

homotopy theory

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abstract nonsense

definition.

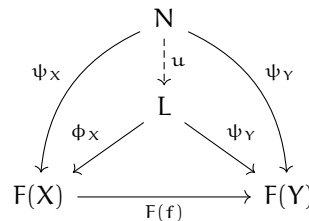
- (Limits, [wiki](#).)

- A *diagram* of shape J in C is a functor from J to C

$$F : J \rightarrow C.$$

The category J is thought of as an index category, and the diagram F is thought of as indexing a collection of objects and morphisms in C patterned on J .

- Let $F : J \rightarrow C$ be a diagram of shape J in a category C . A *cone* to F is an object N in C together with a family $\psi_X : N \rightarrow F(X)$ of morphisms indexed by the objects X of J (so a cone is an object and a bunch of maps from this object to certain objects that are governed by the diagram), so that for every morphism $X \rightarrow Y$ in J , we have $F(f) \circ \psi_X = \psi_Y$ I guess this is what nLab meant when he said that everything in sight commutes).
- A *limit* of the diagram $F : J \rightarrow C$ is a cone (L, ϕ) to F such that for every cone (N, ψ) there exists a *unique* morphism $u : N \rightarrow L$ such that $\phi_X \circ u = \psi_X$ for all X in J .



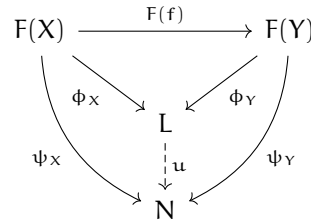
One says that the cone (N, ψ) factors through the cone (L, ϕ) with the unique factorization u . The morphism u is sometimes called the *mediating morphism*.

Limits are also referred to as *universal cones* since they are characterized by a universal property. Limits may also be characterized as terminal objects in the category of cones to F .

It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of F .

- (Colimits, [wiki](#)) The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:
 - A *co-cone* of a diagram $F : J \rightarrow C$ is an object N of C together with a family of morphisms $\psi_X : F(X) \rightarrow N$ (so in the cone we are going from N and now we're going to N) for every object X of J , such that for every morphism $f : X \rightarrow Y$ in J we have $\psi_Y \circ F(f) = \psi_X$ everything in sight commutes.

- A **colimit** of a diagram $F : J \rightarrow C$ is a co-cone (L, ϕ) of F such that for any other co-cone (N, ψ) of F there exists a unique morphism $u : L \rightarrow N$ such that $u \circ \phi_X = \psi_X$ for all X in J .



Colimits are also referred to as **universal co-cones**. They can be characterized as initial objects in the category of co-cones from F .

As with limits, if a diagram F has a colimit then this colimit is unique up to a unique isomorphism.

- An **initial object** in a category C is an object \emptyset such that for any object $x \in C$ there is a unique morphism $\emptyset \rightarrow x$ with source \emptyset and target x .
- For C any category, its **arrow category** $\text{Arr}(C)$ is the category such that
 - an object a of $\text{Arr}(C)$ is a morphism $a : a_0 \rightarrow a_1$ of C ,
 - a morphism $f : a \rightarrow b$ of $\text{Arr}(C)$ is a commutative square

$$\begin{array}{ccc} a_0 & \xrightarrow{f_0} & b_0 \\ a \downarrow & & \downarrow b \\ a_1 & \xrightarrow{f_1} & b_1 \end{array}$$

in C ,

- composition in $\text{Arr}(C)$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

$$\text{Arr}(C) := \text{Funct}(I, C) = [I, C] = C^I$$

for I the interval category $\{0 \rightarrow 1\}$.

- An **equalizer** is a limit

$$\text{eq} \xrightarrow{e} X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y$$

over a parallel pair of morphisms f and g . This means that for $f : X \rightarrow Y$ and $g : X \rightarrow Y$ in a category C , their equalizer, if it exists, is

- an object $\text{eq}(f, g) \in C$,
- a morphism $\text{eq}(f, g) \rightarrow X$

– such that

* pulled back to $\text{eq}(f, g)$ both morphisms become equal:

$$\text{eq}(f, g) \longrightarrow X \xrightarrow{f} Y = [\text{eq}(f, g) \longrightarrow X \xrightarrow{g} Y$$

* and $\text{eq}(f, g)$ is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions f and g between sets X and Y one forms the set Y/\sim of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the the quotient function $p : Y \rightarrow Y/\sim$ satisfies

$$p \circ f = p \circ g.$$

In some category \mathcal{C} , the *coequalizer* $\text{coeq}(f, g)$ of two parallel morphisms f and g between two objects X and Y , if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y \\ & \searrow & \swarrow \\ & \text{coeq}(f, g) & \end{array}$$

Equivalently, in a category \mathcal{C} a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \xrightarrow{p} Z$$

is called a *coequalizer* diagram if

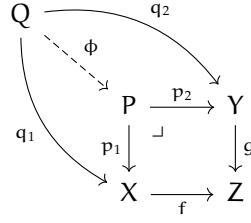
1. $p \circ f = p \circ g$,
2. p is universal for this property: if $q : Y \rightarrow W$ is a morphism of \mathcal{C} such that $q \circ f = q \circ g$, then there is a unique morphism $\phi : Z \rightarrow W$ such that $\phi \circ p = q$

$$\begin{array}{ccccc} X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y & \xrightarrow{p} & Z \\ & & \downarrow q & \nwarrow \phi & \\ & & W & & \end{array}$$

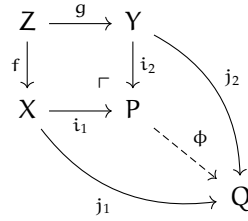
The coequalizer in \mathcal{C} is equivalently an equalizer in the opposite category \mathcal{C}^{op} .

- A *pullback* of the morphisms f and g consists of an object P and two morphisms

$p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ satisfying the following universal property:



- A **pushout** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \rightarrow X$ and $i_2 : P \rightarrow Y$ satisfying the following universal property:



remark. Other names for the pushout are **cofibered product of X and Y** (especially in algebraic categories when i_1 and i_2 are monomorphisms), or **free product of X and Y with Z amalgamated sum**, or more simply an **amalgamation** or **amalgam of X and Y** .

remark. If coproducts exist in some category, then the pushout

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ f \downarrow & \lrcorner & \downarrow i_2 \\ X & \xrightarrow{i_1} & X \amalg_Z Y \end{array}$$

is equivalently the coequalizer

$$X \xrightarrow[i_2 \circ g]{i_1 \circ f} X \amalg Y \longrightarrow X \amalg_Z Y$$

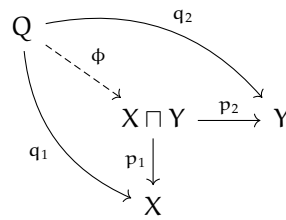
of the two morphisms induced by f and g into the coproduct of X with Y .

example (wiki).

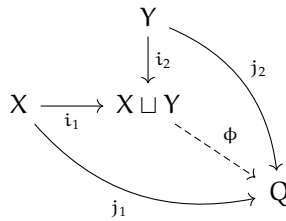
- If X , Y and Z are sets and f, g are functions, the pushout of f and g is the disjoint union of X and Y where elements sharing a common preimage in Z are identified, i.e. $P = (X \amalg Y) / \sim$ where \sim is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

In particular, if X and Y are subsets of some larger set W and Z is their intersection, with f and g the inclusion maps of Z into X and Y , then the pushout can be canonically identified with the union $X \cup Y \subseteq W$.

- The construction of *adjunction spaces* is an example of pushouts in \mathbf{Top} . More precisely, if Z is a subspace of Y and $g : Z \rightarrow Y$ is the inclusion map, we can glue Y to another space X along Z using an *attaching map* $f : Z \rightarrow X$. The result is the *adjunction space* $X \cup_f Y$ which is just the pushout of f and g . More generally, all identification spaces may be regarded as pushouts in this way. See ?? .
- A *product* of X and Y is an object $X \sqcap Y$ and a pair of morphisms $p_1 : X \sqcap Y \rightarrow X$, $p_2 : X \sqcap Y \rightarrow Y$ satisfying the following universal property:



- A *coproduct* of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \rightarrow X \sqcup Y$, $i_2 : Y \rightarrow X \sqcup Y$ satisfying the following universal property:



remark. More generally, for S any set and $F : S \rightarrow \mathcal{C}$ a collection of objects in \mathcal{C} indexed by S , their *coproduct* is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

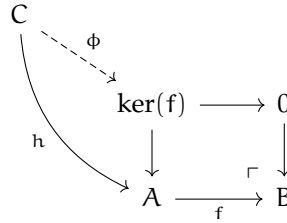
$$F(s) \rightarrow \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

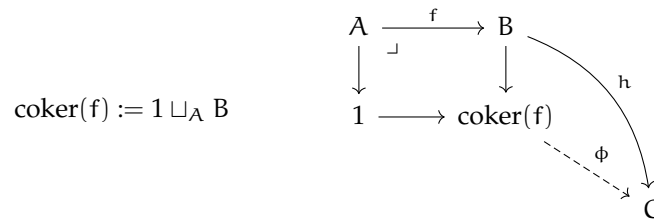
- The *kernel* of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the *kernel* $\ker f$ of a morphism $f : A \rightarrow B$ is the pullback $\ker(f) \rightarrow A$ along f of the unique morphism $0 \rightarrow B$

More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

for every object C and every morphism $h : C \rightarrow A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \rightarrow \ker(f)$ such that $h = p \circ \phi$.



- In a category with a terminal object 1 , the *cokernel* of a morphism $f : A \rightarrow B$ is the pushout (arrows h and ϕ apply if terminal object is zero)

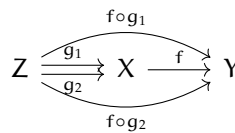


In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\text{coker}(f)$ with the following universal property:

for every object C and every morphism $h : B \rightarrow C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \text{coker}(f) \rightarrow C$ such that $h = \phi \circ i$.

- A morphism $f : X \rightarrow Y$ is a *monomorphism* if for every object Z and every pair of morphisms $g_1, g_2 : Z \rightarrow X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$



Equivalently, f is a monomorphism if for every Z the hom-functor $\text{Hom}(Z, -)$ takes it to an injective function

$$\text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an epimorphism in the opposite category \mathcal{C}^{op} .

- A morphism $f : X \rightarrow Y$ is a **epimorphism** if for every object Z and every pair of morphisms $g_1, g_2 : Y \rightarrow Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{array}{ccccc} & & g_1 \circ f & & \\ & \nearrow & & \searrow & \\ X & \xrightarrow{f} & Y & \xrightarrow{g_1} & Z \\ & \searrow & & \nearrow & \\ & & g_2 \circ f & & \end{array}$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\text{Hom}(-, Z)$ takes it to an injective function

$$\text{Hom}(Y, Z) \xleftarrow{f^*} \text{Hom}(X, Z).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an monomorphism in the opposite category \mathcal{C}^{op} .

- (Retraction.)
 - (wiki) Let X be a topological space and A a subspace of X . Then a continuous map $r : X \rightarrow A$ is a **retraction** if the restriction of r to A is the identity map on A .
 - (nLab) An object A in a category is called a **retract** of an object B if there are morphisms $i : A \rightarrow B$ and $r : B \rightarrow A$ such that $r \circ i = \text{id}_A$. In this case r is called a **retraction of B onto A** and i is called a **section of r** .

$$\text{id} : A \xrightarrow[\text{section}]{i} B \xrightarrow[\text{retraction}]{r} A$$

Hence a **retraction** of a morphism $i : A \rightarrow B$ is a left-inverse and a **section** of a morphism $r : B \rightarrow A$ is a right-inverse.

- (Deformation retract.)
 - (nLab) Let \mathcal{C} be a category equipped with a notion of homotopy between its morphisms. Then a **deformation retraction** of a morphism $i : A \rightarrow X$ is another morphism $r : X \rightarrow A$ such that

?

- (wiki) A continuous map $F : X \times [0, 1] \rightarrow X$ is a **deformation retraction** of a space X into a subspace A if, for every x in X and a in A ,

$$F(x, 0) = x, \quad F(x, 1) \in A \quad \text{and} \quad F(a, 1) = a.$$

In words, a deformation retraction is a homotopy between a retraction and the identity map on X . The subspace A is called a **deformation retract** of X . A deformation retraction is a special case of a homotopy equivalence.

An equivalent definition of deformation retraction is the following. A continuous map $r : X \rightarrow A$ is a **deformation retraction** if it is a retraction and its composition with the inclusion is homotopic to the identity map on X . In this formulation, a deformation retraction carries with it a homotopy between the identity map on X and itself.

- (wiki) If, in the definition of a deformation retraction we add the requirement that

$$F(a, t) = a \quad \forall t \in [0, 1], \forall a \in A,$$

then F is called a **strong deformation retraction**. In words, a strong deformation retraction leaves points in A fixed throughout the homotopy.

example. S^n is a strong deformation retract of $\mathbb{R}^{n+1} \setminus \{0\}$ through $F(x, t) = (1 - t)x + t \frac{x}{\|x\|}$.

- (wiki) The inclusion of a closed subspace A in the space X is a ?? if and only if A is a **neighbourhood deformation retract** of X , meaning that there is a continuous map $u : X \rightarrow [0, 1]$ with $A = u^{-1}(0)$ and a homotopy $H : X \times [0, 1] \rightarrow X$ such that $H(x, 0) = x$ for all $x \in X$, $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$, and $H(x, 1) \in A$ if $u(x) < 1$.

For example, the inclusion of a subcomplex in a CW complex is a cofibration.

elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \rightarrow Y$ continuous maps. An **homotopy** from f to g is a continuous map

$$H : X \times [0, 1] \rightarrow Y, \quad (x, t) \mapsto H(x, t) = H_t(x)$$

such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation \simeq is an equivalence relation on the set of continuous maps $X \rightarrow Y$. A homotopy of maps $H_t : X \rightarrow Y$ is called **relative to** $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top , the **homotopy category** h-Top , where composition of homotopy classes is induced by composition of representing maps. If $f : X \rightarrow Y$ represents an isomorphism in h-Top , then f is called a **homotopy equivalence** or **h-equivalence**. In explicit terms this means $f : X \rightarrow Y$ is a homotopy equivalence if there exists $g : Y \rightarrow X$, a **homotopy inverse of** f , such that gf and fg are both homotopic to the identity. Spaces X and Y are called **homotopy equivalent** or of the same **homotopy type** if there exists a homotopy equivalence $X \rightarrow Y$. A space is **contractible** if it is homotopy equivalent to a point. A map $f : X \rightarrow Y$ is **null homotopic** if it is homotopic to a constant map.

- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the *n-th homotopy group* are homotopy classes of maps $(S^n, s_0) \rightarrow (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \rightarrow (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

Also,

$$\pi_n(X, *) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n / \partial I^n, X]^0$$

where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \rightarrow Y$.

proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the *relative homotopy group* $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n . That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

The elements of such a group are homotopy classes of based maps $D^n \rightarrow X$ which carry the boundary S^{n-1} into A . Two maps f, g are called *homotopic relative to* A if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0, 1] \rightarrow X$ such that, for each p in S^{n-1} and t in $[0, 1]$, the element $F(p, t)$ is in A . Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

remark. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0)$$

where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ to I^{n-1} , or by restricting maps $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$. The map, called the *boundary map*, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called *n-connected* if $\pi_i(X, x_0) = 0$ for $i \leq n$. Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair (X, A) is *n-connected* if $\pi_i(X, A, x_0) = 0$ for $i \leq n$.
- Two pointed spaces (X, x_0) and (Y, y_0) are *n-equivalent* if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i < n$ and surjective for $i = n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on *compactly generated weakly Hausdorff* spaces.

- X is **compactly generated** if, for any subset $C \subset X$, and for all continuous maps $f : K \rightarrow X$ from compact Hausdorff spaces,

if $f^{-1}(C)$ is closed in K , then C is closed.

claim (What I picked up from the lecture). If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is **k-closed**.

From **May**: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi : X \rightarrow Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta_Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a “closed equivalence relation” is compactly generated.

Several other propositions follow in **May**. Now some other notes from the lectures:

In CGWH, $\text{Hom}(X, Y)$ is a space with the compact-open topology. **This is a compactly generated space, $k(\text{Hom}(X, Y))$.**

remark. (Also see [wiki on currying](#))

$\text{Map}(X, Y) :=$ the space of maps $X \rightarrow Y$.

$\text{Map}(X \times Y, Z) \cong \text{Map}(X, \text{Map}(Y, Z))$

$\text{Hom}(X \times Y, Z) \cong \text{Hom}(X, \text{Map}(Y, Z))$

In the last line, product is product in CGWH, not in Top .

The functor $- \times Y$ is left adjoint to $\text{Map}(Y, -)$.

cofibrations

definition.

- ([wiki](#)) In mathematics, in particular in homotopy theory, a continuous map between topological spaces $i : A \rightarrow X$ is a **cofibration** if it has the **homotopy extension property** with respect to all topological spaces S .

That is, i is a cofibration if

- for each topological space S ,
- and for any continuous maps $f, f' : A \rightarrow S$
- and $g : X \rightarrow S$ with $g \circ i = f$,
- for any homotopy $h : A \times I \rightarrow S$ from f to f' ,

there is a continuous map $g' : X \rightarrow S$ and a homotopy $h' : X \times I \rightarrow S$ from g to g' such that

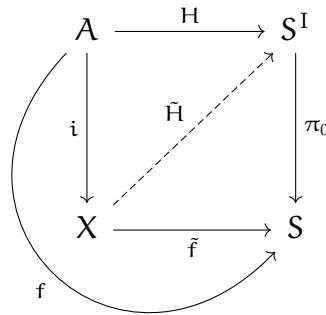
$$h'(i(a), t) = h(a, t) \quad \text{for all } a \in A \text{ and } t \in I.$$

- (wiki) In what follows, let $I = [0, 1]$ denote the unit interval.

A map $i : A \rightarrow X$ is a **cofibration** if for any map $f : A \rightarrow S$ such that there is an extension to X , meaning there is a map $\tilde{f} : X \rightarrow S$ such that $\tilde{f} \circ i = f$, we can extend a homotopy of maps $H : A \times I \rightarrow S$ to a homotopy of maps $\tilde{H} : X \times I \rightarrow S$ where

$$H(a, 0) = f(a)$$

$$\tilde{H}(x, 0) = \tilde{f}(x)$$

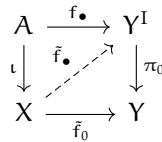


- (wiki) Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the **homotopy extension property** if, given a homotopy $f_\bullet : A \rightarrow Y^I$ and a map $\tilde{f}_0 : X \rightarrow Y$ such that

$$\tilde{f}_0 \circ \iota = f_0$$

(so \tilde{f} is the lift of $f_0 : A \rightarrow Y$) then there exists an **extension** of f_\bullet to a homotopy $\tilde{f}_\bullet : X \rightarrow Y^I$ such that $\tilde{f}_\bullet \circ \iota = f_\bullet$.

That is,



So there's some **currying** to make usual homotopies $f_\bullet : A \times I \rightarrow Y$ look like $f_\bullet : A \rightarrow Y^I$. Or, as said in our lectures, "a homotopy $X \times I \rightarrow Y$ is the same as a map $X \rightarrow \text{Map}(I, Y)$ ".

- (May) A map $i : A \rightarrow X$ is a **cofibration** if it satisfies the **homotopy extension property (HEP)**. This means that if $h \circ i_0 = f \circ i$ in the diagram

$$\begin{array}{ccc}
 A & \xrightarrow{i_0} & A \times I \\
 \downarrow i & \nearrow h & \downarrow i \times \text{id} \\
 & Y & \\
 X & \xrightarrow{i_0} & X \times I \\
 & \nwarrow \tilde{h} & \\
 & &
 \end{array}$$

then there exists \tilde{h} that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map $i : A \rightarrow X$ between topological spaces is a **Hurewicz cofibration** if it satisfies the homotopy extension property.

Let's say it one more time: for any $g : X \rightarrow Y$ and any homotopy $H : A \times I \rightarrow Y$ such that

$$\begin{array}{ccc}
 A \times \{0\} & \longrightarrow & A \times I \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{g} & Y
 \end{array}$$

there is $H' : X \times I \rightarrow Y$,

$$\begin{array}{ccc}
 X \times \{0\} & \longrightarrow & A \times I \\
 \downarrow g & & \downarrow \\
 X \times I & \xrightarrow{H'} & Y
 \end{array}$$

such that

$$\begin{array}{ccc}
 A \times I & & \\
 \downarrow & \searrow H & \\
 X \times I & \xrightarrow{H'} & Y
 \end{array}$$

example. $\partial D^n \rightarrow D$ is a Hurewicz cofibration. **Why?**

exercise. Prove that an inclusion $f : A \rightarrow X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

definition (Mapping cylinder).

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. [make the dashed map unique—up to something maybe](#)). Namely, we can let Y in our original test diagram be the *mapping cylinder*

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of i and i_0 . Indeed, suppose that we can construct a map r that makes the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ \downarrow i & \nearrow & \downarrow i \times \text{id} \\ & Mi & \\ \downarrow & \nwarrow r & \\ X & \xrightarrow{i_0} & X \times I \end{array}$$

By the universal property of the pushouts, given maps f and h in our original test diagram induce a map $Mi \rightarrow Y$, and its composit with r gives a homotopy \tilde{h} that makes the diagram commute. [So just saying that \$Mi\$ is universal.](#)

- (nLab) Given a continuous map $f : X \rightarrow Y$ of topological spaces, one can define its *mapping cylinder* as a pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{(\sigma_0)_*(f)} & \text{Cyl}(f) \end{array}$$

in Top, where $I = [0, 1]$ and $\sigma : X \rightarrow X \times I$ is given by $x \mapsto (x, 0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$(X \times I \amalg Y) / \sim$$

where \sim is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

- (wiki) The *mapping cylinder* of a function f between topological spaces X and Y is the quotient

$$M_f = ([0, 1] \times X \amalg Y) / \sim$$

where \amalg denotes disjoint union, and \sim is the equivalence relation generated by

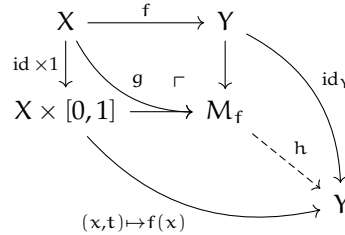
$$(0, x) \sim f(x) \text{ for each } x \in X.$$

That is, the mapping cylinder M_f is obtained by gluing one end of $X \times [0, 1]$ to Y via the map f . Notice that the “top” of the cylinder $\{1\} \times X$ is homeomorphic to X , while the “bottom” is the space $f(X) \subset Y$.

(Dani) So the mapping cylinder is just deforming X to Y putting X inside Y via f .

- (Homework) Let $f : X \rightarrow Y$ be a map. Let $M_f = X \times [0, 1] \cup_f Y$ be the *mapping cylinder of f* , i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$ and of $f : X \rightarrow Y$.

exercise. Let $g : X \rightarrow M_f$ be the map $X \xrightarrow{\cong} X \times \{0\} \rightarrow M_f$. Let $h : M_f \rightarrow Y$ be the map that is induced by $X \times [0, 1] \rightarrow Y : (x, t) \mapsto f(x)$ and $\text{id}_Y : Y \rightarrow Y$. Observe that f is the composition of g and h .



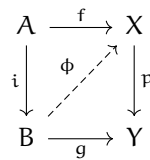
In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\text{colim } A_i) \times B$ is canonically homeomorphic with $\text{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \rightarrow M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

exercise. $X \rightarrow M_f \rightarrow Y$. Prove $X \rightarrow M_f$ is a cofibration.

fibrations

- (nLab) A morphism i has the *left lifting property with respect to a morphism p* and p has the *right lifting property with respect to i* if for each morphisms f and g , if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:



- (nLab) Let C be a category with products and with interval object I . A morphism $E \rightarrow B$ has the *homotopy lifting property* if it has the right lifting property with respect to all morphisms of the form $(\text{id}, 0) : Y \rightarrow Y \times I$.

This means that for all commuting squares

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ \downarrow & \nearrow \sigma & \downarrow p \\ Y \times I & \xrightarrow{F} & B \end{array}$$

there exists a morphism $\sigma : Y \times I \rightarrow E$ such that both triangles in the former diagram commute.

A **fibration** (also called **Hurewicz fibration**) is a mapping $p : E \rightarrow B$ satisfying the homotopy lifting property for all spaces X .

- (Hatcher) A map $p : E \rightarrow B$ is said to have the **homotopy lifting property** with respect to a space X if, given a homotopy $g_t : X \rightarrow B$ and a map $\tilde{g}_0 : X \rightarrow E$ lifting g_0 , so $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \rightarrow E$ lifting g_t .

The **lift extension property for a pair** (Z, A) asserts that every map $X \rightarrow B$ has a lift $Z \rightarrow E$ extending a given lift defined on the subspace $A \subset Z$. The case $(Z, A) = (X \times I, X \times \{0\})$ is the homotopy lifting property.

A **fibration** is a map $p : E \rightarrow B$ having the homotopy property with respect to all spaces X .

Theorem 3 (4.41 Hatcher, Long exact sequence of Serre fibrations, see proposition 18). Suppose $p : E \rightarrow B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, x_0) \rightarrow \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence B is path-connected and there is a long exact sequence

$$\cdots \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \cdots \rightarrow \pi_0(E, x_0) \rightarrow 0$$

The map $p : E \rightarrow B$ is said to have the **homotopy lifting property for a pair** (X, A) if each homotopy $f_t : X \rightarrow B$ lifts to a homotopy $\tilde{g}_t : X \rightarrow E$ starting with a given lift \tilde{g}_0 and extending a given lift $\tilde{g}_t : A \rightarrow E$. In other words, the homotopy lifting property for (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

The point is that the homotopy lifting property for disks is equivalent to the homotopy lifting property for all CW pairs (X, A) . A map $p : E \rightarrow B$ satisfying the homotopy lifting property for disks is sometimes called a **Serre fibration**.

A **fiber bundle** structure on a space E , with fiber F , consists of a projection map $p : E \rightarrow B$ such that each point B has a neighbourhood U for which there is a homeomorphism $h : p^{-1}(U) \rightarrow U \times F$ making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow p & \swarrow \\ & U & \end{array}$$

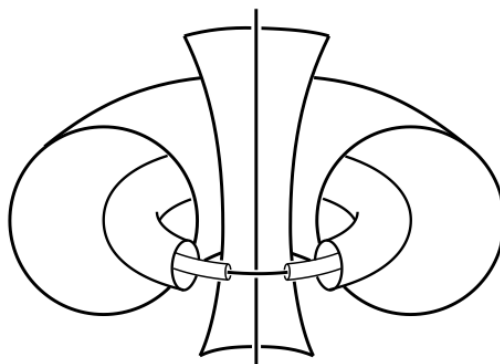
example. Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces $S^n \rightarrow \mathbb{RP}^n$ with fiber S^0 . Over the complex numbers the analog of this is a fiber bundle $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$. Here S^{2n+1} is the unit sphere in \mathbb{C}^{n+1} and \mathbb{CP}^n is viewed as the quotient space of S^{2n+1} under the equivalence relation $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ for $\lambda \in S^1$. The projection $p : S^{2n+1} \rightarrow \mathbb{CP}^n$ sends (z_0, \dots, z_n) to its equivalence class $[z_0, \dots, z_n]$.

To see that the local triviality condition for fibre bundles is satisfied, ...

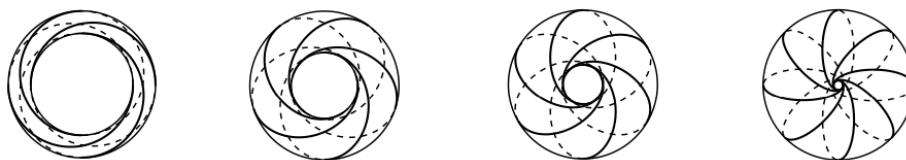
The construction of the bundle $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ also works when $n = \infty$, so there is a fiber bundle $S^1 \rightarrow S^\infty \rightarrow \mathbb{CP}^\infty$.

The case $n = 1$ is particularly interesting since $\mathbb{CP}^1 = S^2$ and bundle becomes $S^1 \rightarrow S^3 \rightarrow S^2$ with fiber, total space, and base all spheres. This is known as the **Hopf bundle**. The projection $S^3 \rightarrow S^2$ can be taken to be $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$. (That is, seeing S^2 as the one-point compactification of \mathbb{C} .)

In polar coordinates we may see S^3 as the union of several tori. Stereographic projection yields the following figure:



The limiting cases T_0 and T_∞ correspond to the unit circle in the xy -plane and the z -axis under the stereographic projection. Each torus T_ρ is a union of circle fibers. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. As ρ goes to 0 or ∞ the fiber circles approach the circles T_0 and T_∞ , which are also fibers. The figure below shows four tori decomposed into fibers:



How could we visualize the projection onto S^2 ? Could it work to think $S^2 = \mathbb{C} \cup \infty$ and just do stereographic projection from 3-space to the plane disregarding one point? What

would that even mean hehe

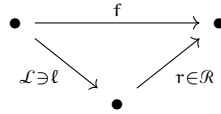
Replacing the field \mathbb{C} by the quaternions \mathbb{H} , the same constructions yield fiber bundles $S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$ over quaternionic projective spaces $\mathbb{H}P^n$. Here the fiber S^3 is the unit quaternions, and S^{4n+3} is the unit sphere in \mathbb{H}^{n+1} . Taking $n = 1$ gives a second Hopf bundle $S^3 \rightarrow S^7 \rightarrow S^4 = \mathbb{H}P^1$.

Another Hopf bundle $S^7 \rightarrow S^{15} \rightarrow S^8 \dots$

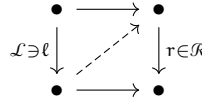
model structures

definition (Riehl). A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{M} is comprised of two classes of morphisms \mathcal{L} and \mathcal{R} so that

1. Every morphism in \mathcal{M} may be factored as a morphism in \mathcal{L} followed by a morphism in \mathcal{R} :

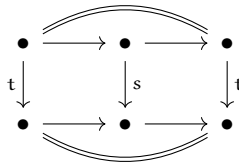


2. The maps in \mathcal{L} have the *left lifting property* with respect to each map in \mathcal{R} and equivalently the maps in \mathcal{R} have the *right lifting property* with respect to each map in \mathcal{L} , that is, any commutative square



admits a diagonal filler as indicated making both triangles commute. When this lift is unique, we say the factorization system is *orthogonal*.

3. The classes \mathcal{L} and \mathcal{R} are each closed under retracts in the arrow category: given a commutative diagram



if s is in that class then so is its retract t .

exercise (3.1.8 from Riehl). Verify that the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} is closed under coproducts, closed under retracts, and contains the isomorphisms.

definition. Given a contravariant functor $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ there is a corresponding category (*of elements of \mathcal{F}*) that lies over \mathcal{C} , that is,

$$\text{el } \mathcal{F} \rightarrow \mathcal{C}$$

given by

Objects: pairs (C, X) where $C \in \text{Obj } \mathcal{C}$ and $X \in \mathcal{F}(C)$.

Morphisms: $f : (C, X) \rightarrow (C', X')$ are morphisms $f : C \rightarrow C'$ such that $\mathcal{F}(f)(X') = X$.

remark. We can use the Yoneda embedding to view \mathcal{C} as a subcategory of $\text{Psh}(\mathcal{C})$,

$$\mathcal{C} \hookrightarrow \text{Psh}(\mathcal{C})$$

And also $\mathcal{F} \in \text{Psh}(\mathcal{C})$. In fact, the element category is just the slice category:

$$\text{el } \mathcal{F} \cong \mathcal{C}/\mathcal{F}.$$

question. Given $\mathcal{D} \rightarrow \mathcal{C}$ is it isomorphic to $\text{el } \mathcal{F} \rightarrow \mathcal{C}$?

definition. $G : \mathcal{D} \rightarrow \mathcal{C}$ is a **discrete fibration** if for any $d \in \mathcal{D}$ and any $f : C \rightarrow G(d)$ there exists a unique lift from f of f to $f' \in \mathcal{D}$ such that the target of f' is d . That is,

$$\begin{array}{ccc} \bullet & \xrightarrow{\exists! f'} & d \\ G \downarrow & & \downarrow G \\ C & \xrightarrow{f} & G(d) \end{array}$$

remark. Given a discrete fibration we may construct a functor $\mathcal{F} : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ simply by defining $\mathcal{F}(C) = G^{-1}(C)$ and if $C \rightarrow C' \rightarrow \dots \rightarrow d$.

definition (Lecture). A **model structure** on a category \mathcal{A} is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called **weak-equivalences**, **cofibrations** and **fibrations** with the following properties:

0. All (finite) small limits and colimits.
1. **(2 of 3)** Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in \mathcal{W} then all of them are.
2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system. That is,
 - (a) Any morphism in \mathcal{A} can be factored as a morphism in \mathcal{B} followed by a morphism in \mathcal{D} .
 - (b) Lifts:

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ f \downarrow & \nearrow \exists & \downarrow g \\ \bullet & \xrightarrow{\quad} & \bullet \end{array}$$

- (c') Notice that the axiom of retracts is not necessary. $r' \in \mathcal{R} \iff$ it satisfies (b) for all $\ell \in \mathcal{L}$.

definition.

- X is **fibrant** if $X \rightarrow \text{pt}$.
- X is **cofibrant** if $X \hookrightarrow X$
- X is **bifibrant** if $0 \hookrightarrow X \twoheadrightarrow \text{pt}$

examples (Two interesting model category structures on CGWH).

1. **Hurewicz model structure** (Strom).

- Cofibrations:= Hurewicz cofibrations.
- Fibrations:= maps $E \rightarrow B$ such that for all spaces X [Photo1].
- Weak equivalences:= homotopy equivalences.

2. **Quillen model structure**. Defined on Top .

- Cofibrations = retracts of relative cell complexes.

- Fibrations = Serre Fibrations:
- $$\begin{array}{ccc} D^n & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ D^n \times I & \longrightarrow & B \end{array}$$

- Weak equivalences: $f : X \rightarrow Y$

Also, we have

- Fibrant: all of Obj Top .
- Cofibrant: $\exists \{\text{CW complexes}\}$.

definition. Given a category \mathcal{C} and a class of morphisms $W \subset \text{Mor } \mathcal{C}$, its **localization** is a category $\mathcal{C}[W^{-1}]$ such that there is a functor $\text{Loc}_W \mathcal{C} \rightarrow \mathcal{C}[W^{-1}]$ that sends weak equivalences to isomorphisms. Also, it satisfies the universal property that for every $F : \mathcal{C} \rightarrow \mathcal{D}$ such that $F(X) \subset \text{Iso}$, the following diagram commutes

$$\begin{array}{ccc} & \mathcal{C}[W^{-1}] & \\ \text{Loc}_W \nearrow & & \searrow \exists! G \\ \mathcal{C} & \xrightarrow{F} & \mathcal{D} \end{array}$$

Theorem 4. Let \mathcal{C} and (\mathcal{C}, W, F) be a model category and $\mathcal{C}[W^{-1}] \cong \text{Ho } \mathcal{C}$ where the latter is given by

- $\text{Ob Ho } \mathcal{C} = \{\text{fibrant-cofibrant-bifibrant objects of } \mathcal{C}\}$.
- $\text{Mor Ho } \mathcal{C} = \text{Mor}_{\mathcal{C}}(X, Y)/\text{homotopy}$.

Let's say what homotopy means

definition. Given two maps

$$X \xrightarrow[f]{g} Y$$

- We say $f \sim_{\text{left}} g$ if for the *cylinder* $\text{Cyl}(X)$ defined by

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{\quad} & X \\ \text{cofibr.} \searrow & & \nearrow \text{trivial fib.} \\ & \text{Cyl}(X) & \end{array}$$

we have that

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f,g)} & Y \times Y \\ & \searrow & \nearrow \exists H \\ & \text{Cyl}(X) & \end{array}$$

- We say $f \sim_{\text{right}} g$ if

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{(f,g)} & Y \times Y \\ \text{dashed} \searrow \exists H & & \nearrow \\ & \text{Path}(Y) & \end{array}$$

claim. Given $X \xrightarrow[f]{g} Y$, if X is cofibrant and Y is fibrant, then $f \sim_{\text{left}} g \iff f \sim_{\text{right}} g$ and \sim is an equivalence relation.

whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A *CW complex* X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called *vertices*) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along *attaching maps* $j : S^n \rightarrow X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying (j, x) with $j(x)$ for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j (see ??). Each resulting map $D^{n+1} \rightarrow X$ is called a *cell*. The

subspace X^n is called the n -*skeleton* of X .

$$\begin{array}{ccc} S^n & \xhookrightarrow{i} & D^{n+1} \\ j \downarrow & \lrcorner & \downarrow \\ X^n & \longrightarrow & X^{n+1} \end{array}$$

lemma 5 (HELP). content...

Theorem 6 (Whitehead, May). If X is a CW complex and $e : Y \rightarrow Z$ is an n -equivalence, then $e_* : [X, Y] \rightarrow [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 7 (Whitehead, May). An n -equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 8 (Whitehead (4.5), Hatcher). If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ for all n , then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y .

exercise (Hatcher 4.1.12). Show that an n -connected, n -dimensional CW complex is contractible.

Solution. Just recall that n -connectedness means that $\pi_i(X) = 0$ for all $i \leq n$, which means that X is contractible by theorem 7. \square

lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \rightarrow X$$

$$\text{Hom}(X, Y) \mapsto \text{Hom}(X', Y)$$

$$\begin{aligned} \text{Hom}(A, B) &\cong \text{Hom}(A, B') \text{ natural in } A \implies \\ \text{Hom}(B, B) &\cong \text{Hom}(B, B') \& \text{Hom}(B', B) \cong \text{Hom}(B', B') \\ &\implies B \cong B'. \end{aligned}$$

- for (\Leftarrow) commutativity of the hypothesis gives us commutativity of the right-most square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an n -equivalence.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 9 (Yoneda).

$$\{\text{Natural transformations } \text{Hom}(-, X) \rightarrow F\} \cong F(X)$$

corollary 10. $(\text{Hom}(-, X) \rightarrow \text{Hom}(-, Y)) \cong \text{Hom}(X, Y)$.

corollary 11. The correspondence $X \mapsto \text{Hom}(-, X)$ is fully faithful, that is, the correspondence $\text{Hom}(X, X') \rightarrow \text{Hom}(\text{Hom}(-, X), \text{Hom}(-, X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. \square

lemma 12. Let $E \times_B X$ be the pullback of

$$\begin{array}{ccc} & E & \\ & \downarrow & \\ X & \xrightarrow{\cong} & B \end{array}$$

be such that $E \rightarrow B$ is an homotopy fibration and $f : X \rightarrow B$ is a homotopy equivalence. Let

$$\begin{array}{ccccc} E \times_B X & \rightarrow & E & \xrightarrow{\cong} & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & B & & B \end{array}$$

be the pullback. Then $E \times_B X \rightarrow E$ is a homotopy equivalence.

Proof. Let $g : B \rightarrow X$ be the homotopy inverse of f .

(Step 1) Construct another pullback

$$\begin{array}{ccccc} E \times_B B & \longrightarrow & X \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{g} & X & \xrightarrow{f} & B \end{array}$$

(Step 2) Construct $E \rightarrow E \times_B B$.

Consider

$$\begin{array}{ccccc} E & \xrightarrow{\text{id}} & E & & \\ \downarrow & & \downarrow & & \\ E \times I & \xrightarrow{f \times \text{id}} & B \times I & \longrightarrow & B? \end{array}$$

And then $E \rightarrow E \times_B B \rightarrow E \times_B X \rightarrow E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{array}{ccc} Z \sqcup Z & \longrightarrow & I \times Z \\ \downarrow f_1 \sqcup f_2 & \swarrow & \downarrow \\ E \times_B X & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\simeq} & B \end{array}$$

□

corollary 13. $B \xrightarrow{f} B$ is homotopy equivalence and $E \rightarrow B$ is a fibration, in

$$\begin{array}{ccc} E \times_B B & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B \end{array}$$

$E \times_B B \rightarrow E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 14. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \\ X & \longrightarrow & X \cup_A B \end{array}$$

be a pushout with $A \rightarrow X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f, g) \rightarrow X \cup_A B$ is a homotopy equivalence.

remark.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ X & & \end{array} \quad \begin{array}{ccc} A & \hookrightarrow & M_f \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_A M_f \cong M(f, g) \end{array}$$

definition.

- The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is

$$\begin{array}{ccc} X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y .

- The *homotopy fiber* if $f : Y \rightarrow Z$ is the pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ \text{pt} & \longrightarrow & Z \end{array}$$

$F \subset Z^I \times_Z Y \rightarrow Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark. The pullback of

$$\begin{array}{ccc} & & Z^I \times_Z Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is the homotopy pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

lemma 15. If $X \rightarrow Z$ is a fibration then for

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{array}{ccccc} X \times_Z & \longrightarrow & Y \\ \downarrow \simeq & & \downarrow \simeq \\ X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Z^I \times_Z Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

□

Finally,

$$\begin{array}{ccccc} \text{hofib } f_1 & \longrightarrow & \text{hofib } f & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

and

$$\begin{array}{ccc} Z & \longrightarrow & F(f) \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & X \end{array} \quad \begin{array}{c} \downarrow \\ X \times_Y Y^I \\ \downarrow \end{array}$$

and an exact sequence

$$\Omega^2 \text{hofib} \rightarrow \Omega^2 X \rightarrow \Omega^2 Y \rightarrow \Omega \text{hofib } f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow \text{hofib } f \rightarrow X \xrightarrow{f} Y$$

lemma 16 (Exactness). $\forall z, [z \text{hofib } f] \rightarrow [Z, X] \rightarrow [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \rightarrow \pi_0(\Omega^2 Y) \rightarrow \pi_0(\Omega \text{hofib } f) \rightarrow \pi_0(\Omega X) \rightarrow \pi_0(\Omega Y) \rightarrow \pi_0(\text{hofib } f) \rightarrow \pi_0(X) \rightarrow \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewicz fibrations. Let's talk about Serre fibrations. Notice that H. fibration \implies S. fibration. What is the most natural example of a Serre fibration?

proposition 17 (also [Hatcher 4.48](#)). Let E be a fiber bundle with fiber F . Then f is a Serre fibration.

Proof. What does it mean to be a Serre fibration? It means that

$$\begin{array}{ccc} I^n & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ I^{n+1} = I^n \times I & \longrightarrow & B \end{array}$$

So if \mathcal{U} is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U .

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{array}{ccc} I^n & \longrightarrow & F \times U \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ I^{n+1} & \longrightarrow & U \end{array}$$

has a lift for every little square because

$$\begin{array}{ccc} X & \longrightarrow & U \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ X \times I & \longrightarrow & \text{pt} \end{array}$$

is always a fibration ([think about this](#)) and because pullbacks of fibrations are fibrations:

$$\begin{array}{ccc} U \times F & \longrightarrow & U \\ \downarrow & & \downarrow \\ F & \longrightarrow & \text{pt} \end{array}$$

. Then we may just add up the squares because

$$\begin{array}{c} D^n \\ \downarrow \\ D^n \times I \end{array}$$

and we're done. □

proposition 18 (Sere fibration long exact sequence, see theorem 3). Let $g : E \rightarrow B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \\ & & \uparrow = & & \uparrow = & & \uparrow \cong & & \uparrow = & & \uparrow = & & \\ \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(E, F) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \end{array}$$

example. We have shown that $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 19. Let X be a CW-complex, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \emptyset$, so

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a pushout (this happens for inclusions, **check it?**).

If (A, C) is n -connected and (B, C) is m -connected, then

$$\pi_i(A, C) \rightarrow \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and surjective for $i = m + n$.

blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \rightarrow Y$ be a map. Consider the pullback

$$\begin{array}{ccc} W(f) & \longrightarrow & Y^I \\ (q, p) \downarrow & & \downarrow (ev_0, ev_1) \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

where

$$\begin{aligned} W(f) &= \{(x, w) \in X \times Y^I \mid f(x) = w(0)\}, \\ q(x, w) &= x, \quad p(x, w) = w(1). \end{aligned}$$

Since (ev_0, ev_1) is a fibration, the maps (q, p) , q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \rightarrow W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \rightarrow (* \times_{\text{ev}_0} X^I \times_{\text{ev}_1} A, \text{pt})$ is the same as a map $I^n \times I \rightarrow X$ that satisfies:

- $I^n \{0\} \cup \partial I^n \times I \rightarrow *$.
- $I^n \times \{1\} \rightarrow A$.

It is fairly straightforward to show that

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \Omega A & \longrightarrow & \Omega X & \longrightarrow & \text{hofib} \longrightarrow A \longrightarrow X \\ \pi_0(\nearrow) = & & \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_{n-1}(\text{hofib}) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(X) \\ & & & & \searrow & \downarrow \cong & \nearrow \\ & & & & & \pi_n(X, A) & \end{array}$$

Theorem 20 (Blakers-Massey 1). Let

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ f \downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

be a homotopy pushout, g is m equivalence, f is n -equivalence and $m, n \geq 0$. Then $Q \rightarrow X \times_P^h Y$ is $(m + n - 1)$ -equivalence.

Theorem 21 (Blakers-Massey 2). P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \emptyset$ (*strict pushout*)

$$\begin{array}{ccc} Q & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & P \end{array}$$

Then $\pi_i(Y, Q) \rightarrow \pi_i(P, X)$ is epi for $i = m + n$ and iso for $0 \leq i < m + n$.

Theorem 22 (Blakers-Massey 3). $P = X \cup Y$, X and Y are open in P , $X \cap Y = Q \neq \emptyset$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a *k-equivalence* if the induced map on the i th homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W : \text{at least } p \text{ coordinates of } x \text{ are } j \text{ the same coordinates of the center of } W\}$

lemma 23. Let W be a cube in \mathbb{R}^d with $\dim W \leq d$. If for all faces W' of ∂W , $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \text{ rel } \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

freudenthal theorem (2 april)

definition. The appropriate analogue of the Cartesian product in the category of based spaces is the *smash product* $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y .

We also have the *suspension of pointed spaces*, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X) / (t, x) \sim (0, y) \sim (1, y) \quad \forall y \in X.$$

Then we have

$$\text{Hom}_{\text{CGWH}_*}(\Sigma X, \Sigma X) \cong \text{Hom}_{\text{CGWH}_*}(X, \Omega \Sigma X)$$

where $\Sigma X = S^1 \wedge X$ and $\Omega \Sigma X = \text{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\text{Map}(S^1, -)$.

So let X be a space. The identity map $\text{id}_{\Sigma X} : \Sigma X \rightarrow \Sigma X$ then induces a map $X \rightarrow \Omega \Sigma X$.

Theorem 24 (Freudenthal). Let X be ℓ -connected space. Then $X \rightarrow \Omega \Sigma X$ is a $(2\ell + 1)$ -equivalence, that is,

$$\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X),$$

is a bijection for $i < 2\ell + 1$ and a surjection for $i = 2\ell + 1$ (May).

Proof 1.

$$\begin{array}{ccccc}
 X & \xrightarrow{(\ell+1)\text{-equiv}} & & & * \\
 & \searrow & & \nearrow & \\
 & & \Omega \Sigma X & & \\
 & \nearrow & \downarrow h_! & \searrow & \\
 * & \xrightarrow{(\ell+1)\text{-equiv}} & \Sigma X & &
 \end{array}$$

□

Proof 2. Consider

$$\begin{array}{ccc}
 X & \longrightarrow & CX \\
 \downarrow & & \downarrow \\
 CX & \longrightarrow & \Sigma X
 \end{array}$$

Then we use relative homotopy long exact sequence with (X, CX) to get $\pi_i(CX, X) \cong \pi_{i-1}(X)$, which is zero for $0 \leq i \leq \ell + 1$. Then use relative homotopy exact sequence for the pair $(\Sigma X, CX)$. then we get that $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$. And then if you use it for $(\Sigma X, X)$ and

But it also turns out that $\pi_i(\Sigma X) = \pi_{i-1}(\Omega \Sigma X)$ because

$$\pi_n(Z) = \text{Hom}_{\mathbf{h}\text{-Top},*}(S^n, Z) = \text{Hom}(S^1 \wedge S^{n-1}, Z) = \text{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX, X) \rightarrow \pi_i(\Sigma X, CX)$ which is isomorphism for $0 \leq i \leq 2\ell + 1$ and surjective for $i = 2\ell + 2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \rightarrow \pi_{i-1}(\Omega \Sigma X)$ for i as desired. \square

corollary 25. If X is ℓ -connected, then ΣX is $(\ell + 1)$ -connected for $\ell \geq 0$.

space	S^0	$\Sigma S^0 = S^1$	$\Sigma^2 S^0 = S^2$	$\Sigma^3 S^0 = S^3$	\dots	$\Sigma^n S^0 = S^n$
conectedness	-1	0	1	2	\dots	$(n - 1)$

corollary 26. S^n is $(n - 1)$ -connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \rightarrow S^2$$

we get

$$0 = \pi_2(S^3) \rightarrow \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \rightarrow \pi_1(S^3) = 0,$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map $S^n \rightarrow S^n$. We get a map $CS^n \rightarrow CS^n$ (in general, for $f : X \rightarrow Y$ we have $(t, x) \mapsto (t, f(x))$ in the cones). We also have $CS^n \rightarrow CS^n/S^n = S^{n+1}$.

Now if we take $\text{id} : S^n \rightarrow S^n$ we shall get $\text{id} : S^{n+1} \rightarrow S^{n+1}$. Think about this like $\pi_1(S^1) \rightarrow \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \rightarrow \pi_i(\Sigma X)$ is surjective because $i = 0$. From Hopf fibration we have $\pi_2(S^2) = \mathbb{Z}$. So we have a surjective map $\mathbb{Z} \rightarrow \mathbb{Z}$. So it's an isomorphism and we conclude that id_{S^2} is a generator of $\pi_2(S^2)$.

corollary 27. Since S^n is $(n - 1)$ -connected, we have

$$\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$$

is isomorphism for $i \leq 2(n - 1) = 2n - 1$ and epimorphism form $i = 2n - 1$. (We just shift the indices of theorem 24 by one.)

corollary 28. $\pi_n(S^n) = \mathbb{Z}$ with id_{S^n} as generator.

corollary 29. $\pi_{n+k}(S^n) \rightarrow \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \leq n - 1$ and epimorphism for $k = n - 1$.

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are $\mathbb{Z}/2$. This is what are called the *kth stable homotopy groups of a sphere*. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \rightarrow S^2$$

we get

$$0 = \pi_i(S^1) \rightarrow \pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2) \rightarrow \pi_{i-1}(S^2) = 0$$

So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

claim (Serre). $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{finite abelian}$. And $\pi_i(S^k)$ is finite abelian in all other cases.

another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{(n-1)\text{-equiv}} & D^n \\ \downarrow 0\text{-equiv} & \lrcorner & \downarrow \\ X & \longrightarrow & X \cup D^n \end{array}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is $(n-1)$ -equivalence by Blakers-Massey. So, by attaching $\sqcup D^n$ we can kill $\pi_{n-1}(X)$, that is, $X \cup (\sqcup D^n)$ has trivial π_{n-1} .

Now notice that

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

that is, $\pi_i(D^n, S^{n-1}) = 0$ for $i \leq n-1$. This implies that $\pi_i(X \cup D^n, X) = 0$ for $i \leq n-1$.

Also by homotopy long exact sequence,

$$\pi_{n-1}(X) \rightarrow \pi_i(X \cup D^n) \text{ is surjective}$$

$$\pi_i(X) \rightarrow \pi_i(X \cup D^n) \text{ is isomorphism for } i \leq n-2.$$

So what we have thus far is

$$\pi_n(X \cup D^n) \longrightarrow \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup D^n) \longrightarrow 0 = \pi_{n-1}(X \cup D^n)$$

Notice that $\pi_n(X \cup D^n, X)$ is not ingeneral cyclic (counterexample $S^1 \cup D^2$ taking universal cover which is real line with spheres on integers, homotopy equivalent to $\bigvee_{\mathbb{Z}} S^2$ which is not finitely generated).

So basically attaching a disk via f will kill $[f]$ inside $\pi_n(X)$ this is called *killing* an element of the homotopy group.

proposition 30. For any CW-complex X , $X^\ell \rightarrow X$ is an ℓ -equivalence.

remark. We have used that for $A \hookrightarrow X$ from long exact sequence of relative homotopy groups we get $\pi_n(X, A) = 0$.

corollary 31. Let $i \leq \ell$ and $g : D^i \rightarrow X$, $g : \partial D^i \rightarrow X^\ell$. Then there is a homotopy rel ∂D^i to a map with $\text{img} \subset X^\ell$.

Theorem 32 (Cellular approximation theorem). Let X and Y be CW-complexes and $\xi : Y \rightarrow X$ be any map. Then ξ is homotopic to a *cellular map*, that is, a map $\psi : Y \rightarrow X$, such that for all ℓ , $\psi Y^\ell \subset X^\ell$.

We also have

proposition 33. Let $n \geq 2$. Then

$$\pi_n \left(\bigvee_{k \in I} S^n \right) = \bigoplus_{k \in I} \pi_n(S^n) = \bigoplus_{k \in I} \mathbb{Z} = \mathbb{Z}^{\oplus I}$$

proposition 34. First notice that for finite I ,

$$X^n = X^{n+1} = \bigvee_{k \in I} S^n$$

by looking carefully. Then

$$\pi_n(X, X^{n+1}) = 0 = \pi_{n+1}(X, X^{n+1})$$

so

$$\bigoplus_{k \in I} \mathbb{Z} = \prod_{k \in I} \pi_n(S^n) = \pi_n(X) = \pi_n(X^{n+1}) = \pi_n(X^n) = \pi_n \left(\bigvee_{k \in I} S^n \right)$$

and for the infinite case it also works, using finite compactness of the CW complex.

postnikov tower and CW-approximation, 9 april

- Let X be a space. Then there is a CW-complex Y and a weak homotopy equivalence from $Y \rightarrow X$.
- Let $A \rightarrow X$ be a map of spaces. Then it can be factored as $A \rightarrow Y \rightarrow X$ where $A \hookrightarrow Y$ is a relative CW-complex, and $Y \rightarrow X$ is a weak homotopy equivalence.

remark. Notice that the second item is the first one with $A = \emptyset$. Then, the second case is a Serre cofibration since it is a construction involving the cofibration $S^{n-1} \hookrightarrow D^n$ (this is a cofibration by definition).

- Let A be a space. Then there is a space $\tau_{\leq n}A$ such that $A \hookrightarrow \tau_{\leq n}A$; $\tau_{\leq n}A$ is obtained by adding cells of $\dim \geq n + 2$. $A \hookrightarrow \tau_{\leq n}A$ is $(n + 1)$ -equivalence and

$$\pi_k(\tau_{\leq n}A) = 0 \quad k > n.$$

Moreover, $A \rightarrow \tau_{\leq n}A$ is unique among morphisms in $\text{Ho}(\text{CGWH})$ from A into spaces with $\pi_k = 0$ for $k > n$.

This is called a *Postnikov tower* and it looks like this:

$$\begin{array}{c}
 \dots \\
 \downarrow \\
 \tau_{\leq 2}A \\
 \downarrow \\
 \tau_{\leq 1}A \\
 \downarrow \\
 \tau_{\leq 0}A \\
 \uparrow \quad \uparrow \\
 A \longrightarrow \tau_{\leq 0}A
 \end{array}$$

The idea is that $\tau_{\leq n}A$ is obtained from A by killing elements of dimension greater than n , that is, by

- attaching $n + 2$ cells that kill all $\pi_{n+1}(A)$,
- attaching $n + 3$ cells that kill all $\pi_{n+2}(A)$,
- attaching $n + 3 \dots$
- attaching $n + n \dots$

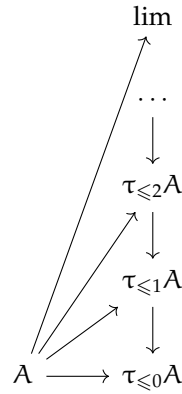
So consider the space X that is obtained from A after attaching cells of dimension $\geq n + 2$, so we have a map $A \rightarrow X$. Consider also a space Y with $\pi_k(Y) = 0$ for $k > n$. Then for any map $A \rightarrow Y$ there is a map $X \rightarrow Y$ that extends $A \rightarrow Y$. This accounts for a bijection

$$[X, Y] \cong [A, Y].$$

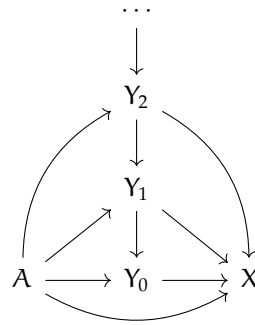
In class we struggled a bit to prove surjectivity, finally using an argument related to the pair $(X \times I, X \times \partial I \cup A \times I)$.

The point is that the spaces in the Postnikov tower are like the original space but with trivial homotopy groups for $k \geq n$.

question. What is the limit of the Posnikov tower?



- Let $A \rightarrow X$ be a map (of CW-complexes (or spaces?)). Then



Proof pending

- We also have the *Whitehead tower*, obtained from the homotopy fiber

$$\text{hofib } f_n \longrightarrow A \xrightarrow{f_n} \tau_{\leq n-1} A$$

which yields

$$\cdots \rightarrow \pi_{k+1}(A) \xrightarrow{\cong} \pi_{k+1}(\tau_{\leq n} A) \rightarrow \pi_k(\text{hofib}) \rightarrow \pi_k(A) \rightarrow \pi_k(\tau_{\leq n} A) \rightarrow \cdots$$

so

$k \leq n-1$	$k = n$	$k \geq n+1$
$\pi_k(\text{hofib } f_n) = 0$	$\pi_n(\text{hofib } f_n) = 0$	$\pi_k(A) = \pi_k(\text{hofib } f_n)$

- Now there's a natural way to construct the following diagram:

$$A \longrightarrow \tau_{\leq n} A \longrightarrow \tau_{\leq n-1} A$$

which yields the bundle

$$\text{hofib} \longrightarrow \tau_{\leq n} A \longrightarrow \tau_{\leq n-1} A$$

and in this case we get

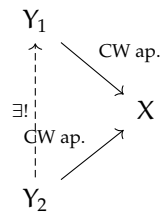
$k \neq n$	$k = n$
$\pi_k(\text{hofib}) = 0$	$\pi_n(\text{hofib}) = \pi_n(A)$

and this is what we call a $K(\pi, n)$ -space (all homotopy groups are trivial but the n th.)

Moore space, $K(\pi, n)$ and Hurewicz theorem, 11&16 apr

Theorem 35 (Uniqueness of CW-approximations). Recall that a CW-approximation of X is a map $f : Z \rightarrow X$ and a CW-complex Z that is a weak homotopy equivalence (induces isomorphisms in all homotopy groups).

We have that



up to homotopy equivalences

lemma 36 (Compression). If the relative homotopy groups of a pair (Y, B) is zero for $n = \dim e$ for every cell $e \in X \setminus A$ then any map $(X, A) \rightarrow (Y, B)$ is homotopic rel A to $(X, A) \rightarrow (B, B)$ (so intuitively we can collapse Y).

Proof. With fibrations (photo)

□

proposition 37. Let $f : X \rightarrow Y$ be an n -equivalence (in [Hatcher](#) stated as weak equivalence but argument is the same). Then f induces an *n -equivalence in homology* $H_i(X, \mathbb{Z}) \rightarrow H_i(Y, \mathbb{Z})$ (an isomorphism for $i < n$ and surjection for $i = n$).

Proof. photo

□

corollary 38. If $f : X \rightarrow Y$ is a weak equivalence, then f induces an isomorphism in $H_*(-, G)$ and $H^*(-, G)$.

Proof. Universal coefficients.

□

definition. Let π be an abelian group. Take

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi$$

a **free resolution**, i.e. $F_1 = \mathbb{Z}^{\oplus J}$ and $F_0 = \mathbb{Z}^{\oplus I}$ are free abelian groups and $\pi = F_0/F_1$. Let's take the corresponding maps

$$\begin{aligned} \bigvee_{j \in S} S^n &\longrightarrow \bigvee_{i \in I} S^n \longrightarrow \text{hocofib } f \\ x_j &\longmapsto \sum a_i y_i \end{aligned}$$

where a_i is the degree of $S^n \rightarrow S^n$. Recall that the homotopy cofiber $\text{hocofib } f$ is the mapping cone of f . It is the **cone of pointed spaces**. We call this space the **Moore space** $M(\pi, n)$ and it is such that

$$\tilde{H}_i(M(\pi, n)) = \begin{cases} 0, & i \neq n \\ \pi, & i = n \end{cases}$$

What do we get in homology? Exactly the sequence of free groups above. So, $H_n(\text{hocofib } f) = \pi$. What do we get in homotopy? **Might be π as well**. Let's prove something stronger:

Theorem 39. Let Y be such that $\pi_i(Y) = 0$ for $i > n$ and $\pi_0(Y) = 0$. Then

$$[\text{hocofib } f, Y] \rightarrow \text{Hom}(\pi_n(\text{hocofib } f), \pi_n(Y))$$

is a bijection.

Proof. Photo

Take

$$\bigvee_I S^n \longrightarrow \bigvee_J S^m \longrightarrow \text{hocofib } f$$

Now apply $[-, Y]$. We get

$$[\bigvee_I S^1] \longleftarrow [\bigvee_J S^1] \longleftarrow [\text{hocofib } f, Y] \longleftarrow 0$$

□

lemma 40. If (X, A) is r -connected, A s -connected for all $r, s \geq 0$, then the map

$$\pi_i(X, A) \rightarrow \pi_i(X/A)$$

induced by the quotient map $X \rightarrow X/A$ is an $(r + s + 1)$ equivalence.

Proof. If (X, A) is r -connected. □

After the lemma we proved that $\pi_n(C_f \cong G$.

Theorem 41. Now consider a Moore space, kill all homotopy groups to get $\tau_{\leq n}(M)$. It is a $K(\pi, n)$ space with cells in $\dim \geq n$, obtained from $\text{hofib } f$ by attaching cells of $\dim \geq n + 2$. Then

$$\begin{array}{ccc} M & \xrightarrow{\quad} & \tau_{\leq n} M \\ & \searrow & \swarrow \text{---} \\ & Y & \end{array} \quad \exists! \text{ up to homotopy}$$

$$[\text{hocofib } f, y] \cong [\tau_{\leq n}(\text{hocofib } f), Y] = \text{Hom}(\pi, \pi_n(Y)).$$

If $\pi_n(Y) = \pi$, then there is a weak equivalence

$$\tau_{\leq n}(\text{hocofib } f) \rightarrow Y.$$

Theorem 42 (Hurewicz). Let X be an $(n - 1)$ -connected space for $n \geq 2$. Then

$$\tilde{H}_i(X) = \begin{cases} 0, & i < n \\ \pi_n(X), & i = n \end{cases}$$

Proof. Photo.

Idea is to construct a Moore space that is **a piece of the CW approximation**. **Why a piece?**
Write and understand why this worked! \square

Theorem 43 (Relative Hurewicz theorem). Let (X, A) be n -connected, A be 1-connected, $n \geq 2$. Then

$$H_i(X, A) = \begin{cases} 0, & i < n \\ \pi_n(X, A), & i = n \end{cases}$$

Proof. Take a CW approximation of (X, A) ,

$$\begin{array}{ccccc} & & Y & & \\ & \nearrow & & \searrow \cong & \\ B & \xrightarrow{\cong} & A & \xrightarrow{\quad} & X \end{array}$$

So the approximation is (B, Y) .

Then we have

$$\begin{aligned} \pi_i(Y, B) &= \pi_i(Y/B), \quad i \leq n \\ H_i(Y, B) &= \tilde{H}_i(Y/B), \quad \forall i \end{aligned}$$

and first line implies that $\pi_1(Y/B) = 0$ for $i < n$. But then we are done, right? \square

Representability of the functor $H^n(-, G)$

remark. Recall the *adjoint relation*

$$\langle \Sigma X, K \rangle = \langle X, \Omega K \rangle$$

where ΣX is the *reduced suspension* of a space X , ΩK is the *loop space* of another space K and the brackets mean homotopy classes of basepointed maps. Choosing $X = S^0$ and $K = M$, the left-hand side becomes $\pi_1(M, m)$ and the right-hand side becomes the path components of $\text{Map}((S^1, 0), (M, m))$.

Now let's do some other interesting remarks.

definition. Let \mathcal{C} be a category. Then $\text{Psh}(\mathcal{C})$ is the category of *presheaves* of \mathcal{C} i.e. the category of functors $\mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ and natural transformations. For any object A in \mathcal{C} there is a presheaf $\text{Hom}_{\mathcal{C}}(-, A)$. A presheaf \mathcal{F} that is isomorphic to $\text{Hom}_{\mathcal{C}}(-, A)$ for some A is called *representable*.

For example, $\text{Hom}_{\text{CW}}(X, K(\pi, n)) \cong H^n(X, \pi)$, that is, H^n is representable.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}^1(H_{n-1}(K(G, n), G)) & \longrightarrow & H^n(K(G, n), G) & \xrightarrow{\cong} & \text{Hom}(H_n(K(G, n)), G) \longrightarrow 0 \\ & & \parallel & & & & \parallel \\ & & 0 & & & & 0 \end{array}$$

There is a special element in $H^n(K(G, N), G)$, the preimage of id_G .

claim.

$$[X, K(G, n)] \cong \tilde{H}^n(X, G)$$

where on the left we have based CW-complexes.

lemma 44 (Yoneda). Let \mathcal{F} be a presheaf, A be an object in \mathcal{C} . Then

$$\text{Nat}(\text{Hom}_{\mathcal{C}}(-, A), \mathcal{F}) \cong \mathcal{F}(A)$$

naturally in A .

Proof. For $f : C \rightarrow A$, we have this commutative diagram:

$$\begin{array}{ccc} \text{id}_A & \longmapsto & f \\ \\ \text{Hom}(A, A) & \longrightarrow & \text{Hom}(C, A) \\ \eta_A \downarrow & & \downarrow \eta_C \\ \mathcal{F}(A) & \longrightarrow & \mathcal{F}(C) \\ \\ \eta_A(\text{id}_A) & \longmapsto & \eta_C(f) \end{array}$$

So, natural transformations $\eta : \text{Hom}(-, A) \rightarrow \mathcal{F}$ are determined by $\eta_A(\text{id}_A)$. And that's it because then the map $\eta \mapsto \eta_A(\text{id}_A)$ is what we are looking for. It remains to write why it is injective, surjective and natural. \square

representability theory

If a functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$ is representable, then it sends colimits to limits and sends weak colimits to weak colimits.

Theorem 45. If $F : \text{Ho CW}_* \rightarrow \text{Sets}$ sends

1. coproducts (=wedges) to products,
2.
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \cup_A C \end{array}$$
 to weak pullback, where $B \cup_A C$ is a CW complex, B and C are CW complexes, $A = B \cap C$,
then F is representable.
(By B with isomorphism $F \cong [-, B]$ given by some $X \in F(B)$.)

definition. A CW-complex B together with a choice of $\gamma \in F(B)$ is a *spherical classifying space* of F if

$$\begin{aligned} \gamma_* : [S^n, B] &\rightarrow F(S^n) \\ f &\mapsto f^*(\gamma) \end{aligned}$$

is an isomorphism for $n > 0$ (because for $n = 0$ S^n is not connected).

proposition 46. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F , then B_1 and B_2 are homotopy equivalent via the map that sends γ_1 to γ_2 .

proposition 47. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F , then $g : B_1 \rightarrow B_2$ is such that $g^*(\gamma_2) = \gamma_1$, then g is a homotopy equivalence.

May 2: after representation theorem

Cohomology operations

$$\text{Nat}(H^n(-, G_1), H^n(-, G_2)) \leftrightarrow [K(G_1, n), K(G_2, n)]$$

example.

1. $x \mapsto x \smile x, H^n \rightarrow H^{2n}$. This is a natural transformation between the H functors.
2. The short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

yields

$$\cdots \longrightarrow H^n(X, \mathbb{Z}/p^2) \longrightarrow H^n(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p) \longrightarrow H^{n+1}(X, \mathbb{Z}/p^2) \longrightarrow \cdots$$

which is natural in X , yielding the **Bockstein homomorphism**

$$H^n(-, \mathbb{Z}/p \rightarrow H^{n+1}(-, \mathbb{Z}/p)$$

spectral sequences

1. *Exact couple:*

$$\begin{array}{ccc} A & \xrightarrow{i} & A \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

It looks like this is just notation for an exact sequence that repeats over and over.

example. Take

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

and take C_\bullet torsion free chain complex. You get

$$0 \longrightarrow C_\bullet \xrightarrow{\times p} C_\bullet \longrightarrow C_\bullet \otimes \mathbb{Z}/p \longrightarrow 0$$

which yields a long exact sequence

$$\begin{array}{ccc} H_n & \xrightarrow{\quad} & H_n \\ & \searrow & \swarrow \\ H_{n-1} & \xrightarrow{\quad} & \cdots \\ & \swarrow & \searrow \\ & H_n & \end{array}$$

Or better yet (notation)

$$\begin{array}{ccc} H_*(C) & \xrightarrow{\quad} & H_*(C) \\ & \searrow & \swarrow \\ & H_*(C, \mathbb{Z}/p) & \end{array}$$

2. *Derived couple of exact couple*

$$\begin{array}{ccc} A' & \xrightarrow{i'} & A' \\ & \swarrow k' & \searrow j' \\ & E' & \end{array}$$

Where $A' = \text{img } i$,

And the exact couple yields a homology $E^1 = H(A, d)$ with differential $d = jk$,

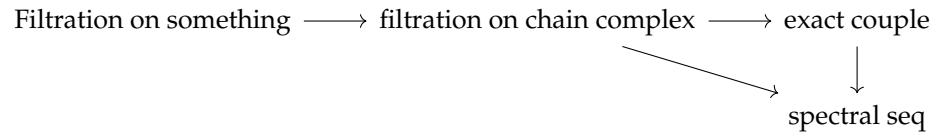
$i' = i|_{A'}$, $j'(ia) := [j(a)]$, $k'([e]) = k(e)$.

We have checked that

- j' and k' are well-defined. i' is well-defined automatically.
- that the image of k' is in fact in A'
- $j'i' = 0$, $k'j' = 0$, $i'k' = 0$.
- that $j'(ia) = 0 \implies ia = i'a'$, $k'[e] = 0 \implies [e] = j'(a')$ and that $i'(a') = 0 \implies a' = k[e]$.

So, we have shown that each exact couple gives a derived couple.

3.



A **filtration** on an abelian group/R-module/chain complex/... C is

$$\dots \subseteq F_n C \subseteq F_{n+1} C \subseteq C, \quad n \in \mathbb{Z}$$

and there is an associated graded $\text{gr } F_\bullet C := \bigoplus_{n \in \mathbb{Z}} F_n C / F_{n-1} C$.

We hope to recover C from $\text{gr } F_\bullet$.

Problem 1. If $\bigcap_{n \in \mathbb{Z}} F_n C \neq 0$ then the map $\bigcap_{n \in \mathbb{Z}} F_n C$ loses some information. We may solve this by asking that

1. $F_n C = 0$ for $n < 0$
2. $C \rightarrow \varinjlim C/F_n C$ is isomorphism.

Problem 2. If $\bigcup_{n \in \mathbb{Z}} F_n C \neq C$ that would be very bad. Then we should ask that $C = \bigcup_{n \in \mathbb{Z}} F_n C$ which is $C = \varinjlim F_n C$.

example. We discussed the cases of $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}$ and $\prod_{n \in \mathbb{N}} \mathbb{Z}$. We found that even though $\bigcap_{n \in \mathbb{Z}} F_n C = 0$ in both cases, we could not recover the desired information (?).

Now, to each Serre fibration corresponds a spectral sequence.

following sss.pdf

X is a filtered chain complex

$$\cdots \subseteq X_{n-1} \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$$

We automatically get

$$\begin{array}{ccccccc}
 H_{p+q}(X_p) & \xrightarrow{\quad} & H_{p+q}(X_p/X_{p-1}) & \xrightarrow{\quad} & H_{p+q-1}(X_{p-1}) & \rightarrow & H_{p+q-1}(X_{p-1}/X_{p-2}) \rightarrow \cdots \\
 \downarrow i & & \downarrow & & \downarrow i & & \downarrow \\
 H_{p+q}(X_{p+1}) & \rightarrow & H_{p+q}(X_{p+1}/X_p) & \rightarrow & H_{p+q-1}(X_p) & \xrightarrow{j} & H_{p+q-1}(X_p/X_{p-1}) \xrightarrow{k} \cdots \\
 \downarrow i & & \downarrow & & \downarrow i & & \downarrow \\
 H_{p+q}(X_{p+2}) & \rightarrow & H_{p+q}(X_{p+2}/X_{p+1}) & \rightarrow & H_{p+q-1}(X_{p+1}) & \rightarrow & H_{p+q-1}(X_{p+1}/X_p) \rightarrow \cdots \\
 \downarrow i & & \downarrow & & \downarrow i & & \downarrow \\
 \cdots & & \cdots & & \cdots & & \cdots
 \end{array}$$

Notice that the red arrows are the induced exact sequence of

$$\begin{array}{ccccccc}
 0 & \longrightarrow & X_{p-1} & \longrightarrow & X_p & \longrightarrow & X_p/X_{p-1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & X_p & \longrightarrow & X_{p+1} & \longrightarrow & X_{p+1}/X_p \longrightarrow 0
 \end{array}$$

Now consider

$$A = \bigoplus_{p,q} H_{p+q}(X_p)$$

and simply call $A_{p,q} = H_{p+q}(X_p)$. Also take

$$E = \bigoplus_{p,q} H_{p+q}(X_p/X_{p-1})$$

and $E_{p,q} := H_{p+q}(X_p/X_{p-1})$.

We have

$$\begin{array}{ccc}
 A & \xrightarrow{i} & A \\
 & \swarrow k \quad \nwarrow j & \\
 & E &
 \end{array}$$

where

map	bidgree
$i : A_{p,q} \rightarrow A_{p+1,q-1}$	$(1, -1)$
$j : A_{p,q} \rightarrow E_{p,q}$	$(0, 0)$
$k : E_{p,q} \rightarrow A_{p-1,q}$	$(-1, 0)$

Now consider the derived couple of this exact couple several times:

$$\begin{array}{ccc} A^2 & \xrightarrow{i_2} & A^2 \\ & \nwarrow k_2 \quad \nearrow j_2 & \\ & E^2 & \end{array} \qquad \begin{array}{ccc} A^3 & \xrightarrow{i_3} & A^3 \\ & \nwarrow k_3 \quad \nearrow j_3 & \\ & E^3 & \end{array}$$

where, going back to definitions

$$E^2 = \ker d_1 / \operatorname{img} d_1 \quad E^3 = \ker d_2 / \operatorname{img} d_2$$

and so on.

And then think about the bidigrees of the other maps. Well they are

map	bidigree
$i_k = i_{k-1} \operatorname{img} i_{k-1}$	$(1, -1)$
j_n	$-(n-1), n-1$
k_n	$(-1, 0)$
$d = j_n k_n$	$(-n, n-1)$

(we thought about this).

Here are some reasonable assumptions:

1. $X_p = 0$ for $p < 0$.
2. $\bigcup_p X_p = X$.
3. $H_{p+q}(X_p/X_{p-1}) = 0$ for $q < 0$.

remark. It will happen that for very large r , $E_{p,q}^r = E_{p,q}^r = \cdots = E_{p,q}^\infty$.

Also it will happen that

$$E_{p,q}^\infty \cong \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

and we will know what

$$\bigoplus_{\substack{p+q \text{ fixed} \\ p \in \mathbb{N}}} \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

is, and understand

$$H_* \quad \dots \subseteq \operatorname{img} H_{p+q}(X_p) \subseteq \operatorname{img} H_{p+q}(X_{p+1}) \subseteq \dots \subseteq H_{p+q}(X),$$

which is induced by

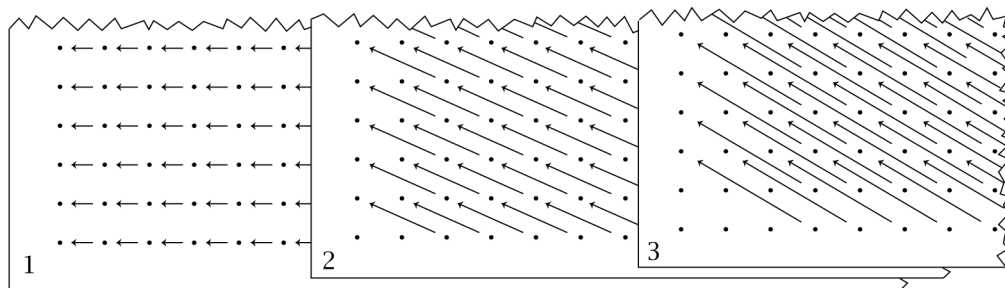
$$X \longrightarrow X_{p+1} \longrightarrow X$$

in our hopes to understand $H_{p+q}(X)$ which is of course just the homology of X .

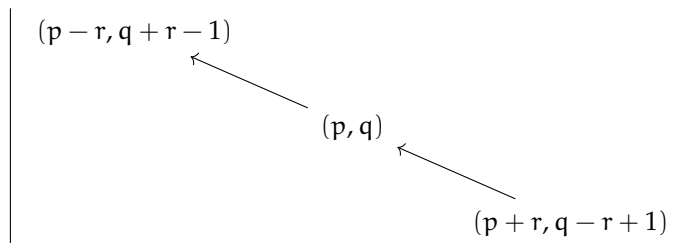
Now let's put the E 's in a diagram like

$$\begin{array}{ccc} \dots & & \\ E_{0,1}^1 & E_{1,1}^1 & E_{2,1}^1 \\ E_{0,0}^1 & E_{1,0}^1 & E_{2,0}^1 \end{array}$$

and if arrows are the differential d with bidegree $(-n, n-1)$, we already have Hatcher's picture



And something interesting will happen when r is very big. **What?** This:



that for a fixed (p, q) there is a large r such that if $p < r$ and $q + 1 < r$ we shall have

$$E_{p,q}^r = E_{p,q}^{r+1} := E_{p,q}^\infty.$$

Serre spectral sequence

Now take a Serre fibration $F \longrightarrow E \longrightarrow B$ with

$$\pi_0(F) = 0 \quad \text{and} \quad \pi_1(B) = 0$$

and it turns out that

$$E_{p,q}^2 = H_p(B, H_q(H)) \implies H_{p,q}(E).$$

example. For $S^1 \rightarrow S^3 \rightarrow S^2$ we have found that the first two pages are

$$\begin{array}{ccccc}
 \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/\text{img } d_2 & & \mathbb{Z} \\
 & \nwarrow \cong & & & & & \\
 \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/\text{img } d_2 \\
 & \nearrow d & & & & & \\
 & & E_2 & & & & E_3
 \end{array}$$

so $E^3 = E^\infty$.

example. We have also done $S^3 \rightarrow S^7 \rightarrow S^4$ and $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$. See **sss**guide.

proofs

Now we will prove that there exists a spectral sequence and that it converges where it converges.

serre spectral sequence

Consider either of

$$\pi_0(B) = 0 \quad \text{or} \quad \begin{cases} \pi_1(B) = 0 \\ \text{or } \pi_1(B) \text{ acts trivially on } H_q(F) \\ \text{or take } H_* \text{ with local coefficients} \end{cases}$$

Theorem 48. There is spectral sequence E^\vee that converges to $H_*(E)$ and such that

$$E_{p,q}^2 = H_p(B, H_q(F, G)).$$

Proof. The construction of the spectral sequence is not complicated. Start with E^1 . Then Do E^2 with

$$\begin{array}{ccc}
 E^k \times F & \longrightarrow & E \\
 \downarrow & \lrcorner & \downarrow \\
 D^k & \longrightarrow & B
 \end{array}$$

□

Consider the following conditions on the big staircase diagram:

- In each A column almost all maps are isomorphisms.
- In each column E almost each entry is 0.
- $E_{p,q}^1 = 0$ for $p < 0$ and $q < 0$.
- $X_p = 0$ for $p < 0$ and $H_n(X_p) \rightarrow H_n(X_{p+1})$ is isomorphism for $p \gg 0$.

And also

- $A_{-\infty, p+q} := A_{p,q}$ for $p \ll 0$.

- $A_{+\infty, p+q} := A_{p,q}$ for $p \gg 0$.

And then

e1. $A_{-\infty, p+q} = 0$.

e2. $A_{+\infty, p+q} = 0$.

claim. $b \implies E_{p,q}^r$ stabilizes for fixed panel q , so $E_{p,q}^\infty$ makes sense.

Now let's check that indices are the way they are in

$$\begin{array}{ccccccccccc}
 E_{p+r-1, q-r+2}^r & \xrightarrow{k} & A_{p+r-2, q-r+2}^r & \xrightarrow{i} & A_{p+r-1, q-r+1}^r & \xrightarrow{j} & E_{p,q}^r & \xrightarrow{k} & A_{p-1, q}^r & \xrightarrow{i} & A_{p, q-1}^r & \xrightarrow{j} & E_{p-r+1, q-r-2}^r \\
 \parallel & & & & & & & & \parallel & & \parallel & & \parallel \\
 0 & & & & & & & & \begin{smallmatrix} 0 \\ \text{if } e1 \end{smallmatrix} & & \begin{smallmatrix} 0 \\ \text{if } e1 \end{smallmatrix} & & 0
 \end{array}$$

Ok, after some other considerations we have concluded that

$E_{p,q}^\infty$ also makes sense and it is a piece of p -graded associative algebra graded of $A_{+\infty, p+q}$ with

$$\begin{aligned}
 E_{p,q}^r &= A_{p+q+r, q-r+q}^r / i A_{p+r-2, q-1+2}^r \\
 &= i^{r-1} (A_{p,q}^1) / i^r (A_{p-1, q+1}^1) \\
 &= F_p A_{+\infty, p+q} / F_{p-1} A_{+\infty, p+q}
 \end{aligned}$$

Something on 14 may

In cover spaces we have homeomorphisms between the fibers. In hurewicz fibrations this may not be true, but we still can have homotopy equivalences. So consider a Hurewicz fibration $f : E \rightarrow B$ and a path $I \rightarrow B$. Then we have:

$$\begin{array}{ccccc}
 F_a & \xhookrightarrow{\quad} & E & & \\
 \downarrow i_0 & \nearrow & \downarrow f & & \\
 F_a \times I & \xrightarrow{\pi} & I & \xrightarrow{p} & B
 \end{array}$$

And we claim that there is an homotopy equivalence

$$F_a \cong F_a \times \{1\} \rightarrow F_b = f^{-1}b$$

To see why, consider two homotopic paths $p_1, p_2 : I \rightarrow B$. Construct the following diagram:

We have (or will?) established:

proposition 49. For any Hurewicz fibration $f : E \rightarrow B$ there is "an action of $\pi_1(B)$ on F_b up to homotopy"

$$\begin{aligned}
 \Pi_1(B) &\rightarrow \text{Ho Top} \\
 b &\mapsto F_b
 \end{aligned}$$

We have an action

$$\pi_1(B) \curvearrowright H_*(F_b)$$

Why do need this? Consider the following case:

$$\begin{array}{ccc} f_i^* E & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f_i} & B \end{array}$$

for $f_1, f_2 : X \rightarrow B$.

serre spectral sequence for cohomology

Suppose

$$F \hookrightarrow X \longrightarrow B$$

Is a Serre fibration with $\pi_0(B) = 0$ and that the action described above $\pi_1(B) \curvearrowright H^*(F; G)$ is trivial.

Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ such that

- a. $d_r : E_r^{p,q} \rightarrow E_r^{p+r, q-r+q}$; $E_{r+1}^{p,q} = \ker d_r / \text{img } d_r$.
- b. $E_\infty^{p,q} \cong F_p^{p+q} / F_{p+1}^{p+q}$; $0 \subset F_{p+q}^{p+q} \subset \dots \subset F_0^{p+q} = H^{p+q}(X, G)$.
- c. $E_2^{p,q} = H^p(B, H^q(F, G))$.
- d. $E_2^{p,q} \times E_2^{s,t} \rightarrow E_2^{p+s, q+t}$, which is given by $(-1)^{qs}$.

$$H^p(B, H^q(F, R)) \times H^s(B, H^t(F, R)) \xrightarrow{\sim} H^{p+s}(B, H(F, R))$$

now supposing that $G = R$ is a ring.

e. This

$$F_p^m \times F_s^n \xrightarrow{\sim} F_{p+s}^{m+n}$$

which induces

$$F_p^m / F_{p+1}^m \times F_s^n / F_{s+1}^n \rightarrow F_{p+s}^{m+n} / F_{p+s+1}^{m+n}$$

and is

$$F_\infty^{p, m-p} \times E_\infty^{s, n-s} \rightarrow E_\infty^{p+s, m+n-p-s}$$

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