homotopy theory

Contents

abstract nonsense

definition.

- **(Limits, wiki.)**
  - A ***diagram*** of shape J in C is a functor from J to C
    $$F : J \to C.$$
    The category J is thought of as an index category, and the diagram F is thought of as indexing a collection of obtects and morphisms in C patterned on J.
  - Let $F : J \to C$ be a diagram of chape J in a category C. A ***cone*** to F is an object N to C together with a family $\psi_X : N \to F(X)$ of morphisms indexed by the objects X of J (so a cone is an object and a bunch of maps from this object to certain objectes that are governed by the diagram), so that for every morphism $X \to Y$ in J, we have $F(f) \circ \psi_X = \psi_Y$ I guess this is what nLab meant when he said that everything in sight commutes).
  - A ***limit*** of the diagram $F : J \to C$ is a cone (L, ϕ) to F such that for every cone (N, ψ) there exists a *unique* morphism $u : N \to L$ such that $\phi_X \circ u = \psi_X$ for all X in J.
    $$\begin{tikzcd} & N \arrow[ddl, bend right, "\psi_X"'] \arrow[d, dashed, "u"] \arrow[ddr, bend left, "\psi_Y"] & \\ & L \arrow[dl, "\phi_X"'] \arrow[dr, "\psi_Y"] & \\ F(X) \arrow[rr, "F(f)"'] & & F(Y) \end{tikzcd}$$
    One says that the cone (N, ψ) factors through the cone (L, ϕ) with the unique factorization u. The morphism u is sometimes called the ***mediating morphism***.

    Limits are also referred to as ***universal cones*** since they are characterized by a universal property. Limits may also be caracterized as terminal objects in the category of cones to F.

    It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of F.
- **(Colimits, wiki)** The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:
  - A ***co-cone*** of a diagram $F : J \to C$ is an object N of C together with a family of morphisms $\psi_X : F(X) \to N$ (so in the cone we are going *from* N and now we're going *to*N) for every object X of J, such that for every morphism $f : X \to Y$ in J we have $\psi_Y \circ F(f) = \psi_X$ everything in sight commutes.
  - A ***colimit*** of a diagram $F : J \to C$ is a co-cone (L, ϕ) of F such that for any other co-cone (N, ψ) of F there exists a unique morphism $u : L \to N$ such that

$u \circ \phi_X = \psi_X$ for all X in J.

$$\begin{tikzcd}
F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\phi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\
& L \arrow[d, dashed, "u"] & \\
& N &
\end{tikzcd}$$

Colimits are also referred to as ***unersal co-cones***. They can be characterized as initial objects in the category of co-cones from F.

As with limits, if a diagram F has a colimit then this colimit is unique up to a unique isomorphism.

- An ***initial object*** in a category $\mathcal{C}$ is an object $\varnothing$ such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source $\varnothing$ and target x.
- For $\mathcal{C}$ any category, its ***arrow category*** $\mathrm{Arr}(\mathcal{C})$ is the category such that
  - an object a of $\mathrm{Arr}(C)$ is a morphism $a : a_0 \to a_1$ of $\mathcal{C}$,
  - a morphism $f : a \to b$ of $\mathrm{Arr}(\mathcal{C})$ is a commutative square

$$\begin{tikzcd}
a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\
a_1 \arrow[r, "f_1"'] & b_1
\end{tikzcd}$$

    in $\mathcal{C}$,
  - composition in $\mathrm{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

  This is isomorphic to the functor category

$$\mathrm{Arr}(\mathcal{C}) := \mathrm{Funct}(I, C) = [I, C] = C^I$$

  for I the intervale category $\{0 \to 1\}$.
- An ***equalizer*** is a limit

$$\mathrm{eq} \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

  over a parallel pair of morphisms f and g. This means that for $f : X \to Y$ and $g : X \to Y$ in a category $\mathcal{C}$, their equalizer, if it exists, is
  - an object $\mathrm{eq}(f, g) \in \mathcal{C}$,
  - a morphism $\mathrm{eq}(f, g) \to X$
  - such that

* pulled back to $\mathrm{eq}(f,g)$ both morphisms become equal:

$$\mathrm{eq}(f,g) \longrightarrow X \xrightarrow{f} Y \quad = \quad [\ \mathrm{eq}(f,g) \longrightarrow X \xrightarrow{g} Y$$

* and $\mathrm{eq}(f,g)$ is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions f and g between sets X and Y one forms the set $Y/\sim$ of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the the quotient function $p : Y \to Y/\sim$ satisfies

$$p \circ f = p \circ g.$$

In some category $\mathcal{C}$, the ***coequalizer*** $\mathrm{coeq}(f,g)$ of two parallel morphisms f and g between two objects X and Y, if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{coeq}(f,g) & \end{tikzcd}$$

Equivalently, in a category $\mathcal{C}$ a diagram

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{p} Z$$

is called a ***coequalizer*** diagram if

1. $p \circ f = p \circ g$,
2. p is universal for this property: if $q : Y \to W$ is a morphism of $\mathcal{C}$ such that $q \circ f = q \circ g$, then there is a unique morphism $\phi : Z \to W$ such that $\phi \circ p = q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] \arrow[d, "q"] & Z \arrow[dl, dashed, "\phi"] \\ & W & \end{tikzcd}$$

The coequalizer in $\mathcal{C}$ is equivalently an equializer in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A ***pullback*** of the morphisms f and g consists of an object P and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Q \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$

- A ***pushout*** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\ X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\phi"] & \\ & & Q \end{tikzcd}$$

remark. Other names for the pushout are ***cofibered product of*** X ***and*** Y (especially in algebraic categories when i_1 and i_2 are monomorphisms), or ***free product of*** X ***and*** Y with Z ***amalgamated sum***, or more simply an ***amalgamation*** or ***amalgam of*** X ***and*** Y.

remark. If coproducts exist in some category, then the pushout

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\ X \arrow[r, "i_1"'] & X \coprod_Z Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is equivalently the coequalizer

$$X \underset{i_2 \circ g}{\overset{i_1 \circ f}{\rightrightarrows}} X \coprod Y \longrightarrow X \coprod_Z Y$$

of the two morphisms induced by f and g into the coproduct of X with Y.

example (wiki).

  - If X, Y and Z are sets and f, g are functions, the pushout of f and g is the disjoint union of X and Y where elements sharing a common preimage in Z are identified, i.e. $P = (X \coprod Y) / \sim$ where $\sim$ is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

    In particular, if X and Y are subsets of some larger set W and Z is their intersection, with f and g the inclusion maps of Z into X and Y, then the pusout can be canonically identified with the union $X \cup Y \subseteq W$.

  - The construcion of ***adjunction spaces*** is an example of pushouts in Top. More precisely, if Z is a subspace of Y and $g : Z \to Y$ is the inclusion map, we can glue Y to another space X along Z using an ***attaching map*** $f : Z \to X$. The result is the ***adjunction space*** $X \cup_f Y$ which is just the pushout of f and g. More generally, all identification spaces may be regarded as pushouts in this way. See **??** .

- A ***product*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \sqcap Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}$$

- A ***coproduct*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$, $i_2 : Y \to X \sqcup Y$ satisfying the following universal property:

$$\begin{tikzcd}
& Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & X \sqcup Y \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

remark. More generally, for S any set and $F : S \to \mathcal{C}$ a collection of objects in $\mathcal{C}$ indexed by S, their ***coproduct*** is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

- The ***kernel*** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the ***kernel*** $\ker f$ of a morphism $f : A \to B$ is the pullback $\ker(f) \to A$ along f of the unique morphism $0 \to B$

  More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

  > for every object C and every morphism $h : C \to A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \to \ker(f)$ such that $h = p \circ \phi$.

$$\begin{tikzcd}
C \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "h"'] & & \\
& \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & 0 \arrow[d] \\
& A \arrow[r, "f"'] & B
\end{tikzcd}$$

- In a category with a terminal object 1, the ***cokernel*** of a morphism $f : A \to B$ is the pushout (arrows h and ϕ apply if terminal object is zero)

$$\operatorname{coker}(f) := 1 \sqcup_A B \qquad \begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \arrow[ddr, bend left, "h"] & \\
1 \arrow[r] & \operatorname{coker}(f) \arrow[dr, dashed, "\phi"] & \\
& & C
\end{tikzcd}$$

  In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\operatorname{coker}(f)$ with the following universal property:

  > for every object C and every morphism $h : B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \operatorname{coker}(f) \to C$ such that $h = \phi \circ i$.

- A morphism $f : X \to Y$ is a ***monomorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Z \to X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$\begin{tikzcd}
Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[rr, bend left, "f \circ g_1"] \arrow[rr, bend right, "f \circ g_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

  Equivalently, f is a monomorphism if for every Z the hom-functor $\operatorname{Hom}(Z, -)$ takes it to an injective function

$$\operatorname{Hom}(Z, X) \xhookrightarrow{f_*} \operatorname{Hom}(Z, Y).$$

  Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A morphism $f : X \to Y$ is a ***epimorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Y \to Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\mathrm{Hom}(-, Z)$ takes it to an injective function

$$\mathrm{Hom}(Y, Z) \xhookrightarrow{f^*} \mathrm{Hom}(X, Z).$$

Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- (Retraction.)
  - (wiki) Let X be a topological space and A a subspace of X. Then a continuous map $r : X \to A$ is a ***retraction*** if the restriction of r to A is the identity map on A.
  - (nLab) An object A in a category is called a ***retract*** of an object B if there are morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{id}_A$. In this case r is called a ***retraction of*** B ***onto*** A and i is called a ***section of*** r.

    $$\mathrm{id} : A \xrightarrow[\text{section}]{i} B \xrightarrow[\text{retraction}]{r} A$$

    Hence a ***retraction*** of a morphism $i : A \to B$ is a left-inverse and a ***section*** of a morphism $r : B \to A$ is a right-inverse.
- (Deformation retract.)
  - (nLab) Let $\mathcal{C}$ be a category equipped with a notion of homotopy between its morphisms. Then a ***deformation retraction*** of a morphism $i : A \to X$ is another morphism $r : X \to A$ such that

    $$?$$

  - (wiki) A continuous map $F : X \times [0, 1] \to X$ is a ***deformation retraction*** of a space X into a subspace A if, for every x in X and a in A,

    $$F(x, 0) = x, \qquad F(x, 1) \in A \quad \text{and} \quad F(a, 1) = a.$$

    In words, a deformation retraction is a homotopy between a retraction and the identity map on X. The subspace A is called a ***deformation retract*** of X. A deformation retraction is a special case of a homotopy equivalence.

    An equivalent definition of deformation retraction is the following. A continuous map $r : X \to A$ is a ***deformation retraction*** if it is a retraction and its compositition with the inclusion is homotopic to the identity map on X.In this formulation, a deformation retraction carries with it a homotopy between the identity map on X and itself.

- (wiki) If, in the definition of a deformation retraction we add the requirement that
$$F(a,t) = a \qquad \forall t \in [0,1],\ \forall a \in A,$$
then F is called a ***strong deformation retraction***. In words, a strong deformation retraction leaves points in A fixed throughout the homotopy.

  example. S^n is a strong deformation retract of $\mathbb{R}^{n+1}\backslash 0$ through $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$.

- (wiki) The inclusion of a closed subspace A in the space X is a **??** if and only if A is a ***neighbourhood deformation retract*** of X, meaning that there is a continuous map $u : X \to [0,1]$ with $A = u^{-1}(0)$ and a homotopy $H : X \times [0,1] \to X$ such that $H(x,0) = x$ for all $x \in X$, $H(a,t) = a$ for all $a \in A$ and $t \in [0,1]$, and $H(x,1) \in A$ if $u(x) < 1$.

  For example, the inclusion of a subcomplex in a CW complex is a cofibration.

elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. An ***homotopy*** from f to g is a continuous map
$$H : X \times [0,1] \to Y, \qquad (x,t) \mapsto H(x,t) = H_t(x)$$
) such that $f(x) = H(x,0)$ and $g(x) = H(x,1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. A homotopy of maps $H_t : X \to Y$ is called ***relative to*** $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top, the ***homotopy category*** h-Top, where comoposition of homotopy classes is induced by composition of representing maps. If $f : X \to Y$ represents an isomorphism in h-Top, then f is called a ***homotopy equivalence*** or h-***equivalence***. In explicit termins this means $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$, a ***homotopy inverse of*** f, such that gf and fg are both homotopic to the identity. Spaces X and Y are called ***homotopy equivalent*** or of the same ***homotopy type*** if there exists a homotopy equivalence $X \to Y$. A space is ***contractible*** if it is homotopy equivalent to a point. A map $f : X \to Y$ is ***null homotopic*** if it is homotopic to a constant map.

- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the n-***th homotopy group*** are homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

  Also,
$$\pi_n(X,*) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n/\partial I^n, X]^0$$

where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \to Y$.

proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the ***relative homotopy group*** $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n. That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

  The elements of such a group are homotopy classes of based maps $D^n \to X$ which carry the boundary S^{n-1} into A. Two maps f, g are called ***homotopic relative to*** A if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0, 1] \to X$ such that, for each p in S^{n-1} and t in $[0, 1]$, the element $F(p, t)$ is in A. Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

  remark. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, x_0)$$

  where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. The map, called the ***boundary map***, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called n***-connected*** if $\pi_i(X, x_0) = 0$ for $i \leqslant n$. Thus 0-connected means path-connected and 1 connected means simply-connected.

- A pair (X, A) is n***-connected*** if $\pi(X, A, x_0) = 0$ for $i \leqslant n$.

- Two pointed spaces (X, x_0) and (Y, y_0) are n***-equivalent*** if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i < n$ and surjective for $i = n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on ***compactly generated wakly Hausdorff*** spaces.

- X is ***compactly generated*** if, for any subset $C \subset X$, and for all continuous maps $f : K \to X$ from compact Housdorff spaces,

$$\text{if } f^{-1}(C) \text{is closed in } K, \text{ then } C \text{ is closed.}$$

claim (What I picked up from the lecture)**.** If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is k-closed.

From May: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi : X \to Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH, $\mathrm{Hom}(X, Y)$ is a space with the compact-open topology. This is a compactly generated space, $\mathbf{k}(\mathrm{Hom}(X, Y))$.

remark. (Also see wiki on currying)

$$\begin{aligned} \mathrm{Map}(X, Y) &:= \text{ the space of maps } X \to Y. \\ \mathrm{Map}(X \times Y, Z) &\cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)) \\ \mathrm{Hom}(X \times Y, Z) &\cong \mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \end{aligned}$$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to $\mathrm{Map}(Y, -)$.

cofibrations

definition.

- (wiki) In mathematics, in particular in homotopy theory, a continuous map between topological spaces $i : A \to X$ is a ***cofibration*** if it has the ***homotopy extension property*** with respect to all topological spaces S.

  That is, i is a cofibration if

  - for each topological space S,
  - and for any continuous maps $f, f' : A \to S$
  - and $g : X \to S$ with $g \circ i = f$,
  - for any homotopy $h : A \times I \to S$ from f to f',

there is a continuous map $g' : X \to S$ and a homotopy $h' : X \times I \to S$ from g to g' such that

$$h'(i(a), t) = h(a, t) \qquad \text{for all } a \in A \text{ and } t \in I.$$

- (wiki) In what follows, let $I = [0, 1]$ denote the unit interval.

  A map $i : A \to X$ is a ***cofibration*** if for any map $f : A \to S$ such that there is an extension to X, meaning there is a map $\tilde{f} : X \to S$ such that $\tilde{f} \circ i = f$, we can extend a homotopy of maps $H : A \times I \to S$ to a homotopy of maps $\tilde{H} : X \times I \to S$ where

$$H(a, 0) = f(a)$$
$$\tilde{H}(x, 0) = \tilde{f}(x)$$

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "i"'] \arrow[dr, bend right=80, "f"'] & S^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}"'] \arrow[ur, dashed, "\tilde{H}"] & S
\end{tikzcd}$$

- (wiki) Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the ***homotopy extension property*** if, given a homotopy $f_\bullet : A \to Y^I$ and a map $\tilde{f}_0 : X \to Y$ such that

$$\tilde{f}_0 \circ \iota = f_0$$

  (so $\tilde{f}$ is the lift of $f_0 : A \to Y$) then there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{f}_\bullet : X \to Y^I$ such that $\tilde{f}_\bullet \circ \iota = f_\bullet$.

  That is,

$$\begin{tikzcd}
A \arrow[r, "f_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}_0"'] \arrow[ur, dashed, "\tilde{f}_\bullet"] & Y
\end{tikzcd}$$

  So there's some currying to make usual homotopies $f_\bullet : A \times I \to Y$ look like $f_\bullet : A \to Y^I$. Or, as said in our lectures, "a homotopy $X \times I \to Y$ is the same as a map $X \to \mathrm{Map}(I, Y)$".

- (May) A map $i : A \to X$ is a ***cofibration*** if it satisfies the ***homotopy extension property (HEP)***. This means that if $h \circ i_0 = f \circ i$ in the diagram

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i \times id"] \\
& Y & \\
X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"]
\end{tikzcd}$$

then there exists $\tilde{h}$ that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map $i : A \to X$ between topological spaces is a ***Hurewicz cofibration*** if it satisfies the homotopy extension property.

  Let's say it one more time: for any $g : X \to Y$ and any homotopy $H : A \times I \to Y$ such that

$$\begin{tikzcd}
A \times \{0\} \arrow[r] \arrow[d] & A \times I \arrow[d] \\
X \arrow[r, "g"'] & Y
\end{tikzcd}$$

  there is $H' : X \times I \to Y$,

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d, "g"] & A \times I \arrow[d] \\
X \times I \arrow[r, "H'"'] & Y
\end{tikzcd}$$

  such that

$$\begin{tikzcd}
A \times I \arrow[d] \arrow[dr, "H"] & \\
X \times I \arrow[r, "H'"'] & Y
\end{tikzcd}$$

example. $\partial D^n \to D$ is a Huerwicz cofibration. Why?

exercise. Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

definition (Mapping cylinder).

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. make the dashed map unique—up to something maybe). Namely, we can let Y in our original test diagram be the ***mapping cylinder***

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of i and i_0. Indeed, suppose that we can construct a map r that makes the following diagram commute

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd, "i \times id"] \\
& Mi & \\
X \arrow[ur] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "r"]
\end{tikzcd}$$

By the universal property of the pushouts, given maps f and h in our original test diagram induce a map $Mi \to Y$, and its comoposite with r gives a homotopy $\tilde{h}$ that makes the diagram commute. So just saying that Mi is universal.

- (nLab) Given a continuous map $f : X \to Y$ of topological spaces, one can define its ***mapping cylinder*** as a pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X \times I \arrow[r, "(\sigma_0)_*(f)"'] & Cyl(f)
\end{tikzcd}$$

in Top, where $I = [0, 1]$ and $\sigma : X \to X \times I$ is given by $x \mapsto (x, 0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$\left(X \times I \coprod Y\right) \Big/ \sim$$

where $\sim$ is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

- (wiki) The ***mapping cylinder*** of a function f between topological spaces X and Y is the quotient

$$M_f = (([0, 1] \times X) \amalg Y) / \sim$$

where $\amalg$ denotes disjoint union, and $\sim$ is the equivalence relation generated by

$$(0, x) \sim f(x) \text{ for each } x \in X.$$

That is, the mapping cylinder M_f is obtained by gluing one end of $X \times [0, 1]$ to Y via the map f. Notice that the "top" of the cylinder $\{1\} \times X$ is homeomorphic to X, while the "bottom" is the space $f(X) \subset Y$.

> (Dani) So the mapping cylinder is just deforming X to Y putting X inside Y via f.

- (Homework) Let $f : X \to Y$ be a map. Let $M_f = X \times [0, 1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \overset{\cong}{\to} X \times \{0\} \hookrightarrow X \times [0, 1]$ and of $f : X \times Y$.

exercise. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \times [0,1] \arrow[r] \arrow[drr, bend right, "(x,t)\mapsto f(x)"'] & M_f \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "h"] & \\
& & Y
\end{tikzcd}
$$

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\operatorname{colim} A_i) \times B$ is canonically homeomorphic with $\operatorname{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

exercise. $X \to M_f \to Y$. Prove $X \to M_f$ is a cofibration.

fibrations

- (nLab) A morphism i has the ***left lifting property with respect to a morphism*** p and p has the ***right lifting property with respect to*** i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\phi"] & Y
\end{tikzcd}
$$

- (nLab) Let C be a category with products and with interval object I. A morphism $E \to B$ has the ***homotopy lifting property*** if it has the right lifting property with respect to all morphisms of the form $(\mathrm{id}, 0) : Y \to Y \times I$.

  This means that for all commuting squares

$$
\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\sigma"] & B
\end{tikzcd}
$$

there exists a morphism $\sigma : Y \times I \to E$ such that both triangles in the former diagram commute.

A ***fibration*** (also called ***Hurewicz fibration***) is a mapping $p : E \to B$ satisfying the homotopy lifting property for all spaces X.

- (Hatcher) A map $p : E \to B$ is said to have the ***homotopy lifting property*** with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, so $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

  The ***lift extension property for a pair*** (Z, A) asserts that every map $X \to B$ has a lift $Z \to E$ extending a given lift defined on the subspace $A \subset Z$. The case $(Z, A) = (X \times I, X \times \{0\})$ is the homotopy lifting property.

  A ***fibration*** is a map $p : E \to B$ having the homotopy property with respect to all spaces X.

  Theorem 3 (4.41 Hatcher, Long exact sequence of Serre fibrations, see proposition 18)**.** Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geqslant 0$. Choose basepoints b_0 nB and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_0(B, b_0)$ is an isomorphism for all $n \geqslant 1$. Hence b is path-connected and there is a long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \overset{p_*}{\to} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

  The map $p : E \to B$ is said to have the ***homotopy lifting property for a pair*** (X, A) if each homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$. In other words, the homotopy lifting property for (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

  The point is that the homotopy lifting property for disjs is equivalent to the homotopy lifting property for all CW pairs (X, A). A map $p : E \to B$ satisfying the homotopy lifting property for disks is sometimes called a ***Serre fibration***.

  A ***fiber bundle*** structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point B has a neighbourhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the following diagram commute

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

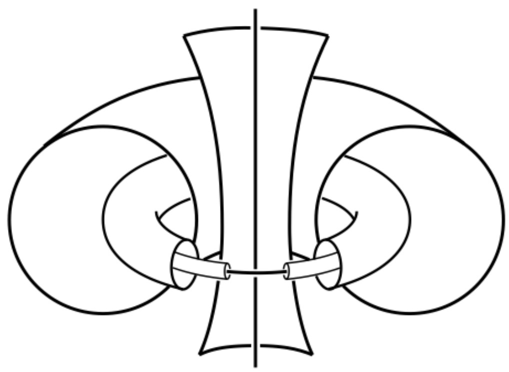
example. Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces $S^n \to \mathbb{R}P^n$ with fiber S^0. Over the complex numbers the analog of this is a fiber bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. Here S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$ is viewed as the quotient space of S^{2n+1} under the equivalence relation $(z_0, \ldots, z_n) \sim \lambda(z_0, \ldots, z_n)$ for $\lambda \in S^1$. The projection $p : S^{2n+1} \to \mathbb{C}P^n$ sends $(z_0, \ldots, z_n)$ to its equivalence class $[z_0, \ldots, z_n]$.

To see that the local triviality condition for fibre bundles is satisfied, ...

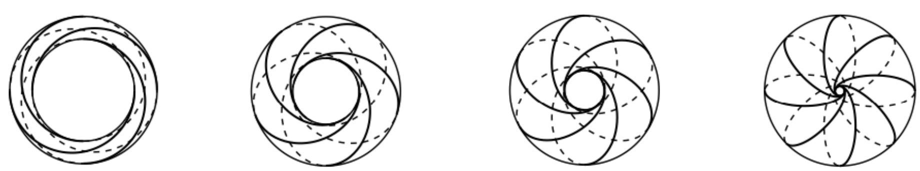
The constructino of the bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ also works when $n = \infty$, so there is a fiber bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$.

The case $n = 1$ is particularly interesting since $\mathbb{C}P^1 = S^2$ and bundle becomes $S^1 \to S^3 \to S^2$ with fiber, total space, and base all speres. This is known as the ***Hopf bundle***. The projection $S^3 \to S^2$ can be taken to be $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$. (That is, seeing S^2 as the one-point compactification of $\mathbb{C}$.)

In polar coordinates we may see S^3 as the union of several tori. Stereorgraphic projection yields the following figure:



The limiting cases T_0 and T_∞ correspond to the unit circle in the xy-plane and the z-axis under the stereographic projection. Each torus T_ρ is aunion of circle fibers. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. As ρ goes to 0 or ∞ the fiber circles approach the circles T_0 and T_∞, which are also fibers. The figure below shows four tori decomposed into fibers:



How could we visualize the projection onto S^2? Could it work to think $S^2 = \mathbb{C} \cup \infty$ and just do stereographic projection from 3-space to the plane disregarding one point? What would that even mean hehe

Replacing the field $\mathbb{C}$ by the quaternions $\mathbb{H}$, the same constructions yield fiber bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ over quaternionic projective spaces $\mathbb{H}P^n$. Here the fiber S^3 is the unit quaternions, and S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1}$. Taking $n = 1$ gives a second Hopf bundle $S^3 \to S^7 \to S^4 = \mathbb{H}P^1$.

Another Hopf bundle $S^7 \to S^{15} \to S^8$...

model structures

definition (Riehl)**.** A ***weak factorization system*** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised o two clases of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

1. Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$:

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

2. The maps in $\mathcal{L}$ have the ***left lifting property*** with respect to each map in $\mathcal{R}$ and equivalently the maps in $\mathcal{R}$ have the ***right lifting property*** with respect to each map in $\mathcal{L}$, that is, any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

admits a diagonal filler as indicated making both triangles commute. When this lift is unique, we say the factorization system is ***orthogonal***.

3. The classes $\mathcal{L}$ and $\mathcal{R}$ are each closed under retracts in the arrow category: given a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "t"'] \arrow[rr, bend left, equal] & \bullet \arrow[r] \arrow[d, "s"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r] \arrow[rr, bend right, equal] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

if s is in that class then so is its retract t.

exercise (3.1.8 from Riehl)**.** Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

definition. Given a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \text{Sets}$ there is a corresponding category (***of elements of*** $\mathcal{F}$) that lies over $\mathcal{C}$, that is,

$$\text{el}\,\mathcal{F} \to \mathcal{C}$$

given by

Objects: pairs (C, X) where $C \in \text{Obj}\,\mathcal{C}$ and $X \in \mathcal{F}(X)$.

Morphisms: $f : (C, X) \to (C', C')$ are morphisms $f : C \to C'$ such that $\mathcal{F}(f)(X') = X$.

remark. We can use the Yoneda embedding to view $\mathcal{C}$ as a subcategory of $\mathrm{Psh}(\mathcal{C})$,

$$\mathcal{C} \hookrightarrow \mathrm{Psh}(\mathcal{C})$$

And also $\mathcal{F} \in \mathrm{Psh}(\mathcal{C})$. In fact, the element category is just the slice category:

$$\mathrm{el}\,\mathcal{F} \cong \mathcal{C}/\mathcal{F}.$$

question. Given $\mathcal{D} \to \mathcal{C}$ is it isomorphic to $\mathrm{el}\,\mathcal{F} \to \mathcal{C}$?

definition. $G : \mathcal{D} \to \mathcal{C}$ is a ***discrete fibration*** if for any $d \in \mathcal{D}$ and any $f : C \to G(d)$ there exists a unique lift from f' of f to $f \in \mathcal{D}$ such that the target ot f' is is d. That is,

$$\begin{tikzcd} \bullet \arrow[r, dashed, "\exists! f'"] \arrow[d, "G"'] & d \arrow[d, "G"] \\ C \arrow[r, "f"'] & G(d) \end{tikzcd}$$

remark. Given a discrete fibration we may construct a functor $\mathcal{F} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Sets}$ simply by defining $\mathcal{F}(C) = G^{-1}(C)$ and if $C \to C' \cdots \to d$.

definition (Lecture)**.** A ***model structure*** on a category $\mathcal{A}$ is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called ***weak-equivalences***, ***cofibrations*** and ***fibrations*** with the following properties:

0. All (finite) small limits an colimits.
1. **(2 of 3)** Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in $\mathcal{W}$ then all of them are.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system. That is,
   (a) Any morphism in $\mathcal{A}$ can be factored as a morphism in $\mathcal{B}$ followed by a morphism in $\mathcal{D}$.
   (b) Lifts:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \bullet \end{tikzcd}$$

   (c') Notice that the aciom of retracts is not necessary. $r' \in \mathcal{R} \iff$ it satisfies (b) for all $\ell \in \mathcal{L}$.

definition.

- X is ***fibrant*** if $X \twoheadrightarrow \mathrm{pt}$.
- X is ***cofibrant*** if $X \rightarrowtail\!\!\!\twoheadrightarrow X$

- X is ***bifibrant*** if $0 \rightarrowtail X \twoheadrightarrow \mathrm{pt}$

examples (Two interesting model category structures on CGWH)**.**

1. ***Hurewicz model structure*** (Strom).
   - Cofibrations:= Huerwicz cofibrations.
   - Fibrations:= maps $E \to B$ such that for all spaces X [Photo1].
   - Weak equivalences:= homotopy equivalences.
2. ***Quillen model structure***. Defined on Top.
   - Cofibrations = retracts of relative cell complexes.
   - Fibrations = Serre Fibrations: $\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$
   - Weak equivalences: $f : X \to Y$

   Also, we have
   - Fibrant: all of Obj Top.
   - Cofibrant: $\exists$\{CW complexes\}.

definition. Given a category $\mathcal{C}$ and a class of morphisms $W \subset \operatorname{Mor}\mathcal{C}$, its ***localization*** is a category $\mathcal{C}[W^{-1}]$ such that there is a functor $\operatorname{Loc}_W \mathcal{C} \to \mathcal{C}[W^{-1}]$ that sends weak equivalences to isomorphisms. Also, its satisfies the universal property that for every $F : \mathcal{C} \to \mathcal{D}$ such that $F(X) \subset \mathrm{Iso}$, the following diagram commutes

$$\begin{tikzcd} & \mathcal{C}[W^{-1}] \arrow[dr, dashed, "\exists! G"] & \\ \mathcal{C} \arrow[ur, "\operatorname{Loc}_W"] \arrow[rr, "F"] & & \mathcal{D} \end{tikzcd}$$

Theorem 4. Let $\mathcal{C}$ and (C, W, F) be a model category and $\mathcal{C}[W^{-1}] \cong \operatorname{Ho}\mathcal{C}$ where the latter is given by

- $\operatorname{Ob}\operatorname{Ho}\mathcal{E} = \{\text{fibrant-cofibrant-bifibrant objects of } \mathcal{E}\}$.
- $\operatorname{Mor}\operatorname{Ho}\mathcal{E} = \operatorname{Mor}_{\mathcal{E}}(X, Y)/\text{homotopy}$.

Let's say what homotopy means

definition. Given two maps

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

- We say $f \underset{\text{left}}{\sim} g$ if for the ***cylinder*** $\mathrm{Cyl}(X)$ defined by

$$\begin{tikzcd} X \sqcup X \arrow[rr] \arrow[dr, tail, "\text{cofibr.}"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, two heads, "\text{trivial fib.}"'] & \end{tikzcd}$$

we have that

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr] & & Y \times Y \\ & \mathrm{Cyl}(X) \arrow[ur, dashed, "\exists H"'] & \end{tikzcd}$$

- We say $f \underset{\text{right}}{\sim} g$ if

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr, dashed, "\exists H"'] & & Y \times Y \\ & \mathrm{Path}(Y) \arrow[ur] & \end{tikzcd}$$

claim. Given $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, if X is cofibrant and Y is fibrant, then $f \underset{\text{left}}{\sim} g \iff f \underset{\text{right}}{\sim} g$ and $\sim$ is an equivalence relation.

whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A ***CW complex*** X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called ***vertices***) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along ***attaching maps*** $j : S^n \to X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying (j, x) with $j(x)$ for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j (see **??**). Each resulting map $D^{n+1} \to X$ is called a ***cell***. The subspace X^n is called the n-***skeleton*** of X.

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\ulcorner"] & D^{n+1} \arrow[d] \\ X^n \arrow[r] & X^{n+1} \end{tikzcd}$$

lemma 5 (HELP). content...

Theorem 6 (Whitehead, May). If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 7 (Whitehead, May). An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 8 (Whitehead (4.5), Hatcher). If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12). Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 7. $\square$

lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \to X$$

$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(X', Y)$$

$$\begin{gathered} \mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, B') \text{ natual in } A \implies \\ \mathrm{Hom}(B, B) \cong \mathrm{Hom}(B, B') \& \mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', B') \\ \implies B \cong B'. \end{gathered}$$

- for ($\Longleftarrow$) commutativity of the hypotesis gives us commutativiuty of the rightmost square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an n-equivalence.
- π_n of loop space is the same as π_{n+1} of original space.

- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 9 (Yoneda)**.**

$$\{\text{Natural transformations } \operatorname{Hom}(-,X) \to F\} \cong F(X)$$

corollary 10. $(\operatorname{Hom}(-,X) \to \operatorname{Hom}(-,Y)) \cong \operatorname{Hom}(X,Y)$.

corollary 11. The correspondence $X \mapsto \operatorname{Hom}(-,X)$ is fully faithful, that is, the correspondence $\operatorname{Hom}(X,X') \to \operatorname{Hom}(\operatorname{Hom}(-,X),\operatorname{Hom}(-,X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. □

lemma 12. Let $E \times_B X$ be the pullback of

$$\begin{tikzcd} & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be such that $E \to B$ is an homotopy fibration and $f : X \to B$ is a homotopy equivalence. Let

$$\begin{tikzcd} E \times_B X \to E \arrow[r, dash, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be the pullback. Then $E \times_B X \to E$ is a homotopy equivalence.

Proof. Let $g : B \to X$ be the homotopy inverse of f.

(Step 1) Construct another pullback

$$\begin{tikzcd} E \times_B B \arrow[r] \arrow[d, two heads] & X \times_B E \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] \\ B \arrow[r, "g"'] & X \arrow[r, "f"'] & B \end{tikzcd}$$

(Step 2) Constuct $E \to E \times_B B$.

Consider

$$\begin{tikzcd} E \arrow[rr, "\mathrm{id}"] \arrow[d] & & E \arrow[d, two heads] \\ E \times I \arrow[r, "f \times \mathrm{id}"'] \arrow[urr, dashed] & B \times I \arrow[r] & B? \end{tikzcd}$$

And then $E \to E \times_B B \to E \times_B X \to E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{tikzcd} Z \sqcup Z \arrow[r] \arrow[d, "f_1 \sqcup f_2"'] \arrow[rdd, bend left] & I \times Z \arrow[d] \arrow[dl, dashed] \arrow[dd, bend left] \\ E \times_B X \arrow[r] \arrow[d] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

$\square$

corollary 13. $B \xrightarrow{f} B$ is homotopy equivalence and $E \to B$ is a fibration, in

$$\begin{tikzcd} E \times_B B \arrow[r] \arrow[d] & E \arrow[d] \\ B \arrow[r, "f"'] & B \end{tikzcd}$$

$E \times_B B \to E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 14. Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "g"] & B \arrow[d] \\ X \arrow[r] & X \cup_A B \end{tikzcd}$$

be a pushout with $A \to X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f, g) \to X \cup_A B$ is a homotopy equivalence.

remark.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & M_f \arrow[d] \\ X \arrow[r] & X \cup_A M_f \cong M(f, g) \end{tikzcd}$$

definition.

- The ***homotopy pullback*** of a diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is

$$\begin{tikzcd} X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y.

- The ***homotopy fiber*** if $f : Y \to Z$ is the pullback of

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ pt \arrow[r] & Z \end{tikzcd}$$

$F \subset Z^I \times_Z Y \to Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark. The pullback of

$$\begin{tikzcd} & Z^I \times_Z Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is the homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

lemma 15. If $X \to Z$ is a fibration then for

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r, two heads] & Z \end{tikzcd}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{tikzcd}
X \times_Z \arrow[r] \arrow[d, "\simeq"] & Y \arrow[d, "\simeq"] \\
X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r, two heads] \arrow[d] & Z^I \times_Z Y \arrow[d, two heads] \\
X \arrow[r] & Z
\end{tikzcd}$$

∎

Finally,

$$\begin{tikzcd}
\text{hofib}\, f_1 \arrow[r] \arrow[d] & \text{hofib}\, f \arrow[r] \arrow[d, two heads] & * \arrow[d] \\
* \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

and

$$\begin{tikzcd}
Z \arrow[r] \arrow[dd] & F(f) \arrow[d] \arrow[dl, dash] \arrow[dd, bend left, two heads] \\
 & X \times_Y Y^I \arrow[d] \\
X \times I \arrow[r] & X
\end{tikzcd}$$

and an exact sequence

$$\Omega^2 \text{hofib} \to \Omega^2 X \to \Omega^2 Y \to \Omega\, \text{hofib}\, f \to \Omega X \to \Omega Y \to \text{hofib}\, f \to X \xrightarrow{f} Y$$

lemma 16 (Exactness)**.** $\forall z$, $[z\, \text{hofib}\, f] \to [Z, X] \to [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \to \pi_0(\Omega^2 Y) \to \pi_0(\Omega\, \text{hofib}\, f) \to \pi_0(\Omega X) \to \pi_0(\Omega Y) \to \pi_0(\text{hofib}\, f) \to \pi_0(X) \to \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewickz fibrations. Let's talk about Serre fibrations. Notice that H. fibration $\implies$ S. fibration. What is the most natural example of a Serre fibration?

proposition 17 (also Hatcher 4.48)**.** Let E be a fiber bundle with fiber F. Then f is a Serre fibration.

Proof. What sdoes it mean to be a Serre fibration? It means that

$$\begin{tikzcd}
I^n \arrow[r] \arrow[d] & E \arrow[d] \\
I^{n+1} = I^n \times I \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

So if $\mathcal{U}$ is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U.

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{tikzcd}
I^n \arrow[r] \arrow[d] & F \times U \arrow[d, two heads] \\
I^{n+1} \arrow[r] \arrow[ur, dashed] & U
\end{tikzcd}$$

has a lift for every little square because

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & U \arrow[d] \\
X \times I \arrow[r] \arrow[ur, dashed] & pt
\end{tikzcd}$$

is always a fibration (think about this) and because pullbacks of fibrations are fibrations:

$$\begin{tikzcd}
U \times F \arrow[r, two heads] \arrow[d, two heads] & U \arrow[d, two heads] \\
F \arrow[r, two heads] & pt
\end{tikzcd}$$

. Then we may just add up the squares because

$$\begin{tikzcd}
D^n \arrow[d] \\
D^n \times I
\end{tikzcd}$$

and we're done. □

proposition 18 (Sere fibration long exact sequence, see theorem 3). Let $g : E \to B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(E) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(F) \arrow[r] \arrow[u, "="] & \pi_n(E) \arrow[r] \arrow[u, "="] & \pi_n(E, F) \arrow[r] \arrow[u, dashed, "\cong"] & \pi_{n-1}(F) \arrow[r] \arrow[u, "="] & \pi_{n-1}(E) \arrow[r] \arrow[u, "="] & \cdots
\end{tikzcd}$$

example. We have shown that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 19. Let X be a CW-comples, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \varnothing$, so

$$\begin{tikzcd} C \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

is a pushout (this happens for inclusions, check it?).

If (A, C) is n-connected and (B, C) is m-connected, then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and sujerctive for $i = m + n$.

blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \to Y$ be a map. Consider the pullback

$$\begin{tikzcd} W(f) \arrow[r] \arrow[d, "(q,p)"'] & Y^I \arrow[d, "(ev_0, ev_1)"] \\ X \times Y \arrow[r, "f \times id"'] & Y \times Y \end{tikzcd}$$

where

$$W(f) = \{(x, w) \in X \times Y^I | f(x) = w(0)\},$$
$$q(x, w) = x, \quad p(x, w) = w(1).$$

Since (ev_0, ev_1) is a fibration, the maps (q, p), q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \to W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \to (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \to X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \to *$.
- $I^n \times \{1\} \to A$.

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow \text{hofib} \longrightarrow A \longrightarrow X$$

$$\begin{tikzcd} \pi_0(\nearrow) = & \pi_n(A) \arrow[r] & \pi_n(X) \arrow[r] \arrow[dr] & \pi_{n-1}(\mathrm{hofib}) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \\ & & & \pi_n(X,A) \arrow[ur] & & \end{tikzcd}$$

Theorem 20 (Blakers-Massey 1)**.** Let

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

be a homotopy pushout, g is m equivalence, f is n-equivalence and $m, n \geqslant 0$. Then $Q \to X \times_P^h$ is $(m+n-1)$-equivalence.

Theorem 21 (Blakers-Massey 2)**.** P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \varnothing$ (***strict pushout***)

$$\begin{tikzcd} Q \arrow[r, tail] \arrow[d, tail] & Y \arrow[d, tail] \\ X \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & X \end{tikzcd}$$

Then $\pi_i(Y, \mathbb{Q}) \to \pi_i(P, X)$ is epi for $i = m+n$ and iso for $0 \leqslant i < m+n$.

Theorem 22 (Blakers-Massey 3)**.** $P = X \cup Y$, X and Y are open in P, $X \cap Y = Q \neq \varnothing$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a k-***equivalence*** if the induced map on the ith homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W :$ at least p coordinates of x are ¡ the same coordinates of the center of $W\}$

lemma 23. Let W be a cube in $\mathbb{R}^d$ with $\dim W \leqslant d$. If for all faces W' of ∂W, $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \ \mathrm{rel}\, \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

freudenthal theorem (2 april)

definition. The appropriate analogue of the Cartesian product in the category of based spaces is the ***smash product*** $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y.

We also have the ***suspension of pointed spaces***, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X)/(t,x) \sim (0,y) \sim (1,y) \; \forall y \in X.$$

Then we have

$$\mathrm{Hom}_{\mathrm{CGWH}_*}(\Sigma X, \Sigma X) \cong \mathrm{Hom}_{\mathrm{CGWH}_*}(X, \Omega\Sigma X)$$

where $\Sigma X = S \wedge X$ and $\Omega\Sigma X = \mathrm{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\mathrm{Map}(S^1, -)$.

So let X be a space. The identity map $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ then induces a map $X \to \Omega\Sigma X$.

Theorem 24 (Freudenthal)**.** Let X be ℓ-connected space. Then $X \to \Omega\Sigma X$ is a $(2\ell+1)$-equivalence, that is,

$$\pi_i(X) \to \pi_{i+1}(\Sigma X),$$

is a bijection for $i < 2\ell + 1$ and a surjection for $i = 2\ell + 1$ (May).

Proof 1.

$$\begin{tikzcd}
X \arrow[rr, "(\ell+1)\text{-equiv}"] \arrow[dd, "(\ell+1)\text{-equiv}"'] \arrow[dr] & & * \arrow[dd] \\
& \Omega\Sigma X \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner h"] & \\
* \arrow[rr] & & \Sigma X \arrow[ul, phantom, "h \ulcorner"]
\end{tikzcd}$$

$\square$

Proof 2. Consider

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X
\end{tikzcd}$$

Then we use relative homotopy long exact sequence with (X, CX) to get $\pi_i(CX, X) \cong \pi_{i-i}(X)$, which is zero for $0 \leqslant i \leqslant \ell + 1$. Then use relative homotopy exact sequence for the pair $(\Sigma X, CX)$. then we get that $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$. And then if you use it for $(\Sigma X, X)$ and

But it also turns out that $\pi_i(\Sigma X) = \pi_{i-1}(\Omega\Sigma X)$ because

$$\pi_n(Z) = \mathrm{Hom}_{\text{h-Top},*}(S^n, Z) = \mathrm{Hom}(S^1 \wedge S^{n-1}, Z) = \mathrm{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX, X) \to \pi_i(\Sigma X, CX)$ which is isomorphism for $0 \leqslant i \leqslant 2\ell + 1$ and surjective for $i = 2\ell + 2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \to \pi_{i-1}(\Omega\Sigma X)$ for i as desired. $\square$

corollary 25. If X is ℓ-connected, then ΣX is $(\ell+1)$-connected for $\ell \geqslant 0$.

space	S^0	$\Sigma S^0 = S^1$	$\Sigma^2 S^0 = S^2$	$\Sigma^3 S^0 = S^3$	$\cdots$	$\Sigma^n S^0 = S^n$
conectedness	-1	0	1	2	$\cdots$	$(n-1)$

corollary 26. S^n is $(n-1)$-connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_2(S^3) \to \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \to \pi_1(S^3) = 0,$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map $S^n \to S^n$. We get a map $CS^n \to CS^n$ (in general, for $f : X \to Y$ we have $(t, x) \mapsto (t, f(x))$ in the cones). We also have $CS^n \to CS^n/S^n = S^{n+1}$.

Now if we take $\mathrm{id} : S^n \to S^n$ we shall get $\mathrm{id} : S^{n+1} \to S^{n+1}$. Think about this like $\pi_1(S^1) \to \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective because $i = 0$. From Hopf fibration we have $\pi_2(S^2) = \mathbb{Z}$. So we have a surjective map $\mathbb{Z} \to \mathbb{Z}$. So it's an isomorphism and we conclude that id_{S^2} is a generator of $\pi_2(S^2)$.

corollary 27. Since S^n is $(n-1)$-connected, we have

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is isomorphism for $i \leqslant 2(n-1) = 2n-1$ and epimorphism form $i = 2n-1$. (We just shift the indices of theorem 24 by one.)

corollary 28. $\pi_n(S^n) = \mathbb{Z}$ with id_{S^n} as generator.

corollary 29. $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \leqslant n-1$ and epimorphism for $k = n-1$.

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are $\mathbb{Z}/2$. This is what are called the k***th stable homotopy groups of a sphere***. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_i(S^1) \to \pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2) \to \pi_{i-1}(S^2) = 0$$

So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

claim (Serre)**.** $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$finite abelian. And $\pi_i(S^k)$ is finite abelian in all other cases.

another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{tikzcd} S^{n-1} \arrow[r, "(n-1)\text{-equiv}"] \arrow[d, "0\text{-equiv}"'] & D^n \arrow[d] \\ X \arrow[r] & X \cup D^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is $(n-1)$-equivalence by Blakers-Massey. So, by attaching $\sqcup D^n$ we can kill $\pi_{n-1}(X)$, that is, $X \cup (\sqcup D^n)$ has trivial π_{n-1}.

Now notice that

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

that is, $\pi_i(D^n, S^{n-1}) = 0$ for $i \leqslant n-1$. This implies that $\pi_i(X \cup D^n, X) = 0$ for $i \leqslant n-1$.

Also by homotopy long exact sequence,

$$\pi_{n-1}(X) \to \pi_i(X \cup D^n) \text{ is sujrective}$$

$$\pi_i(X) \to \pi_i(X \cup D^n) \text{ is isomorphism for } i \leqslant n-2.$$

So what we have thus far is

$$\pi_n(X \cup D^n) \longrightarrow \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup D^n) \longrightarrow 0 = \pi_{n-1}(X \cup D^n)$$

Notice that $\pi_n(X \cup D^n, X)$ is not ingeneral cyclic (counterexample $S^1 \cup D^2$ taking unieversal cover which is real line with spheres on integers, homotopy equivalent to $\bigvee_{\mathbb{Z}} S^2$ which is not finitely generated).

> So basically attaching a disk via f will kill $[f]$ inside $\pi_n(X)$ this is called ***killing*** an element of the homotopy group.

proposition 30. For any CW-complex X, $X^\ell \to X$ is an ℓ-equivalence.

remark. We have used that for $A \hookrightarrow X$ from long exact sequence of relative homotopy groups we get $\pi_n(X, A) = 0$.

corollary 31. Let $i \leqslant \ell$ and $g : D^i \to X$, $g : \partial D^i \to X^\ell$. Then there is a homotopy rel ∂D^i to a map with img $\subset X^\ell$.

Theorem 32 (Cellular approximation theorem)**.** Let X and Y be CW-complexes and $\xi : Y \to X$ be any map. Then ξ is homotopic to a ***cellular map***, that is, a map $\psi : Y \to X$, such that for all ℓ, $\psi Y^0 \subset X^\ell$.

We also have

proposition 33. Let $n \geqslant 2$. Then

$$\pi_n\left(\bigvee_{k\in I} S^n\right) = \bigoplus_{k\in I} \imath_n(S^n) = \bigoplus_{k\in I} \mathbb{Z} = Z^{\oplus I}$$

proposition 34. First notice that for finite I,

$$X^n = X^{n+1} = \bigvee_{k\in I} S^n$$

by looking carefully. Then

$$\pi_n(X, X^{n+1}) = 0 = \pi_{n+1}(X, X^{n+1})$$

so

$$\bigoplus_{k\in I} \mathbb{Z} = \prod_{k\in I} \pi_n(S^n) = \pi_n(X) = \pi_n(X^{n+1}) = \pi_n(X^n) = \pi_n\left(\bigvee_{k\in I} S^n\right)$$

and for the infinite case it also works, using finite compactness of the CW complex.

postnikov tower and CW-approximation, 9 abril

- Let X be a space. Then there is a CW-complex Y and a weak homotopy equivalence from $Y \to X$.
- Let $A \to X$ be a map of spaces. Then it can be factored as $A \to Y \to X$ where $A \hookrightarrow Y$ is a relative CW-complex, and $Y \to X$ is a weak homotopy equivalence.

  remark. Notice that the second item is the first one with $A = \varnothing$. Then, the second case is a Serre cofibration since it is a construction involving the cofibration $S^{n-1} \hookrightarrow D^n$ (this is a cofibration by definition).

- Let A be a space. Then there is a space $\tau_{\leqslant n}A$ such that $A \hookrightarrow \tau_{\leqslant n}A$; $\tau_{\leqslant n}A$ is obtained by adding cells of dim $\geqslant n+2$. $A \hookrightarrow \tau_{n\leqslant n}A$ is $(n+1)$-equivalence and

  $$\pi_k(\tau_{\leqslant n}A) = 0 \qquad k > n.$$

  Moreover, $A \to \tau_{\leqslant n}A$ is unique among morphisms in $\mathrm{Ho}(CGWH)$ from A into spaces with $\pi_k = 0$ for $k > n$.

This is called a ***Postnikow tower*** and it looks like this:

$$\begin{tikzcd}
 & \dots \arrow[d] \\
 & \tau_{\leqslant 2}A \arrow[d] \\
 & \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

The idea is that $\tau_{\leqslant n}A$ is obtained from A by killing elements of dimension greater than n, that is, by

- attaching $n+2$ cells that kill all $\pi_{n+1}(A)$,
- attaching $n+3$ cells that kill all $\pi_{n+2}(A)$,
- attaching $n+3 \dots$
- attaching $n+n \dots$

So consider the space X that is obtained from A after attaching cells of dimension $\geqslant n+2$, so we have a map $A \to X$. Consider also a space Y with $\pi_k(Y) = 0$ for $k > n$. Then for any map $A \to Y$ there is a map $X \to Y$ that extends $A \to Y$. This accounts for a bijection

$$[X, Y] \cong [A, Y].$$

In class we struggled a bit to prove surjectivity, finally using an argument related to the pair $(X \times I, X \times \partial I \cup A \times I)$.

> The point is that the spaces in the Postnikov tower are like the original space but with trivial homotopy groups for $k \geqslant n$.

question. What is the limit of the Posnikov tower?

$$\begin{tikzcd}
 & \lim \\
 & \dots \arrow[d] \\
 & \tau_{\leqslant 2}A \arrow[d] \\
 & \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuuur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

- Let $A \to X$ be a map (of CW-complexes (or spaces?)). Then

$$\begin{tikzcd}
 & \cdots \arrow[d] & \\
 & Y_2 \arrow[d] \arrow[ddr, bend left] & \\
 & Y_1 \arrow[d] \arrow[dr] & \\
A \arrow[uur, bend left] \arrow[ur] \arrow[r] \arrow[rr, bend right] & Y_0 \arrow[r] & X
\end{tikzcd}$$

Proof pending

- We also have the ***Whitehead tower***, obtained from the homotopy fiber

$$\text{hofib}\, f_n \longrightarrow A \xrightarrow{f_n} \tau_{\leqslant n-1}A$$

which yields

$$\cdots \longrightarrow \pi_{k+1}(A) \xrightarrow{\cong} \pi_{k+1}(\tau_{\leqslant n}A) \longrightarrow \pi_k(\text{hofib}) \longrightarrow \pi_k(A) \longrightarrow \pi_k(\tau_{\leqslant n}A) \longrightarrow \cdots$$

so

$k \leqslant n-1$	$k = n$	$k \geqslant n+1$
$\pi_k(\text{hofib}\, f_n) = 0$	$\pi_n(\text{hofib}\, f_n) = 0$	$\pi_k(A) = \pi_k(\text{hofib}\, f_n)$

- Now there's a natural way to construct the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend right] & \tau_{\leqslant n}A \arrow[r] & \tau_{\leqslant n-1}A
\end{tikzcd}$$

which yields the bundle

$$\text{hofib} \longrightarrow \tau_{\leqslant n}A \longrightarrow \tau_{\leqslant n-1}A$$

and in this case we get

$k \neq n$	$k = n$
$\pi_k(\text{hofib}) = 0$	$\pi_n(\text{hofib}) = \pi_n(A)$

and this is what we call a $K(\pi, n)$-***space*** (all homotopy groups are trivial but the nth.)

Moore space, $K(\pi, n)$ and Hurewicz theorem, 11&16 apr

Theorem 35 (Uniqueness of CW-approximations)**.** Recall that a CW-approximation of X is a map $f : Z \to X$ and a CW-complex Z that is a weak homotopy equivalence (induces isomorphisms in all homotopy groups).

We have that

$$\begin{tikzcd} Y_1 \arrow[dr, "\text{CW ap.}"] & \\ & X \\ Y_2 \arrow[uu, dashed, "\exists!"] \arrow[ur, "\text{CW ap.}"'] & \end{tikzcd}$$

up to homotopy equivalences

lemma 36 (Compression)**.** If the relative homotopy groups of a pair (Y, B) is zero for $n = \dim e$ for every cell $e \in X \backslash A$ then any map $(X, A) \to (Y, B)$ is homotopic rel A to $(X, A) \to (B, B)$ (so intuitively we can collapse Y).

Proof. With fibrations (photo) □

proposition 37. Let $f : X \to Y$ be an n-equivalence (in Hatcher stated as weak equivalence but argument is the same). Then f induces an n-***equivalence in homology*** $H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$ (an isomorphism for $i < n$ and surjection for $i = n$).

Proof. photo □

corollary 38. If $f : X \to Y$ is a weak equivalence, then f induces an isomorphism in $H_*(-, G)$ and $H^*(-, G)$.

Proof. Universal coefficients. □

definition. Let π be an abelian group. Take

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi$$

a ***free resolution***, i.e. $F_1 = Z^{\oplus J}$ and $F_0 = Z^{\oplus I}$ are free abelian groups and $\pi = F_0/F_1$. Let's take the corresponding maps

$$\bigvee_{j \in S} S^n \longrightarrow \bigvee_{i \in I} S^n \longrightarrow \text{hocofib } f$$

$$x_j \longmapsto \sum a_i y_i$$

where a_i is the degree of $S^n \to S^n$. Recall that the homotopy cofiber $\operatorname{hocofib} f$ is the mapping cone of f. It is the ***cone of pointed spaces***. We call this space the ***Moore space*** $M(\pi, n)$ and it is such that

$$\tilde{H}_i(M(\pi, n)) = \begin{cases} 0, & i \neq n \\ \pi, & i = n \end{cases}$$

What do we get in homology? Exactly the sequence of free groups above. So, $H_n(\operatorname{hocofib} f) = \pi$. What do we get in homotopy? Might be π as well. Let's prove something stronger:

Theorem 39. Let Y be such that $\pi_i(Y) = 0$ for $i > n$ and $\pi_0(Y) = 0$. Then

$$[\operatorname{hocofib} f, Y] \to \operatorname{Hom}(\pi_n(\operatorname{hocofib} f), \pi_n(Y))$$

is a bijection.

Proof. Photo

Take

$$\bigvee_I S^n \longrightarrow \bigvee_J S^m \longrightarrow \operatorname{hocofib} f$$

Now apply $[-, Y]$. We get

$$\left[\bigvee_I S^1\right] \longleftarrow \left[\bigvee_J S^1\right] \longleftarrow [\operatorname{hocofib} f, Y] \longleftarrow 0$$

$\square$

lemma 40. If (X, A) is r-connected, A s-connected for all $r, s \geqslant 0$, then the map

$$\pi_i(X, A) \to \pi_i(X/A)$$

induced by the quotient map $X \to X/A$ is an $(r + s + 1)$ equivalence.

Proof. If (X, A) is r-connected. $\square$

After the lemma we proved that $\pi_n(C_f \cong G$.

Theorem 41. Now consider a Moore space, kill all homotopy groups to get $\tau_{\leqslant n}(M)$. It is a $K(\pi, n)$ space with cells in dim $\geqslant n$, obtained from $\operatorname{hofib} f$ by attaching cells of dim $\geqslant n + 2$. Then

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \tau_{\leqslant n} M \arrow[dl, dashed, "\exists! \text{ up to homotopy}"] \\ & Y & \end{tikzcd}$$

$$[\operatorname{hocofib} f, y] \cong [\tau_{\leqslant n}(\operatorname{hocofib} f), Y] = \operatorname{Hom}(\pi, \pi_n(Y)).$$

If $\pi_n(Y) = \pi$, then there is a weak equivalence

$$\tau_{\leqslant n}(\operatorname{hocofib} f) \to Y.$$

Theorem 42 (Hurewicz)**.** Let X be an $(n-1)$-connected space for $n \geqslant 2$. Then

$$\tilde{H}_i(X) = \begin{cases} 0, & i < n \\ \pi_n(X), & i = n \end{cases}$$

Proof. Photo.

Idea is to construct a Moore space that is a piece of the CW approximation. Why a piece? Write and understand why this worked! □

Theorem 43 (Relative Hurewicz theorem)**.** Let (X, A) be n-connected, A be 1-connected, $n \geqslant 2$. Then

$$H_i(X, A) = \begin{cases} 0, & i < n \\ \pi_n(X, A), & i = n \end{cases}$$

Proof. Take a CW approximation of (X, A),

$$\begin{tikzcd} & Y \arrow[dr, "\cong"] & \\ B \arrow[ur] \arrow[r, "\cong"'] & A \arrow[r] & X \end{tikzcd}$$

So the approximation is (B, Y).

Then we have

$$\begin{aligned} \pi_i(Y, B) &= \pi_i(Y/B), \quad i \leqslant n \\ H_i(Y, B) &= \tilde{H}_i(Y/B), \quad \forall i \end{aligned}$$

and first line implies that $\pi_1(Y/B) = 0$ for $i < n$. But then we are done, right? □

Representability of the functor $H^n(-, G)$

remark. Recall the ***adjoint relation***

$$\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$$

where ΣX is the ***reduced suspension*** of a space X, ΩK is the ***loop space*** of another space K and the brackets mean homotopy classes of basepointed maps. Choosing $X = S^0$ and $K = M$, the left-hand side becomes $\pi_1(M, m)$ and the right-hand side becomes the path components of $\mathrm{Map}((S^1, 0), (M, m))$.

Now let's do some other interesting remarks.

definition. Let $\mathcal{C}$ be a category. Then $\mathrm{Psh}(\mathcal{C})$ is the category of ***presheaves*** of $\mathcal{C}$ i.e. the category of functors $\mathcal{C}^{op} \to \mathrm{Sets}$ and natural transformations. For any object A in $\mathcal{C}$ there is a presheaf $\mathrm{Hom}_{\mathcal{C}}(-, A)$. A presheaf $\mathcal{F}$ that is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, A)$ for some A is called ***representable***.

For example, $\mathrm{Hom}_{CW}(X, K(\pi, n)) \cong H^n(X, \pi)$, that is, H^n is representable.

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(K(G,n),G)) \arrow[r] \arrow[d, equal] & H^n(K(G,n),G) \arrow[r, "\cong"] & \mathrm{Hom}(H_n(K(G,n)),G) \arrow[r] \arrow[d, equal] & 0 \\ & 0 & & 0 & \end{tikzcd}$$

There is a special element in $\mathbb{H}^n(K(G,N), G)$, the preimage if id_G.

claim.

$$[X, K(G,n) \cong \tilde{H}^n(X,G)]$$

where on the left we have based CW-complexes.

lemma 44 (Yoneda). Let $\mathcal{F}$ be a presheaf, A be an object in $\mathcal{C}$. Then

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-,A), \mathcal{F}) \cong \mathcal{F}(A)$$

naturally in A.

Proof. For $f : C \to A$, we have this commutative diagram:

$$\begin{tikzcd} \mathrm{id}_A \arrow[r, mapsto] & f \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hom}(A,A) \arrow[r] \arrow[d, "\eta_A"'] & \mathrm{Hom}(C,A) \arrow[d, "\eta_C"] \\ \mathcal{F}(A) \arrow[r] & \mathcal{F}(C) \end{tikzcd}$$

$$\begin{tikzcd} \eta_A(\mathrm{id}_A) \arrow[r, mapsto] & \eta_C(f) \end{tikzcd}$$

So, natural transformations $\eta : \mathrm{Hom}(-,A) \to \mathcal{F}$ are determined by $\eta_A(\mathrm{id}_A)$. And that's it because then the map $\eta \mapsto \eta_A(\mathrm{id}_A)$ is what we are looking for. It remains to write why it is injective, surjective and natural. □

representability theory

If a functor $F : \mathcal{C}^{op} \to \mathrm{Sets}$ is representable, then it sends colimits to limits and sends weak colimits to weak colimits.

Theorem 45. If $F : \mathrm{Ho}\,\mathrm{CW}_* \to \mathrm{Sets}$ sends

1. coproducts (=wedges) to products,
2. $\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \cup_A C \end{tikzcd}$ to weak pullback, where $B \cup_A C$ is a CW complex, B and C are CW complexes, $A = B \cap C$,

then F is representable.

(By B with isomorphism $F \cong [-, B]$ given by some $X \in F(B)$.)

definition. A CW-complex B together with a choice of $\gamma \in F(B)$ is a ***spherical classifying space*** of F if

$$\begin{aligned} \gamma_* : [S^n, B] &\to F(S^n) \\ f &\mapsto f^*(\gamma) \end{aligned}$$

is an isomorphism for $n > 0$ (because for $n = 0$ S^n is not connected).

proposition 46. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F, then B_1 and B_2 are homotopy equivalent via the map that sends γ_1 to γ_2.

proposition 47. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F, then $g : B_1 \to B_2$ is such that $g^*/(\gamma_2) = \gamma_1$, then f is a homotopy equivalence.

May 2: cohomology operations

$$\mathrm{Nat}(H^n(-, G_1), H^n(-, G_2)) \leftrightarrow [K(G_1, n), K(G_2, n)]$$

example.

1. $x \mapsto x \smile x$, $H^n \to H^{2n}$. This is a natural transformation between the H functors.
2. The short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

yields

$$\cdots \longrightarrow H^n(X, \mathbb{Z}/p^2) \longrightarrow H^n(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p^2) \longrightarrow \cdots$$

which is natural in X, yielding the ***Bockstein homomorphism***

$$H^n(-, \mathbb{Z}/p \to H^{n+1}(-, \mathbb{Z}/p)$$

also May 2: spectral sequences

1. ***Exact couple***: $$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

It looks like this is just notation for an exact sequence that repeats over and over.

example. Take

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

and take $C_\bullet$ torsion free chain complex. You get

$$0 \longrightarrow C_\bullet \xrightarrow{\times p} C_\bullet \longrightarrow C_\bullet \otimes \mathbb{Z}/p \longrightarrow 0$$

which yields a long exact sequence

$$\begin{tikzcd} H_n \arrow[rrr] & & & H_n \arrow[dl] \\ & H_{n-1} \arrow[r] & \dots & \\ & & H_n \arrow[ul] & \end{tikzcd}$$

Or better yet (notation)

$$\begin{tikzcd} H_*(C) \arrow[rr] & & H_*(C) \arrow[dl] \\ & H_*(C, \mathbb{Z}_p) \arrow[ul] & \end{tikzcd}$$

2. ***Derived couple of exact couple***

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Where $A' = \operatorname{img} i$,

And the exact couple yields a homology $E^1 = H(A,)d$ with differential $d = jk$,

$i' = i|_{A'}, j'(ia) := [j(a)], k'([e]) = k(e)$.

We have checked that

- j' and k' are well-defined. i' is well-defined automatically.
- that the image of k' is in fact in A'
- $j'i' = 0, k'j' = 0, i'k' = 0$.
- that $j'(ai) = 0 \implies ia = i'a', k'[e] = 0 \implies [e] = j'(a')$ and that $i'(a') = 0 \implies a' = k[e]$.

So, we have shown that each exact couple gives a derived couple.

3.

$$\begin{tikzcd}
\text{Filtration on something} \arrow[r] & \text{filtration on chain complex} \arrow[r] \arrow[dr] & \text{exact couple} \arrow[d] \\
& & \text{spectral seq}
\end{tikzcd}$$

A ***filtration*** on an abelian group/R-module/chain complex/... C is

$$\ldots \subseteq F_n C \subseteq F_{n+1} C \subseteq C, \qquad n \in \mathbb{Z}$$

and there is an associated graded $\operatorname{gr} F_\bullet C := \bigoplus_{n \in \mathbb{Z}} F_n C / F_{n-1} C$.

We hope to recover C from $\operatorname{gr} F_\bullet$.

Problem 1. If $\bigcap_{n \in \mathbb{Z}} F_n C \neq 0$ then the map $\bigcap_{n \in \mathbb{Z}} F_n C$ looses some information. We may solve this by asking that

1. $F_n C = 0$ for $n < 0$
2. $C \to \lim C/F_n C$ is isomorphism.

Problem 2. If $\bigcup_{n \in \mathbb{Z}} F_n C \neq C$ that would be very bad. Then we should ask that $C = \bigcup_{n \in \mathbb{Z}} F_n C$ which is $C = \operatorname{colim} F_n C$.

example. We discussed the cases of $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ and $\prod_{n \in \mathbb{N}} \mathbb{Z}$. We found that even though $\bigcap_{n \in \mathbb{Z}} F_n C = 0$ in both cases, we could not recover the desired information (?).

Now, to each Serre fibration corresponds a spectral sequence.

spectral sequences cont.

X is a filtered chain complex

$$\cdots \subseteq X_{n-1} \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$$

We automatically get

$$\begin{tikzcd}
H_{p+q}(X_p) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_p/X_{p-1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p-1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p-1}/X_{p-2}) \arrow[r] \arrow[d] & \cdots \\
H_{p+q}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+1}/X_p) \arrow[r] \arrow[d] & H_{p+q-1}(X_p) \arrow[r, "j"'] \arrow[d, "i"'] & H_{p+q-1}(X_p/X_{p-1}) \arrow[r, "k"'] \arrow[d] & \cdots \\
H_{p+q}(X_{p+2}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+2}/X_{p+1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p+1}/X_p) \arrow[r] \arrow[d] & \cdots \\
\cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Notice that the red arrows are the induced exact sequence of

$$\begin{tikzcd}
0 \arrow[r] & X_{p-1} \arrow[r] \arrow[d] & X_p \arrow[r] \arrow[d] & X_p/X_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_p \arrow[r] & X_{p+1} \arrow[r] & X_{p+1}/X_p \arrow[r] & 0
\end{tikzcd}$$

Now consider

$$A = \bigoplus_{p+q} H_{p+q}(X_p)$$

and simply call $A_{p,1} = H_{p+q}(X_p)$. Also take

$$E = \bigoplus_{p,q} H_{p+q}(X_p/X_{p-1})$$

and $E_{p,q} := H_{p+q}(X_p/X_{p-1})$.

We have

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

where

map	**bidigree**
$i : A_{p,q} \to A_{p+1,q-1}$	$(1, -1)$
$j : A_{p,q} \to E_{p,q}$	$(0, 0)$
$k : E_{p,q} \to A_{p-1,q}$	$(-1, 0)$

Now consider the derived couple of this exact couple several times:

$$\begin{tikzcd}
A^2 \arrow[rr, "i_2"] & & A^2 \arrow[dl, "j_2"] \\
& E^2 \arrow[ul, "k_2"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^3 \arrow[rr, "i_3"] & & A^3 \arrow[dl, "j_3"] \\
& E^3 \arrow[ul, "k_3"] &
\end{tikzcd}$$

where, going back to definitions

$$E^2 = \ker d_1 / \operatorname{img} d_1 \qquad E^3 = \ker d_2 / \operatorname{img} d_2$$

and so on.

And then think about the bidigrees of the other maps. Well they are

map	**bidigree**
$i_k = i_{k-1}\vert_{\operatorname{img} i_{k-1}}$	$(1, -1)$
j_n	$(-(n-1), n-1)$
k_n	$(-1, 0)$
$d = j_n k_n$	$(-n, n-1)$

(we thought about this).

Here are some reasonable assumtions:

1. $X_p = 0$ for $p < 0$.
2. $\bigcup_p X_p = X$.
3. $H_{p+q}(X_p/X_{p-1}) = 0$ for $q < 0$.

remark. It will happen that for very large r, $E^r_{p,q} = E^r_{p,q} = \cdots = E^\infty_{p,q}$.

Also it will happen that

$$E^\infty_{p,q} \cong \operatorname{img} H_{p+q}(X_p)/\operatorname{img} H_{p+q}(X_{p-1})$$

and we will know what

$$\bigoplus_{\substack{p+q \text{ fixed} \\ p\in\mathbb{N}}} \operatorname{img} H_{p+q}(X_p)/\operatorname{img} H_{p+q}(X_{p-1})$$

is, and understand

$$H_* \qquad \ldots \subseteq \operatorname{img} H_{p+q}(X_p) \subseteq \operatorname{img} H_{p+q}(X_{p+1}) \subseteq \ldots \subseteq H_{p+q}(X),$$

which is induced by

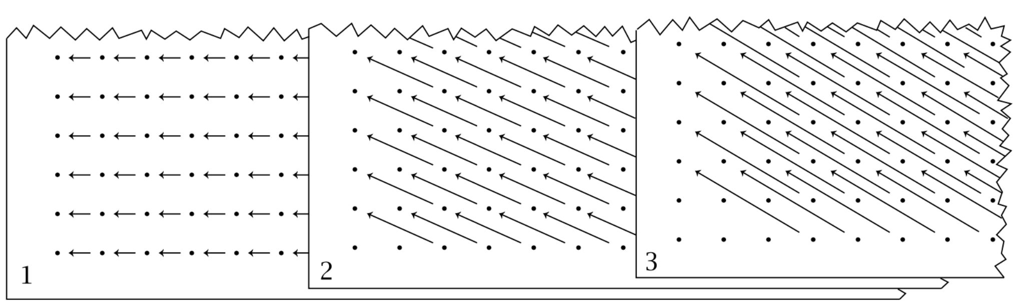
$$X \longrightarrow X_{p+1} \longrightarrow X$$

in our hopes to understand $H_{p+q}(X)$ which is of course just the homology of X.

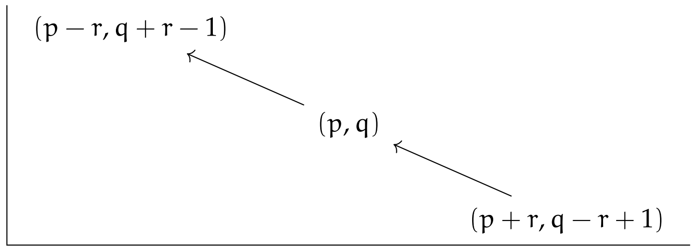
Now let's put the E's in a diagram like

$$\begin{array}{|lll}
\ldots & & \\
E^1_{0,1} & E^1_{1,1} & E^1_{2,1} \\
E^1_{0,0} & E^1_{1,0} & E^1_{2,0} \\
\hline
\end{array}$$

and if arrows are the differential d with bidegree $(-n, n-1)$, we already have Hatcher's picture



And something interesting will happen when r is very big. What? This:



that for a fixed (p, q) there is a large r such that if $p < r$ and $q + 1 < r$ we shall have

$$E^r_{p,q} = E^{r+1}_{p,q} := E^\infty_{p,q}.$$

serre spectral sequence

Now take a Serre fibration $F \longrightarrow E \longrightarrow B$ with

$$\pi_0(F) = 0 \quad \text{and} \quad \pi_1(B) = 0$$

and it turns out that

$$E^2_{p,q} = H_p(B, H_q(H)) \implies H_{p,q}(E).$$

example. For $S^1 \to S^3 \to S^2$ we have found that the first two pages are

$$\begin{array}{ccc} \mathbb{Z} & & \mathbb{Z} \\ & \overset{\cong}{\underset{d}{\nwarrow}} & \\ \mathbb{Z} & & \mathbb{Z} \\ & E_2 & \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{Z}/\operatorname{img} d_2 & & \mathbb{Z} \\ \mathbb{Z} & & \mathbb{Z}/\operatorname{img} d_2 \\ & E_3 & \end{array}$$

so $E^3 = E^\infty$.

example. We have also done $S^3 \to S^7 \to S^4$ and $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. See **sssguide**.

Now we will prove that there exists a spectral sequence and that it converges where it converges. Consider either of

$$\pi_0(B) = 0 \qquad\qquad \text{or} \qquad\qquad \begin{cases} \pi_1(B) = 0 \\ \text{or } \pi_1(B) \text{ acts trivially on } H_q(F) \\ \text{or take } H_* \text{ with local coefficients} \end{cases}$$

Theorem 48. There is spectral sequence E^ν that converges to $H_*(E)$ and such that

$$E^2_{p,q} = H_p(B, H_q(F, G)).$$

Proof. The construction of the spectral sequence is not complicated. Start with E^1. Then Do E^2 with

$$\begin{tikzcd} E^k \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ D^k \arrow[r] & B \end{tikzcd}$$

□

Consider the following conditions on the big staircase diagram:

a. In each A column almost all maps are isomorphisms.

b. In each column E almost each entry is 0.

c. $E^1_{p,q} = 0$ for $p < 0$ and $q < 0$.

d. $X_p = 0$ for $p < 0$ and $H_n(X_p) \to H_n(X_{p+1})$ is isomorphism for $p \gg 0$.

And also

- $A_{-\infty,p+q} := A_{p,q}$ for $p \ll 0$.
- $A_{+\infty,p+q} := A_{p,q}$ for $p \gg 0$.

And then

e1. $A_{-\infty,p+q} = 0$.

e2. $A_{+\infty,p+q} = 0$.

claim. b $\implies$ $E^r_{p,q}$ stabilizes for fixed panel q, so $E^\infty_{p,q}$ makes sense.

Now let's check that indices are the way they are in

$$\underset{\substack{\| \\ 0}}{E^r_{p+r-1,q-r+2}} \xrightarrow{k} A^r_{p+r-2,q-r+2} \xrightarrow{i} A^r_{p+r-1,q-r+1} \xrightarrow{j} E^r_{p,q} \xrightarrow{k} \underset{\substack{\| \\ 0 \\ \text{if e1}}}{A^r_{p-1,q}} \xrightarrow{i} \underset{\substack{\| \\ 0 \\ \text{if e1}}}{A^r_{p,q-1}} \xrightarrow{j} \underset{\substack{\| \\ 0}}{E^r_{p-r+1,q-r-2}}$$

Ok, after some other considerations we have concluded that

> $E^\infty_{p,q}$ also makes sense and it is a piece of p-graded associative algebra graded of $A_{+\infty,p+q}$ with

$$\begin{aligned} E^r_{p,q} &= A^r_{p+q+r,q-r+q}/iA^r_{p+r-2,q-1+2} \\ &= i^{r-1}(A^1_{p,q})/i^r(A^1_{p-1,q+1}) \\ &= F_pA_{+\infty,p+q}/F_{p-1}A_{+\infty,p+q} \end{aligned}$$

14 may

In cover spaces we have homeomorphisms between the fibers. In hurewicz fibrations this may not be true, but we still can have homotopy equivalences. So consider a Hurewicz fibration $f : E \to B$ and a path $I \to B$. Then we have:

$$\begin{tikzcd} F_a \arrow[rr, hook] \arrow[d, "i_0"] & & E \arrow[d, two heads, "f"] \\ F_a \times I \arrow[urr, dash] \arrow[r, "\pi"'] & I \arrow[r, "p"] & B \end{tikzcd}$$

And we claim that there is an homotopy equivalence

$$F_a \cong F_a \times \{1\} \to F_b = f^{-1}b$$

To see why, consider two homotopic paths $p_1, p_2 : I \to B$. Construct the following diagram:

[see sss.pdf]

We have (or will?) established:

proposition 49. For any Hurewicz fibration $f : E \to B$ there is "an action of $\pi_1(B)$ on F_b up to homotopy"

$$\Pi_1(B) \to \mathrm{Ho\,Top}$$
$$b \mapsto F_b$$

We have an action

$$\pi_1(B) \curvearrowright H_*(F_b)$$

> Today I've forgotten to prove that the action of $\pi_1(B)$ on fibre is actually an action: that the action of $x \cdot y$ is the composition of the action of y and of x

> The fact that fibrations over disks are always trivial is easy to prove. The algebraic analogue of this is the Quillen-Suslin theorem

Why do need this? Consider the following case:

$$\begin{tikzcd} f_i^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads] \\ X \arrow[r, "f_i"'] & B \end{tikzcd}$$

for $f_1, f_2 : X \to B$.

serre spectral sequence for cohomology

Suppose

$$F \hookrightarrow X \longrightarrow B$$

Is a Serre fibration with $\pi_0(B) = 0$ and that the action described above $\pi_1(B) \curvearrowright H^*(F; G)$ is trivial.

Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ such that

a. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+q}$; $\qquad E_{r+1}^{p,q} = \ker d_r / \operatorname{img} d_r$.

b. $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$; $\qquad 0 \subset F_{p+q}^{p+q} \subset \ldots \subset F_0^{p+q} = H^{p+q}(X, G)$.

c. $E_2^{p,q} = H^p(B, H^q(F, G))$.

d. $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$, which is given by $(-1)^{qs}$.

$$H^p(B, H^q(F, R)) \times H^s(B, H^t(F, R)) \xrightarrow{\smile} H^{p+s}(B, H(F, R))$$

now supposing that $G = R$ is a ring.

e. This

$$F_p^m \times F_s^n \xrightarrow{\smile} F_{p+s}^{m+n}$$

which induces

$$F_p^m / F_{p+1}^m \times F_s^n / F_{s+1}^n \to F_{p+s}^{m+n} / F_{p+s+1}^{m+n}$$

and is

$$F_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$$

example (Cohomology of base space). We have computed the cohomology of $\mathbb{C}P^n$ using Serre spectral sequence.

example (Cohomology of Étale space). Consider the Hopf fibration

$$S^1 \longrightarrow S^3 \longrightarrow S^2$$

Its easy to see that the second page is

$$\begin{array}{c|ccc} 1 & \mathbb{Z} & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \mathbb{Z} \\ \hline & 0 & 1 & 2 \end{array}$$

We notice immediately that $E_3 = E_\infty$, since the $0'$s in the diagram remain to future pages. To find E_3, it turns out that it can be shown that d_2 is an isomorphism (recall that d_2 goes two to the right and one down). This makes the upper-left and lower-right groups zero on the third page:

$$\begin{array}{c|ccc} 1 & & & \mathbb{Z} \\ 0 & \mathbb{Z} & & \\ \hline & 0 & 1 & 2 \end{array}$$

It is now possible to read off the cohomology of the total space S^3 by assembling along the diagonals. In this case, we have

$$H^n(S^3) = \bigoplus_{s+t=n} E_\infty^{s,t}.$$

(This is what leads to considering the diagonals in the diagram.) This gives a $\mathbb{Z}$ in dimension 0 from $E_\infty^{0,0}$ and $\mathbb{Z}$ in dimension 3 from $E^{2,1}$ as expected.

example (Cohomology of fiber)**.** Consider the fibration

$$\Omega S^3 \longrightarrow \underset{\substack{\simeq \\ \mathrm{pt}}}{PS^3} \longrightarrow S^3$$

We use the following

claim. On $E^3, E^4, \ldots$ are equal when the differentials on the second page are 0.

Proof. It's because

$$E_r^{p,q} = 0 \implies E_r^{p,q} = 0$$

We have $E_2^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, so we have $E_3^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, and finally $d_r = 0$ for $r \geqslant 3$. More finally,

$$d_r : E^{p,q} \to E^{p+q} \to E^{p+r,q-r+1}$$

for $-r+1 \geqslant -2$. $\square$

We discovered that

$$\begin{array}{|cccc}
\cdots & & & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
\hline
\end{array}$$

We have also computed ring structure.

example (Cohomology of Ètale espace)**.** Consider the fibration

$$SU(n-1) \longrightarrow SU(n) \longrightarrow S^{2n-1}$$

The particular case of

$$SU(3) \longrightarrow SU(4) \longrightarrow S^{2\cdot 4-1}$$

yields

$$\begin{array}{c|cccccccc}
 & \cdots & & & & & & & \cdots \\
8 & \mathbb{Z}a_3a_5 & & & & & & & \mathbb{Z}a_7a_5a_3 \\
7 & & & & & & & & \\
6 & & & & & & & & \\
5 & \mathbb{Z}a_5 & & & & & & & \mathbb{Z}a_7a_5 \\
4 & & & & & & & & \\
3 & \mathbb{Z}a_3 & & & & & & & \mathbb{Z}a_7a_3 \\
2 & & & & & & & & \\
1 & & & & & & & & \\
0 & \mathbb{Z} & & & & & & & \mathbb{Z}a_3 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$$

Giving $E_2 = E_\infty$ since all differentials are zero. It follows that the cohomology ring is $\Lambda(a_3, a_5, a_7)$. The case of

$$SU(2) = S^3 \longrightarrow SU(3) \longrightarrow S^{2\cdot 3-1} = S^5$$

yields, too, that $E_2 = E_\infty$

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Theorem 50. Let $F \to X \to B$ be a Serre fibration. Then there is a spectral sequence that converges to $H_n(X, G)$. If the action of $\pi_1(B)$ on $H_*(F, G)$ is trivial, then

$$H_2^{p,q} = H_p(B, H_q(F, G)).$$

More generally, if the action is not trivial, then $E_2^{p,q} = H_p(B, \mathcal{L}_q)$ where $\mathcal{L}$ is a local system coming from $\pi_1(B) \curvearrowright F$.

We are not going to prove this theorem.

definition. Action of $\pi_1(B)$ on $H_*(F, G)$ for Serre fibration.

$$\begin{tikzcd}
F \arrow[d] & F \arrow[d] & F \arrow[d] \\
i^*p^*E \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & p^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\
0 \arrow[r, "i", "\simeq"'] & I \arrow[r, "p"] & B
\end{tikzcd}$$

Now considering the long exact sequence of homotopy, we get:

$$\begin{tikzcd}
\pi_i(F) \arrow[r] \arrow[d, "\cong"] & \pi_i(p^*E) \arrow[r] \arrow[d, "\cong"] & \pi_i(X) \arrow[d, "\cong"] \\
\pi_i(F) \arrow[r] & \pi_i(i^*p^*E) \arrow[r] & \pi_i(Y)
\end{tikzcd}$$

because, in general, for an homotopy equivalence $f : X \to Y$, the pullback is an homotopy equivalence:

$$\begin{tikzcd}
F \arrow[r, "="] \arrow[d] & F \arrow[d] \\
f^*E \arrow[r, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\
X \arrow[r, "\simeq"] & Y
\end{tikzcd}$$

Then we take pullback of a larger diagram to prove that the action is associative (that it is, in fact, an action) via homoloy groups.

Proof.

1. We may assume that B is a CW-complex with one 0-cell, and $X \to B$ is a Hurewicz fibration.

   Proof.

   $$\begin{tikzcd}
   & F \arrow[d] \\
   & X \arrow[d] \\
   C \arrow[r] & B
   \end{tikzcd}$$

   There is a CW approximation $\alpha : C \to B$ with C having exaclty one 0-cell that is sent to the basepoint of B. Now we say

   > Every space can factored as weak equivalence followed by Hurewicz fibration. (Path space is behind the scenes.)

   We obtain

   $$\begin{tikzcd}
   p^{-1}_{\alpha^* p} e_0 \arrow[d] & F \arrow[l, "\simeq"'] \arrow[r] & F \arrow[d] \\
   E^{\alpha_{0,p}} \arrow[d] & \alpha^* X \arrow[l, "\simeq"'] \arrow[r, "\cong"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d] \\
   C \arrow[r, equal] & C \arrow[r] & B
   \end{tikzcd}$$

   Anyway, that's the CW-complex replacing B. $\square$

2. There is a nice filtration on X. Take the filtration

   $$\varnothing \subseteq \operatorname{sk}_0 B \subseteq \operatorname{sk}_1 B \subseteq \cdots \subseteq B$$

   which induces

   $$\varnothing \subseteq X^0 := p^{-1} \operatorname{sk}_0 B \subseteq X^1 := p^{-1} \operatorname{sk}_1 B \subseteq \cdots \subseteq X$$

   and it turns out that

(a) $\bigcup_i X^i = X$.

(b) $\mathrm{sk}_n B \to B$ is an n-equivalence (we already know this from other constructions). Then $X^n \hookrightarrow X$ induces isomorphisms on H_i for $i < n-1$ by Hurewicz theorem and five-lemma.

This means that we get a filtration on $C_*(X)$ and on $H_*(X)$. From b. we get that in each column $A'_{\bullet,p+q}$ only a finite number of maps are not isomorphisms. When we discussed spectral sequences, we say that this corresponds to a filtration

$$\varnothing \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

that converges to H_*.

Namely, we have a spectral sequence such that

$$E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$$

and

$$E^\infty_{p,q} = \text{the } p\text{-th graded piece of } H_{p+q}(X).$$

This means that

$$\begin{aligned} B^{p-1} \hookrightarrow B^p \quad & \text{is a } (p-1)\text{-equivalence} \\ X^{p-1} \hookrightarrow X^p \quad & \text{is a } (p-1)\text{-equivalence} \\ H_{p+q}(X_p, X_{p-1}) = 0 \quad & q < 0. \end{aligned}$$

In conclusion, we have the first-quadrant of the spectral sequence $E^1_{p,q} = H_{p+q}(X_p, X_{p-1})$, which proves the first statement in the theorem.

(c) Now suppose the action of $\pi_1(B)$ on $H_*(F, G)$ is trivial. Look at the CW structure of things

$$\begin{tikzcd} \widetilde{S^{p-1}} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \widetilde{D^p} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & X^p \arrow[d] \\ S^{p-1} \arrow[r] & D^p \arrow[r, "\alpha"] & B^p \end{tikzcd}$$

We also proved that

claim. $\bigoplus_\alpha H_*(\widetilde{D^p}, \widetilde{S^{p-1}}) \to H_*(X^p, X^{p-1})$ is an isomorphism.

And then for this we did

$$\begin{tikzcd}
u \quad x \quad o & x & x \\
S^{p-1} \times F_\alpha \times 0 \arrow[r] \arrow[d] & F_\alpha \arrow[r, hook] & \widetilde{D^p_\alpha} \arrow[d] \\
S^{p-1} \times F_\alpha \times I \arrow[r, "\mathrm{proj}"] & S^{p-1} \times I \arrow[r, "(u,t)\mapsto t\cdot u"] & D^p \\
u \quad x \quad o & u \quad o & 0
\end{tikzcd}$$

and used Künneth formula:

By definition, $(X, A)\times$

$\square$

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In the end we would like to prove that homotopy groups of sphere are finitely generated and moreover all except two kinds (which?) are in fact finite.

definition. A ***Serre class*** $\mathcal{C}$ of abelian groups is a class of abelian groups closed under the operations of taking subgroups, quotients, and forming extensions. That is, for any short exact sequence of abelian groups

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

with A and C in $\mathcal{C}$, then B is also in $\mathcal{C}$.

example.

1. The class of finitely-generated groups.
2. Torsion groups. From algebraic geometry you might be acquainted with

   $$M \otimes_{\mathbb{Z}} \mathbb{Q} = S^{-1}M, \quad S = \mathbb{Z}\backslash\{0\}$$

   More generally,

   $$M \otimes_R S^{-1}R = S^{-1}M$$

   Why do we care? Because **localization is an exact functor**, equivalently, $S^{-1}R$ is a flat R-module. Notice that

   $$M \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \iff M \text{ is a torsion group.}$$

   Which follows simply from the fact that

   $$\forall m, s, \quad \frac{m}{s} = \frac{0}{1} \iff \forall m, x\ \exists s' :\ s'm = 0.$$

So

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

implies

$$0 \longrightarrow A \otimes \mathbb{Q} \longrightarrow B \otimes \mathbb{Q} \longrightarrow C \otimes \mathbb{Q} \longrightarrow 0$$

so $A \otimes \mathbb{Q}$ and $B \otimes \mathbb{Q}$ are zero iff $B \otimes \mathbb{Q}$ is zero.

$\mathcal{C}_1$ and $\mathcal{C}_1$ are Serre classes then $\mathcal{C}_1 \cap \mathcal{C}_2$ is a Serre class.

Finite abelian groups.

Torsion groups such that the order of any element is coprime to any $p \in P$ for some subset P of primes. For example,

- $P = \{p\}$ then we get the *condition* p *does not divide the orders of ements*.
- P = all primes except p = all $s \neq 0$ such that $(p, s) = 1$. Then we get *the order of elements are powers of* p. It turns out that $\mathbb{Z}_{(p)} = p^{-1}\mathbb{Z}$, where (p) is the ideal generated by p. Then

$$M \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = 0 \iff M \text{ does not have } p\text{-torsion}$$

that is, the orders are not divisible by p.

I think we concluded that localization of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}_{(p)}$ does not change anything.

So again

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

implies

$$0 \longrightarrow A \otimes \mathbb{Z}_{(p)} \longrightarrow B \otimes \mathbb{Z}_{(p)} \longrightarrow C \otimes \mathbb{Z}_{(p)} \longrightarrow 0$$

so $A \otimes \mathbb{Z}_{(p)}$ and $B \otimes \mathbb{Z}_{(p)}$ are zero iff $B \otimes \mathbb{Z}_{(p)}$ is zero.

claim. For a simply connected space it is equivalent that $H_i \in \mathcal{C}$ for $0 < i < n$ and that $\pi_i \in \mathcal{C}$ for $0 < i < n$.

proposition 51. Suppose $\mathcal{C}$ is a Serre class and

$$A \longrightarrow B \longrightarrow C$$

and $A, C \in \mathcal{C}$. Then $B \in \mathcal{C}$.

Proof.

$$\begin{tikzcd}
 & A \arrow[d] \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \ker \arrow[r] \arrow[d] & B \arrow[r] \arrow[dr] & \mathrm{img} \arrow[r] \arrow[d] & 0 \\
 & 0 & & C &
\end{tikzcd}$$

$\square$

remark.

1. If $C_\bullet$ is a chain complex, all C_n is in $\mathcal{C}$, then H_n is in $\mathcal{C}$.
2. $F_\bullet A$ a filtration, A in $\mathcal{C}$ then each graded piece of $\operatorname{gr} A$ is in $\mathcal{C}$.
3. $F_\bullet A$ finite filtration, i.e. $0 \subset F_0A \subset \ldots \subset F_nA = A$, and each graded piece $\operatorname{gr} A$ is in $\mathcal{C}$, then A is in $\mathcal{C}$. This follows from the fact that $F_0A = F_0A/0$, F_1A/F_0A, F_2A/F_1A $\ldots$ are all in $\mathcal{C}$.
4. First-quadrant of spectral sequence that converges to $H_*(C_\bullet)$. Suppose $E^r_{p,q}$ is in $\mathcal{C}$ for all p, q and some fixed r. Then $E^{r+1}_{p,q}$ is in $\mathcal{C}$ implies that $E^\infty_{p,q}$ is in $\mathcal{C}$, which in turn implies that $H_*(C_\bullet)$ is in $\mathcal{C}$.

definition. $f : A \to B$ is a

- mod $\mathcal{C}$ ***monomorphism*** if $\ker f$ is in $\mathcal{C}$.
- mod $\mathcal{C}$ ***epimorphism*** if $\operatorname{coker} f$ is in $\mathcal{C}$.
- mod $\mathcal{C}$ ***isomorphism*** if both $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{C}$.

proposition 52.

1. A monomorphism mod $\mathcal{C}$, epimorphism mod $\mathcal{C}$ and isomorphism mod $\mathcal{C}$ are closed under composition.
2. Isomorphisms mod $\mathcal{C}$ satisfy 2-out-of-3.

Proof.

$$
\begin{tikzcd}
& & & 0 \arrow[dr] & & & & 0 & & & \\
& & & & \ker\beta \arrow[rr] \arrow[dr] & & \operatorname{coker}\alpha \arrow[ur] \arrow[ddrr] & & & & \\
& & & & & B \arrow[ur] \arrow[dr, "\beta"] & & & & & \\
0 \arrow[rr] & & \ker\beta\alpha \arrow[rr] \arrow[uurr] & & A \arrow[ur, "\alpha"] \arrow[rr, "\beta\alpha"] & & C \arrow[rr] \arrow[dr] & & \operatorname{coker}\beta\alpha \arrow[rr] \arrow[dl] & & 0 \\
& & & \ker\alpha \arrow[ul] \arrow[ur] & & & & \operatorname{coker}\beta \arrow[dl] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 \arrow[ul] & & 0 & & 0 & &
\end{tikzcd}
$$

□

definition. A Serre class $\mathcal{C}$ is

- a ***Serre ideal*** if for all A, B abelian groups, $A \in \mathcal{C}$ implies $A \otimes B$ and $\operatorname{Tor}(A, B)$ are in $\mathcal{C}$, and

- a ***Serre ring*** if for all A, B in $\mathcal{C}$, $A \otimes B$ and $\operatorname{Tor}(A, B)$ are in $\mathcal{C}$.

examples.

1. Finitely-generated groups are a Serre ring.
2. Torsion groups is a Serre ideal. $\mathbb{Q} \otimes A = 0 \implies \mathbb{Q} \otimes A \otimes B = 0$ so

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow B \longrightarrow 0$$

   implies

$$0 \longrightarrow F_1 \otimes A \longrightarrow F_0 \otimes A \longrightarrow \operatorname{Tor}(B, A) \longrightarrow 0$$

   implies

$$0 \longrightarrow F_1 \otimes A \otimes \mathbb{Q} \longrightarrow F_0 \otimes A \otimes \mathbb{Q} \longrightarrow \operatorname{Tor}(B, A) \otimes \mathbb{Q} \longrightarrow 0$$

   so $F_0 \otimes A \otimes \mathbb{Q}$ and $\operatorname{Tor}(B, A) \otimes \mathbb{Q}$ are zero iff $F_1 \otimes A \otimes \mathbb{Q}$ is zero.

3. Finite groups is a Serre ring.
4. Intersection of Serre rings is Serre ring.
5. $M \otimes \mathbb{Z}_{(p)}$ is a Serre ideal.

proposition 53. $F \to X \to B$ Serre fibration. $\pi_0(B) = 0$, $\pi_1(B) \curvearrowright H_A(F)$ trivial. Then if 2 out of 3 among F, X, B have H_n in $\mathcal{C}$ for $n > 0$, then the third one does too.

Proof.

(X, B in $\mathcal{C}$.) By universal coefficients theorem on homology,

$$E^2_{p,q} = H_p(B, H_q(F)) \cong H_p(B) \otimes H_q(F) \oplus \operatorname{Tor}(H_{p-1}(B), H_q(F))$$

lemma 54 (that we will need later)**.** Notice that if for all $0 \leqslant k \leqslant p$, $H_p(B) \in \mathcal{C}$, for all q, $H_q(F) \in \mathcal{C} \implies E^2_{p,q} \in \mathcal{C}$.

Anyway from universal coefficients we get that $E^3, E^4, \dots E^\infty$ are in $\mathcal{C}$, because every subgroup and quotient of objects in a Serre class are also in the Serre class. Since $E^\infty_{p,q} = F_p H_{p,q}(E)/F_{p-1} H_{p+q}(E)$ is $\mathcal{C}$, we get that $E^\infty_{0,q} \in \mathcal{C}$, so that we have a filtration $0 \to F_{p-1} \to \dots \to F_p/F_{p-1} \to 0$ satisfying some property that allows us to conclude that $H_{p+q}(E) = E_{p+q} H_{p+q}(E) \in \mathcal{C}$. Then H_* is in $\mathcal{C}$.

(F, E in $\mathcal{C}$.) Since $H_n(E) \in \mathcal{C}$, then $F_p H_{p+q}(E)/F_{p-1} H_{p+1}(E) = E^\infty_{p,q} \in \mathcal{C}$. Then we have

$$0 \longrightarrow E^{r+1}_{k,0} \longrightarrow E^r_{k,0} \longrightarrow E^r_{k-1,r-1}$$

And again, if for all $r \geqslant 2$, $E^r_{k-r,r-1} \in \mathcal{C}$ and $E^\infty_{k,0} \in \mathcal{C}$, then for all r, $E^r_{k,0} \in \mathcal{C}$. This follows from going all the way to infinity to make sure the two nontrivial groups not in the middle on the last sequence are in $\mathcal{C}$, and then go back one by one to make show that actually all of them are in $\mathcal{C}$.

In conclusion, $E^2_{k,0} \in \mathcal{C}$, and then $H_0(B) = \mathbb{Z}$ and $E^2_{0,q} \in \mathcal{C}$, and then $H_1(B) \in \mathcal{C} \implies E^2_{1,q} \in \mathcal{C}$. So its just some inductions over and over.

(B, E in $\mathcal{C}$.) Again we have that $E^\infty_{p,q} \in \mathcal{C}$.

$$E^r_{r,q-r+1} \longrightarrow E^r_{0,q} \longrightarrow E^{r+1}_{0,q} \longrightarrow 0$$

Now we do induction again. We get $E^2_{r,q-r+1} = H_r(B, H_{q-r+1}(F))$. Now the left-hand-side of this equality is in $\mathcal{C}$, so $H_k(F) \in \mathcal{C}$ for all $0 < k < q$. (Technical proof, try to read the details...) $\square$

Let's consider further examples.

examples.

- **(Wang sequence)** Consider a fibration over S^n of the form $F \to E \to S^n$ for $n \geqslant 2$. We get

$$\begin{array}{c|ccc}
 & \dots & & \dots \\
2 & H_2(F) & & H_2(F) \\
1 & H_2(F) & & H_1(F) \\
0 & H_0(F) & & H_0(F)a_3 \\
\hline
 & 0 & \dots & n
\end{array}$$

So we only get nontrivial differentials in the nth page. We get

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{n,q} \arrow[r] \arrow[d, equal] & E^n_{n,q} \arrow[r] \arrow[d, equal] & E^n_{0,q+n-1} \arrow[r] \arrow[d, equal] \arrow[dr] & E^\infty_{0,q+n-1} \arrow[r] & 0 \\
0 \arrow[r] & E^{n+1}_{n,q} \arrow[ur] & E^2_{n,q} & E^2_{0,q+n-1} & E^{n+1}_{0,q+n-1} \arrow[r] & 0
\end{tikzcd} \tag{1}$$

Also we get

$$\begin{tikzcd}
0 \arrow[r] & E^\infty_{0,q+n-1} \arrow[r] \arrow[d, equal] & H_{q+n-1}(E) \arrow[r] & E^\infty_{n,q-1} \arrow[r] \arrow[d, equal] & 0 \\
 & F_0 H_{q+n-1} & & F_n H_{q+n-1}/F_{n-1}H_{q+n-1} &
\end{tikzcd} \tag{2}$$

And from those two we get

$$\begin{tikzcd}
\cdots \arrow[r] & H_{q+n}(E) \arrow[r] \arrow[d, two heads] & E^2_{n,q} = H_q(F) \arrow[r] \arrow[dl] \arrow[d, equal] & E^2_{0,q+n-1} = H_{q+n-1}(F) \arrow[r] \arrow[d, equal] \arrow[dr] & H_{q+n-1}(E) \arrow[r] & 0 \\
0 \arrow[r] & E^{n+1}_{n,q} \arrow[ur] & E^2_{n,q} & E^2_{0,q+n-1} & E^{n+1}_{0,q+n-1} \arrow[r] & 0
\end{tikzcd} \tag{3}$$

So in the end, we have

$$\cdots \longrightarrow H_{q+n}(E) \longrightarrow H_q(F) \longrightarrow H_{q+n-1}(E) \longrightarrow \cdots$$

- **(Gysin.)** Nos consider $S^n \to E \to B$ for $n \geqslant 1$ and trivial action on homology of F. As before we get

$$0 \longrightarrow E^\infty_{p,0} \longrightarrow E^{n+1}_{p,0} \longrightarrow E^{n+1}$$

Spectral sequences allow us to reconstruct $H_*(\text{gr}\, C_\bullet)$ from $\text{gr}\, C_\bullet$.

Spans and multi-valued maps

Consider this picture

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A & & B
\end{tikzcd}$$

Such that for all $c \in C$, $f(c)$ is ***seat*** to $g(c)$. The whole thing is called a ***span***.

Now let

$$\begin{aligned}
&D = \text{img}\, C \to A \\
&\ I\, \text{img of } f^{-1} \text{ under } C \to B \\
&D \to B/I
\end{aligned}$$

Now from Serre spectral sequence we shall get that

$$\begin{tikzcd}
& H_n(E,F) \arrow[dl] \arrow[dr] & \\
H_n(B,pt) & & H_{n-1}(F)
\end{tikzcd}$$

self-study day 4 june

Some questions:

1. What is this whole filtration business? I still don't understand what the filtration is and how it works. It's very important because that's the way we get the homology of the Etále space out of the spectral sequence but why how.
2. Is the action of π_1 on π_n the one I studied today in Hatcher? The idea of that action is *do a loop, do the map of the sphere (the element of π_n, then do the loop backwards)*.
3. Weren't we going to study transgression next?

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