \text{hofib } f_1 & \longrightarrow & \text{hofib } f & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

and

$$\begin{array}{ccc} Z & \longrightarrow & F(f) \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & X \times_Y Y^I \end{array}$$

and an exact sequence

$$\Omega^2 \text{hofib} \rightarrow \Omega^2 X \rightarrow \Omega^2 Y \rightarrow \Omega \text{hofib } f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow \text{hofib } f \rightarrow X \xrightarrow{f} Y$$

lemma 14 (Exactness). $\forall z, [z \text{hofib } f] \rightarrow [Z, X] \rightarrow [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \rightarrow \pi_0(\Omega^2 Y) \rightarrow \pi_0(\Omega \operatorname{hofib} f) \rightarrow \pi_0(\Omega X) \rightarrow \pi_0(\Omega Y) \rightarrow \pi_0(\operatorname{hofib} f) \rightarrow \pi_0(X) \rightarrow \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

21 march (Serre fibration long exact sequence)

We've been talking a lot about Hurewicz fibrations. Let's talk about Serre fibrations. Notice that H. fibration \implies S. fibration. What is the most natural example of a Serre fibration?

proposition 15. Let E be a fiber bundle with fiber F . Then f is a Serre fibration.

Proof. What does it mean to be a Serre fibration? It means that

$$\begin{array}{ccc} I^n & \xrightarrow{\quad} & E \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} = I^n \times I & \longrightarrow & B \end{array}$$

So if \mathcal{U} is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U .

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{array}{ccc} I^n & \xrightarrow{\quad} & F \times U \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} & \longrightarrow & U \end{array}$$

has a lift for every little square because

$$\begin{array}{ccc} X & \xrightarrow{\quad} & U \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & \text{pt} \end{array}$$

is always a fibration (**think about this**) and because pullbacks of fibrations are fibrations:

$$\begin{array}{ccc} U \times F & \longrightarrow & U \\ \downarrow & & \downarrow \\ F & \longrightarrow & \text{pt} \end{array}$$

. Then we may just add up the squares because

$$\begin{array}{c} D^n \\ \downarrow \\ D^n \times I \end{array}$$

and we're done. \square

proposition 16 (Construction of homotopy long exact sequence from relative homotopy long exact sequence). Let $g : E \rightarrow B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \cong \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(E, F) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \end{array}$$

example. We have shown that $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 17. Let X be a CW-complex, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \emptyset$, so

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a pushout (this happens for inclusions, **check it?**).

If (A, C) is n -connected and (B, C) is m -connected, then

$$\pi_i(A, C) \rightarrow \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and surjective for $i = m + n$.

26 march (Blakers-Massey)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \rightarrow Y$ be a map. Consider the pullback

$$\begin{array}{ccc} W(f) & \longrightarrow & Y^I \\ (q, p) \downarrow & & \downarrow (ev_0, ev_1) \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

where

$$W(f) = \{(x, w) \in X \times Y^I \mid f(x) = w(0)\},$$

$$q(x, w) = x, \quad p(x, w) = w(1).$$

Since (ev_0, ev_1) is a fibration, the maps (q, p) , q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \rightarrow W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \rightarrow (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \rightarrow X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \rightarrow *$.
- $I^n \times \{1\} \rightarrow A$.

It is fairly straightforward to show that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega A & \longrightarrow & \Omega X & \longrightarrow & \text{hofib} \longrightarrow A \longrightarrow X \\ \pi_0(\nearrow) = & & \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_{n-1}(\text{hofib}) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(X) \\ & & & & \searrow & \downarrow \cong & \nearrow \\ & & & & & \pi_n(X, A) & \end{array}$$

Theorem 18 (Blakers-Massey 1). Let

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ f \downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

be a homotopy pushout, g is m equivalence, f is n -equivalence and $m, n \geq 0$. Then $Q \rightarrow X \times_P^h Y$ is $(m + n - 1)$ -equivalence.

Theorem 19 (Blakers-Massey 2). P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \emptyset$ (*strict pushout*)

$$\begin{array}{ccc} Q & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & X \end{array} \quad \lrcorner$$

Then $\pi_i(Y, Q) \rightarrow \pi_i(P, X)$ is epi for $i = m + n$ and iso for $0 \leq i < m + n$.

Theorem 20 (Blakers-Massey 3). $P = X \cup Y$, X and Y are open in P , $X \cap Y = Q \neq \emptyset$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a *k-equivalence* if the induced map on the i th homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W : \text{at least } p \text{ coordinates of } x \text{ are the same coordinates of the center of } W\}$

lemma 21. Let W be a cube in \mathbb{R}^d with $\dim W \leq d$. If for all faces W' of ∂W , $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \text{ rel } \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

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