algebraic topology exercises

Contents

Homework 1

0 Preliminaries

In the category of sets there is a bijection $\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$ that depends naturally on X, Y and Z. The notions related to this bijection are "Cartesian closed category", "currying" and "internal Hom".

Definition. A category $\mathcal{C}$ is ***Cartesian closed*** if:

1. $\mathcal{C}$ has all finite products (Caveat: some require that $\mathcal{C}$ has all finite limits)
2. For any object Y the functor $- \times Y$ has a right adjoint, which we will denote by $\mathrm{Map}(Y, -)$ or by $-^Y$.

Remark. By section 3 here, the second property above implies that we get a functor $\mathrm{Map}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, and moreover we get natural isomorphisms $\mathrm{Hom}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Hom}(X \times Y, Z)$ and $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \cong \mathrm{Map}(X \times Y, Z)$.

Lemma (Yoneda, wiki)**.** Let F be a functor from a locally small category $\mathcal{C}$ to Set. Then for each object X of $\mathcal{C}$, the natural transformations $\mathrm{Nat}(\mathrm{Hom}(X, -), F)$ are in one-to-one correspondence with the elements of $F(X)$, that is

$$\mathrm{Nat}(\mathrm{Hom}(X, -), F) \cong F(X)$$

Moreover, this isomorphism is natural in A and F when both sides are regarded as functors from $\mathcal{C} \times \mathrm{Set}^{\mathcal{C}}$ to Set. ($\mathrm{Set}^{\mathcal{C}}$ denotes de category of functors from $\mathcal{C}$ to Set.)

There is a contravariant version of Yoneda lemma asserting that if F is a contravariant functor from $\mathcal{C}$ to Set,

$$\mathrm{Nat}(\mathrm{Hom}(-, X), F) \cong F(X).$$

Corollary. $\mathrm{Nat}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, Y)) \cong \mathrm{Hom}(X, Y)$.

Remark. The correspondence $X \mapsto \mathrm{Hom}(-, X)$ is fully faithful, that is, the correspondence $\mathrm{Hom}(X, X') \to \mathrm{Nat}(\mathrm{Hom}(-, X), \mathrm{Hom}(-, X'))$ is injective and bijective.

Exercise (a). Let $\mathcal{C}$ be any category. Show that if for some objects X and X' we have $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(X', Y)$ for all objects Y, with isomorphisms being natural in Y, then $X \cong X'$. Dually, if $\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, X')$ naturally in Y, then also $X \cong X'$.

Solution. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. The dual statement follows from the analogue formulation of Yoneda lemma. □

Exercise (b). Let $\mathcal{C}$ be a Cartesian closed category and pt be the terminal object. Show that for any object X we have $X \cong \mathrm{Map}(\mathrm{pt}, X)$.

Solution. Using item (a) with X and $X' = \mathrm{Map}(\mathrm{pt}, X)$, it suffices to show that

$$\mathrm{Hom}(Y, X) \cong \mathrm{Hom}(Y, \mathrm{Map}(\mathrm{pt}, X))$$

for all objects Y and isomorphisms natural in Y.

Since $\mathcal{C}$ is Cartesian closed, we have isomorphisms natural in Y

$$\mathrm{Hom}(Y, \mathrm{Map}(\mathrm{pt}, X)) \cong \mathrm{Hom}(Y \times \mathrm{pt}, X) \cong \mathrm{Hom}(Y, X)$$

since pt is a terminal object. Indeed:

Claim. In a Cartesian closed category $\mathcal{C}$ with terminal object pt, we have that $Y \times \mathrm{pt} \cong Y$ for any object Y.

Proof of claim. **(From StackExchange)** The universal property of the product $Y \times \mathrm{pt}$ shows that the maps id_Y and $t_Y : Y \to \mathrm{pt}$ must factor through some $u : Y \to Y \times \mathrm{pt}$, making $\pi_1 \circ u = \mathrm{id}_Y$.

$$\begin{tikzcd}
Y \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "t_Y"'] & & \\
& Y \times \mathrm{pt} \arrow[r, "\pi_1"] \arrow[d, "t_{Y \times \mathrm{pt}}"] & Y \\
& \mathrm{pt} &
\end{tikzcd}$$

It is also true that $u \circ \pi_1 = \mathrm{id}_{Y \times \mathrm{pt}}$, since

- $\pi_1 \circ u \circ \pi_1 = \mathrm{id}_Y \circ \pi_1 = \pi_1$ and

- $t_{Y\times pt} \circ u \circ \pi_1 = t_{Y\times pt}$

so by uniqueness of the universal property we get that $u \circ \pi_1 = \mathrm{id}_{Y\times pt}$.

$$\begin{tikzcd}
Y \times pt \arrow[dr, "\pi_1"] \arrow[drrr, bend left, "\pi_1"] \arrow[ddrr, bend right, "t_{Y\times pt}"'] & & & \\
& Y \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathrm{id}_Y"] \arrow[ddr, bend right, "t_Y"'] & & \\
& & Y \times pt \arrow[r, "\pi_1"] \arrow[d, "t_{Y\times pt}"] & Y \\
& & pt &
\end{tikzcd}$$

□

□

1 Based spaces and smash product

Definition. The appropriate analogue of the Cartesian product in the category of based spaces is the ***smash product*** $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y.

Exercise. For a based space (X, x_0) let ΣX be $[0,1] \times X/\{1\} \times X \cup \{0\} \times X \cup [0,1] \times \{x_0\}$. Check that $\Sigma X \cong S^1 \wedge X$. In particular $S^n \cong S^1 \wedge S^{n-1} \cong (S^1)^{\wedge n}$.

Remark. Another way of defining the reduced suspension ΣX (I think) is

$$\Sigma X = (I \times X)/(t, x) \sim (0, y) \sim (1, y)\ \forall y \in X.$$

Proof. To see that $\Sigma X \cong S^1 \wedge X$ simply notice that "both spaces are the quotient $X \times I$ with $X \times \partial I \cup \{x_0\} \times I$ collapsed to a point" (Hatcher, ex. 0.10). This is clear for ΣX. For $X \wedge S^1$, notice that collapsing $X \times \partial I$ to a point in $X \times I$ ammounts to taking $X \times S^1$ and collapsing one copy of X to a point. Further, collapsing $x_0 \times I$ to a point ammounts to collapsing the copy of S^1 in $X \vee S^1$ to a point.

Let's try induction on n. If $n = 2$, the smash product $S^1 \wedge S^1$ is easily seen to be S^2 since it consists on collapsing the boundary $S^1 \vee S^1$ of the square whose quotient yields $S^1 \times S^1$. For the inductive step Still incomplete... □

2 Mapping cylinders and Hurewicz cofibrations

Definition (wikipedia). Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the ***homotopy extension property*** if for any space Y, any homotopy $g_\bullet : A \to Y^I$ and any map $\tilde{g}_0 : X \to Y$ such that $\tilde{g}_0 \circ \iota = g_0$, there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{g}_\bullet : X \to Y^I$ such that $\tilde{g}_\bullet \circ \iota = g_\bullet$.

$$\begin{tikzcd} A \arrow[r, "g_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\ X \arrow[r, "\tilde{g}_0"'] \arrow[ur, dashed, "\tilde{g}_\bullet"] & Y \end{tikzcd}$$

A ***Hurewicz cofibration*** is a map $\iota : A \to X$ satisfying the homotopy extension property.

Exercise (a). Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

Remark. A little late I noticed the comment on Telegram that we may assume A to be a closed subspace. Maybe I wouldn't have tried the solution following Miller if I had knew this earlier, hehe— still it was nice to see two different solutions.

Solution following Hatcher. ($\Longrightarrow$) According to the former definition, choose $Y = (X \times \{0\}) \cup (A \times I)$. The inclusion $A \times I \hookrightarrow Y$ is an homotopy $g_\bullet$ from A to Y. Also, the inclusion $X \times \{0\} \hookrightarrow Y$ is an extension $\tilde{g}_0$. Then there exists an extension $\tilde{g}_\bullet$ of the whole homotopy, which is just a map from $X \times I$ to Y. We have thus produced a retraction:

$$\begin{tikzcd} (X \times \{0\}) \cup (A \times I) \arrow[r, "\mathrm{id}"] \arrow[d, hook] & (X \times \{0\}) \cup (A \times I) = Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

($\Longleftarrow$) Now suppose that $(X \times \{0\}) \cup (A \times I)$ is a retract of $X \times I$. Let Y be any space, $g_\bullet : A \to Y$ an homotopy and $\tilde{g}_0$ a map such that $\tilde{g}_0 = g_0 \circ f$.

The homotopy $g_\bullet$ along with $\tilde{g}_0$ yield a map $\varphi : (A \times I) \cup (X \times \{0\}) \to Y \cup (X \times \{0\})$. The key observation is that if A is closed in X, then this map is continuous by the gluing lemma. Then we simply compose the given retraction r with this map to obtain the homotopy extension:

$$\begin{tikzcd} A \times I \arrow[rr, "g_\bullet"] \arrow[dd] & & Y \\ & (X \times \{0\}) \cup (A \times I) \arrow[ur, "\varphi"] & \\ X \times I \arrow[ur, "r"] & & \end{tikzcd}$$

A complicated argument in Hatcher's appendix shows that such a function is continuous even without the assumption that A is closed. ☐

Solution following Miller. The homotopy extension property may be defined as a map $\iota : A \to X$ such that for any solid-arrow diagram as below, a dotted blue arrow exists making the whole diagram commute:

$$
\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d] & X \arrow[d] \arrow[ddr, bend left] \\
A \times I \arrow[r] \arrow[drr, bend right] & X \times I \arrow[dr, dashed, blue] \\
& & Y
\end{tikzcd}
$$

Now consider the pushout corresponding to ι and the inclusion $A \to A \times I$. By the universal property of the pushout, the former diagram must factor by the pushout, and we get the following diagram:

$$
\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & X \arrow[d] \arrow[ddr, bend left] \arrow[dddrr, bend left] \\
A \times I \arrow[r] \arrow[drr, bend right] \arrow[ddrrr, bend right] & X \times \{0\} \sqcup_A (A \times I) \arrow[dr, dashed, "\exists!"] \\
& & X \times I \arrow[dr, dashed, blue] \\
& & & Y
\end{tikzcd}
$$

The implication ($\implies$) of our exercise again follows by choosing $Y = (X \times \{0\}) \cup (A \times I)$. For the implication ($\impliedby$) it appears that we have the same problem as before: we need to construct the blue dashed arrow from the rest of the diagram (using that the black dashed arrow has a left inverse), but it seems that the natural thing to do is defining this function from the two pieces just like before, and we must make sure it is continuous. $\square$

Definition. Let $f : X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f : X \to Y$. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

Remark. In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\operatorname{colim} A_i) \times B$ is canonically homeomorphic with $\operatorname{colim}(A_i \times B)$.

Exercise.

b. Show that h is a deformation retract, and in particular is a homotopy equivalence.

c. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

b. We have that

$$\begin{tikzcd}
X\times\{0\} \arrow[r, "f"] \arrow[d, "\mathrm{id}\times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X\times I \arrow[r] \arrow[drr, bend right, "(x,t)\mapsto f(x)"'] & M_f \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "h"] & \\
& & Y
\end{tikzcd}$$

We must show that there is a homotopy between the identity map on M_f and a retraction from M_f to Y. So we want $h : M_f \times I \to M_f$ such that

$$h(-,0) = \mathrm{id}_{M_f}, \quad \operatorname{img} h(-,1) \subset Y \quad \text{and} \quad h(-,1)|_Y = \mathrm{id}_Y$$

Since M_f is a pullback, we can see it as a colimit, that is

$$M_f = \operatorname{colim}(\ X \times I \longleftarrow X \longrightarrow Y\ )$$

and, since colimits commute with products in CGWH, we get

$$M_f \times I = \operatorname{colim}(\ X \times I \times I \longleftarrow X \times I \longrightarrow Y \times I\ )$$

that is,

$$\begin{tikzcd}
X\times\{0\}\times I \arrow[r] \arrow[d] & Y\times I \arrow[d, "?"] \arrow[ddr, bend left, "(y,s)\mapsto y"] & \\
X\times I\times I \arrow[r, "?"'] \arrow[drr, bend right, "(x,t,s)\mapsto f(x)"'] & M_f\times I \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed] & \\
& & M_f
\end{tikzcd}$$

[I certainly got stuck in concluding...]

c. [Also in progress...] Consider the following lifting problem:

$$\begin{tikzcd}
X \arrow[r, "H"] \arrow[d, "g"'] & Z^I \arrow[d, "\pi_0"] \\
M_f \arrow[r, "h"'] \arrow[ur, dashed] & Z
\end{tikzcd}$$

□

3 Path spaces and fibrations

Exercise.

a. Show that $\mathrm{Map}(I, Y)$ deformation retracts on $\mathrm{Map}(\mathrm{pt}, Y)$. Most likely you'll have to find a correct map $I \times I \to I$. Also show that $\mathrm{Map}(I, Y) \to \mathrm{Map}(\mathrm{pt}, Y)$ is a Hurewicz fibration. The key map will be of the form $I \times I \to I \times I$.

Solution.

a. **(**$\mathrm{Map}(I, Y) \to \mathrm{Map}(\mathrm{pt}, Y)$ **is a Hurewicz fibration.)** Let A be any space. We must show that for any homotopy H and lift h_0 there exists an homotopy $\tilde{H}$ as in the following diagram:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "h_0"] \arrow[d, hook] & \mathrm{Map}(I, Y) \arrow[d, "p"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & \mathrm{Map}(0, Y)
\end{tikzcd}$$

From the isomorphism $\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ we may rewrite the problem as

$$\begin{tikzcd}
(A \times \{0\}) \times I \arrow[dr, "H_0"] & \\
(A \times I) \times I \arrow[r, dashed, "\tilde{H}"] & Y \\
(A \times I) \times \{0\} \arrow[ur, "H"'] &
\end{tikzcd}$$

So we define the dashed arrow by

$$(a, s, t) \mapsto \begin{cases} H_0(a, 0, s - t) & \text{when } s - t \geqslant 0 \\ H(a, t - s, 0) & \text{when } s - t \leqslant 0 \end{cases}$$

so that when $s = t$ the functions coincide, when $s = 0$ we get H and when $t = 0$ we get H_0.

($\mathrm{Map}(I, Y)$ **deformation retracts on** $\mathrm{Map}(\mathrm{pt}, Y)$**.)** We must show there is a homotopy

$$h : \mathrm{Map}(I, Y) \times I \longrightarrow \mathrm{Map}(I, Y)$$

such that

$$h(-, 0) = \mathrm{id}_{\mathrm{Map}(I,Y)}, \qquad h(-, 1) \subset \mathrm{Map}(\mathrm{pt}, Y)$$

$$\text{and} \qquad h(-, 1)|_{\mathrm{Map}(\mathrm{pt},Y)} = \mathrm{id}_{\mathrm{Map}(\mathrm{pt},Y)}\,.$$

Consider the map

$$\begin{aligned} I \times I &\to I \\ (s, t) &\mapsto s - st \end{aligned}$$

Our deformation retract may be written like

$$\begin{aligned} h : \mathrm{Map}(I, Y) \times I &\longrightarrow \mathrm{Map}(I, Y) \\ (f(s), t) &\longmapsto f(s - st) \end{aligned}$$

Then for $t = 0$ we have the identity on $\mathrm{Map}(I, Y)$, and when $t = 1$ we have ev_0.

□

Let $f : X \to Y$ be a map. Let E_f be the pullback of $f : X \to Y$ and of $\mathrm{ev}_0 : \mathrm{Map}(I, Y) \to Y$. Let $h : X \to E_f$ be the map that sends x to $(x, \mathrm{const}(f(x)))$, where $\mathrm{const}(f(x)) : I \to Y$ is the constant path at $f(x)$. Let $g : E_f \to Y$ be the composition of projection map $E_f \to \mathrm{Map}(I, Y)$ with $\mathrm{ev}_1 : \mathrm{Map}(I, Y) \to Y$.

Homework 1.5

1 Exercise on model categories

Exercise (3.1.8 from Riehl)**.** Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. **(Coproducts.)** Suppose the maps $\ell_i : A_i \to B_i$ are in $\mathcal{L}$. Then their coproduct in the arrow category is the obvious map $\coprod A_i \to \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \to \coprod \ell_i$ such that for any other object $m : A \to B$ in the arrow category and a map $g : \ell \to m$, the following diagram is completed uniquely:

$$\begin{tikzcd} \ell_i \arrow[r, "f_i"] \arrow[rr, bend right, "g"'] & \coprod \ell_i \arrow[r, dashed, "\exists!"] & m \end{tikzcd} \qquad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{tikzcd} A_i \arrow[r, "\ell_i"] \arrow[d, "f_i^1"'] & B_i \arrow[d, "f_i^2"] \\ \coprod A_i \arrow[r, "\coprod \ell_i"] \arrow[d, dashed, "\exists!"'] & \coprod B_i \arrow[d, dashed, "\exists!"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{tikzcd}
\coprod A_i \arrow[r] \arrow[d, "\coprod \ell_i"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
\coprod B_i \arrow[r] & \bullet
\end{tikzcd}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr, dashed] & \coprod B_i \arrow[r] & \bullet
\end{tikzcd}$$

which in turn means we have a unique map

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

by the universal property of the coproduct $\coprod B_i$.

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts $\coprod A_i$ and $\coprod B_i$ since, in general,

$$\mathrm{Hom}\left(\coprod X_i, Y\right) \cong \prod \mathrm{Hom}(X_i, Y).$$

More explicitly, we now that the red paths in the following diagrams are the same:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & {\color{red}\bullet} \arrow[d, red, "r\in\mathcal{R}"] \\
{\color{red}B_i} \arrow[r, red] \arrow[urr] & {\color{red}\coprod B_i} \arrow[r] \arrow[ur, dashed, red] & {\color{red}\bullet}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
{\color{red}B_i} \arrow[r, red] \arrow[urr] & {\color{red}\coprod B_i} \arrow[r, red] \arrow[ur, dashed] & {\color{red}\bullet}
\end{tikzcd}$$

and also

$$\begin{tikzcd}
{\color{red}A_i} \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & {\color{red}\coprod A_i} \arrow[r] \arrow[d, red] & {\color{red}\bullet} \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & {\color{red}\coprod B_i} \arrow[r] \arrow[ur, dashed, red] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{\color{red}A_i} \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & {\color{red}\coprod A_i} \arrow[r, red] \arrow[d] & {\color{red}\bullet} \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}$$

so the conclusion follows from the former comment.

(Closed under retracts.) Let us at least state what a retract of a morphism g should be in the arrow category. Recall that a retract is just

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r] & X \end{tikzcd}$$

So in the arrow category we get

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"] \arrow[rr, bend left] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

$\square$

2 Hatcher's exercise on Whitehead's theorem

Theorem 1 (Whitehead, May). If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 2 (Whitehead, May). An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 3 (Whitehead (4.5), Hatcher). If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

Exercise (Hatcher 4.1.12). Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 2. $\square$

Homework 2

Definition (H-space, Hatcher p. 281). X is an ***H-space***, 'H' standing for Hopf, if there is a continuous multiplication map $\mu : X \times X \to X$ and an identitity element $e \in X$ such that the two maps $X \to X$ given by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity through maps $(X, e) \to (X, e)$.

Exercise (4.1.3). For an H–space (X, x_0) with multiplication $\mu : X \times X \to X$, show that the group operation in $\pi_n(X, x_0)$ can also be defined by the rule $(f + g)(x) = \mu(f(x), g(x))$.

Solution. According to the Eckmann-Hilton argument, we may show that $\pi_n(X, x_0)$ with the usual operation $+$ and the operation $\oplus$ given by $(f \oplus g)(x) = \mu(f(x), g(x))$ coincide if we manage to show that for all $a, b, c, d \in \pi_n(X, x_0)$

$$(a + b) \oplus (c + d) = (a \oplus c) + (b \oplus d).$$

This follows from definitions. Recall that for $f, g \in \pi_n(X, x_0)$,

$$(f + g)(s_1, s_2, ..., s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [1/2, 1] \end{cases}$$

so

$$\begin{aligned}
(a \oplus c) + (b \oplus d) &= \begin{cases} (a \oplus c)(2s_1, s_2, \ldots, s_n) & s_1 \in [0, 1/2] \\ (b \oplus d)(2s_1 - 1, s_2, \ldots, s_n) & s_1 \in [1/2, 1] \end{cases} \\
&= \begin{cases} \mu(a(2s_1, s_2, \ldots, s_n), c(2s_1, s_2, \ldots, s_n)) & s_1 \in [0, 1/2] \\ \mu(b(2s_1 - 1, s_2, \ldots, s_n), d(2s_1 - 1, s_2, \ldots, s_n)) & s_1 \in [1/2, 1] \end{cases} \\
&= \mu(a + b, c + d) \\
&= (a + b) \oplus (c + d)
\end{aligned}$$

□

Exercise (4.1.19). Consider the equivalence relation $\simeq_w$ generated by weak homotopy equivalence: $X \simeq_w Y$ if there are spaces $X = X_1, X_2, \ldots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \simeq_w Y$ iff X and Y have a common CW approximation.

Solution. ($\Longleftarrow$) Suppose Z is a common CW approximation of X and Y, that is, Z is a CW complex and there are weak homotopy equivalences $Z \to X$ and $Z \to Y$. Then the sequence of spaces $X = X_1$, $Z = X_2$ and $Y = X_3$ shows that $X \simeq_w Y$.

($\Longrightarrow$) Suppose Z is a CW approximation of X and let's show it can be made (somehow) into a CW approximation of Y. There is a weak homotopy equivalence $Z \to X$, and also a weak homotopy equivalence either $X = X_1 \to X_2$ or $X = X_1 \leftarrow X_2$. I wonder if this implies that the composition $Z \to X = X_1 \to X_2$ is also a weak homotopy equivalence □

Exercise (4.2.1). Use homotopy groups to show that there is no retraction $\mathbb{R}P^n \to \mathbb{R}P^k$ for $n > k > 0$.

Solution (in progress...) Suppose there is a retraction

$$\begin{tikzcd} \mathbb{R}P^k \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{R}P^n \arrow[r] & \mathbb{R}P^k \end{tikzcd}$$

it induces isomorphisms

$$\begin{tikzcd}
\pi_i(\mathbb{R}P^k) \arrow[r] \arrow[rr, bend right, "\cong"'] & \pi_i(\mathbb{R}P^n) \arrow[r] & \pi_i(\mathbb{R}P^k)
\end{tikzcd}$$

and then just notice that that map is zero. Indeed, recall that the homotopy groups of a covering space are the same as the base (because the homotopy groups of the fibers are trivial), so we have that $\pi_k(\mathbb{R}P^k) \to \pi_k(\mathbb{R}P^n) = 0 \to \pi_k(\mathbb{R}P^k)$ will be zero. □

Exercise (4.2.2)**.** Show the action of $\pi_1(\mathbb{R}P^n)$ on $\pi_n(\mathbb{R}P^n) \cong \mathbb{Z}$ is trivial for n odd and nontrivial for n even.

Exercise (4.2.8)**.** Show that the suspension of an acyclic CW complex is contractible.

Solution. (Warning: there are acyclic spaces with non-trivial homotopy groups.) Let's try to use Hurewicz theorem. Recall that by Freudenthal suspension theorem (coro 4.24) that if X is n-connected, then $\pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k \leqslant 2n$. This makes π_1 of the suspension trivial. □

Exercise (4.2.12)**.** Show that a map $f : X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ to the universal covers induces an isomorphism on homology. [The latter condition can be restated in terms of homology with local coefficientes as saying that $f_* : H_*(X; \mathbb{Z}[\pi_1 X]) \to H_*(Y; \mathbb{Z}[\pi_1 Y])$ is an isomorphism].

Exercise (4.2.13)**.** Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leqslant n$. [Pass to universal covers and use homology.]

Solution. Let X and Y be n-dimensional CW complexes and $f : X \to Y$ such that $f_* : \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i \leqslant n$. Let's try to use Hurewicz theorem, which states that a map between simply connected CW complexes is a homotopy equivalence if it induces isomorphisms on all homology groups.

Consider the universal covers $\tilde{X}$ and $\tilde{Y}$, which are simply connected and also have CW structures. By prop. 4.1, the cover projections induce isomorphisms in the homotopy groups for all $i \geqslant 2$. By StackExchange there is a unique lift $\tilde{f}$ to the universal covers making the diagram on the left commute, and by functoriality the diagram on the right also commutes.

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \tilde{Y} \arrow[d, "q"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi_i(\tilde{X}) \arrow[r, "\tilde{f}_*"] \arrow[d, "p_*"', "\cong"] & \pi_i(\tilde{Y}) \arrow[d, "q_*", "\cong"'] \\
\pi_i(X) \arrow[r, "\cong", "f_*"'] & \pi_i(Y)
\end{tikzcd}
\qquad i \geqslant 2$$

We conclude that $\tilde{f}$ is a weak homotopy equivalence, and by prop. 4.21 it induces isomorphisms on homology groups. Finally, by Hurewicz theorem (coro. 4.33) it is an homotopy equivalence and so is f. $\square$

Exercise (4.2.15). Show that a closed simply connected 3-manifold is homotopy equivalent to S^3.

Solution. Since both S^3 and M are simply connected, by Whitehead's theorem it suffices to construct a map $M \to S^3$ that induces isomorphisms on $\pi_n(X, x_0)$. To construct the map first notice that M is 2-connected. To see that $\pi_2(M) = 0$ we notice that $H^2(M) \cong H_1(M) \cong \pi_1^{ab}(X) \cong 0$ by Poincaré duality. By Universal Coefficient Theorem (?), we see that (the free-torsion part is the same in homology and cohomology, yielding) $H_2(M) = 0$ too. Now we use Hurewicz theorem, which tells us that the first non-zero homotopy group is isomorphic to the first non-zero homology group via the Hurewicz map $h : \pi_3(M) \cong H_3(M)$. Further, since M is simply-connected, it is orientable by prop. 3.25, and by thm 3.26 $H_3(M) \cong \mathbb{Z}$.

The generator of $\pi_3(M)$ is the map we need to apply Whitehead's theorem. Indeed, it is a map $f : S^3 \to M$ such that $h[f] = f_*(\alpha)$ with α a generator of $H_n(D^n, \partial D^n)$, is a generator of $H_3(M)$ by definition of the Hurewicz map. In other words, f_* maps generator to generator and thus is an isomorphism. Since the other homotopy groups are zero, we are done. $\square$

Exercise (4.2.31). For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ this yields isomorphisms

$$\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$$
$$\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$$

Thus $\pi_7(S^4)$ and $\pi_5(S^{15}) \oplus \pi_{n-1}(S^7)$ contain $\mathbb{Z}$ summands.

Solution. Consider the long exact sequence in homotopy,

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow \pi_{i-1}(E) \longrightarrow \cdots .$$

This yields

$$0 \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \pi_{i-1}(F) \longrightarrow 0 .$$

We want tto show that this sequence splits, show we may show that there is a backward arrow on the right part of the diagram.

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & F \arrow[d] \\
D^n \arrow[r, dashed] \arrow[d] & E \arrow[d] \\
D^n/S^{n-1} \cong S^n \arrow[r] & B
\end{tikzcd}$$

□

Exercise (4.2.32). Show that if $S^k \longrightarrow S^m \longrightarrow S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$. [Look at the long exact sequence of homotopy groups.]

Solution (in progress...) From the previous exercise we have

$$\pi_i(S^n) = \pi_i(S^m) \oplus \pi_{i-1}(S^k).$$

Notice that the inclusion of the fiber in the total space *is* homotopic to a constant map because this is a fibration, ie. there are local neighbourhoods in tha base where the preimage looks like $\mathbb{R}^n \times S^n$, implying that $n + k = m$, that is, $k < m$. So $\pi_k(S^m) = 0$.

Now if we take $i = n$, we get that

$$\mathbb{Z} \cong \pi_n(S^m) \oplus \pi_{n-1}(S^k).$$

Now observe that

- If $k = 0$ and $n = m > 1$ then $S^n = S^m$ is simply-connceted and there is no non-trivial covering $S^m \to S^n$.
- $k = 0$ and $m = n = 1$, then there is $S^0 \to S^1 \to S^1$.
- $k > 0$ then $n < m$, so $\pi_n(S^m) = 0$ and then $\mathbb{Z} \cong \pi_{n-1}(S^k)$. This means that $n - 1 \geqslant k$.

Now choose $i = k + 1$. We get that

$$\pi_{k+1}(S^n) \cong \pi_{k+1}(S^m) \oplus \mathbb{Z}.$$

This means that, since $m = n + k$ because fiber bundle, we have $n + k \geqslant 2k + 1 > k + 1$. This implies that $\pi_{k+1}(S^m) = 0$. Finally $\pi_{k+1}(S^n) \cong \mathbb{Z} \implies k + 1 \geqslant n$. □

Exercise (4.2.34). Let $p : S^3 \to S^2$ be the Hopf bundle and let $q : T^3 \to S^3$ be the quotient map collapsing the complement of a ball in the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ to a point. Show that $pq : T^3 \to S^2$ induces the trivial map on π_* and $\widetilde{H}_*$, but is not homotopic to a constant map.

Solution. First let's show that pq induces a trivial map on π_* and $\widetilde{H}_*$. Recall that the product behaves good in homotopy groups, so that $\pi_1(T^3) \cong \mathbb{Z}^3$ and $\pi_i(T^3) \cong 0$ for $i > 1$.

Now, notice that fiber bundles are Hurewicz fibrations (over second countable manifolds). This gives us a lift

$$\begin{tikzcd}
T^3 \arrow[r] \arrow[d] & S^3 \arrow[d] \\
T^3 \times I \arrow[r] \arrow[ur, dashed] & S^2
\end{tikzcd}$$

We get a map $g : T^3 \to S^3$ that factors through the fiber

$$\begin{tikzcd}
g : H_3(T^3) \arrow[rr] \arrow[dr] & & H_3(S^3) \arrow[dl] \\
& H_3(S^1) &
\end{tikzcd}$$

which makes $f_* : H_3(T^3) \to H_3(S^3)$ an isomorphism.

$\square$

Exercise. There is a fiber sequence $U(n) \hookrightarrow U(n+1) \to U(n+1)/U(n) \cong S^{2n+1}$. Use this to show that $\pi_k(U(n)) \to \pi_k(U(n+1))$ is isomorphism for $n > k/2$. Compute $\pi_k(U(n))$ for $n \geqslant 2$ and $k = 1, 2, 3$. In fact, if k is even then $\pi_k(U(N)) = 0$ and if k is odd then $\pi_k(U(N)) = \mathbb{Z}$, where again $N > k/2$. These equalities are known as Bott periodicity.

Solution. The required isomorphisms $\pi_k(U(n)) \to \pi_k(U(n+1))$ follow simply from the fact that S^{2n+1} is $2n+1$-connected: in the long homotopy sequence of the fiber bundle we have

$$\pi_{k+1}(S^{2n+1}) \longrightarrow \pi_k(U(n)) \longrightarrow \pi_k(U(n+1)) \longrightarrow \pi_k(S^{2n+1})$$

so when $2n+1 > k+1 \iff n > k/2$ the homotopy groups of the spheres vanish and we have an isomorphism.

The group $\pi_1(U(n))$ is isomorphic to $\mathbb{Z}$. This follows from the fact that $U(1)$ is homeomorphic to a circle and by induction using the former isomorphism $\pi_1(U(n)) \cong \pi_1(U(n+1))$. We also have $\pi_2(U(1)) = 0$, so that again by induction we get $\pi_2(U(n)) = 0$. Finally, a similar argument shows $\pi_3(U(n)) = 0$. $\square$

cohomology ring of $\mathbb{C}P^n$

Exercise. Show that

$$H^\bullet(\mathbb{C}P^n) \cong \mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where α has degree 2.

Proof. The CW structure of $\mathbb{C}P^n$ consists of one cell for every even dimension. This gives us the following chain complex:

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow \mathbb{Z}, \qquad \text{if } n \text{ is even}$$

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \dots \longrightarrow 0, \qquad \text{if } n \text{ is odd}$$

which yields the cohomology

$$H^i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & i = 0, 2, 4, \dots 2n \\ 0, & i \text{ otherwise} \end{cases}$$

so that

$$H^\bullet(\mathbb{C}P^n) = H^0(\mathbb{C}P^n) \oplus H^2(\mathbb{C}P^n) \oplus \dots \oplus H^{2n}(\mathbb{C}P^n)$$

This means that the underlying group of the cohomology ring is the same as that of

$$\mathbb{Z}[\alpha]/(\alpha^{n+1})$$

where α has degree 2. To show that these groups are also isomorphic as algebras we can use Poincaré duality as follows.

Consider the case $n = 2$, where we may immediately multiply the generator of second cohomology group with itself:

$$\begin{aligned} H^2(\mathbb{C}P^2) \times H^2(\mathbb{C}P^2) &\to H^4(\mathbb{C}P^2) \\ (\alpha, \alpha) \qquad &\mapsto \alpha \smile \alpha = \alpha^2 \end{aligned}$$

By Poincaré duality this map is a nondegenerate symmetric bilinear form, so it must map generator to a generator. The fact that the product of the generator in degree 2 is the generator of degree 4 yields an homomorphism

$$\begin{aligned} \varphi : \mathbb{Z}[\alpha] &\to H^\bullet(\mathbb{C}P^n) \\ \alpha &\mapsto \alpha \in H^2(\mathbb{C}P^n) \end{aligned}$$

with kernel (α^{n+1}) as desired.

Now the case of $\mathbb{C}P^3$ is:

$$\begin{aligned} H^2(\mathbb{C}P^3) \times H^4(\mathbb{C}P^3) &\to H^6(\mathbb{C}P^3) \\ (\alpha, \alpha^2) \qquad &\mapsto \alpha \smile \alpha^2 = \alpha^3 \end{aligned}$$

which also maps generator to generator, producing the desired algebra isomorphism. Notice we have used the group isomorphism $H^4(\mathbb{C}P^3) \approx H^4(\mathbb{C}P^2)$ when denoting the generator of $H^4(\mathbb{C}P^3)$ as α^2. Such an isomorphism is induced by inclusion $\mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n$ via relative cohomology exact sequence.

The case for dimension n follows by induction. $\square$

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