

homotopy theory

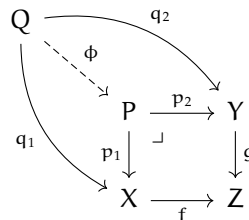
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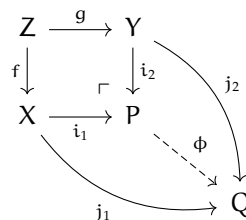
1 abstract nonsense

definition.

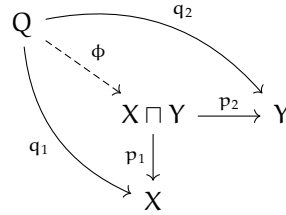
- A *pullback* of the morphisms f and g consists of an object P and two morphisms $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ satisfying the following universal property:



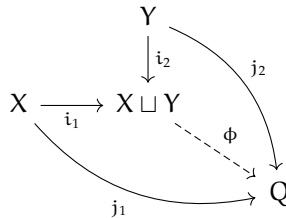
- A *pushout* of the morphisms f and g consists of an object P and two morphisms $i_1 : X \rightarrow P$ and $i_2 : Y \rightarrow P$ satisfying the following universal property:



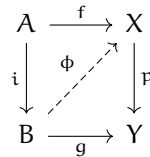
- A **product** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcup Y \rightarrow X$, $p_2 : X \sqcup Y \rightarrow Y$ satisfying the following universal property:



- A **coproduct** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \rightarrow X \sqcup Y$, $i_2 : Y \rightarrow X \sqcup Y$ satisfying the following universal property:



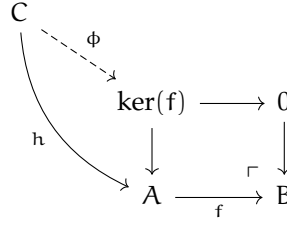
- A morphism i has the **left lifting property with respect to a morphism** p and p has the **right lifting property with respect to** i if for each morphisms f and g , if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:



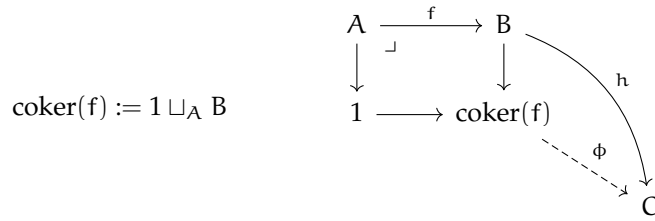
- The **kernel** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the **kernel** $\ker f$ of a morphism $f : A \rightarrow B$ is the pullback $\ker(f) \rightarrow A$ along f of the unique morphism $0 \rightarrow B$

More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

for every object C and every morphism $h : C \rightarrow A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \rightarrow \ker(f)$ such that $h = p \circ \phi$.



- In a category with a terminal object 1, the *cokernel* of a morphism $f : A \rightarrow B$ is the pushout (arrows h and ϕ apply if terminal object is zero)

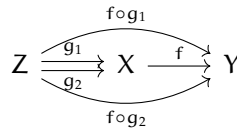


In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\text{coker}(f)$ with the following universal property:

for every object C and every morphism $h : B \rightarrow C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \text{coker}(f) \rightarrow C$ such that $h = \phi \circ i$.

- A morphism $f : X \rightarrow Y$ is a *monomorphism* if for every object Z and every pair of morphisms $g_1, g_2 : Z \rightarrow X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$



Equivalently, f is a monomorphism if for every Z the hom-functor $\text{Hom}(Z, -)$ takes it to an injective function

$$\text{Hom}(Z, X) \xrightarrow{f_*} \text{Hom}(Z, Y).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an epimorphism in the opposite category \mathcal{C}^{op} .

- A morphism $f : X \rightarrow Y$ is a *epimorphism* if for every object Z and every pair of morphisms $g_1, g_2 : Y \rightarrow Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{array}{ccccc}
 & & g_1 \circ f & & \\
 & \nearrow & & \nwarrow & \\
 X & \xrightarrow{f} & Y & \xrightarrow{g_1} & Z \\
 & \searrow & & \nearrow & \\
 & & g_2 \circ f & &
 \end{array}$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\text{Hom}(-, Z)$ takes it to an injective function

$$\text{Hom}(Y, Z) \xleftarrow{f^*} \text{Hom}(X, Z).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an monomorphism in the opposite category \mathcal{C}^{op} .

derived categories

We want to construct a category where weak homotopy equivalences are isomorphisms.

definition. Let \mathcal{C} be a category and $\mathcal{W} \subset \text{Mor}(\mathcal{C})$. The *localization of \mathcal{C} at \mathcal{W}* is another category $\mathcal{C}[\mathcal{W}^{-1}]$ and a functor $L : \dots$ so that...

Theorem 1.1 (Uniqueness). Up to equivalence

Theorem 1.2 (Localization by parts). $\mathcal{T} \subset \mathcal{S} \subset \text{Mor}(\mathcal{C})$, $\tilde{\mathcal{S}} = L_{\mathcal{T}}(\mathcal{S})$.

2 elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \rightarrow Y$ continuous maps. An *homotopy* from f to g is a continuous map

$$H : X \times [0, 1] \rightarrow Y, \quad (x, t) \mapsto H(x, t) = H_t(x)$$

) such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation \simeq is an equivalence relation on the set of continuous maps $X \rightarrow Y$. A homotopy of maps $H_t : X \rightarrow Y$ is called *relative to* $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top , the *homotopy category* h-Top , where composition of homotopy classes is induced by composition of representing maps. If $f : X \rightarrow Y$ represents an isomorphism in h-Top , then f is called a *homotopy equivalence* or *h-equivalence*. In explicit terms this means $f : X \rightarrow Y$ is a homotopy equivalence if there exists $g : Y \rightarrow X$, a *homotopy inverse of f* , such that gf and fg are both homotopic to the identity. Spaces X and Y are called *homotopy equivalent* or of the same *homotopy type* if there exists a homotopy equivalence $X \rightarrow Y$. A space is *contractible* if it is homotopy equivalent to a point. A map $f : X \rightarrow Y$ is *null homotopic* if it is homotopic to a constant map.

- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the *n-th homotopy group* are homotopy classes of maps $(S^n, s_0) \rightarrow (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \rightarrow (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

Also,

$$\pi_n(X, *) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n / \partial I^n, X]^0$$

where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $f : X \rightarrow Y$.

proposition 2.1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the *relative homotopy group* $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n . That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, *)].$$

The elements of such a group are homotopy classes of based maps $D^n \rightarrow X$ which carry the boundary S^{n-1} into A . Two maps f, g are called *homotopic relative to A* if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0, 1] \rightarrow X$ such that, for each p in S^{n-1} and t in $[0, 1]$, the element $F(p, t)$ is in A . Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

remark 2.1. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(A, x_0) \xrightarrow{\partial} \pi_{n-1}(X, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0)$$

where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ to I^{n-1} , or by restricting maps $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$. The map, called the *boundary map*, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called *n-connected* if $\pi_i(X, x_0) = 0$ for $i \leq n$. Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair (X, A) is *n-connected* if $\pi_i(X, A, x_0) = 0$ for $i \leq n$.
- Two pointed spaces (X, x_0) and (Y, y_0) are *n-equivalent* if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i \leq n$.

3 the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on *compactly generated weakly Hausdorff* spaces.

- X is **compactly generated** if, for any subset $C \subset X$, and for all continuous maps $f : K \rightarrow X$ from compact Hausdorff spaces,

if $f^{-1}(C)$ is closed in K , then C is closed.

claim. If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is **k-closed**.

In CGWH, $\text{Hom}(X, Y)$ is a space with the compact-open topology. **This is a compactly generated space, $\mathbf{k}(\text{Hom}(X, Y))$.**

$$\begin{aligned}\text{Map}(X, Y) &:= \text{the space of maps } X \rightarrow Y. \\ \text{Map}(X \times Y, Z) &\cong \text{Map}(X, \text{Map}(Y, Z)) \\ \text{Hom}(X \times Y, Z) &\cong \text{Hom}(X, \text{Map}(Y, Z))\end{aligned}$$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to $\text{Map}(Y, -)$.

- A **homotopy** $X \times I \rightarrow Y$ is the same as a map $X \rightarrow \text{Map}(I, Y)$.
- A map $A \rightarrow X$ is a **Hurewicz cofibration** for any $g : X \rightarrow Y$ and any homotopy $H : A \times I \rightarrow Y$ such that

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & A \times I \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

there is $H : X \times I \rightarrow Y$,

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & A \times I \\ \downarrow g & & \downarrow \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

$$\begin{array}{ccc} A \times I & & \\ \downarrow & \searrow H & \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

(This is a standard notion, you can look it up.)

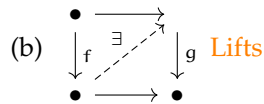
example. $\partial D^n \rightarrow D$ is a Hurewicz cofibration. **Why?**

definition. A **model structure** on a category \mathcal{A} is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called **weak-equivalences**, **cofibrations** and **fibrations** with the following properties:

1. Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in \mathcal{W} then all of them are.

2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system.

(a) Any morphism in \mathcal{A} can be factored as a morphism in \mathcal{B} followed by a morphism in \mathcal{D} .



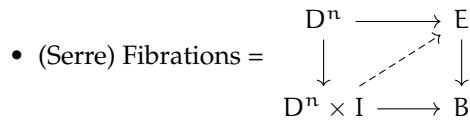
Two interesting model category structures on CGWH.

1. Hurewicz model structure (Strom).

- Cofibrations:= Hurewicz cofibrations.
- Fibrations:= maps $E \rightarrow B$ such that for all spaces X [Photo1].
- Weak equivalences:= homotopy equivalences.

2. Quillen model structure.

- Cofibrations = retracts of relative cell complexes.



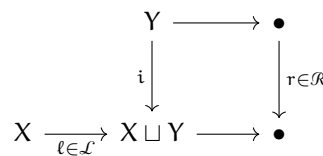
- Weak equivalences: $f : X \rightarrow Y$

Hurewicz cofibration $f : A \rightarrow X$ in CGWH.

- f is injective ...
- $f : A \rightarrow X$ is a cofibration ...

exercise (3.1.8 from Riehl's "Homotopical categories: ..."). Verify that the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. (Coproducts.) Let i be the coproduct of a morphism $\ell \in \mathcal{L}$. Consider the following lifting problem of i with respect to a morphism $r \in \mathcal{R}$:



It suffices to find a surjective map $s : X \rightarrow Y$ making the outer polygon in the following diagram commute:

$$\begin{array}{ccccc}
 X & \xrightarrow{s} & Y & \xrightarrow{p} & \bullet \\
 \searrow \ell \in \mathcal{L} & & \downarrow i & \nearrow \phi & \downarrow r \in \mathcal{R} \\
 & & X \sqcup Y & \xrightarrow{q} & \bullet
 \end{array}$$

In this case we have a lift ϕ such that $\phi\ell = ps$ and $r\phi = q$. Since $\ell = is$, we have that $\phi is = ps$ and since s is surjective $\phi i = p$, so ϕ is the desired lift.

How can we find such a map s ?

Also pending: retracts, isomorphisms. □

Blakers-Massey excision theorem (relies on technical lemma, proof from Tom Dieck's book)
 \implies Cellular approximation. Also \implies Freudenthal theorem.

definition (Mapping cylinder).

exercise. $X \rightarrow M_f \rightarrow Y$. Prove $X \rightarrow M_f$ is a cofibration.

4 lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \rightarrow X$$

$$\text{Hom}(X, Y) \mapsto \text{Hom}(X', Y)$$

$$\begin{aligned}
 &\text{Hom}(A, B) \cong \text{Hom}(A, B') \text{ natural in } A \implies \\
 &\text{Hom}(B, B) \cong \text{Hom}(B, B') \& \text{Hom}(B', B) \cong \text{Hom}(B', B') \\
 &\implies B \cong B'.
 \end{aligned}$$

- for (\Leftarrow) commutativity of the hypothesis gives us commutativity of the right-most square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- Some good concepts are pushouts, coproducts, direct limits.
- What we did? Prove the bijection between the homotopic sets given an n -equivalence.

- Defined smash.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 4.1 (Yoneda).

$$\{\text{Natural transformations } \text{Hom}(-, X) \rightarrow F\} \cong F(X)$$

corollary 4.2. $(\text{Hom}(-, X) \rightarrow \text{Hom}(-, Y)) \cong \text{Hom}(X, Y)$.

corollary 4.3. The correspondence $X \mapsto \text{Hom}(-, X)$ is fully faithful, that is, the correspondence $\text{Hom}(X, X') \rightarrow \text{Hom}(\text{Hom}(-, X), \text{Hom}(-, X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. \square

lemma 4.4. Let $E \times_B X$ be the pullback of

$$\begin{array}{ccc} & E & \\ & \downarrow & \\ X & \xrightarrow{\simeq} & B \end{array}$$

be such that $E \rightarrow B$ is an homotopy fibration and $f : X \rightarrow B$ is a homotopy equivalence. Let

$$\begin{array}{ccccc} E \times_B X & \rightarrow & E & \xrightarrow{\simeq} & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\simeq} & B & & B \end{array}$$

be the pullback. Then $E \times_B X \rightarrow E$ is a homotopy equivalence.

Proof. Let $g : B \rightarrow X$ be the homotopy inverse of f .

(Step 1) Construct another pullback

$$\begin{array}{ccccc} E \times_B B & \longrightarrow & X \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{g} & X & \xrightarrow{f} & B \end{array}$$

(Step 2) Construct $E \rightarrow E \times_B B$.

Consider

$$\begin{array}{ccccc} E & \xrightarrow{\text{id}} & E & & \\ \downarrow & & \downarrow & \nearrow \text{dashed} & \\ E \times I & \xrightarrow{f \times \text{id}} & B \times I & \longrightarrow & B? \end{array}$$

And then $E \rightarrow E \times_B B \rightarrow E \times_B X \rightarrow E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{array}{ccc} Z \sqcup Z & \longrightarrow & I \times Z \\ \downarrow f_1 \sqcup f_2 & \nearrow \text{dashed} & \downarrow \\ E \times_B X & \longrightarrow & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\cong} & B \end{array}$$

□

corollary 4.5. $B \xrightarrow{f} B$ is homotopy equivalence and $E \rightarrow B$ is a fibration, in

$$\begin{array}{ccc} E \times_B B & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B \end{array}$$

$E \times_B B \rightarrow E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 4.6. Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \\ X & \longrightarrow & X \cup_A B \end{array}$$

be a pushout with $A \rightarrow X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f, g) \rightarrow X \cup_A B$ is a homotopy equivalence.

remark 4.1.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ X & & \end{array} \quad \begin{array}{ccc} A & \hookrightarrow & M_f \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \cup_A M_f \cong M(f, g) \end{array}$$

definition.

- The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is

$$\begin{array}{ccc} X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y .

- The *homotopy fiber* if $f : Y \rightarrow Z$ is the pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ \text{pt} & \longrightarrow & Z \end{array}$$

$F \subset Z^I \times_Z Y \rightarrow Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark 4.2. The pullback of

$$\begin{array}{ccc} & & Z^I \times_Z Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is the motopy pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

lemma 4.7. If $X \rightarrow Z$ is a fibration then for

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \twoheadrightarrow & Z \end{array}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{array}{ccc}
 X \times_Z Y & \xrightarrow{\quad} & Y \\
 \downarrow \simeq & & \downarrow \simeq \\
 X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \twoheadrightarrow & Z^I \times_Z Y \\
 \downarrow & & \downarrow \\
 X & \xrightarrow{\quad} & Z
 \end{array}$$

□

Finally,

$$\begin{array}{ccccc}
 \text{hofib } f_1 & \longrightarrow & \text{hofib } f & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & X & \xrightarrow{f} & Y
 \end{array}$$

and

$$\begin{array}{ccc}
 Z & \xrightarrow{\quad} & F(f) \\
 \downarrow & \nearrow & \downarrow \\
 X \times I & \xrightarrow{\quad} & X
 \end{array}
 \quad
 \begin{array}{c}
 X \times_Y Y^I \\
 \downarrow \\
 X
 \end{array}$$

and an exact sequence

$$\Omega^2 \text{hofib } f \rightarrow \Omega^2 X \rightarrow \Omega^2 Y \rightarrow \Omega \text{hofib } f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow \text{hofib } f \rightarrow X \xrightarrow{f} Y$$

lemma 4.8 (Exactness). $\forall z, [z \text{hofib } f] \rightarrow [Z, X] \rightarrow [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \rightarrow \pi_0(\Omega^2 Y) \rightarrow \pi_0(\Omega \text{hofib } f) \rightarrow \pi_0(\Omega X) \rightarrow \pi_0(\Omega Y) \rightarrow \pi_0(\text{hofib } f) \rightarrow \pi_0(X) \rightarrow \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

21 march (Serre fibration long exact sequence)

We've been talking a lot about Hurewicz fibrations. Let's talk about Serre fibrations. Notice that H. fibration \implies S. fibration. What is the most natural example of a Serre fibration?

proposition 4.9. Let E be a fiber bundle with fiber F . Then f is a Serre fibration.

Proof. What does it mean to be a Serre fibration? It means that

$$\begin{array}{ccc} I^n & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} = I^n \times I & \longrightarrow & B \end{array}$$

So if \mathcal{U} is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U .

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{array}{ccc} I^n & \longrightarrow & F \times U \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} & \longrightarrow & U \end{array}$$

has a lift for every little square because

$$\begin{array}{ccc} X & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & \text{pt} \end{array}$$

is always a fibration (**think about this**) and because pullbacks of fibrations are fibrations:

$$\begin{array}{ccc} U \times F & \longrightarrow & U \\ \downarrow & & \downarrow \\ F & \longrightarrow & \text{pt} \end{array}$$

. Then we may just add up the squares because

$$\begin{array}{c} D^n \\ \downarrow \\ D^n \times I \end{array}$$

and we're done. □

proposition 4.10 (Construction of homotopy long exact sequence from relative homotopy long exact sequence). Let $g : E \rightarrow B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$.

Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{array}{ccccccccc}
 \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(E, F) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots
 \end{array}$$

example. We have shown that $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 4.11. Let X be a CW-complex, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \emptyset$, so

$$\begin{array}{ccc}
 C & \longrightarrow & A \\
 \downarrow & \lrcorner & \downarrow \\
 B & \longrightarrow & X
 \end{array}$$

is a pushout (this happens for inclusions, **check it?**).

If (A, C) is n -connected and (B, C) is m -connected, then

$$\pi_i(A, C) \rightarrow \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and surjective for $i = m + n$.

26 march (Blakers-Massey)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \rightarrow Y$ be a map. Consider the pullback

$$\begin{array}{ccc}
 W(f) & \longrightarrow & Y^I \\
 (q,p) \downarrow & & \downarrow (ev_0, ev_1) \\
 X \times Y & \xrightarrow{f \times id} & Y \times Y
 \end{array}$$

where

$$\begin{aligned}
 W(f) &= \{(x, w) \in X \times Y^I \mid f(x) = w(0)\}, \\
 q(x, w) &= x, \quad p(x, w) = w(1).
 \end{aligned}$$

Since (ev_0, ev_1) is a fibration, the maps (q, p) , q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \rightarrow W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \rightarrow (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \rightarrow X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \rightarrow *$.
- $I^n \times \{1\} \rightarrow A$.

It is fairly straightforward to show that

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \Omega A & \longrightarrow & \Omega X & \longrightarrow & \text{hofib} \longrightarrow A \longrightarrow X \\
 \pi_0(\nearrow) = & & \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_{n-1}(\text{hofib}) \longrightarrow \pi_{n-1}(A) \longrightarrow \pi_{n-1}(X) \\
 & & & & \searrow & \downarrow \cong & \nearrow \\
 & & & & & \pi_n(X, A) &
 \end{array}$$

Theorem 4.12 (Blakers-Massey 1). Let

$$\begin{array}{ccc}
 Q & \xrightarrow{g} & Y \\
 f \downarrow & & \downarrow \\
 X & \longrightarrow & P
 \end{array}$$

be a homotopy pushout, g is m equivalence, f is n -equivalence and $m, n \geq 0$. Then $Q \rightarrow X \times_P Y$ is $(m + n - 1)$ -equivalence.

Theorem 4.13 (Blakers-Massey 2). P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \emptyset$ (*strict pushout*)

$$\begin{array}{ccc}
 Q & \hookrightarrow & Y \\
 \downarrow & & \downarrow \\
 X & \hookrightarrow & P
 \end{array}$$

Then $\pi_i(Y, Q) \rightarrow \pi_i(P, X)$ is epi for $i = m + n$ and iso for $0 \leq i < m + n$.

Theorem 4.14 (Blakers-Massey 3). $P = X \cup Y$, X and Y are open in P , $X \cap Y = Q \neq \emptyset$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a *k-equivalence* if the induced map on the i th homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W : \text{at least } p \text{ coordinates of } x \text{ are } j \text{ the same coordinates of the center of } W\}$

lemma 4.15. Let W be a cube in \mathbb{R}^d with $\dim W \leq d$. If for all faces W' of ∂W , $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \text{ rel } \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.