algebraic topology exercises

Contents

Homework 1

0 Preliminaries

In the category of sets there is a bijection $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ that depends naturally on X, Y and Z. The notions related to this bijection are "Cartesian closed category", "currying" and "internal Hom".

Definition. A category $\mathcal{C}$ is ***Cartesian closed*** if:

1. $\mathcal{C}$ has all finite products (Caveat: some require that $\mathcal{C}$ has all finite limits)
2. For any object Y the functor $- \times Y$ has a right adjoint, which we will denote by $\mathrm{Map}(Y, -)$ or by $-^Y$.

Remark. By section 3 here, the second property above implies that we get a functor $\mathrm{Map}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, and moreover we get natural isomorphisms $\mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$ and $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Map}(X \times Y, Z)$.

Lemma (Yoneda, wiki)**.** Let F be a functor from a locally small category $\mathcal{C}$ to Set. Then for each object X of $\mathcal{C}$, the natural transformations $\mathrm{Nat}(\mathrm{Hom}(X, -), F)$ are in one-to-one correspondence with the elements of $F(X)$, that is

$$\mathrm{Nat}(\mathrm{Hom}(X, -), F) \cong F(X)$$

Moreover, this isomorphism is natural in A and F when both sides are regarded as functors from $\mathcal{C} \times \mathrm{Set}^{\mathcal{C}}$ to Set. ($\mathrm{Set}^{\mathcal{C}}$ denotes de category of functors from $\mathcal{C}$ to Set.)

There is a contravariant version of Yoneda lemma asserting that if F is a contravariant functor from $\mathcal{C}$ to Set,

$$\mathrm{Nat}(\mathrm{Hom}(-, X), F) \cong F(X).$$

Corollary. $\mathrm{Nat}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \cong \mathrm{Hom}(X, Y)$.

Remark. The correspondence $X \mapsto \mathrm{Hom}(-, X)$ is fully faithful, that is, the correspondence $\mathrm{Hom}(X, X') \to \mathrm{Nat}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, X'))$ is injective and bijective.

Exercise (a). Let $\mathcal{C}$ be any category. Show that if for some objects X and X' we have $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X', Y)$ for all objects Y, with isomorphisms being natural in Y, then $X \cong X'$. Dually, if $\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, X')$ naturally in Y , then also $X \cong X'$.

Solution. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. The dual statement follows from the analogue formulation of Yoneda lemma. □

Exercise. Let $\mathcal{C}$ be a Cartesian closed category and pt be the terminal object. Show that for any object X we have $X \cong \mathrm{Map}(\mathrm{pt}, X)$.

Solution. Using item (a) with X and $X' = \mathrm{Map}(\mathrm{pt}, X)$, it suffices to show that

$$\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, \mathrm{Map}(\mathrm{pt}, X))$$

for all objects Y and isomorphisms natural in Y.

Since $\mathcal{C}$ is Cartesian closed, we have isomorphisms natural in Y

$$\mathrm{Hom}(Y, \mathrm{Map}(\mathrm{pt}, X)) \cong \mathrm{Hom}(Y \times \mathrm{pt}, X) \cong \mathrm{Hom}(Y, X)$$

since pt is a terminal object. Indeed:

Claim. In a Cartesian closed category $\mathcal{C}$ with terminal object pt, we have that $Y \times \mathrm{pt} \cong Y$ for any object Y.

Proof of claim. **(From StackExchange)** The universal property of the product $Y \times \mathrm{pt}$ shows that the maps id_Y and $t_Y : Y \to \mathrm{pt}$ must factor through some $u : Y \to Y \times \mathrm{pt}$, making $\pi_1 \circ u = \mathrm{id}_Y$.

$$\begin{tikzcd}
Y \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "t_Y"'] & & \\
& Y \times \mathrm{pt} \arrow[r, "\pi_1"] \arrow[d, "t_{Y \times \mathrm{pt}}"] & Y \\
& \mathrm{pt} &
\end{tikzcd}$$

It is also true that $u \circ \pi_1 = \mathrm{id}_{Y \times \mathrm{pt}}$, since

- $\pi_1 \circ u \circ \pi_1 = \mathrm{id}_Y \circ \pi_1 = \pi_1$ and

- $t_{Y\times pt} \circ u \circ \pi_1 = t_{Y\times pt}$

so by uniqueness of the universal property we get that $u \circ \pi_1 = \mathrm{id}_{Y\times pt}$.

$$\begin{tikzcd}
Y \times pt \arrow[dr, "\pi_1"] \arrow[drrr, bend left, "\pi_1"] \arrow[ddrr, bend right, "t_{Y\times pt}"'] & & & \\
& Y \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "t_Y"'] & & \\
& & Y \times pt \arrow[r, "\pi_1"] \arrow[d, "t_{Y\times pt}"] & Y \\
& & pt &
\end{tikzcd}$$

□

□

1 Based spaces and smash product

[In progress...]

2 Mapping cylinders and Hurewicz cofibrations

Definition (wikipedia). Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the ***homotopy extension property*** if for any space Y, any homotopy $g_\bullet : A \to Y^I$ and any map $\tilde{g}_0 : X \to Y$ such that $\tilde{g}_0 \circ \iota = g_0$, there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{g}_\bullet : X \to Y^I$ such that $\tilde{g}_\bullet \circ \iota = g_\bullet$.

$$\begin{tikzcd}
A \arrow[r, "g_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{g}_0"'] \arrow[ur, dashed, "\tilde{g}_\bullet"] & Y
\end{tikzcd}$$

A ***Hurewicz cofibration*** is a map $\iota : A \to X$ satisfying the homotopy extension property.

Exercise (a). Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

Remark. A little late I noticed the comment on Telegram that we may assume A to be a closed subspace. Maybe I wouldn't have tried the solution following Miller if I had knew this earlier, hehe— still it was nice to see two different solutions.

Solution following Hatcher. ($\implies$) According to the former definition, choose $Y = (X \times \{0\}) \cup (A \times I)$. The inclusion $A \times I \hookrightarrow Y$ is an homotopy $g_\bullet$ from A to Y. Also, the inclusion

$X \times \{0\} \hookrightarrow Y$ is an extension $\tilde{g}_0$. Then there exists an extension $\tilde{g}_\bullet$ of the whole homotopy, which is just a map from $X \times I$ to Y. We have thus produced a retraction:

$$\begin{tikzcd}
(X \times \{0\}) \cup (A \times I) \arrow[r, "\mathrm{id}"] \arrow[d, hook] & (X \times \{0\}) \cup (A \times I) = Y \\
X \times I \arrow[ur, dashed] &
\end{tikzcd}$$

$(\Longleftarrow)$ Now suppose that $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$. Let Y be any space, $g_\bullet : A \to Y$ an homotopy and $\tilde{g}_0$ a map such that $\tilde{g}_0 = g_0 \circ f$.

The homotopy $g_\bullet$ along with $\tilde{g}_0$ yield a map $\varphi : (A \times I) \cup (X \times \{0\}) \to Y \cup (X \times \{0\})$. The key observation is that if A is closed in X, then this map is continuous by the gluing lemma. Then we simply compose the given retraction r with this map to obtain the homotopy extension:

$$\begin{tikzcd}
A \times I \arrow[rr, "g_\bullet"] \arrow[dd] & & Y \\
& (X \times \{0\}) \cup (A \times I) \arrow[ur, "\varphi"'] & \\
X \times I \arrow[ur, "r"'] & &
\end{tikzcd}$$

A complicated argument in Hatcher's appendix shows that such a function is continuous even without the assumption that A is closed. $\square$

Solution following Miller. The homotopy extension property may be defined as a map $\iota : A \to X$ such that for any solid-arrow diagram as below, a dotted blue arrow exists making the whole diagram commute:

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] & \\
A \times I \arrow[r] \arrow[drr, bend right] & X \times I \arrow[dr, dashed, blue] & \\
& & Y
\end{tikzcd}$$

Now consider the pushout corresponding to ι and the inclusion $A \to A \times I$. By the universal property of the pushout, the former diagram must factor by the pushout, and we get the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "\iota"] \arrow[d] & & X \arrow[d] \arrow[drr, bend left] \arrow[ddrrr, bend left] & & \\
A \times I \arrow[rr] \arrow[urr, phantom, "\ulcorner", very near end] \arrow[drrr, bend right] \arrow[ddrrrr, bend right] & & X \times \{0\} \sqcup_A (A \times I) \arrow[dr, dashed, "\exists!"] & & \\
& & & X \times I \arrow[dr, dashed, blue] & \\
& & & & Y
\end{tikzcd}$$

The implication ($\implies$) of our exercise again follows by choosing $Y = (X \times \{0\}) \cup (A \times I)$. For the implication ($\impliedby$) it appears that we have the same problem as before: we need to construct the blue dashed arrow from the rest of the diagram (using that the black dashed arrow has a left inverse), but it seems that the natural thing to do is defining this function from the two pieces just like before, and we must make sure it is continuous. $\square$

Definition. Let $f : X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f : X \to Y$. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

Remark. In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\operatorname{colim} A_i) \times B$ is canonically homeomorphic with $\operatorname{colim}(A_i \times B)$.

Exercise.

b. Show that h is a deformation retract, and in particular is a homotopy equivalence.

c. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

b. We have that

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \times I \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] \arrow[drr, bend right, "(x,t) \mapsto f(x)"'] & M_f \arrow[dr, dashed, "h"] & \\
& & Y
\end{tikzcd}$$

We must show that there is a homotopy between the identity map on M_f and a retraction from M_f to Y. So we want $h : M_f \times I \to M_f$ such that

$$h(-,0) = \mathrm{id}_{M_f}, \quad \operatorname{img} h(-,1) \subset Y \quad \text{and} \quad h(-,1)|_Y = \mathrm{id}_Y$$

Since M_f is a pullback, we can see it as a colimit, that is

$$M_f = \operatorname{colim}(\ X \times I \longleftarrow X \longrightarrow Y\ )$$

and, since colimits commute with products in CGWH, we get

$$M_f \times I = \operatorname{colim}(\ X \times I \times I \longleftarrow X \times I \longrightarrow Y \times I\ )$$

that is,

$$\begin{tikzcd}
X \times \{0\} \times I \arrow[r] \arrow[d] & Y \times I \arrow[d, "?"] \arrow[ddr, bend left, "{(y,s)\mapsto y}"] & \\
X \times I \times I \arrow[r, "?"'] \arrow[drr, bend right, "{(x,t,s)\mapsto f(x)}"'] & M_f \times I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & M_f
\end{tikzcd}$$

[I certainly got stuck in concluding…]

c. [Also in progress…] Consider the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "H"] \arrow[d, "g"'] & Z^I \arrow[d, "\pi_0"] \\
M_f \arrow[r, "h"'] \arrow[ur, dashed] & Z
\end{tikzcd}$$

□

3 Path spaces and fibrations

Exercise.

a. Show that $\mathrm{Map}(I, Y)$ deformation retracts on $\mathrm{Map}(pt, Y)$. Most likely you'll have to find a correct map $I \times I \to I$. Also show that $\mathrm{Map}(I, Y) \to \mathrm{Map}(pt, Y)$ is a Hurewicz fibration. The key map will be of the form $I \times I \to I \times I$.

Solution.

a. **($\mathrm{Map}(I, Y) \to \mathrm{Map}(pt, Y)$ is a Hurewicz fibration.)** Let A be any space. We must show that for any homotopy H and lift h_0 there exists an homotopy $\tilde{H}$ as in the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "h_0"] \arrow[d, hook] & \mathrm{Map}(I, Y) \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & \mathrm{Map}(0, Y)
\end{tikzcd}$$

From the isomorphism $\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ we may rewrite the problem as

$$
\begin{tikzcd}
(A \times \{0\}) \times I \arrow[dr, "H_0"] & \\
(A \times I) \times I \arrow[r, dashed, "\tilde{H}"] & Y \\
(A \times I) \times \{0\} \arrow[ur, "H"'] &
\end{tikzcd}
$$

So we define the dashed arrow by

$$(a, s, t) \mapsto \begin{cases} H_0(a, 0, s-t) & \text{when } s - t \geqslant 0 \\ H(a, t-s, 0) & \text{when } s - t \leqslant 0 \end{cases}$$

so that when $s = t$ the functions coincide, when $s = 0$ we get H and when $t = 0$ we get H_0.

($\mathrm{Map}(I, Y)$ **deformation retracts on** $\mathrm{Map}(pt, Y)$**.)** We must show there is a homotopy

$$h : \mathrm{Map}(I, Y) \times I \longrightarrow \mathrm{Map}(I, Y)$$

such that

$$h(-, 0) = \mathrm{id}_{\mathrm{Map}(I,Y)}, \qquad h(-, 1) \subset \mathrm{Map}(pt, Y)$$

$$\text{and} \qquad h(-, 1)|_{\mathrm{Map}(pt,Y)} = \mathrm{id}_{\mathrm{Map}(pt,Y)} .$$

Consider the map

$$\begin{aligned} I \times I &\to I \\ (s, t) &\mapsto s - st \end{aligned}$$

Our deformation retract may be written like

$$\begin{aligned} h : \mathrm{Map}(I, Y) \times I &\longrightarrow \mathrm{Map}(I, Y) \\ (f(s), t) &\longmapsto f(s - st) \end{aligned}$$

Then for $t = 0$ we have the identity on $\mathrm{Map}(I, Y)$, and when $t = 1$ we have ev_0.

$\square$

Let $f : X \to Y$ be a map. Let E_f be the pullback of $f : X \to Y$ and of $ev_0 : \mathrm{Map}(I, Y) \to Y$. Let $h : X \to E_f$ be the map that sends x to $(x, \mathrm{const}(f(x)))$, where $\mathrm{const}(f(x)) : I \to Y$ is the constant path at $f(x)$. Let $g : E_f \to Y$ be the composition of projection map $E_f \to \mathrm{Map}(I, Y)$ with $ev_1 : \mathrm{Map}(I, Y) \to Y$.

Homework 1.5

1 Exercise on model categories

Exercise (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. **(Coproducts.)** Suppose the maps $\ell_i : A_i \to B_i$ are in $\mathcal{L}$. Then their coproduct in the arrow category is the obvious map $\coprod A_i \to \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \to \coprod \ell_i$ such that for any other object $m : A \to B$ in the arrow category and a map $g : \ell \to m$, the following diagram is completed uniquely:

$$\begin{tikzcd} \ell_i \arrow[r, "f_i"] \arrow[rr, bend right, "g"'] & \coprod \ell_i \arrow[r, dashed, "\exists!"] & m \end{tikzcd} \qquad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{tikzcd} A_i \arrow[r, "\ell_i"] \arrow[d, "f_i^1"'] & B_i \arrow[d, "f_i^2"] \\ \coprod A_i \arrow[r, "\coprod \ell_i"] \arrow[d, dashed, "\exists!"'] & \coprod B_i \arrow[d, dashed, "\exists!"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{tikzcd} \coprod A_i \arrow[r] \arrow[d, "\coprod \ell_i"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "\mathcal{L} \ni \ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ B_i \arrow[r] \arrow[urr, dashed] & \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

which in turn means we have a unique map

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

by the universal property of the coproduct $\coprod B_i$.

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts $\coprod A_i$ and $\coprod B_i$ since, in general,

$$\mathrm{Hom}\left(\coprod X_i, Y\right) \cong \prod \mathrm{Hom}(X_i, Y).$$

More explicitly, we now that the red paths in the following diagrams are the same:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \color{red}\bullet \arrow[d, red, "r\in\mathcal{R}"] \\
\color{red}B_i \arrow[r, red] \arrow[urr] & \color{red}\coprod B_i \arrow[r] \arrow[ur, dashed, red] & \color{red}\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
\color{red}B_i \arrow[r, red] \arrow[urr] & \color{red}\coprod B_i \arrow[r, red] \arrow[ur, dashed] & \color{red}\bullet
\end{tikzcd}$$

and also

$$\begin{tikzcd}
\color{red}A_i \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \color{red}\coprod A_i \arrow[r] \arrow[d, red] & \color{red}\bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \color{red}\coprod B_i \arrow[r] \arrow[ur, dashed, red] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\color{red}A_i \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \color{red}\coprod A_i \arrow[r, red] \arrow[d] & \color{red}\bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

so the conclusion follows from the former comment.

(Closed under retracts.) Let us at least state what a retract of a morphism g should be in the arrow category. Recall that a retract is just

$$\begin{tikzcd}
X \arrow[r] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r] & X
\end{tikzcd}$$

So in the arrow category we get

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[rr, bend left] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

$\square$

2 Hatcher's exercise on Whitehead's theorem

Theorem 1 (Whitehead, May)**.** If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 2 (Whitehead, May)**.** An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 3 (Whitehead (4.5), Hatcher)**.** If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

Exercise (Hatcher 4.1.12)**.** Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 2. $\square$

Homework 2

Exercise (4.1.3)**.** For an H–space (X, x_0) with multiplication $\mu : X \times X \to X$, show that the group operation in $\pi_n(X, x_0)$ can also be defined by the rule $(f + g)(x) = \mu(f(x), g(x))$.

Exercise (4.1.19)**.** Consider the equivalence relation $\simeq_w$ generated by weak homotopy equivalence: $X \simeq_w Y$ if there are spaces $X = X_1, X_2, \dots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \simeq_w Y$ iff X and Y have a common CW approximation.

Solution. ($\Longleftarrow$) Suppose Z is a common CW approximation of X and Y, that is, Z is a CW complex and there are weak homotopy equivalences $Z \to X$ and $Z \to Y$. Then the sequence of spaces $X = X_1$, $Z = X_2$ and $Y = X_3$ shows that $X \simeq_w Y$.

($\Longrightarrow$) Suppose Z is a CW approximation of X and let's show it can be made (somehow) into a CW approximation of Y. There is a weak homotopy equivalence $Z \to X$, and also a weak homotopy equivalence either $X = X_1 \to X_2$ or $X = X_1 \leftarrow X_2$. I wonder if this implies that the composition $Z \to X = X_1 \to X_2$ is also a weak homotopy equivalence $\square$

Exercise (4.2.1)**.** Use homotopy groups to show that there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ for $n > k > 0$.

Solution (in progress...) Suppose there is a retraction

$$\begin{tikzcd} \mathbb{R}P^k \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{R}P^n \arrow[r] & \mathbb{R}P^k \end{tikzcd}$$

it induces isomorphisms

$$\begin{tikzcd} \pi_i(\mathbb{R}P^k) \arrow[r] \arrow[rr, bend right, "\cong"'] & \pi_i(\mathbb{R}P^n) \arrow[r] & \pi_i(\mathbb{R}P^k) \end{tikzcd}$$

If we had that this is a Serre fibration, we would get that

$$\text{content...}$$

□

Exercise (4.2.13). Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leqslant n$. [Pass to universal covers and use homology.]

Solution. Let X and Y be n-dimensional CW complexes and $f : X \to Y$ such that $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leqslant n$. Let's try to use Hurewicz theorem, which states that a map between simply connected CW complexes is a homotopy equivalence if it induces isomorphisms on all homology groups.

Consider the universal covers $\tilde{X}$ and $\tilde{Y}$, which are simply connected and also have CW structures. By prop. 4.1, the cover projections induce isomorphisms in the homotopy groups for all $i \geqslant 2$. By StackExchange there is a unique lift $\tilde{f}$ to the universal covers making the diagram on the left commute, and by functoriality the diagram on the right also commutes.

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \tilde{Y} \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} \pi_i(\tilde{X}) \arrow[r, "\tilde{f}_*"] \arrow[d, "p_*"', "\cong"] & \pi_i(\tilde{Y}) \arrow[d, "q_*", "\cong"'] \\ \pi_i(X) \arrow[r, "\cong", "f_*"'] & \pi_i(Y) \end{tikzcd} \qquad i \geqslant 2$$

We conclude that $\tilde{f}$ is a weak homotopy equivalence, and by prop. 4.21 it induces isomorphisms on homology groups. Finally, by Hurewicz theorem (coro. 4.33) it is an homotopy equivalence and so is f. □

Exercise. There is a fiber sequence $U(n) \hookrightarrow U(n+1) \to U(n+1)/U(n) \cong S^{2n+1}$. Use this to show that $\pi_k(U(n)) \to \pi_k(U(n+1))$ is isomorphism for $n > k/2$. Compute $\pi_k(U(n))$ for $n \geqslant 2$ and $k = 1, 2, 3$. In fact, if k is even then $\pi_k(U(N)) = 0$ and if k is odd then $\pi_k(U(N)) = \mathbb{Z}$, where again $N > k/2$. These equalities are known as Bott periodicity.

Solution. The required isomorphisms $\pi_k(U(n)) \to \pi_k(U(n+1))$ follow simply from the fact that S^{2n+1} is $2n+1$-connected: in the long homotopy sequence of the fiber bundle we have

$$\pi_{k+1}(S^{2n+1}) \longrightarrow \pi_k(U(n)) \longrightarrow \pi_k(U(n+1)) \longrightarrow \pi_k(S^{2n+1})$$

so when $2n+1 > k+1 \iff n > k/2$ the homotopy groups of the spheres vanish and we have an isomorphism.

The group $\pi_1(U(n))$ is isomorphic to $\mathbb{Z}$. This follows from the fact that $U(1)$ is homeomorphic to a circle and by induction using the former isomorphism $\pi_1(U(n)) \cong \pi_1(U(n+1))$. We also have $\pi_2(U(1)) = 0$, so that again by induction we get $\pi_2(U(n)) = 0$. Finally, a similar argument shows $\pi_3(U(n)) = 0$. □

Exercise (4.2.32). Show that if $S^k \longrightarrow S^m \longrightarrow S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$. [Look at the long exact sequence of homotopy groups.]

Solution (in progress...) The sequence of homotopy groups is

$$\cdots \longrightarrow \pi_i(S^k) \longrightarrow \pi_i(S^m) \longrightarrow \pi_i(S^n) \longrightarrow \pi_{i-1}(S^k) \longrightarrow \pi_{i-1}(S^m) \longrightarrow \cdots$$

so that if $i \leqslant k$ we have $\pi_i(S^m) = \pi_i(S^n)$, □

cohomology ring of $\mathbb{C}P^n$

Exercise. Show that

$$H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where α has degree 2.

Proof. The CW structure of $\mathbb{C}P^n$ consists of one cell for every even dimension. This gives us the following chain complex:

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z}, \qquad \text{if } n \text{ is even}$$

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow 0, \qquad \text{if } n \text{ is odd}$$

which yields the cohomology

$$H^i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots 2n \\ 0, & i \text{ otherwise} \end{cases}$$

so that

$$H^\bullet(\mathbb{C}P^n) = H^0(\mathbb{C}P^n) \oplus H^2(\mathbb{C}P^n) \oplus \dots \oplus H^{2n}(\mathbb{C}P^n)$$

This means that the underlying group of the cohomology ring is the same as that of

$$\mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where α has degree 2. To show that these groups are also isomorphic as algebras we can use Poincaré duality as follows.

Consider the case $n = 2$, where we may immediately multiply the generator of second cohomology group with itself:

$$\begin{aligned} H^2(\mathbb{C}P^2) \times H^2(\mathbb{C}P^2) &\to H^4(\mathbb{C}P^2) \\ (\alpha, \alpha) \qquad &\mapsto \alpha \smile \alpha = \alpha^2 \end{aligned}$$

By Poincaré duality this map is bilinear, so it must map generator to a generator. The fact that the product of the generator in degree 2 is the generator of degree 4 yields an homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}[\alpha] &\to H^\bullet(\mathbb{C}P^n) \\ \alpha &\mapsto \alpha \in H^2(\mathbb{C}P^n) \end{aligned}$$

with kernel (α^{n+1}) as desired.

Now the case of $\mathbb{C}P^3$ is:

$$\begin{aligned} H^2(\mathbb{C}P^3) \times H^4(\mathbb{C}P^3) &\to H^6(\mathbb{C}P^3) \\ (\alpha, \alpha^2) \qquad &\mapsto \alpha \smile \alpha^2 = \alpha^3 \end{aligned}$$

which also maps generator to generator, producing the desired algebra isomorphism. Notice we have used the group isomorphism $H^4(\mathbb{C}P^3) \approx H^4(\mathbb{C}P^2)$ when denoting the generator of $H^4(\mathbb{C}P^3)$ as α^2. Such an isomorphism is induced by inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ via relative cohomology exact sequence.

The case for dimension n follows by induction. $\square$

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