# algebraic topology exercises

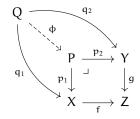
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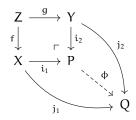
#### some definitions

#### definition.

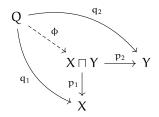
- An *initial object* in a category C is an object  $\varnothing$  such that for any object  $x \in C$  there is a unique morphism  $\varnothing \to x$  with source  $\varnothing$  and target x.
- A *pullback* of the morphisms f and g consists of an object P and two morphisms  $p_1: P \to X$  and  $p_2: P \to Y$  satisfying the following universal property:



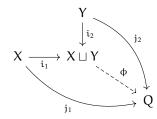
• A *pushout* of the morphisms f and g consists of an object P and two morphisms  $i_1 : P \to X$  and  $i_2 : P \to Y$  satisfying the following universal property:



• A *product* of X and Y is an object  $X \sqcup Y$  and a pair of morphisms  $p_1 : X \sqcap Y \to X$ ,  $p_2 : X \sqcap Y \to Y$  satisfying the following universal property:



• A *coproduct* of X and Y is an object  $X \sqcup Y$  and a pair of morphisms  $i_1 : X \to X \sqcup Y$ ,  $i_2 : Y \to X \sqcup Y$  satisfying the following universal property:



**remark.** More generally, for S any set and  $F: S \to C$  a collection of objects in the category C indexed by S, their *coproduct* is an object  $\coprod_{s \in S} F(s)$  equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from F(s).

• A morphism i has the *left lifting property with respect to a morphism* p and p has the *right lifting property with respect to* i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists φ (I think not necessarily unique) completing the diagram:

$$\begin{array}{ccc}
A & \xrightarrow{f} & X \\
\downarrow & & \downarrow & \downarrow p \\
B & \xrightarrow{g} & Y
\end{array}$$

- For *C* any category, its *arrow category* Arr(*C*) is the category such that
  - an object a of Arr(C) is a morphism  $a : a_0 \to a_1$  of C,

– a morphism  $f : a \rightarrow b$  of Arr(C) is a commutative square

$$\begin{array}{ccc}
a_0 & \xrightarrow{f_0} & b_0 \\
a\downarrow & & \downarrow b \\
a_1 & \xrightarrow{f_1} & b_1
\end{array}$$

in  $\mathcal{C}$ ,

 composition in Arr(*C*) is given simply by placing commutative squares side by side to get a commutative oblong.

This is osomorphic to the functor category

$$Arr(C) := Funct(I, C) = [I, C] = C^{I}$$

for I the interval category  $\{0 \rightarrow 1\}$ .

# exercise on mapping cylinder and Hurewicz cofibrations

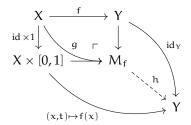
exercise. Let  $f: X \to Y$  be a map. Let  $M_f = X \times [0,1] \cup_f Y$  be the *mapping cylinder of* f, i.e. the pushout of  $X \stackrel{\cong}{\to} X \times \{0\} \hookrightarrow X \times [0,1]$  and of  $f: X \times Y$ . Let  $g: X \to M_f$  be the map  $X \stackrel{\cong}{\to} X \times \{1\} \to M_f$ . Let  $h: M_f \to Y$  be the map that is induced by  $X \times [0,1] \to Y$ :  $(x,t) \mapsto f(x)$  and  $id_Y: Y \to Y$ . Observe that f is the composition of g and h.

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e.  $(\operatorname{colim} A_i) \times B$  is canonically homeomorphic with  $\operatorname{colim}(A_i \times B)$ .

- 1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
- 2. Show that  $g: X \to M_f$  is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

1. We have that



We must show that there is a homotopy between the identity map on  $M_f$  and a retraction from  $M_f$  to Y. So we want  $h: M_f \times [0,1] \to M_f$  such that

$$h(-,0)=id_{\mathbf{M}_f},\quad img\,h(-,1)\subset Y\quad and\quad h(-,1)|_Y=id_Y$$

The fact that h is a deformation retract is consequence of this diagram. But I still can't see why it must be a homotopy equivalence...

2. Consider the following lifting problem:

$$\begin{matrix} X & \xrightarrow{H} & Z^I \\ g \downarrow & & \downarrow \pi_0 \\ M_f & \xrightarrow{h} & Z \end{matrix}$$

Looks OK but why should the dashed arrow exist...?

# path spaces and fibrations

exercise.

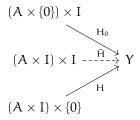
a. Show that Map(I,Y) deformation retracts on Map(pt,Y). Most likely you'll have to find a correct map  $I \times I \to I$ . Also show that  $Map(I,Y) \to Map(pt,Y)$  is a Hurewicz fibration. The key map will be of the form  $I \times I \to I \times I$ .

Solution.

a. (Map(I,Y)  $\rightarrow$  Map(pt,Y) **is a Hurewicz fibration.)** Let A be any space. We must show that for any homotopy H and lift  $h_0$  there exists an homotopy  $\tilde{H}$  as in the following diagram:

$$\begin{array}{c} A \times \{0\} \xrightarrow{h_0} Map(I,Y) \\ \downarrow \qquad \qquad \downarrow^{\tilde{H}} \qquad \downarrow^{p} \\ A \times I \xrightarrow{H} Map(0,Y) \end{array}$$

From the isomorphism  $Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$  we may rewrite the problem as



So we define the dashed arrow by

$$(\mathfrak{a},s,t) \mapsto \begin{cases} H_0(\mathfrak{a},0,s-t) & \text{ when } s-t \geqslant 0 \\ H(\mathfrak{a},t-s,0) & \text{ when } s-t \leqslant 0 \end{cases}$$

so that when s=t the functions coincide, when s=0 we get H and when t=0 we get  $H_0$ .

(Map(I,Y) **deformation retracts on** Map(pt,Y)**.)** We must show there is a homotopy

$$h: Map(I, Y) \times I \longrightarrow Map(I, Y)$$

such that

$$h(-,0) = id_{Map(I,Y)}, \qquad h(-,1) \subset Map(pt,Y)$$

and 
$$\mathsf{h}(-,1)|_{Map(pt,Y)} = id_{Map(pt,Y)}\,.$$

Consider the map

$$\begin{split} I \times I &\to I \\ (s,t) &\mapsto s - st \end{split}$$

Our deformation retract may be written like

$$h: Map(I,Y) \times I \longrightarrow Map(I,Y)$$
  
 $(f(s),t) \longmapsto f(s-st)$ 

Then for t=0 we have the identity on Map(I,Y), and when t=1 we have  $ev_0$ .

### exercise on model categories

exercise (3.1.8 from Riehl). Verify that the class of morphisms  $\mathcal L$  characterized by the left lifting property against a fixed class of morphisms  $\mathcal R$  is closed under coproducts, closed under retracts, and contains the isomorphisms.

*Solution.* (*Coproducts.*) Comment from Sergey: Coproduct of morphisms  $A_i \to B_i$  in a category  $\mathcal{C}$  is the obvious morphism  $\sqcup A_i \to \sqcup B_i$ . (Because in this construction morphisms  $A_i \to B_i$  are seen as objects of what's called the arrow category of the category  $\mathcal{C}$ )

Suppose the maps  $\ell_i: A_i \to B_i$  are in  $\mathcal{L}$ . Then their coproduct in the arrow category is the obvious map  $\coprod A_i \to \coprod B_i$ .

Explicitly, their coproduct is an arrow  $\coprod \ell_i$  and a collection of maps  $f_i : \ell_i \to \coprod \ell_i$  such that for any other object  $m : A \to B$  in the arrow category and a map  $g : \ell \to m$ , the following diagram is completed uniquely:

$$\ell_i \xrightarrow{f_i} \coprod \ell_i \xrightarrow{\exists !} m \quad \forall i$$

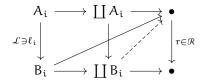
So we conclude that the source of  $\coprod \ell_i$  is  $\coprod A_i$  and its target  $\coprod B_i$ . Indeed, we really looking at

$$\begin{array}{ccc} A_i & \stackrel{\ell_i}{\longrightarrow} & B_i \\ f_i^1 & & \downarrow f_i^2 \\ \coprod A_i & \stackrel{\coprod \ell_i}{\longrightarrow} & \coprod B_i \\ \exists ! & & \downarrow \exists ! \\ A & \stackrel{m}{\longrightarrow} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism  $r \in \mathcal{R}$ :

Since  $\ell_i \in \mathcal{L}$ , we have maps

which in turn means we have a unique map



by the universal property of the coproduct  $\prod B_i$ .

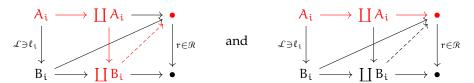
To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts  $\coprod A_i$  and  $\coprod B_i$  since, in general,

$$\text{Hom}\left(\coprod X_{i},Y\right)\cong\prod\text{Hom}(X_{i},Y).$$

More explicitly, we now that the paths in red in the following diagrams are the same:

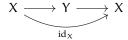


and also

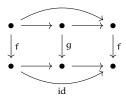


so the conclusion follows from the former comment.

**(Closed under retracts.)** Let us at least state what a retract of a morphism g should be in the arrow category. Recall that a retract is just



So in the arrow category we get



#### Hatcher's exercise on Whitehead's theorem

**Theorem 1** (Whitehead, May). If X is a CW complex and  $e: Y \to Z$  is an n-equivalence, then  $e_*: [X, Y] \to [X, Z]$  is a bijection if dim X < n and surjection if dim X = n.

**Theorem 2** (Whitehead, May). An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

**Theorem 3** (Whitehead (4.5), Hatcher). If a map  $f: X \to Y$  between connected CW complexes induces isomorphisms  $f_*: \pi_n(X) \to \pi_n(Y)$  for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex  $X \hookrightarrow Y$ , the conclusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12). Show that an n-connected, n-dimensional CW complex is contractible.

*Solution.* Just recall that n-connectedness means that  $\pi_i(X) = 0$  for all  $i \leq n$ , which means that X is contractible by theorem 2.

# References

- [1] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000 (cit. on p. 9).
- [2] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832 (cit. on p. 9).
- [3] Emily Riehl. *Homotopical categories: from model categories to*  $(\infty,1)$ *-categories.* 2020. arXiv: 1904.00886 [math.AT] (cit. on p. 7).