# homotopy theory

## Contents



## abstract nonsense

**definition.**

- **(Limits, wiki.)**
  - A ***diagram*** of shape J in C is a functor from J to C
  $$F : J \to C.$$
  The category J is thought of as an index category, and the diagram F is thought of as indexing a collection of obtects and morphisms in C patterned on J.
  - Let $F : J \to C$ be a diagram of chape J in a category C. A ***cone*** to F is an object N to C together with a family $\psi_X : N \to F(X)$ of morphisms indexed by the objects X of J (so a cone is an object and a bunch of maps from this object to certain objectes that are governed by the diagram), so that for every morphism $X \to Y$ in J, we have $F(f) \circ \psi_X = \psi_Y$ I guess this is what nLab meant when he said that everything in sight commutes).
  - A ***limit*** of the diagram $F : J \to C$ is a cone $(L, \phi)$ to F such that for every cone $(N, \psi)$ there exists a *unique* morphism $u : N \to L$ such that $\phi_X \circ u = \psi_X$ for all X in J.
  $$\begin{tikzcd} & N \arrow[ddl, bend right, "\psi_X"'] \arrow[d, dashed, "u"] \arrow[ddr, bend left, "\psi_Y"] & \\ & L \arrow[dl, "\phi_X"'] \arrow[dr, "\psi_Y"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$
  One says that the cone $(N, \psi)$ factors through the cone $(L, \phi)$ with the unique factorization $u$. The morphism $u$ is sometimes called the ***mediating morphism***.

  Limits are also referred to as ***universal cones*** since they are characterized by a universal property. Limits may also be caracterized as terminal objects in the category of cones to F.

  It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of F.
- **(Colimits, wiki)** The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:
  - A ***co-cone*** of a diagram $F : J \to C$ is an object N of C together with a family of morphisms $\psi_X : F(X) \to N$ (so in the cone we are going *from* N and now we're going *to*N) for every object X of J, such that for every morphism $f : X \to Y$ in J we have $\psi_Y \circ F(f) = \psi_X$ everything in sight commutes.
  - A ***colimit*** of a diagram $F : J \to C$ is a co-cone $(L, \phi)$ of F such that for any other co-cone $(N, \psi)$ of F there exists a unique morphism $u : L \to N$ such that

$u \circ \phi_X = \psi_X$ for all $X$ in $J$.

$$\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\phi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}$$

Colimits are also referred to as ***unersal co-cones***. They can be characterized as initial objects in the category of co-cones from $F$.

As with limits, if a diagram $F$ has a colimit then this colimit is unique up to a unique isomorphism.

- An ***initial object*** in a category $\mathcal{C}$ is an object $\varnothing$ such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source $\varnothing$ and target $x$.
- For $\mathcal{C}$ any category, its ***arrow category*** $\mathrm{Arr}(\mathcal{C})$ is the category such that
  - an object $a$ of $\mathrm{Arr}(C)$ is a morphism $a : a_0 \to a_1$ of $\mathcal{C}$,
  - a morphism $f : a \to b$ of $\mathrm{Arr}(\mathcal{C})$ is a commutative square

$$\begin{tikzcd}
a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\
a_1 \arrow[r, "f_1"'] & b_1
\end{tikzcd}$$

    in $\mathcal{C}$,
  - composition in $\mathrm{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

  This is isomorphic to the functor category

$$\mathrm{Arr}(\mathcal{C}) := \mathrm{Funct}(I, C) = [I, C] = C^I$$

  for I the intervale category $\{0 \to 1\}$.
- An ***equalizer*** is a limit

$$\begin{tikzcd}
eq \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

  over a parallel pair of morphisms $f$ and $g$. This means that for $f : X \to Y$ and $g : X \to Y$ in a category $\mathcal{C}$, their equalizer, if it exists, is
  - an object $eq(f, g) \in \mathcal{C}$,
  - a morphism $eq(f, g) \to X$
  - such that

* pulled back to $\mathrm{eq}(f,g)$ both morphisms become equal:

$$\mathrm{eq}(f,g) \longrightarrow X \xrightarrow{f} Y \quad = \quad [\ \mathrm{eq}(f,g) \longrightarrow X \xrightarrow{g} Y$$

* and $\mathrm{eq}(f,g)$ is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions $f$ and $g$ between sets $X$ and $Y$ one forms the set $Y/\sim$ of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the the quotient function $p : Y \to Y/\sim$ satisfies

$$p \circ f = p \circ g.$$

In some category $\mathcal{C}$, the ***coequalizer*** $\mathrm{coeq}(f,g)$ of two parallel morphisms $f$ and $g$ between two objects $X$ and $Y$, if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{coeq}(f,g) & \end{tikzcd}$$

Equivalently, in a category $\mathcal{C}$ a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

is called a ***coequalizer*** diagram if

1. $p \circ f = p \circ g$,
2. $p$ is universal for this property: if $q : Y \to W$ is a morphism of $\mathcal{C}$ such that $q \circ f = q \circ g$, then there is a unique morphism $\phi : Z \to W$ such that $\phi \circ p = q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] \arrow[d, "q"] & Z \arrow[dl, dashed, "\phi"] \\ & W & \end{tikzcd}$$

The coequalizer in $\mathcal{C}$ is equivalently an equializer in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A ***pullback*** of the morphisms $f$ and $g$ consists of an object $P$ and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

- A ***pushout*** of the morphisms $f$ and $g$ consists of an object $P$ and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

**remark.** Other names for the pushout are ***cofibered product of*** $X$ ***and*** $Y$ (especially in algebraic categories when $i_1$ and $i_2$ are monomorphisms), or ***free product of*** $X$ ***and*** $Y$ with $Z$ ***amalgamated sum***, or more simply an ***amalgamation*** or ***amalgam of*** $X$ ***and*** $Y$.

**remark.** If coproducts exist in some category, then the pushout

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\
X \arrow[r, "i_1"'] & X \coprod_Z Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is equivalently the coequalizer

$$X \underset{i_2 \circ g}{\overset{i_1 \circ f}{\rightrightarrows}} X \coprod Y \longrightarrow X \coprod_Z Y$$

of the two morphisms induced by $f$ and $g$ into the coproduct of $X$ with $Y$.

**example (wiki).**

- If $X$, $Y$ and $Z$ are sets and $f, g$ are functions, the pushout of $f$ and $g$ is the disjoint union of $X$ and $Y$ where elements sharing a common preimage in $Z$ are identified, i.e. $P = (X \coprod Y) / \sim$ where $\sim$ is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

  In particular, if $X$ and $Y$ are subsets of some larger set $W$ and $Z$ is their intersection, with $f$ and $g$ the inclusion maps of $Z$ into $X$ and $Y$, then the pusout can be canonically identified with the union $X \cup Y \subseteq W$.

- The construcion of ***adjunction spaces*** is an example of pushouts in Top. More precisely, if $Z$ is a subspace of $Y$ and $g : Z \to Y$ is the inclusion map, we can glue $Y$ to another space $X$ along $Z$ using an ***attaching map*** $f : Z \to X$. The result is the ***adjunction space*** $X \cup_f Y$ which is just the pushout of $f$ and $g$. More generally, all identification spaces may be regarded as pushouts in this way. See **??** .

- A ***product*** of $X$ and $Y$ is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \sqcap Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}$$

- A ***coproduct*** of $X$ and $Y$ is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$, $i_2 : Y \to X \sqcup Y$ satisfying the following universal property:

$$\begin{tikzcd}
& Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & X \sqcup Y \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

  **remark.** More generally, for $S$ any set and $F : S \to \mathcal{C}$ a collection of objects in $\mathcal{C}$ indexed by $S$, their ***coproduct*** is an object

$$\coprod_{s \in S} F(s)$$

  equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

  such that this is universal among objects with maps from $F(s)$.

- The ***kernel*** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object $0$ and pullbacks, the ***kernel*** $\ker f$ of a morphism $f : A \to B$ is the pullback $\ker(f) \to A$ along $f$ of the unique morphism $0 \to B$

  More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

  > for every object $C$ and every morphism $h : C \to A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \to \ker(f)$ such that $h = p \circ \phi$.

$$
\begin{tikzcd}
C \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "h"'] & & \\
& \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}
$$

- In a category with a terminal object 1, the ***cokernel*** of a morphism $f : A \to B$ is the pushout (arrows $h$ and $\phi$ apply if terminal object is zero)

$$
\operatorname{coker}(f) := 1 \sqcup_A B
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & B \arrow[d] \arrow[ddr, bend left, "h"] & \\
1 \arrow[r] & \operatorname{coker}(f) \arrow[dr, dashed, "\phi"] & \\
& & C
\end{tikzcd}
$$

  In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\operatorname{coker}(f)$ with the following universal property:

  > for every object $C$ and every morphism $h : B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \operatorname{coker}(f) \to C$ such that $h = \phi \circ i$.

- A morphism $f : X \to Y$ is a ***monomorphism*** if for every object $Z$ and every pair of morphisms $g_1, g_2 : Z \to X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$
\begin{tikzcd}
Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[rr, bend left, "f \circ g_1"] \arrow[rr, bend right, "f \circ g_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

  Equivalently, $f$ is a monomorphism if for every $Z$ the hom-functor $\operatorname{Hom}(Z, -)$ takes it to an injective function

$$\operatorname{Hom}(Z, X) \xhookrightarrow{f_*} \operatorname{Hom}(Z, Y).$$

  Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A morphism $f : X \to Y$ is a ***epimorphism*** if for every object $Z$ and every pair of morphisms $g_1, g_2 : Y \to Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

Equivalently, $f$ is a epimorphism if for every $Z$ the hom-functor $\mathrm{Hom}(-, Z)$ takes it to an injective function

$$\mathrm{Hom}(Y, Z) \xhookrightarrow{f^*} \mathrm{Hom}(X, Z).$$

Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- (Retraction.)
  - (wiki) Let $X$ be a topological space and $A$ a subspace of $X$. Then a continuous map $r : X \to A$ is a ***retraction*** if the restriction of $r$ to $A$ is the identity map on $A$.
  - (nLab) An object $A$ in a category is called a ***retract*** of an object $B$ if there are morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{id}_A$. In this case $r$ is called a ***retraction of*** $B$ ***onto*** $A$ and $i$ is called a ***section of*** $r$.

    $$\mathrm{id} : A \xrightarrow[\text{section}]{i} B \xrightarrow[\text{retraction}]{r} A$$

    Hence a ***retraction*** of a morphism $i : A \to B$ is a left-inverse and a ***section*** of a morphism $r : B \to A$ is a right-inverse.
- (Deformation retract.)
  - (nLab) Let $\mathcal{C}$ be a category equipped with a notion of homotopy between its morphisms. Then a ***deformation retraction*** of a morphism $i : A \to X$ is another morphism $r : X \to A$ such that

    ?

  - (wiki) A continuous map $F : X \times [0, 1] \to X$ is a ***deformation retraction*** of a space $X$ into a subspace $A$ if, for every $x$ in $X$ and $a$ in $A$,

    $$F(x, 0) = x, \qquad F(x, 1) \in A \quad \text{and} \quad F(a, 1) = a.$$

    In words, a deformation retraction is a homotopy between a retraction and the identity map on $X$. The subspace $A$ is called a ***deformation retract*** of $X$. A deformation retraction is a special case of a homotopy equivalence.

    An equivalent definition of deformation retraction is the following. A continuous map $r : X \to A$ is a ***deformation retraction*** if it is a retraction and its composititon with the inclusion is homotopic to the identity map on X.In this formulation, a deformation retraction carries with it a homotopy between the identity map on $X$ and itself.

– (wiki) If, in the definition of a deformation retraction we add the requirement that

$$F(a,t) = a \qquad \forall t \in [0,1],\ \forall a \in A,$$

then F is called a ***strong deformation retraction***. In words, a strong deformation retraction leaves points in A fixed throughout the homotopy.

**example.** $S^n$ is a strong deformation retract of $\mathbb{R}^{n+1}\backslash 0$ through $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$.

– (wiki) The inclusion of a closed subspace A in the space X is a **??** if and only if A is a ***neighbourhood deformation retract*** of X, meaning that there is a continuous map $u : X \to [0,1]$ with $A = u^{-1}(0)$ and a homotopy $H : X \times [0,1] \to X$ such that $H(x,0) = x$ for all $x \in X$, $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$, and $H(x,1) \in A$ if $u(x) < 1$.

For example, the inclusion of a subcomplex in a CW complex is a cofibration.

## elementary concepts

**definition.**

- Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. An ***homotopy*** from f to g is a continuous map

  $$H : X \times [0,1] \to Y, \qquad (x,t) \mapsto H(x,t) = H_t(x)$$

  ) such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. A homotopy of maps $H_t : X \to Y$ is called ***relative to*** $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top, the ***homotopy category*** h-Top, where comoposition of homotopy classes is induced by composition of representing maps. If $f : X \to Y$ represents an isomorphism in h-Top, then f is called a ***homotopy equivalence*** or h-***equivalence***. In explicit termins this means $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$, a ***homotopy inverse of*** f, such that $gf$ and $fg$ are both homotopic to the identity. Spaces X and Y are called ***homotopy equivalent*** or of the same ***homotopy type*** if there exists a homotopy equivalence $X \to Y$. A space is ***contractible*** if it is homotopy equivalent to a point. A map $f : X \to Y$ is ***null homotopic*** if it is homotopic to a constant map.

- Let $(X, x_0)$ be a pointed topological space and $s_0 \in S^n$. The elements of the n***-th homotopy group*** are homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

  Also,

  $$\pi_n(X,*) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n/\partial I^n, X]^0$$

where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \to Y$.

**proposition 1.** $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let $A$ be a subspace of $X$ and $x_0 \in A$. The elements of the ***relative homotopy group*** $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where $J^{n-1}$ is the union of all but one face of $I^n$. That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

  The elements of such a group are homotopy classes of based maps $D^n \to X$ which carry the boundary $S^{n-1}$ into $A$. Two maps $f, g$ are called ***homotopic relative to*** $A$ if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0, 1] \to X$ such that, for each $p$ in $S^{n-1}$ and $t$ in $[0, 1]$, the element $F(p, t)$ is in $A$. Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

  **remark.** This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair $(X, A, x)$ we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, x_0)$$

  where $i$ and $j$ are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map $\partial$ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to $I^{n-1}$, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. The map, called the ***boundary map***, is a homomorphism when $n > 1$.

- A space $X$ with basepoint $x_0$ is called $n$***-connected*** if $\pi_i(X, x_0) = 0$ for $i \leqslant n$. Thus 0-connected means path-connected and 1 connected means simply-connected.

- A pair $(X, A)$ is $n$***-connected*** if $\pi(X, A, x_0) = 0$ for $i \leqslant n$.

- Two pointed spaces $(X, x_0)$ and $(Y, y_0)$ are $n$***-equivalent*** if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i < n$ and surjective for $i = n$.

## the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on ***compactly generated wakly Hausdorff*** spaces.

- $X$ is ***compactly generated*** if, for any subset $C \subset X$, and for all continuous maps $f : K \to X$ from compact Housdorff spaces,

$$\text{if } f^{-1}(C) \text{is closed in } K, \text{ then } C \text{ is closed.}$$

**claim** (What I picked up from the lecture)**.** If $X$ is compactly generated, then $X$ is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is k-closed.

From May: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

**proposition 2.** If $X$ is compactly generated and $\pi : X \to Y$ is a quotient map, then $Y$ is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH, $\mathrm{Hom}(X, Y)$ is a space with the compact-open topology. This is a compactly generated space, $\mathbf{k}(\mathrm{Hom}(X, Y))$.

**remark.** (Also see wiki on currying)

$$\begin{aligned}
\mathrm{Map}(X, Y) &:= \text{ the space of maps } X \to Y. \\
\mathrm{Map}(X \times Y, Z) &\cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\
\mathrm{Hom}(X \times Y, Z) &\cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z))
\end{aligned}$$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to $\mathrm{Map}(Y, -)$.

## cofibrations

**definition.**

- (wiki) In mathematics, in particular in homotopy theory, a continuous map between topological spaces $i : A \to X$ is a ***cofibration*** if it has the ***homotopy extension property*** with respect to all topological spaces $S$.

  That is, $i$ is a cofibration if

  - for each topological space $S$,
  - and for any continuous maps $f, f' : A \to S$
  - and $g : X \to S$ with $g \circ i = f$,
  - for any homotopy $h : A \times I \to S$ from $f$ to $f'$,

there is a continuous map $g' : X \to S$ and a homotopy $h' : X \times I \to S$ from $g$ to $g'$ such that

$$h'(i(a), t) = h(a, t) \qquad \text{for all } a \in A \text{ and } t \in I.$$

- ([wiki](wiki)) In what follows, let $I = [0, 1]$ denote the unit interval.

  A map $i : A \to X$ is a ***cofibration*** if for any map $f : A \to S$ such that there is an extension to $X$, meaning there is a map $\tilde{f} : X \to S$ such that $\tilde{f} \circ i = f$, we can extend a homotopy of maps $H : A \times I \to S$ to a homotopy of maps $\tilde{H} : X \times I \to S$ where

$$H(a, 0) = f(a)$$
$$\tilde{H}(x, 0) = \tilde{f}(x)$$

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "i"'] \arrow[dr, bend right=90, "f"'] & S^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}"'] \arrow[ur, dashed, "\tilde{H}"] & S
\end{tikzcd}$$

- ([wiki](wiki)) Let $X$ be a topological space and let $A \subset X$. We say that the pair $(X, A)$ has the ***homotopy extension property*** if, given a homotopy $f_\bullet : A \to Y^I$ and a map $\tilde{f}_0 : X \to Y$ such that

$$\tilde{f}_0 \circ \iota = f_0$$

  (so $\tilde{f}$ is the lift of $f_0 : A \to Y$) then there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{f}_\bullet : X \to Y^I$ such that $\tilde{f}_\bullet \circ \iota = f_\bullet$.

  That is,

$$\begin{tikzcd}
A \arrow[r, "f_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}_0"'] \arrow[ur, dashed, "\tilde{f}_\bullet"] & Y
\end{tikzcd}$$

  So there's some [currying](currying) to make usual homotopies $f_\bullet : A \times I \to Y$ look like $f_\bullet : A \to Y^I$. Or, as said in our lectures, "a homotopy $X \times I \to Y$ is the same as a map $X \to \mathrm{Map}(I, Y)$".

- (May) A map $i : A \to X$ is a ***cofibration*** if it satisfies the ***homotopy extension property (HEP)***. This means that if $h \circ i_0 = f \circ i$ in the diagram

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i \times id"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}$$

  then there exists $\tilde{h}$ that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map $i : A \to X$ between topological spaces is a ***Hurewicz cofibration*** if it satisfies the homotopy extension property.

  Let's say it one more time: for any $g : X \to Y$ and any homotopy $H : A \times I \to Y$ such that

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d] & A \times I \arrow[d] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

  there is $H' : X \times I \to Y$,

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d, "g"] & A \times I \arrow[d] \\
X \times I \arrow[r, "H'"'] & Y
\end{tikzcd}$$

  such that

$$\begin{tikzcd}
A \times I \arrow[d] \arrow[dr, "H"] & \\
X \times I \arrow[r, "H'"'] & Y
\end{tikzcd}$$

**example.** $\partial D^n \to D$ is a Huerwicz cofibration. Why?

**exercise.** Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

**definition** (Mapping cylinder)**.**

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. make the dashed map unique—up to something maybe). Namely, we can let $Y$ in our original test diagram be the ***mapping cylinder***

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of $i$ and $i_0$. Indeed, suppose that we can construct a map $r$ that makes the following diagram commute

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd, "i \times id"] \\
& Mi & \\
X \arrow[rr, "i_0"'] \arrow[ur] & & X \times I \arrow[ul, dashed, "r"]
\end{tikzcd}$$

By the universal property of the pushouts, given maps $f$ and $h$ in our original test diagram induce a map $Mi \to Y$, and its comoposite with $r$ gives a homotopy $\tilde{h}$ that makes the diagram commute. So just saying that $Mi$ is universal.

- (nLab) Given a continuous map $f : X \to Y$ of topological spaces, one can define its ***mapping cylinder*** as a pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X \times I \arrow[r, "(\sigma_0)_*(f)"'] & Cyl(f)
\end{tikzcd}$$

in Top, where $I = [0, 1]$ and $\sigma : X \to X \times I$ is given by $x \mapsto (x, 0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$\left(X \times I \coprod Y\right) \Big/ \sim$$

where $\sim$ is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

- (wiki) The ***mapping cylinder*** of a function $f$ between topological spaces $X$ and $Y$ is the quotient

$$M_f = (([0, 1] \times X) \amalg Y)/\sim$$

where $\amalg$ denotes disjoint union, and $\sim$ is the equivalence relation generated by

$$(0, x) \sim f(x) \text{ for each } x \in X.$$

That is, the mapping cylinder $M_f$ is obtained by gluing one end of $X \times [0, 1]$ to $Y$ via the map $f$. Notice that the "top" of the cylinder $\{1\} \times X$ is homeomorphic to $X$, while the "bottom" is the space $f(X) \subset Y$.

> (Dani) So the mapping cylinder is just deforming $X$ to $Y$ putting $X$ inside $Y$ via $f$.

- (Homework) Let $f : X \to Y$ be a map. Let $M_f = X \times [0, 1] \cup_f Y$ be the ***mapping cylinder of*** $f$, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$ and of $f : X \times Y$.

**exercise.** Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that $f$ is the composition of $g$ and $h$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \times [0,1] \arrow[r] \arrow[drr, bend right, "(x,t)\mapsto f(x)"'] \arrow[ur, phantom, "\ulcorner"] & M_f \arrow[dr, dashed, "h"] & \\
& & Y
\end{tikzcd}$$

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\operatorname{colim} A_i) \times B$ is canonically homeomorphic with $\operatorname{colim}(A_i \times B)$.

1. Show that $h$ is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

**exercise.** $X \to M_f \to Y$. Prove $X \to M_f$ is a cofibration.

## fibrations

- (nLab) A morphism $i$ has the ***left lifting property with respect to a morphism*** $p$ and $p$ has the ***right lifting property with respect to*** $i$ if for each morphisms $f$ and $g$, if the outer square in the following diagram commutes, there exists $\phi$ (I think not necessarily unique) completing the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\phi"] & Y
\end{tikzcd}$$

- (nLab) Let $C$ be a category with products and with interval object $I$. A morphism $E \to B$ has the ***homotopy lifting property*** if it has the right lifting property with respect to all morphisms of the form $(\mathrm{id}, 0) : Y \to Y \times I$.

  This means that for all commuting squares

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\sigma"] & B
\end{tikzcd}$$

there exists a morphism $\sigma : Y \times I \to E$ such that both triangles in the former diagram commute.

A ***fibration*** (also called ***Hurewicz fibration***) is a mapping $p : E \to B$ satisfying the homotopy lifting property for all spaces $X$.

- (Hatcher) A map $p : E \to B$ is said to have the ***homotopy lifting property*** with respect to a space $X$ if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting $g_0$, so $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ lifting $g_t$.

  The ***lift extension property for a pair*** $(Z, A)$ asserts that every map $X \to B$ has a lift $Z \to E$ extending a given lift defined on the subspace $A \subset Z$. The case $(Z, A) = (X \times I, X \times \{0\})$ is the homotopy lifting property.

  A ***fibration*** is a map $p : E \to B$ having the homotopy property with respect to all spaces $X$.

  **Theorem 3** (4.41 Hatcher, Long exact sequence of Serre fibrations, see proposition 18)**.** Suppose $p : E \to B$ has the homotopy lifting property with respect to disks $D^k$ for all $k \geqslant 0$. Choose basepoints $b_0 \, nB$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_0(B, b_0)$ is an isomorphism for all $n \geqslant 1$. Hence $b$ is path-connected and there is a long exact sequence

  $$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

  The map $p : E \to B$ is said to have the ***homotopy lifting property for a pair*** $(X, A)$ if each homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$. In other words, the homotopy lifting property for $(X, A)$ is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

  The point is that the homotopy lifting property for disjs is equivalent to the homotopy lifting property for all CW pairs $(X, A)$. A map $p : E \to B$ satisfying the homotopy lifting property for disks is sometimes called a ***Serre fibration***.

  A ***fiber bundle*** structure on a space $E$, with fiber $F$, consists of a projection map $p : E \to B$ such that each point $B$ has a neighbourhood $U$ for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the following diagram commute

  $$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

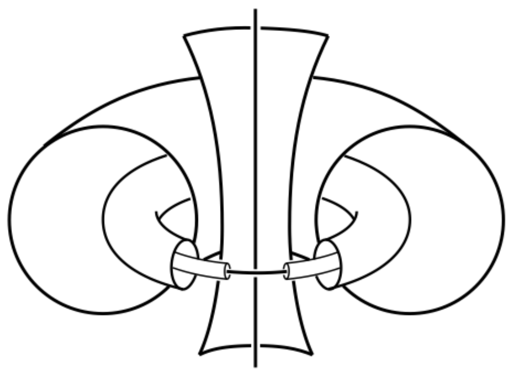
**example.** Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces $S^n \to \mathbb{R}P^n$ with fiber $S^0$. Over the complex numbers the analog of this is a fiber bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. Here $S^{2n+1}$ is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$ is viewed as the quotient space of $S^{2n+1}$ under the equivalence relation $(z_0, \ldots, z_n) \sim \lambda(z_0, \ldots, z_n)$ for $\lambda \in S^1$. The projection $p : S^{2n+1} \to \mathbb{C}P^n$ sends $(z_0, \ldots, z_n)$ to its equivalence class $[z_0, \ldots, z_n]$.

To see that the local triviality condition for fibre bundles is satisfied, ...

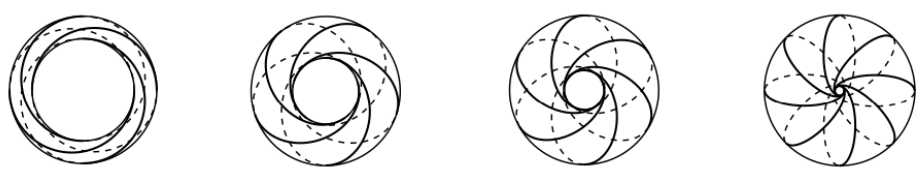
The constructino of the bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ also works when $n = \infty$, so there is a fiber bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$.

The case $n = 1$ is particularly interesting since $\mathbb{C}P^1 = S^2$ and bundle becomes $S^1 \to S^3 \to S^2$ with fiber, total space, and base all speres. This is known as the ***Hopf bundle***. The projection $S^3 \to S^2$ can be taken to be $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$. (That is, seeing $S^2$ as the one-point compactification of $\mathbb{C}$.)

In polar coordinates we may see $S^3$ as the union of several tori. Stereorgraphic projection yields the following figure:



The limiting cases $T_0$ and $T_\infty$ correspond to the unit circle in the $xy$-plane and the $z$-axis under the stereographic projection. Each torus $T_\rho$ is aunion of circle fibers. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. As $\rho$ goes to 0 or $\infty$ the fiber circles approach the circles $T_0$ and $T_\infty$, which are also fibers. The figure below shows four tori decomposed into fibers:



How could we visualize the projection onto $S^2$? Could it work to think $S^2 = \mathbb{C} \cup \infty$ and just do stereographic projection from 3-space to the plane disregarding one point? What would that even mean hehe

Replacing the field $\mathbb{C}$ by the quaternions $\mathbb{H}$, the same constructions yield fiber bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ over quaternionic projective spaces $\mathbb{H}P^n$. Here the fiber $S^3$ is the unit quaternions, and $S^{4n+3}$ is the unit sphere in $\mathbb{H}^{n+1}$. Taking $n = 1$ gives a second Hopf bundle $S^3 \to S^7 \to S^4 = \mathbb{H}P^1$.

Another Hopf bundle $S^7 \to S^{15} \to S^8$...

## model structures

**definition** (Riehl). A ***weak factorization system*** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised o two clases of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

1. Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$:
$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

2. The maps in $\mathcal{L}$ have the ***left lifting property*** with respect to each map in $\mathcal{R}$ and equivalently the maps in $\mathcal{R}$ have the ***right lifting property*** with respect to each map in $\mathcal{L}$, that is, any commutative square
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$
admits a diagonal filler as indicated making both triangles commute. When this lift is unique, we say the factorization system is ***orthogonal***.

3. The classes $\mathcal{L}$ and $\mathcal{R}$ are each closed under retracts in the arrow category: given a commutative diagram
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "t"'] \arrow[rr, bend left, equal] & \bullet \arrow[r] \arrow[d, "s"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r] \arrow[rr, bend right, equal] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
if $s$ is in that class then so is its retract $t$.

**exercise** (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

**definition.** Given a contravariant functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ there is a corresponding category (***of elements of*** $\mathcal{F}$) that lies over $\mathcal{C}$, that is,
$$\mathrm{el}\,\mathcal{F} \to \mathcal{C}$$
given by

Objects: pairs $(C, X)$ where $C \in \mathrm{Obj}\,\mathcal{C}$ and $X \in \mathcal{F}(X)$.

Morphisms: $f : (C, X) \to (C', C')$ are morphisms $f : C \to C'$ such that $\mathcal{F}(f)(X') = X$.

**remark.** We can use the Yoneda embedding to view $\mathcal{C}$ as a subcategory of $\mathrm{Psh}(\mathcal{C})$,

$$\mathcal{C} \hookrightarrow \mathrm{Psh}(\mathcal{C})$$

And also $\mathcal{F} \in \mathrm{Psh}(\mathcal{C})$. In fact, the element category is just the slice category:

$$\mathrm{el}\,\mathcal{F} \cong \mathcal{C}/\mathcal{F}.$$

**question.** Given $\mathcal{D} \to \mathcal{C}$ is it isomorphic to $\mathrm{el}\,\mathcal{F} \to \mathcal{C}$?

**definition.** $G : \mathcal{D} \to \mathcal{C}$ is a ***discrete fibration*** if for any $d \in \mathcal{D}$ and any $f : C \to G(d)$ there exists a unique lift from $f'$ of $f$ to $f \in \mathcal{D}$ such that the target ot $f'$ is is $d$. That is,

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\exists! f'"] \arrow[d, "G"'] & d \arrow[d, "G"] \\ C \arrow[r, "f"'] & G(d) \end{tikzcd}$$

**remark.** Given a discrete fibration we may construct a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ simply by defining $\mathcal{F}(C) = G^{-1}(C)$ and if $C \to C' \cdots \to d$.

**definition** (Lecture). A ***model structure*** on a category $\mathcal{A}$ is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called ***weak-equivalences***, ***cofibrations*** and ***fibrations*** with the following properties:

0. All (finite) small limits an colimits.
1. **(2 of 3)** Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in $\mathcal{W}$ then all of them are.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system. That is,
   (a) Any morphism in $\mathcal{A}$ can be factored as a morphism in $\mathcal{B}$ followed by a morphism in $\mathcal{D}$.
   (b) Lifts:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \bullet \end{tikzcd}$$

   (c') Notice that the aciom of retracts is not necessary. $r' \in \mathcal{R} \iff$ it satisfies (b) for all $\ell \in \mathcal{L}$.

**definition.**

- $X$ is ***fibrant*** if $X \twoheadrightarrow \mathrm{pt}$.
- $X$ is ***cofibrant*** if $X \rightarrowtail\!\!\twoheadrightarrow X$

- X is ***bifibrant*** if $0 \rightarrowtail X \twoheadrightarrow \mathrm{pt}$

**examples** (Two interesting model category structures on CGWH)**.**

1. ***Hurewicz model structure*** (Strom).
   - Cofibrations:= Huerwicz cofibrations.
   - Fibrations:= maps $E \to B$ such that for all spaces X [Photo1].
   - Weak equivalences:= homotopy equivalences.
2. ***Quillen model structure***. Defined on Top.
   - Cofibrations = retracts of relative cell complexes.
   - Fibrations = Serre Fibrations: $$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$
   - Weak equivalences: $f : X \to Y$

   Also, we have
   - Fibrant: all of Obj Top.
   - Cofibrant: $\exists$\{CW complexes\}.

**definition.** Given a category $\mathcal{C}$ and a class of morphisms $W \subset \operatorname{Mor}\mathcal{C}$, its ***localization*** is a category $\mathcal{C}[W^{-1}]$ such that there is a functor $\mathrm{Loc}_W\,\mathcal{C} \to \mathcal{C}[W^{-1}]$ that sends weak equivalences to isomorphisms. Also, its satisfies the universal property that for every $F : \mathcal{C} \to \mathcal{D}$ such that $F(X) \subset \mathrm{Iso}$, the following diagram commutes

$$\begin{tikzcd} & \mathcal{C}[W^{-1}] \arrow[dr, dashed, "\exists! G"] & \\ \mathcal{C} \arrow[ur, "\mathrm{Loc}_W"] \arrow[rr, "F"] & & \mathcal{D} \end{tikzcd}$$

**Theorem 4.** Let $\mathcal{C}$ and $(C, W, F)$ be a model category and $\mathcal{C}[W^{-1}] \cong \operatorname{Ho}\mathcal{C}$ where the latter is given by

- $\operatorname{Ob}\operatorname{Ho}\mathcal{E} = \{$fibrant-cofibrant-bifibrant objects of $\mathcal{E}\}$.
- $\operatorname{Mor}\operatorname{Ho}\mathcal{E} = \operatorname{Mor}_{\mathcal{E}}(X, Y)/\text{homotopy}$.

Let's say what homotopy means

**definition.** Given two maps

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

- We say $f \underset{\text{left}}{\sim} g$ if for the ***cylinder*** $\mathrm{Cyl}(X)$ defined by

$$\begin{tikzcd} X \sqcup X \arrow[rr] \arrow[dr, tail, "\text{cofibr.}"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, two heads, "\text{trivial fib.}"'] & \end{tikzcd}$$

  we have that

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr] & & Y \times Y \\ & \mathrm{Cyl}(X) \arrow[ur, dashed, "\exists H"'] & \end{tikzcd}$$

- We say $f \underset{\text{right}}{\sim} g$ if

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr, dashed, "\exists H"'] & & Y \times Y \\ & \mathrm{Path}(Y) \arrow[ur] & \end{tikzcd}$$

**claim.** Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, if $X$ is cofibrant and $Y$ is fibrant, then $f \underset{\text{left}}{\sim} g \iff f \underset{\text{right}}{\sim} g$ and $\sim$ is an equivalence relation.

## whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

**definition (May).**

1. A ***CW complex*** $X$ is a space $X$ which is the union of an expanding sequence of subspaces $X^n$ such that, inductively, $X^0$ is a discrete set of points (called ***vertices***) and $X^{n+1}$ is the pushout obtained from $X^n$ by attaching disks $D^{n+1}$ along ***attaching maps*** $j : S^n \to X^n$. Thus $X^{n+1}$ is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying $(j, x)$ with $j(x)$ for $x \in S^n$, where $J_{n+1}$ is the discrete set of such attaching maps $j$ (see **??**). Each resulting map $D^{n+1} \to X$ is called a ***cell***. The subspace $X^n$ is called the $n$-***skeleton*** of $X$.

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "j"'] & D^{n+1} \arrow[d] \\ X^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^{n+1} \end{tikzcd}$$

**lemma 5** (HELP). content...

**Theorem 6** (Whitehead, May). If $X$ is a CW complex and $e : Y \to Z$ is an $n$-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

**Theorem 7** (Whitehead, May). An $n$-equivalence between CW complexes of dimension less than $n$ is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

**Theorem 8** (Whitehead (4.5), Hatcher). If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all $n$, then $f$ is a homotopy equivalence. In case $f$ is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: $X$ is a deformation retract of $Y$.

**exercise** (Hatcher 4.1.12). Show that an $n$-connected, $n$-dimensional CW complex is contractible.

*Solution.* Just recall that $n$-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that $X$ is contractible by theorem 7. □

## lecture notes

### 14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \to X$$

$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(X', Y)$$

$$\begin{gathered} \mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, B') \text{ natual in } A \implies \\ \mathrm{Hom}(B, B) \cong \mathrm{Hom}(B, B') \& \mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', B') \\ \implies B \cong B'. \end{gathered}$$

- for ( $\Longleftarrow$ ) commutativity of the hypotesis gives us commutativiuty of the rightmost square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an $n$-equivalence.
- $\pi_n$ of loop space is the same as $\pi_{n+1}$ of original space.

- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

## 18 mar

**lemma 9** (Yoneda)**.**

$$\{\text{Natural transformations } \operatorname{Hom}(-, X) \to F\} \cong F(X)$$

**corollary 10.** $(\operatorname{Hom}(-, X) \to \operatorname{Hom}(-, Y)) \cong \operatorname{Hom}(X, Y)$.

**corollary 11.** The correspondence $X \mapsto \operatorname{Hom}(-, X)$ is fully faithful, that is, the correspondence $\operatorname{Hom}(X, X') \to \operatorname{Hom}(\operatorname{Hom}(-, X), \operatorname{Hom}(-, X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

*Solution of exercise 1.* The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. □

**lemma 12.** Let $E \times_B X$ be the pullback of

$$\begin{tikzcd} & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be such that $E \to B$ is an homotopy fibration and $f : X \to B$ is a homotopy equivalence. Let

$$\begin{tikzcd} E \times_B X \to E \arrow[r, dash, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be the pullback. Then $E \times_B X \to E$ is a homotopy equivalence.

*Proof.* Let $g : B \to X$ be the homotopy inverse of $f$.

**(Step 1)** Construct another pullback

$$\begin{tikzcd} E \times_B B \arrow[r] \arrow[d, two heads] & X \times_B E \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] \\ B \arrow[r, "g"'] & X \arrow[r, "f"'] & B \end{tikzcd}$$

**(Step 2)** Constuct $E \to E \times_B B$.

Consider

$$\begin{tikzcd} E \arrow[r, "\mathrm{id}"] \arrow[d] & & E \arrow[d, two heads] \\ E \times I \arrow[urr, dashed] \arrow[r, "f \times \mathrm{id}"'] & B \times I \arrow[r] & B? \end{tikzcd}$$

And then $E \to E \times_B B \to E \times_B X \to E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{tikzcd} Z \sqcup Z \arrow[r] \arrow[d, "f_1 \sqcup f_2"'] \arrow[ddr, bend left] & I \times Z \arrow[d] \arrow[dl, dashed] \arrow[dd, bend left] \\ E \times_B X \arrow[r] \arrow[d] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

$\square$

**corollary 13.** $B \xrightarrow{f} B$ is homotopy equivalence and $E \to B$ is a fibration, in

$$\begin{tikzcd} E \times_B B \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r, "f"'] & B \end{tikzcd}$$

$E \times_B B \to E$ is a homotopy equivalence.

**exercise.** If $fg$ is an isomorphism and $f$ and $g$ have right inverses, then $f$ and $g$ are isomorphisms.

**lemma 14.** Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "g"] & B \arrow[d] \\ X \arrow[r] & X \cup_A B \end{tikzcd}$$

be a pushout with $A \to X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f, g) \to X \cup_A B$ is a homotopy equivalence.

**remark.**

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & M_f \arrow[d] \\ X \arrow[r] & X \cup_A M_f \cong M(f, g) \end{tikzcd}$$

**definition.**

- The ***homotopy pullback*** of a diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  is

$$\begin{tikzcd} X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting $x$ and $y$.

- The ***homotopy fiber*** if $f : Y \to Z$ is the pullback of

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ pt \arrow[r] & Z \end{tikzcd}$$

  $F \subset Z^I \times_Z Y \to Z$, where $F$ is the space of paths starting at $x$ and ending at the same point $f(y)$.

**remark.** The pullback of

$$\begin{tikzcd} & Z^I \times_Z Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is the homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

**lemma 15.** If $X \to Z$ is a fibration then for

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r, two heads] & Z \end{tikzcd}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

*Proof.*

$$\begin{tikzcd}
X \times_Z \arrow[r] \arrow[d, dash, "\simeq"] & Y \arrow[d, "\simeq"] \\
X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r, two heads] \arrow[d] & Z^I \times_Z Y \arrow[d, two heads] \\
X \arrow[r] & Z
\end{tikzcd}$$

□

Finally,

$$\begin{tikzcd}
\operatorname{hofib} f_1 \arrow[r] \arrow[d] & \operatorname{hofib} f \arrow[r] \arrow[d, two heads] & * \arrow[d] \\
* \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
Z \arrow[r] \arrow[dd] & F(f) \arrow[d] \arrow[dd, bend left, two heads] \arrow[ddl, dash] \\
 & X \times_Y Y^I \arrow[d] \\
X \times I \arrow[r] & X
\end{tikzcd}$$

and an exact sequence

$$\Omega^2 \operatorname{hofib} \to \Omega^2 X \to \Omega^2 Y \to \Omega \operatorname{hofib} f \to \Omega X \to \Omega Y \to \operatorname{hofib} f \to X \xrightarrow{f} Y$$

**lemma 16** (Exactness)**.** $\forall z$, $[z \operatorname{hofib} f] \to [Z, X] \to [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \to \pi_0(\Omega^2 Y) \to \pi_0(\Omega \operatorname{hofib} f) \to \pi_0(\Omega X) \to \pi_0(\Omega Y) \to \pi_0(\operatorname{hofib} f) \to \pi_0(X) \to \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

## Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewickz fibrations. Let's talk about Serre fibrations. Notice that H. fibration $\implies$ S. fibration. What is the most natural example of a Serre fibration?

**proposition 17** (also Hatcher 4.48)**.** Let E be a fiber bundle with fiber F. Then f is a Serre fibration.

*Proof.* What sdoes it mean to be a Serre fibration? It means that

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & E \arrow[d] \\ I^{n+1} = I^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

So if $\mathcal{U}$ is a covering of $B$ such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some $U$.

Then we subdivide $I^{n+1}$ in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & F \times U \arrow[d, two heads] \\ I^{n+1} \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

has a lift for every little square because

$$\begin{tikzcd} X \arrow[r] \arrow[d] & U \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & pt \end{tikzcd}$$

is always a fibration (think about this) and because pullbacks of fibrations are fibrations:

$$\begin{tikzcd} U \times F \arrow[r, two heads] \arrow[d, two heads] & U \arrow[d, two heads] \\ F \arrow[r, two heads] & pt \end{tikzcd}$$

. Then we may just add up the squares because

$$\begin{tikzcd} D^n \arrow[d] \\ D^n \times I \end{tikzcd}$$

and we're done. $\square$

**proposition 18** (Sere fibration long exact sequence, see theorem 3)**.** Let $g : E \to B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(E) \arrow[r] & \cdots \\ \cdots \arrow[r] & \pi_n(F) \arrow[r] \arrow[u, "="] & \pi_n(E) \arrow[r] \arrow[u, "="] & \pi_n(E, F) \arrow[r] \arrow[u, dashed, "\cong"] & \pi_{n-1}(F) \arrow[r] \arrow[u, "="] & \pi_{n-1}(E) \arrow[r] \arrow[u, "="] & \cdots \end{tikzcd}$$

**example.** We have shown that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

**Theorem 19.** Let $X$ be a CW-comples, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \varnothing$, so

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout (this happens for inclusions, check it?).

If $(A, C)$ is $n$-connected and $(B, C)$ is $m$-connected, then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and sujerctive for $i = m + n$.

## blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \to Y$ be a map. Consider the pullback

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d, "{(q,p)}"'] & Y^I \arrow[d, "{(ev_0, ev_1)}"] \\ X \times Y \arrow[r, "f \times id"'] & Y \times Y \end{tikzcd}$$

where

$$W(f) = \{(x, w) \in X \times Y^I | f(x) = w(0)\},$$
$$q(x, w) = x, \quad p(x, w) = w(1).$$

Since $(ev_0, ev_1)$ is a fibration, the maps $(q, p)$, $q$ and $p$ are fibrations.

Now suppose $f$ is a pointed map with base points $*$. Then $W(f) \to W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

**definition.** By $(I^n, \partial I^n) \to (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \to X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \to *$.
- $I^n \times \{1\} \to A$.

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow \text{hofib} \longrightarrow A \longrightarrow X$$

$$\begin{tikzcd}
\pi_0(\nearrow) = & \pi_n(A) \arrow[r] & \pi_n(X) \arrow[r] \arrow[dr] & \pi_{n-1}(\mathrm{hofib}) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \\
& & & \pi_n(X,A) \arrow[ur] & &
\end{tikzcd}$$

**Theorem 20** (Blakers-Massey 1)**.** Let

$$\begin{tikzcd}
Q \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\
X \arrow[r] & P
\end{tikzcd}$$

be a homotopy pushout, $g$ is $m$ equivalence, $f$ is $n$-equivalence and $m, n \geqslant 0$. Then $Q \to X \times_P^h$ is $(m+n-1)$-equivalence.

**Theorem 21** (Blakers-Massey 2)**.** $P$ is a CW-complex, $X$, $Y$ subcomplexes, $X \cap Y = Q \neq \varnothing$ (***strict pushout***)

$$\begin{tikzcd}
Q \arrow[r, tail] \arrow[d, tail] & Y \arrow[d, tail] \\
X \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & X
\end{tikzcd}$$

Then $\pi_i(Y, \mathbb{Q}) \to \pi_i(P, X)$ is epi for $i = m+n$ and iso for $0 \leqslant i < m+n$.

**Theorem 22** (Blakers-Massey 3)**.** $P = X \cup Y$, $X$ and $Y$ are open in $P$, $X \cap Y = Q \neq \varnothing$.

We proved the third version based on Tom Dieck's proof.

**definition.**

- A map is a $k$-***equivalence*** if the induced map on the $i$th homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W :$ at least $p$ coordinates of $x$ are ¡ the same coordinates of the center of $W\}$

**lemma 23.** Let $W$ be a cube in $\mathbb{R}^d$ with $\dim W \leqslant d$. If for all faces $W'$ of $\partial W$, $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \ \mathrm{rel}\, \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

## freudenthal theorem (2 april)

**definition.** The appropriate analogue of the Cartesian product in the category of based spaces is the ***smash product*** $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs $(x, y)$ such that either $x$ is the basepoint of $X$ or $y$ is the basepoint of $Y$.

We also have the ***suspension of pointed spaces***, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X)/(t,x) \sim (0,y) \sim (1,y) \ \forall y \in X.$$

Then we have

$$\mathrm{Hom}_{\mathrm{CGWH}_*}(\Sigma X, \Sigma X) \cong \mathrm{Hom}_{\mathrm{CGWH}_*}(X, \Omega\Sigma X)$$

where $\Sigma X = S \wedge X$ and $\Omega\Sigma X = \mathrm{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\mathrm{Map}(S^1, -)$.

So let $X$ be a space. The identity map $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ then induces a map $X \to \Omega\Sigma X$.

**Theorem 24** (Freudenthal)**.** Let $X$ be $\ell$-connected space. Then $X \to \Omega\Sigma X$ is a $(2\ell+1)$-equivalence, that is,

$$\pi_i(X) \to \pi_{i+1}(\Sigma X),$$

is a bijection for $i < 2\ell + 1$ and a surjection for $i = 2\ell + 1$ (May).

*Proof 1.*

$$\begin{tikzcd}
X \arrow[rr, "(\ell+1)\text{-equiv}"] \arrow[dd, "(\ell+1)\text{-equiv}"'] \arrow[dr] & & * \arrow[dd] \\
& \Omega\Sigma X \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner h", near start] & \\
* \arrow[rr] & & \Sigma X \arrow[ul, phantom, "h\ulcorner", near start]
\end{tikzcd}$$

$\square$

*Proof 2.* Consider

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X
\end{tikzcd}$$

Then we use relative homotopy long exact sequence with $(X, CX)$ to get $\pi_i(CX, X) \cong \pi_{i-i}(X)$, which is zero for $0 \leqslant i \leqslant \ell + 1$. Then use relative homotopy exact sequence for the pair $(\Sigma X, CX)$. then we get that $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$. And then if you use it for $(\Sigma X, X)$ and

But it also turns out that $\pi_i(\Sigma X) = \pi_{i-1}(\Omega\Sigma X)$ because

$$\pi_n(Z) = \mathrm{Hom}_{\text{h-Top},*}(S^n, Z) = \mathrm{Hom}(S^1 \wedge S^{n-1}, Z) = \mathrm{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX, X) \to \pi_i(\Sigma X, CX)$ which is isomorphism for $0 \leqslant i \leqslant 2\ell + 1$ and surjective for $i = 2\ell + 2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \to \pi_{i-1}(\Omega\Sigma X)$ for $i$ as desired. $\square$

**corollary 25.** If $X$ is $\ell$-connected, then $\Sigma X$ is $(\ell+1)$-connected for $\ell \geqslant 0$.

| | | | | | | |
|---|---|---|---|---|---|---|
| space | $S^0$ | $\Sigma S^0 = S^1$ | $\Sigma^2 S^0 = S^2$ | $\Sigma^3 S^0 = S^3$ | $\cdots$ | $\Sigma^n S^0 = S^n$ |
| conectedness | $-1$ | $0$ | $1$ | $2$ | $\cdots$ | $(n-1)$ |

**corollary 26.** $S^n$ is $(n-1)$-connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_2(S^3) \to \pi_2(S^2) \overset{\cong}{\twoheadrightarrow} \pi_1(S^1) \to \pi_1(S^3) = 0,$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map $S^n \to S^n$. We get a map $CS^n \to CS^n$ (in general, for $f : X \to Y$ we have $(t, x) \mapsto (t, f(x))$ in the cones). We also have $CS^n \to CS^n/S^n = S^{n+1}$.

Now if we take $\mathrm{id} : S^n \to S^n$ we shall get $\mathrm{id} : S^{n+1} \to S^{n+1}$. Think about this like $\pi_1(S^1) \to \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective because $i = 0$. From Hopf fibration we have $\pi_2(S^2) = \mathbb{Z}$. So we have a surjective map $\mathbb{Z} \to \mathbb{Z}$. So it's an isomorphism and we conclude that $\mathrm{id}_{S^2}$ is a generator of $\pi_2(S^2)$.

**corollary 27.** Since $S^n$ is $(n-1)$-connected, we have

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is isomorphism for $i \leqslant 2(n-1) = 2n-1$ and epimorphism form $i = 2n-1$. (We just shift the indices of theorem 24 by one.)

**corollary 28.** $\pi_n(S^n) = \mathbb{Z}$ with $\mathrm{id}_{S^n}$ as generator.

**corollary 29.** $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \leqslant n-1$ and epimorphism for $k = n-1$.

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are $\mathbb{Z}/2$. This is what are called the $k$***th stable homotopy groups of a sphere***. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_i(S^1) \to \pi_i(S^3) \overset{\cong}{\twoheadrightarrow} \pi_i(S^2) \to \pi_{i-1}(S^2) = 0$$

So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

**claim** (Serre)**.** $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$finite abelian. And $\pi_i(S^k)$ is finite abelian in all other cases.

## another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{tikzcd} S^{n-1} \arrow[r, "(n-1)\text{-equiv}"] \arrow[d, "0\text{-equiv}"'] \arrow[dr, phantom, "\ulcorner"] & D^n \arrow[d] \\ X \arrow[r] & X \cup D^n \end{tikzcd}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is $(n-1)$-equivalence by Blakers-Massey. So, by attaching $\sqcup D^n$ we can kill $\pi_{n-1}(X)$, that is, $X \cup (\sqcup D^n)$ has trivial $\pi_{n-1}$.

Now notice that

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

that is, $\pi_i(D^n, S^{n-1}) = 0$ for $i \leqslant n-1$. This implies that $\pi_i(X \cup D^n, X) = 0$ for $i \leqslant n-1$.

Also by homotopy long exact sequence,

$$\pi_{n-1}(X) \to \pi_i(X \cup D^n) \text{ is sujrective}$$

$$\pi_i(X) \to \pi_i(X \cup D^n) \text{ is isomorphism for } i \leqslant n-2.$$

So what we have thus far is

$$\pi_n(X \cup D^n) \longrightarrow \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup D^n) \longrightarrow 0 = \pi_{n-1}(X \cup D^n)$$

Notice that $\pi_n(X \cup D^n, X)$ is not ingeneral cyclic (counterexample $S^1 \cup D^2$ taking unieversal cover which is real line with spheres on integers, homotopy equivalent to $\bigvee_{\mathbb{Z}} S^2$ which is not finitely generated).

> So basically attaching a disk via f will kill [f] inside $\pi_n(X)$ this is called ***killing*** an element of the homotopy group.

**proposition 30.** For any CW-complex X, $X^\ell \to X$ is an $\ell$-equivalence.

**remark.** We have used that for $A \hookrightarrow X$ from long exact sequence of relative homotopy groups we get $\pi_n(X, A) = 0$.

**corollary 31.** Let $i \leqslant \ell$ and $g : D^i \to X$, $g : \partial D^i \to X^\ell$. Then there is a homotopy rel $\partial D^i$ to a map with img $\subset X^\ell$.

**Theorem 32** (Cellular approximation theorem)**.** Let X and Y be CW-complexes and $\xi : Y \to X$ be any map. Then $\xi$ is homotopic to a ***cellular map***, that is, a map $\psi : Y \to X$, such that for all $\ell$, $\psi Y^0 \subset X^\ell$.

We also have

**proposition 33.** Let $n \geqslant 2$. Then

$$\pi_n\left(\bigvee_{k\in I} S^n\right) = \bigoplus_{k\in I} \imath_n(S^n) = \bigoplus_{k\in I} \mathbb{Z} = Z^{\oplus I}$$

**proposition 34.** First notice that for finite $I$,

$$X^n = X^{n+1} = \bigvee_{k\in I} S^n$$

by looking carefully. Then

$$\pi_n(X, X^{n+1}) = 0 = \pi_{n+1}(X, X^{n+1})$$

so

$$\bigoplus_{k\in I} \mathbb{Z} = \prod_{k\in I} \pi_n(S^n) = \pi_n(X) = \pi_n(X^{n+1}) = \pi_n(X^n) = \pi_n\left(\bigvee_{k\in I} S^n\right)$$

and for the infinite case it also works, using finite compactness of the CW complex.

## postnikov tower and CW-approximation, 9 abril

- Let $X$ be a space. Then there is a CW-complex $Y$ and a weak homotopy equivalence from $Y \to X$.
- Let $A \to X$ be a map of spaces. Then it can be factored as $A \to Y \to X$ where $A \hookrightarrow Y$ is a relative CW-complex, and $Y \to X$ is a weak homotopy equivalence.

  **remark.** Notice that the second item is the first one with $A = \varnothing$. Then, the second case is a Serre cofibration since it is a construction involving the cofibration $S^{n-1} \hookrightarrow D^n$ (this is a cofibration by definition).

- Let $A$ be a space. Then there is a space $\tau_{\leqslant n}A$ such that $A \hookrightarrow \tau_{\leqslant n}A$; $\tau_{\leqslant n}A$ is obtained by adding cells of dim $\geqslant n+2$. $A \hookrightarrow \tau_{n \leqslant n}A$ is $(n+1)$-equivalence and

  $$\pi_k(\tau_{\leqslant n}A) = 0 \qquad k > n.$$

  Moreover, $A \to \tau_{\leqslant n}A$ is unique among morphisms in $\mathrm{Ho}(CGWH)$ from $A$ into spaces with $\pi_k = 0$ for $k > n$.

This is called a ***Postnikow tower*** and it looks like this:

$$\begin{tikzcd}
 & \cdots \arrow[d] \\
 & \tau_{\leqslant 2}A \arrow[d] \\
 & \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

The idea is that $\tau_{\leqslant n}A$ is obtained from $A$ by killing elements of dimension greater than $n$, that is, by

- attaching $n+2$ cells that kill all $\pi_{n+1}(A)$,
- attaching $n+3$ cells that kill all $\pi_{n+2}(A)$,
- attaching $n+3\ldots$
- attaching $n+n\ldots$

So consider the space $X$ that is obtained from $A$ after attaching cells of dimension $\geqslant n+2$, so we have a map $A \to X$. Consider also a space $Y$ with $\pi_k(Y) = 0$ for $k > n$. Then for any map $A \to Y$ there is a map $X \to Y$ that extends $A \to Y$. This accounts for a bijection

$$[X, Y] \cong [A, Y].$$

In class we struggled a bit to prove surjectivity, finally using an argument related to the pair $(X \times I, X \times \partial I \cup A \times I)$.

> The point is that the spaces in the Postnikov tower are like the original space but with trivial homotopy groups for $k \geqslant n$.

**question.** What is the limit of the Posnikov tower?

$$\begin{tikzcd}
 & \lim \\
 & \cdots \arrow[d] \\
 & \tau_{\leqslant 2}A \arrow[d] \\
 & \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuuur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

- Let $A \to X$ be a map (of CW-complexes (or spaces?)). Then

$$
\begin{tikzcd}
 & \cdots \arrow[d] & \\
 & Y_2 \arrow[d] \arrow[ddr, bend left] & \\
 & Y_1 \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[uur, bend left] \arrow[rr, bend right] & Y_0 \arrow[r] & X
\end{tikzcd}
$$

  Proof pending

- We also have the ***Whitehead tower***, obtained from the homotopy fiber

$$\operatorname{hofib} f_n \longrightarrow A \xrightarrow{f_n} \tau_{\leqslant n-1} A$$

  which yields

$$\cdots \longrightarrow \pi_{k+1}(A) \xrightarrow{\cong} \pi_{k+1}(\tau_{\leqslant n} A) \longrightarrow \pi_k(\operatorname{hofib}) \longrightarrow \pi_k(A) \longrightarrow \pi_k(\tau_{\leqslant n} A) \longrightarrow \cdots$$

  so

| $k \leqslant n-1$ | $k = n$ | $k \geqslant n+1$ |
|---|---|---|
| $\pi_k(\operatorname{hofib} f_n) = 0$ | $\pi_n(\operatorname{hofib} f_n) = 0$ | $\pi_k(A) = \pi_k(\operatorname{hofib} f_n)$ |

- Now there's a natural way to construct the following diagram:

$$
\begin{tikzcd}
A \arrow[r] \arrow[rr, bend right] & \tau_{\leqslant n} A \arrow[r] & \tau_{\leqslant n-1} A
\end{tikzcd}
$$

  which yields the bundle

$$\operatorname{hofib} \longrightarrow \tau_{\leqslant n} A \longrightarrow \tau_{\leqslant n-1} A$$

  and in this case we get

| $k \neq n$ | $k = n$ |
|---|---|
| $\pi_k(\operatorname{hofib}) = 0$ | $\pi_n(\operatorname{hofib}) = \pi_n(A)$ |

  and this is what we call a $K(\pi, n)$-***space*** (all homotopy groups are trivial but the $n$th.)

## Moore space, $K(\pi, n)$ and Hurewicz theorem, 11&16 apr

**Theorem 35 (Uniqueness of CW-approximations).** Recall that a CW-approximation of $X$ is a map $f : Z \to X$ and a CW-complex $Z$ that is a weak homotopy equivalence (induces isomorphisms in all homotopy groups).

We have that

$$\begin{tikzcd} Y_1 \arrow[dr, "\text{CW ap.}"] & \\ & X \\ Y_2 \arrow[uu, dashed, "\exists!"] \arrow[ur, "\text{CW ap.}"'] & \end{tikzcd}$$

up to homotopy equivalences

**lemma 36 (Compression).** If the relative homotopy groups of a pair $(Y, B)$ is zero for $n = \dim e$ for every cell $e \in X \backslash A$ then any map $(X, A) \to (Y, B)$ is homotopic rel $A$ to $(X, A) \to (B, B)$ (so intuitively we can collapse $Y$).

*Proof.* With fibrations (photo) $\square$

**proposition 37.** Let $f : X \to Y$ be an $n$-equivalence (in Hatcher stated as weak equivalence but argument is the same). Then $f$ induces an ***$n$-equivalence in homology*** $H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$ (an isomorphism for $i < n$ and surjection for $i = n$).

*Proof.* photo $\square$

**corollary 38.** If $f : X \to Y$ is a weak equivalence, then $f$ induces an isomorphism in $H_*(-, G)$ and $H^*(-, G)$.

*Proof.* Universal coefficients. $\square$

**definition.** Let $\pi$ be an abelian group. Take

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi$$

a ***free resolution***, i.e. $F_1 = Z^{\oplus J}$ and $F_0 = Z^{\oplus I}$ are free abelian groups and $\pi = F_0/F_1$. Let's take the corresponding maps

$$\begin{aligned} \textstyle\bigvee_{j \in S} S^n &\longrightarrow \textstyle\bigvee_{i \in I} S^n \longrightarrow \operatorname{hocofib} f \\ x_j &\longmapsto \textstyle\sum a_i y_i \end{aligned}$$

where $a_i$ is the degree of $S^n \to S^n$. Recall that the homotopy cofiber hocofib f is the mapping cone of f. It is the ***cone of pointed spaces***. We call this space the ***Moore space*** $M(\pi, n)$ and it is such that

$$\tilde{H}_i(M(\pi, n)) = \begin{cases} 0, & i \neq n \\ \pi, & i = n \end{cases}$$

What do we get in homology? Exactly the sequence of free groups above. So, $H_n(\text{hocofib } f) = \pi$. What do we get in homotopy? Might be $\pi$ as well. Let's prove something stronger:

**Theorem 39.** Let Y be such that $\pi_i(Y) = 0$ for $i > n$ and $\pi_0(Y) = 0$. Then

$$[\text{hocofib } f, Y] \to \text{Hom}(\pi_n(\text{hocofib } f), \pi_n(Y))$$

is a bijection.

*Proof.* Photo

Take

$$\bigvee_I S^n \longrightarrow \bigvee_J S^m \longrightarrow \text{hocofib } f$$

Now apply $[-, Y]$. We get

$$\left[\bigvee_I S^1\right] \longleftarrow \left[\bigvee_J S^1\right] \longleftarrow [\text{hocofib } f, Y] \longleftarrow 0$$

$\square$

**lemma 40.** If $(X, A)$ is r-connected, A s-connected for all $r, s \geqslant 0$, then the map

$$\pi_i(X, A) \to \pi_i(X/A)$$

induced by the quotient map $X \to X/A$ is an $(r + s + 1)$ equivalence.

*Proof.* If $(X, A)$ is r-connected. $\square$

After the lemma we proved that $\pi_n(C_f \cong G$.

**Theorem 41.** Now consider a Moore space, kill all homotopy groups to get $\tau_{\leqslant n}(M)$. It is a $K(\pi, n)$ space with cells in dim $\geqslant n$, obtained from hofib f by attaching cells of dim $\geqslant n + 2$. Then

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \tau_{\leqslant n} M \arrow[dl, dashed, "\exists! \text{ up to homotopy}"] \\ & Y & \end{tikzcd}$$

$$[\text{hocofib } f, y] \cong [\tau_{\leqslant n}(\text{hocofib } f), Y] = \text{Hom}(\pi, \pi_n(Y)).$$

If $\pi_n(Y) = \pi$, then there is a weak equivalence

$$\tau_{\leqslant n}(\text{hocofib } f) \to Y.$$

**Theorem 42** (Hurewicz)**.** Let $X$ be an $(n-1)$-connected space for $n \geqslant 2$. Then

$$\tilde{H}_i(X) = \begin{cases} 0, & i < n \\ \pi_n(X), & i = n \end{cases}$$

*Proof.* Photo.

Idea is to construct a Moore space that is a piece of the CW approximation. Why a piece? Write and understand why this worked! $\square$

**Theorem 43** (Relative Hurewicz theorem)**.** Let $(X, A)$ be $n$-connected, $A$ be 1-connected, $n \geqslant 2$. Then

$$H_i(X, A) = \begin{cases} 0, & i < n \\ \pi_n(X, A), & i = n \end{cases}$$

*Proof.* Take a CW approximation of $(X, A)$,

$$\begin{tikzcd} & Y \arrow[dr, "\cong"] & \\ B \arrow[ur] \arrow[r, "\cong"'] & A \arrow[r] & X \end{tikzcd}$$

So the approximation is $(B, Y)$.

Then we have

$$\begin{aligned} \pi_i(Y, B) &= \pi_i(Y/B), \quad i \leqslant n \\ H_i(Y, B) &= \tilde{H}_i(Y/B), \quad \forall i \end{aligned}$$

and first line implies that $\pi_1(Y/B) = 0$ for $i < n$. But then we are done, right? $\square$

## Representability of the functor $H^n(-, G)$

**remark.** Recall the ***adjoint relation***

$$\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$$

where $\Sigma X$ is the ***reduced suspension*** of a space $X$, $\Omega K$ is the ***loop space*** of another space $K$ and the brackets mean homotopy classes of basepointed maps. Choosing $X = S^0$ and $K = M$, the left-hand side becomes $\pi_1(M, m)$ and the right-hand side becomes the path components of $\mathrm{Map}((S^1, 0), (M, m))$.

Now let's do some other interesting remarks.

**definition.** Let $\mathcal{C}$ be a category. Then $\mathrm{Psh}(\mathcal{C})$ is the category of ***presheaves*** of $\mathcal{C}$ i.e. the category of functors $\mathcal{C}^{op} \to \mathrm{Sets}$ and natural transformations. For any object $A$ in $\mathcal{C}$ there is a presheaf $\mathrm{Hom}_{\mathcal{C}}(-, A)$. A presheaf $\mathcal{F}$ that is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, A)$ for some $A$ is called ***representable***.

For example, $\mathrm{Hom}_{CW}(X, K(\pi, n)) \cong H^n(X, \pi)$, that is, $H^n$ is representable.

$$0 \longrightarrow \underset{\substack{\shortparallel \\ 0}}{\mathrm{Ext}^1(H_{n-1}(K(G,n),G))} \longrightarrow H^n(K(G,n),G) \xrightarrow{\cong} \underset{\substack{\shortparallel \\ 0}}{\mathrm{Hom}(H_n(K(G,n)),G)} \longrightarrow 0$$

There is a special element in $\mathbb{H}^n(K(G,N),G)$, the preimage if $\mathrm{id}_G$.

**claim.**

$$[X, K(G,n) \cong \tilde{H}^n(X,G)]$$

where on the left we have based CW-complexes.

**lemma 44** (Yoneda)**.** Let $\mathcal{F}$ be a presheaf, $A$ be an object in $\mathcal{C}$. Then

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,A),\mathcal{F}) \cong \mathcal{F}(A)$$

naturally in $A$.

*Proof.* For $f : C \to A$, we have this commutative diagram:

$$\mathrm{id}_A \longmapsto f$$

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[r] \arrow[d, "\eta_A"'] & \mathrm{Hom}(C,A) \arrow[d, "\eta_C"] \\
\mathcal{F}(A) \arrow[r] & \mathcal{F}(C)
\end{tikzcd}$$

$$\eta_A(\mathrm{id}_A) \longmapsto \eta_C(f)$$

So, natural transformations $\eta : \mathrm{Hom}(-,A) \to \mathcal{F}$ are determined by $\eta_A(\mathrm{id}_A)$. And that's it because then the map $\eta \mapsto \eta_A(\mathrm{id}_A)$ is what we are looking for. It remains to write why it is injective, surjective and natural. □

## representability theory

If a functor $F : \mathcal{C}^{op} \to \mathrm{Sets}$ is representable, then it sends colimits to limits and sends weak colimits to weak colimits.

**Theorem 45.** If $F : \mathrm{Ho}\,CW_* \to \mathrm{Sets}$ sends

1. coproducts (=wedges) to products,

2. $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \cup_A C \end{array}$ to weak pullback, where $B \cup_A C$ is a CW complex, B and C are CW complexes, $A = B \cap C$,

   then F is representable.

   (By B with isomorphism $F \cong [-, B]$ given by some $X \in F(B)$.)

**definition.** A CW-complex B together with a choice of $\gamma \in F(B)$ is a ***spherical classifying space*** of F if

$$\begin{aligned} \gamma_* : [S^n, B] &\to F(S^n) \\ f &\mapsto f^*(\gamma) \end{aligned}$$

is an isomorphism for $n > 0$ (because for $n = 0$ $S^n$ is not connected).

**proposition 46.** If $(B_1, \gamma_1)$ and $(B_2, \gamma_2)$ are two classifying spaces for F, then $B_1$ and $B_2$ are homotopy equivalent via the map that sends $\gamma_1$ to $\gamma_2$.

**proposition 47.** If $(B_1, \gamma_1)$ and $(B_2, \gamma_2)$ are two classifying spaces for F, then $g : B_1 \to B_2$ is such that $g^*/(\gamma_2) = \gamma_1$, then f is a homotopy equivalence.

## May 2: after representation theorem

### Cohomology operations

$$\mathrm{Nat}(H^n(-, G_1), H^n(-, G_2)) \leftrightarrow [K(G_1, n), K(G_2, n)]$$

**example.**

1. $x \mapsto x \smile x$, $H^n \to H^{2n}$. This is a natural transformation between the H functors.
2. The short exact sequence

   $$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

   yields

   $$\cdots \longrightarrow H^n(X, \mathbb{Z}/p^2) \longrightarrow H^n(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p^2) \longrightarrow \cdots$$

   which is natural in X, yielding the ***Bockstein homomorphism***

   $$H^n(-, \mathbb{Z}/p \to H^{n+1}(-, \mathbb{Z}/p)$$

**spectral sequences**

1. ***Exact couple***: $$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

   It looks like this is just notation for an exact sequence that repeats over and over.

   **example.** Take

   $$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

   and take $C_\bullet$ torsion free chain complex. You get

   $$0 \longrightarrow C_\bullet \xrightarrow{\times p} C_\bullet \longrightarrow C_\bullet \otimes \mathbb{Z}/p \longrightarrow 0$$

   which yields a long exact sequence

   $$\begin{tikzcd} H_n \arrow[rr] & & H_n \arrow[ddl] \\ & H_{n-1} \arrow[r] & \dots \\ & & H_n \arrow[ul] \end{tikzcd}$$

   Or better yet (notation)

   $$\begin{tikzcd} H_*(C) \arrow[rr] & & H_*(C) \arrow[dl] \\ & H_*(C, \mathbb{Z}_p) \arrow[ul] & \end{tikzcd}$$

2. ***Derived couple of exact couple***

   $$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

   Where $A' = \operatorname{img} i$,

   And the exact couple yields a homology $E^1 = H(A, )d$ with differential $d = jk$,

   $i' = i|_{A'}$, $j'(ia) := [j(a)]$, $k'([e]) = k(e)$.

   We have checked that

- $j'$ and $k'$ are well-defined. $i'$ is well-defined automatically.
- that the image of $k'$ is in fact in $A'$
- $j'i' = 0$, $k'j' = 0$, $i'k' = 0$.
- that $j'(ai) = 0 \implies ia = i'a'$, $k'[e] = 0 \implies [e] = j'(a')$ and that $i'(a') = 0 \implies a' = k[e]$.

So, we have shown that each exact couple gives a derived couple.

3.

$$\begin{tikzcd} \text{Filtration on something} \arrow[r] & \text{filtration on chain complex} \arrow[r] \arrow[dr] & \text{exact couple} \arrow[d] \\ & & \text{spectral seq} \end{tikzcd}$$

A ***filtration*** on an abelian group/R-module/chain complex/... C is

$$\ldots \subseteq F_n C \subseteq F_{n+1} C \subseteq C, \qquad n \in \mathbb{Z}$$

and there is an associated graded gr $F_\bullet C := \bigoplus_{n\in\mathbb{Z}} F_n C/F_{n-1}C$.

We hope to recover C from gr$F_\bullet$.

**Problem 1.** If $\bigcap_{n\in\mathbb{Z}} F_n C \neq 0$ then the map $\bigcap_{n\in\mathbb{Z}} F_n C$ looses some information. We may solve this by asking that

1. $F_n C = 0$ for $n < 0$
2. $C \to \lim C/F_n C$ is isomorphism.

**Problem 2.** If $\bigcup_{n\in\mathbb{Z}} F_n C \neq C$ that would be very bad. Then we should ask that $C = \bigcup_{n\in\mathbb{Z}} F_n C$ which is $C = \operatorname{colim} F_n C$.

**example.** We discussed the cases of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ and $\prod_{n\in\mathbb{N}} \mathbb{Z}$. We found that even though $\bigcap_{n\in\mathbb{Z}} F_n C = 0$ in both cases, we could not recover the desired information (?).

Now, to each Serre fibration corresponds a spectral sequence.

## spectral sequences cont.

X is a filtered chain complex

$$\cdots \subseteq X_{n-1} \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$$

We automatically get

$$\begin{tikzcd}
H_{p+q}(X_p) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_p/X_{p-1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p-1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p-1}/X_{p-2}) \arrow[r] \arrow[d] & \cdots \\
H_{p+q}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+1}/X_p) \arrow[r] \arrow[d] & H_{p+q-1}(X_p) \arrow[r, "j"'] \arrow[d, "i"'] & H_{p+q-1}(X_p/X_{p-1}) \arrow[r, "k"'] \arrow[d] & \cdots \\
H_{p+q}(X_{p+2}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+2}/X_{p+1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p+1}/X_p) \arrow[r] \arrow[d] & \cdots \\
\cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Notice that the red arrows are the induced exact sequence of

$$\begin{tikzcd}
0 \arrow[r] & X_{p-1} \arrow[r] \arrow[d] & X_p \arrow[r] \arrow[d] & X_p/X_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_p \arrow[r] & X_{p+1} \arrow[r] & X_{p+1}/X_p \arrow[r] & 0
\end{tikzcd}$$

Now consider

$$A = \bigoplus_{p+q} H_{p+q}(X_p)$$

and simply call $A_{p,1} = H_{p+q}(X_p)$. Also take

$$E = \bigoplus_{p,q} H_{p+q}(X_p/X_{p-1})$$

and $E_{p,q} := H_{p+q}(X_p/X_{p-1})$.

We have

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

where

| **map** | **bidigree** |
|---|---|
| $i : A_{p,q} \to A_{p+1,q-1}$ | $(1, -1)$ |
| $j : A_{p,q} \to E_{p,q}$ | $(0, 0)$ |
| $k : E_{p,q} \to A_{p-1,q}$ | $(-1, 0)$ |

Now consider the derived couple of this exact couple several times:

$$\begin{tikzcd}
A^2 \arrow[rr, "i_2"] & & A^2 \arrow[dl, "j_2"] \\
& E^2 \arrow[ul, "k_2"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^3 \arrow[rr, "i_3"] & & A^3 \arrow[dl, "j_3"] \\
& E^3 \arrow[ul, "k_3"] &
\end{tikzcd}$$

where, going back to definitions

$$E^2 = \ker d_1 / \operatorname{img} d_1 \qquad E^3 = \ker d_2 / \operatorname{img} d_2$$

and so on.

And then think about the bidigrees of the other maps. Well they are

| **map** | **bidigree** |
|---|---|
| $i_k = i_{k-1}\|_{\operatorname{img} i_{k-1}}$ | $(1,-1)$ |
| $j_n$ | $(-(n-1), n-1)$ |
| $k_n$ | $(-1,0)$ |
| $d = j_n k_n$ | $(-n, n-1)$ |

(we thought about this).

Here are some reasonable assumtions:

1. $X_p = 0$ for $p < 0$.
2. $\bigcup_p X_p = X$.
3. $H_{p+q}(X_p/X_{p-1}) = 0$ for $q < 0$.

**remark.** It will happen that for very large $r$, $E^r_{p,q} = E^r_{p,q} = \cdots = E^\infty_{p,q}$.

Also it will happen that

$$E^\infty_{p,q} \cong \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

and we will know what

$$\bigoplus_{\substack{p+q \text{ fixed} \\ p \in \mathbb{N}}} \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

is, and understand

$$H_* \qquad \ldots \subseteq \operatorname{img} H_{p+q}(X_p) \subseteq \operatorname{img} H_{p+q}(X_{p+1}) \subseteq \ldots \subseteq H_{p+q}(X),$$

which is induced by

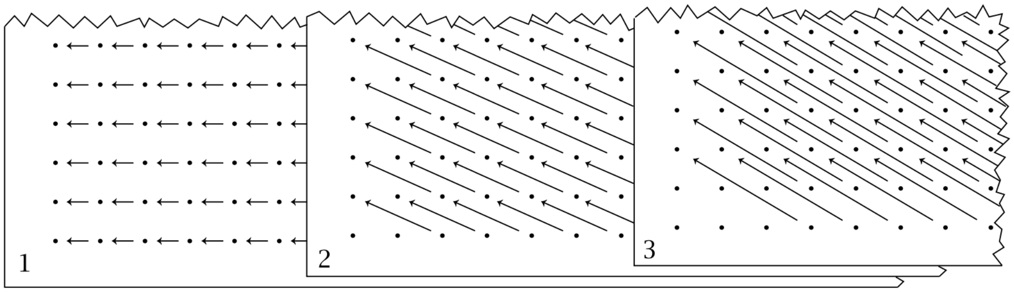
$$X \longrightarrow X_{p+1} \longrightarrow X$$

in our hopes to understand $H_{p+q}(X)$ which is of course just the homology of $X$.

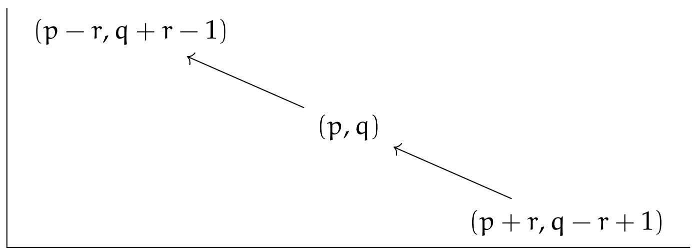
Now let's put the E's in a diagram like

$$\begin{array}{|lll}
\cdots & & \\
E^1_{0,1} & E^1_{1,1} & E^1_{2,1} \\
E^1_{0,0} & E^1_{1,0} & E^1_{2,0} \\
\hline
\end{array}$$

and if arrows are the differential $d$ with bidegree $(-n, n-1)$, we already have Hatcher's picture




And something interesting will happen when $r$ is very big. What? This:




that for a fixed $(p, q)$ there is a large $r$ such that if $p < r$ and $q + 1 < r$ we shall have

$$E^r_{p,q} = E^{r+1}_{p,q} := E^\infty_{p,q}.$$

## serre spectral sequence

Now take a Serre fibration $F \longrightarrow E \longrightarrow B$ with

$$\pi_0(F) = 0 \quad \text{and} \quad \pi_1(B) = 0$$

and it turns out that

$$E^2_{p,q} = H_p(B, H_q(H)) \implies H_{p,q}(E).$$

**example.** For $S^1 \to S^3 \to S^2$ we have found that the first two pages are

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \arrow[ul, "d", "\cong"'] \end{tikzcd} \qquad\qquad \begin{array}{cc} \mathbb{Z}/\operatorname{img} d_2 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}/\operatorname{img} d_2 \end{array}$$

$$E_2 \qquad\qquad\qquad\qquad E_3$$

so $E^3 = E^\infty$.

**example.** We have also done $S^3 \to S^7 \to S^4$ and $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. See **sssguide**.

Now we will prove that there exists a spectral sequence and that it converges where it converges. Consider either of

$$\pi_0(B) = 0 \qquad \text{or} \qquad \begin{cases} \pi_1(B) = 0 \\ \text{or } \pi_1(B) \text{ acts trivially on } H_q(F) \\ \text{or take } H_* \text{ with local coefficients} \end{cases}$$

**Theorem 48.** There is spectral sequence $E^\nu$ that converges to $H_*(E)$ and such that

$$E^2_{p,q} = H_p(B, H_q(F,G)).$$

*Proof.* The construction of the spectral sequence is not complicated. Start with $E^1$. Then Do $E^2$ with

$$\begin{tikzcd} E^k \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ D^k \arrow[r] & B \end{tikzcd}$$

□

Consider the following conditions on the big staircase diagram:

a. In each $A$ column almost all maps are isomorphisms.

b. In each column $E$ almost each entry is 0.

c. $E^1_{p,q} = 0$ for $p < 0$ and $q < 0$.

d. $X_p = 0$ for $p < 0$ and $H_n(X_p) \to H_n(X_{p+1})$ is isomorphism for $p \gg 0$.

And also

- $A_{-\infty,p+q} := A_{p,q}$ for $p \ll 0$.
- $A_{+\infty,p+q} := A_{p,q}$ for $p \gg 0$.

And then

e1. $A_{-\infty,p+q} = 0$.

e2. $A_{+\infty,p+q} = 0$.

**claim.** b $\implies$ $E^r_{p,q}$ stabilizes for fixed panel $q$, so $E^\infty_{p,q}$ makes sense.

Now let's check that indices are the way they are in

$$\underset{\underset{0}{\shortparallel}}{E^r_{p+r-1,q-r+2}} \xrightarrow{k} A^r_{p+r-2,q-r+2} \xrightarrow{i} A^r_{p+r-1,q-r+1} \xrightarrow{j} E^r_{p,q} \xrightarrow{k} \underset{\underset{\text{if e1}}{\underset{0}{\shortparallel}}}{A^r_{p-1,q}} \xrightarrow{i} \underset{\underset{\text{if e1}}{\underset{0}{\shortparallel}}}{A^r_{p,q-1}} \xrightarrow{j} \underset{\underset{0}{\shortparallel}}{E^r_{p-r+1,q-r-2}}$$

Ok, after some other considerations we have concluded that

$E^{\infty}_{p,q}$ also makes sense and it is a piece of p-graded associative algebra graded of $A_{+\infty,p+q}$ with

$$\begin{aligned} E^r_{p,q} &= A^r_{p+q+r,q-r+q}/iA^r_{p+r-2,q-1+2} \\ &= i^{r-1}(A^1_{p,q})/i^r(A^1_{p-1,q+1}) \\ &= F_p A_{+\infty,p+q}/F_{p-1}A_{+\infty,p+q} \end{aligned}$$

## 14 may

In cover spaces we have homeomorphisms between the fibers. In hurewicz fibrations this may not be true, but we still can have homotopy equivalences. So consider a Hurewicz fibration $f : E \to B$ and a path $I \to B$. Then we have:

$$\begin{tikzcd} F_a \arrow[rr, hook] \arrow[d, "i_0"] & & E \arrow[d, two heads, "f"] \\ F_a \times I \arrow[r, "\pi"'] \arrow[urr, dash] & I \arrow[r, "p"] & B \end{tikzcd}$$

And we claim that there is an homotopy equivalence

$$F_a \cong F_a \times \{1\} \to F_b = f^{-1}b$$

To see why, consider two homotopic paths $p_1, p_2 : I \to B$. Construct the following diagram:

[see sss.pdf]

We have (or will?) established:

**proposition 49.** For any Hurewicz fibration $f : E \to B$ there is "an action of $\pi_1(B)$ on $F_b$ up to homotopy"

$$\begin{aligned} \Pi_1(B) &\to \mathrm{Ho\,Top} \\ b &\mapsto F_b \end{aligned}$$

We have an action

$$\pi_1(B) \curvearrowright H_*(F_b)$$

Today I've forgotten to prove that the action of $\pi_1(B)$ on fibre is actually an action: that the action of $x \cdot y$ is the composition of the action of $y$ and of $x$

The fact that fibrations over disks are always trivial is easy to prove. The algebraic analogue of this is the Quillen-Suslin theorem

Why do need this? Consider the following case:

$$\begin{tikzcd} f_i^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads] \\ X \arrow[r, "f_i"'] & B \end{tikzcd}$$

for $f_1, f_2 : X \to B$.

## serre spectral sequence for cohomology

Suppose

$$F \hookrightarrow X \longrightarrow B$$

Is a Serre fibration with $\pi_0(B) = 0$ and that the action described above $\pi_1(B) \curvearrowright H^*(F; G)$ is trivial.

Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ such that

a. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+q}$;  $E_{r+1}^{p,q} = \ker d_r / \operatorname{img} d_r$.

b. $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$;  $0 \subset F_{p+q}^{p+q} \subset \ldots \subset F_0^{p+q} = H^{p+q}(X, G)$.

c. $E_2^{p,q} = H^p(B, H^q(F, G))$.

d. $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$, which is given by $(-1)^{qs}$.

$$H^p(B, H^q(F, R)) \times H^s(B, H^t(F, R)) \xrightarrow{\smile} H^{p+s}(B, H(F, R))$$

now supposing that $G = R$ is a ring.

e. This

$$F_p^m \times F_s^n \xrightarrow{\smile} F_{p+s}^{m+n}$$

which induces

$$F_p^m/F_{p+1}^m \times F_s^n/F_{s+1}^n \to F_{p+s}^{m+n}/F_{p+s+1}^{m+n}$$

and is

$$F_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$$

**example** (Cohomology of base space)**.** We have computed the cohomology of $\mathbb{C}P^n$ using Serre spectral sequence.

**example** (Cohomology of Étale space)**.** Consider the Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

Its easy to see that the second page is

$$\begin{array}{c|ccc} 1 & \mathbb{Z} & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \mathbb{Z} \\ \hline & 0 & 1 & 2 \end{array}$$

We notice immediately that $E_3 = E_\infty$, since the 0's in the diagram remain to future pages. To find $E_3$, it turns out that it can be shown that $d_2$ is an isomorphism (recall that $d_2$ goes two to the right and one down). This makes the upper-left and lower-right groups zero on the third page:

$$\begin{array}{c|ccc} 1 & & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \\ \hline & 0 & 1 & 2 \end{array}$$

It is now possible to read off the cohomology of the total space $S^3$ by assembling along the diagonals. In this case, we have

$$H^n(S^3) = \bigoplus_{s+t=n} E_\infty^{s,t}.$$

(This is what leads to considering the diagonals in the diagram.) This gives a $\mathbb{Z}$ in dimension 0 from $E_\infty^{0,0}$ and $\mathbb{Z}$ in dimension 3 from $E^{2,1}$ as expected.

**example** (Cohomology of fiber). Consider the fibration

$$\Omega S^3 \longrightarrow \underset{\substack{\simeq \\ \mathrm{pt}}}{PS^3} \longrightarrow S^3$$

We use the following

**claim.** On $E^3, E^4, \ldots$ are equal when the differentials on the second page are 0.

*Proof.* It's because

$$E_r^{p,q} = 0 \implies E_r^{p,q} = 0$$

We have $E_2^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, so we have $E_3^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, and finally $d_r = 0$ for $r \geqslant 3$. More finally,

$$d_r : E^{p,q} \to E^{p+q} \to E^{p+r,q-r+1}$$

for $-r+1 \geqslant -2$. □

We discovered that

$$\begin{array}{|cccc}
\cdots & & & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
\hline
\end{array}$$

We have also computed ring structure.

**example** (Cohomology of Ètale espace). Consider the fibration

$$SU(n-1) \longrightarrow SU(n) \longrightarrow S^{2n-1}$$

The particular case of

$$SU(3) \longrightarrow SU(4) \longrightarrow S^{24-1}$$

yields

$$
\begin{array}{c|cccccccc}
 & \cdots & & & & & & & \cdots \\
8 & \mathbb{Z}a_3a_5 & & & & & & & \mathbb{Z}a_7a_5a_3 \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & \mathbb{Z}a_5 & & & & & & & \mathbb{Z}a_7a_5 \\
4 & & & & & & & & \\
3 & \mathbb{Z}a_3 & & & & & & & \mathbb{Z}a_7a_3 \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & \mathbb{Z} & & & & & & & \mathbb{Z}a_3 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}
$$

Giving $E_2 = E_\infty$ since all differentials are zero. It follows that the cohomology ring is $\Lambda(a_3, a_5, a_7)$. The case of

$$SU(2) = S^3 \longrightarrow SU(3) \longrightarrow S^{2\cdot 3-1} = S^5$$

yields, too, that $E_2 = E_\infty$

## 21 may

**Theorem 50.** Let $F \to X \to B$ be a Serre fibration. Then there is a spectral sequence that converges to $H_n(X, G)$. If the action of $\pi_1(B)$ on $H_*(F, G)$ is trivial, then

$$H_2^{p,q} = H_p(B, H_q(F, G)).$$

More generally, if the action is not trivial, then $E_2^{p,q} = H_p(B, \mathcal{L}_q)$ where $\mathcal{L}$ is a local system coming from $\pi_1(B) \curvearrowright F$.

We are not going to prove this theorem.

**definition.** Action of $\pi_1(B)$ on $H_*(F, G)$ for Serre fibration.

$$
\begin{tikzcd}
F \arrow[d] & F \arrow[d] & F \arrow[d] \\
i^*p^*E \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & p^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\
0 \arrow[r, "i", "\simeq"'] & I \arrow[r, "p"] & B
\end{tikzcd}
$$

Now considering the long exact sequence of homotopy, we get:

$$
\begin{tikzcd}
\pi_i(F) \arrow[r] \arrow[d, "\cong"] & \pi_i(p^*E) \arrow[r] \arrow[d, "\cong"] & \pi_i(X) \arrow[d, "\cong"] \\
\pi_i(F) \arrow[r] & \pi_i(i^*p^*E) \arrow[r] & \pi_i(Y)
\end{tikzcd}
$$

because, in general, for an homotopy equivalence $f : X \to Y$, the pullback is an homotopy equivalence:

$$\begin{tikzcd} F \arrow[r, "="] \arrow[d] & F \arrow[d] \\ f^*E \arrow[r, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & Y \end{tikzcd}$$

Then we take pullback of a larger diagram to prove that the action is associative (that it is, in fact, an action) via homoloy groups.

*Proof.*

1. We may assume that $B$ is a CW-complex with one 0-cell, and $X \to B$ is a Hurewicz fibration.

   *Proof.*

$$\begin{tikzcd} & F \arrow[d] \\ & X \arrow[d] \\ C \arrow[r] & B \end{tikzcd}$$

   There is a CW approximation $\alpha : C \to B$ with $C$ having exaclty one 0-cell that is sent to the basepoint of $B$. Now we say

   > Every space can factored as weak equivalence followed by Hurewicz fibration. (Path space is behind the scenes.)

   We obtain

$$\begin{tikzcd} p^{-1}_{\alpha^* p} e_0 \arrow[d] & F \arrow[l, "\simeq"'] \arrow[r] & F \arrow[d] \\ E^{\alpha_{0,p}} \arrow[d] & \alpha^* X \arrow[l, "\simeq"'] \arrow[r, "\cong"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\ C \arrow[r, equal] & C \arrow[r] & B \end{tikzcd}$$

   Anyway, that's the CW-complex replacing $B$. □

2. There is a nice filtration on $X$. Take the filtration

$$\varnothing \quad \subseteq \quad \mathrm{sk}_0 B \quad \subseteq \quad \mathrm{sk}_1 B \quad \subseteq \quad \cdots \quad \subseteq \quad B$$

   which induces

$$\varnothing \quad \subseteq \quad X^0 := p^{-1}\,\mathrm{sk}_0 B \quad \subseteq \quad X^1 := p^{-1}\,\mathrm{sk}_1 B \quad \subseteq \quad \cdots \qquad \subseteq \quad X$$

   and it turns out that

(a) $\bigcup_i X^i = X$.

(b) $\mathrm{sk}_n B \to B$ is an $n$-equivalence (we already know this from other constructions). Then $X^n \hookrightarrow X$ induces isomorphisms on $H_i$ for $i < n-1$ by Hurewicz theorem and five-lemma.

This means that we get a filtration on $C_*(X)$ and on $H_*(X)$. From b. we get that in each column $A'_{\bullet,p+q}$ only a finite number of maps are not isomorphisms. When we discussed spectral sequences, we say that this corresponds to a filtration

$$\varnothing \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

**that converges to** $H_*$.

Namely, we have a spectral sequence such that

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$$

and

$$E^\infty_{p,q} = \text{the } p\text{-th graded piece of } H_{p+q}(X).$$

This means that

$$\begin{aligned} B^{p-1} &\hookrightarrow B^p \quad \text{is a } (p-1)\text{-equivalence} \\ X^{p-1} &\hookrightarrow X^p \quad \text{is a } (p-1)\text{-equivalence} \\ H_{p+q}(X_p, X_{p-1}) &= 0 \quad q < 0. \end{aligned}$$

In conclusion, we have the first-quadrant of the spectral sequence $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$, which proves the first statement in the theorem.

(c) Now suppose the action of $\pi_1(B)$ on $H_*(F, G)$ is trivial. Look at the CW structure of things

$$\begin{tikzcd} \widetilde{S^{p-1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \widetilde{D^p} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & X^p \arrow[d] \\ S^{p-1} \arrow[r] & D^p \arrow[r, "\alpha"] & B^p \end{tikzcd}$$

We also proved that

**claim.** $\bigoplus_\alpha H_*(\widetilde{D^p}, \widetilde{S^{p-1}}) \to H_*(X^p, X^{p-1})$ is an isomorphism.

And then for this we did

$$\begin{tikzcd}
u \quad x \quad o & x & x \\
S^{p-1} \times F_\alpha \times 0 \arrow[r] \arrow[d] & F_\alpha \arrow[r, hook] & \widetilde{D^p_\alpha} \arrow[d] \\
S^{p-1} \times F_\alpha \times I \arrow[r, "\mathrm{proj}"] & S^{p-1} \times I \arrow[r, "(u,t)\mapsto t\cdot u"] & D^p \\
u \quad x \quad o & u \quad o & 0
\end{tikzcd}$$

and used Künneth formula:

By definition, $(X, A) \times$

$\square$

## 23 may

In the end we would like to prove that homotopy groups of sphere are finitely generated and moreover all except two kinds (which?) are in fact finite.

**definition.** A ***Serre class*** $\mathcal{C}$ of abelian groups is a class of abelian groups closed under the operations of taking subgroups, quotients, and forming extensions. That is, for any short exact sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with A and C in $\mathcal{C}$, then B is also in $\mathcal{C}$.

**example.**

1. The class of finitely-generated groups.
2. Torsion groups. From algebraic geometry you might be acquainted with

$$M \otimes_{\mathbb{Z}} \mathbb{Q} = S^{-1}M, \quad S = \mathbb{Z}\backslash\{0\}$$

More generally,

$$M \otimes_R S^{-1}R = S^{-1}M$$

Why do we care? Because **localization is an exact functor**, equivalently, $S^{-1}R$ is a flat R-module. Notice that

$$M \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \iff M \text{ is a torsion group.}$$

Which follows simply from the fact that

$$\forall m, s, \quad \frac{m}{s} = \frac{0}{1} \iff \forall m, x\ \exists s' :\ s'm = 0.$$

So

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

implies

$$0 \longrightarrow A \otimes \mathbb{Q} \longrightarrow B \otimes \mathbb{Q} \longrightarrow C \otimes \mathbb{Q} \longrightarrow 0$$

so $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ are zero iff $B \otimes \mathbb{Q}$ is zero.

$\mathcal{C}_1$ and $\mathcal{C}_1$ are Serre classes then $\mathcal{C}_1 \cap \mathcal{C}_2$ is a Serre class.

Finite abelian groups.

Torsion groups such that the order of any element is coprime to any $p \in P$ for some subset $P$ of primes. For example,

- $P = \{p\}$ then we get the *condition* $p$ *does not divide the orders of ements*.
- $P$ = all primes except $p$ = all $s \neq 0$ such that $(p, s) = 1$. Then we get *the order of elements are powers of* $p$. It turns out that $\mathbb{Z}_{(p)} = p^{-1}\mathbb{Z}$, where $(p)$ is the ideal generated by $p$. Then

  $$M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = 0 \iff M \text{ does not have } p\text{-torsion}$$

  that is, the orders are not divisible by $p$.

  I think we concluded that localization of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_{(p)}$ does not change anything.

  So again

  $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

  implies

  $$0 \longrightarrow A \otimes \mathbb{Z}_{(p)} \longrightarrow B \otimes \mathbb{Z}_{(p)} \longrightarrow C \otimes \mathbb{Z}_{(p)} \longrightarrow 0$$

  so $A \otimes \mathbb{Z}_{(p)}$ and $B \otimes \mathbb{Z}_{(p)}$ are zero iff $B \otimes \mathbb{Z}_{(p)}$ is zero.

**claim.** For a simply connected space it is equivalent that $H_i \in \mathcal{C}$ for $0 < i < n$ and that $\pi_i \in \mathcal{C}$ for $0 < i < n$.

**proposition 51.** Suppose $\mathcal{C}$ is a Serre class and

$$A \longrightarrow B \longrightarrow C$$

and $A, C \in \mathcal{C}$. Then $B \in \mathcal{C}$.

*Proof.*

$$\begin{tikzcd}
& A \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \ker \arrow[r] \arrow[d] & B \arrow[r] \arrow[dr] & \mathrm{img} \arrow[r] \arrow[d] & 0 \\
& 0 & & C &
\end{tikzcd}$$

$\square$

**remark.**

1. If $C_\bullet$ is a chain complex, all $C_n$ is in $\mathcal{C}$, then $H_n$ is in $\mathcal{C}$.
2. $F_\bullet A$ a filtration, $A$ in $\mathcal{C}$ then each graded piece of $\operatorname{gr} A$ is in $\mathcal{C}$.
3. $F_\bullet A$ finite filtration, i.e. $0 \subset F_0A \subset \ldots \subset F_nA = A$, and each graded piece $\operatorname{gr} A$ is in $\mathcal{C}$, then $A$ is in $\mathcal{C}$. This follows from the fact that $F_0A = F_0A/0$, $F_1A/F_0A$, $F_2A/F_1A$ $\ldots$ are all in $\mathcal{C}$.
4. First-quadrant of spectral sequence that converges to $H_*(C_\bullet)$. Suppose $E^r_{p,q}$ is in $\mathcal{C}$ for all $p, q$ and some fixed $r$. Then $E^{r+1}_{p,q}$ is in $\mathcal{C}$ implies that $E^\infty_{p,q}$ is in $\mathcal{C}$, which in turn implies that $H_*(C_\bullet)$ is in $\mathcal{C}$.

**definition.**

- $f : A \to B$ is ***monomorphism*** $\mod \mathcal{C}$ if $\ker f$ is in $\mathcal{C}$.
- $f : A \to B$ is ***epimorphism*** $\mod \mathcal{C}$ if $\operatorname{coker} f$ is in $\mathcal{C}$.
- $f : A \to B$ is ***isomorphism*** $\mod \mathcal{C}$ if both $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{C}$.

**proposition 52.**

1. A monomorphism $\mod \mathcal{C}$, epimorphism $\mod \mathcal{C}$ and isomorphism $\mod \mathcal{C}$ are closed under composition.
2. Isomorphisms $\mod \mathcal{C}$ satisfy 2-out-of-3.

*Proof.*

$$\begin{tikzcd}
 & & 0 \arrow[dr] & & & & 0 & & \\
 & & & \ker\beta \arrow[rr] \arrow[dr] & & \operatorname{coker}\alpha \arrow[ur] \arrow[ddrr] & & & \\
 & & & & B \arrow[ur] \arrow[dr, "\beta"] & & & & \\
0 \arrow[r] & \ker\beta\alpha \arrow[uurr] \arrow[rr] & & A \arrow[ur, "\alpha"] \arrow[rr, "\beta\alpha"] & & C \arrow[rr] \arrow[dr] & & \operatorname{coker}\beta\alpha \arrow[r] \arrow[dl] & 0 \\
 & & \ker\alpha \arrow[ul] \arrow[ur] & & & & \operatorname{coker}\beta \arrow[dl] \arrow[dr] & & \\
 & 0 \arrow[ur] & & 0 \arrow[ul] & & 0 & & 0 &
\end{tikzcd}$$

□

**definition.** A Serre class $\mathcal{C}$ is

- a ***Serre ideal*** if for all $A, B$ abelian groups, $A \in \mathcal{C}$ implies $A \otimes B$ and $\operatorname{Tor}(A, B)$ are in $\mathcal{C}$, and

- a ***Serre ring*** if for all $A, B$ in $\mathcal{C}$, $A \otimes B$ and $\mathrm{Tor}(A, B)$ are in $\mathcal{C}$.

**examples.**

1. Finitely-generated groups are a Serre ring.
2. Torsion groups is a Serre ideal. $\mathbb{Q} \otimes A = 0 \implies \mathbb{Q} \otimes A \otimes B = 0$ so

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

implies

$$0 \longrightarrow F_1 \otimes A \longrightarrow F_0 \otimes A \longrightarrow \mathrm{Tor}(B, A) \longrightarrow 0$$

implies

$$0 \longrightarrow F_1 \otimes A \otimes \mathbb{Q} \longrightarrow F_0 \otimes A \otimes \mathbb{Q} \longrightarrow \mathrm{Tor}(B, A) \otimes \mathbb{Q} \longrightarrow 0$$

so $F_0 \otimes A \otimes \mathbb{Q}$ and $\mathrm{Tor}(B, A) \otimes \mathbb{Q}$ are zero iff $F_1 \otimes A \otimes \mathbb{Q}$ is zero.

3. Finite groups is a Serre ring.
4. Intersection of Serre rings is Serre ring.
5. $M \otimes \mathbb{Z}_{(p)}$ is a Serre ideal.

**proposition 53.** $F \to X \to B$ Serre fibration. $\pi_0(B) = 0$, $\pi_1(B) \curvearrowright H_A(F)$ trivial. Then if 2 out of 3 among $F, X, B$ have $H_n$ in $\mathcal{C}$ for $n > 0$, then the third one does too.

*Proof.* **($X, B$ in $\mathcal{C}$.)** Notice

$$E^2_{p,q} = H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}(H_{p-1}(B), H_q(F))$$

which means that if $E^3, E^4, \ldots E^\infty$ are in $\mathcal{C}$, then $H_*$ is in $\mathcal{C}$.

**($F, B$ in $\mathcal{C}$.)** $\square$

# References

[1] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000 (cit. on pp. 16, 22, 26, 36).

[2] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832 (cit. on pp. 11, 13, 21, 22, 30).

[3] Emily Riehl. *Homotopical categories: from model categories to* $(\infty, 1)$*-categories*. 2020. arXiv: `1904.00886 [math.AT]` (cit. on p. 18).