homotopy theory

Contents

abstract nonsense

definition.

- **(Limits, wiki.)**
  - A ***diagram*** of shape J in C is a functor from J to C
    $$F : J \to C.$$
    The category J is thought of as an index category, and the diagram F is thought of as indexing a collection of obtects and morphisms in C patterned on J.
  - Let $F : J \to C$ be a diagram of chape J in a category C. A ***cone*** to F is an object N to C together with a family $\psi_X : N \to F(X)$ of morphisms indexed by the objects X of J (so a cone is an object and a bunch of maps from this object to certain objectes that are governed by the diagram), so that for every morphism $X \to Y$ in J, we have $F(f) \circ \psi_X = \psi_Y$ I guess this is what nLab meant when he said that everything in sight commutes).
  - A ***limit*** of the diagram $F : J \to C$ is a cone (L, ϕ) to F such that for every cone (N, ψ) there exists a *unique* morphism $u : N \to L$ such that $\phi_X \circ u = \psi_X$ for all X in J.
    $$\begin{tikzcd}
     & N \arrow[d, dashed, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\
     & L \arrow[dl, "\phi_X"'] \arrow[dr, "\psi_Y"] & \\
    F(X) \arrow[rr, "F(f)"'] & & F(Y)
    \end{tikzcd}$$
    One says that the cone (N, ψ) factors through the cone (L, ϕ) with the unique factorization u. The morphism u is sometimes called the ***mediating morphism***.

    Limits are also referred to as ***universal cones*** since they are characterized by a universal property. Limits may also be caracterized as terminal objects in the category of cones to F.

    It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of F.
- **(Colimits, wiki)** The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:
  - A ***co-cone*** of a diagram $F : J \to C$ is an object N of C together with a family of morphisms $\psi_X : F(X) \to N$ (so in the cone we are going *from* N and now we're going *to*N) for every object X of J, such that for every morphism $f : X \to Y$ in J we have $\psi_Y \circ F(f) = \psi_X$ everything in sight commutes.

- A ***colimit*** of a diagram $F : J \to C$ is a co-cone (L, ϕ) of F such that for any other co-cone (N, ψ) of F there exists a unique morphism $u : L \to N$ such that $u \circ \phi_X = \psi_X$ for all X in J.

$$\begin{tikzcd} F(X) \arrow[rr, "F(f)"] \arrow[dr, "\phi_X"'] \arrow[ddr, bend right, "\psi_X"'] & & F(Y) \arrow[dl, "\phi_Y"] \arrow[ddl, bend left, "\psi_Y"] \\ & L \arrow[d, dashed, "u"] & \\ & N & \end{tikzcd}$$

Colimits are also referred to as ***unersal co-cones***. They can be characterized as initial objects in the category of co-cones from F.

As with limits, if a diagram F has a colimit then this colimit is unique up to a unique isomorphism.

- An ***initial object*** in a category $\mathcal{C}$ is an object $\varnothing$ such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source $\varnothing$ and target x.
- For $\mathcal{C}$ any category, its ***arrow category*** $\mathrm{Arr}(\mathcal{C})$ is the category such that
  - an object a of $\mathrm{Arr}(C)$ is a morphism $a : a_0 \to a_1$ of $\mathcal{C}$,
  - a morphism $f : a \to b$ of $\mathrm{Arr}(\mathcal{C})$ is a commutative square

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\ a_1 \arrow[r, "f_1"'] & b_1 \end{tikzcd}$$

  in $\mathcal{C}$,
  - composition in $\mathrm{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

  This is isomorphic to the functor category

$$\mathrm{Arr}(\mathcal{C}) := \mathrm{Funct}(I, C) = [I, C] = C^I$$

  for I the intervale category $\{0 \to 1\}$.
- An ***equalizer*** is a limit

$$eq \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

  over a parallel pair of morphisms f and g. This means that for $f : X \to Y$ and $g : X \to Y$ in a category $\mathcal{C}$, their equalizer, if it exists, is
  - an object $eq(f, g) \in \mathcal{C}$,
  - a morphism $eq(f, g) \to X$

– such that

  * pulled back to $\mathrm{eq}(f,g)$ both morphisms become equal:

$$\mathrm{eq}(f,g) \longrightarrow X \xrightarrow{f} Y \quad = \quad [\ \mathrm{eq}(f,g) \longrightarrow X \xrightarrow{g} Y$$

  * and $\mathrm{eq}(f,g)$ is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions f and g between sets X and Y one forms the set $Y/\sim$ of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the the quotient function $p : Y \to Y/\sim$ satisfies

$$p \circ f = p \circ g.$$

In some category $\mathcal{C}$, the ***coequalizer*** $\mathrm{coeq}(f,g)$ of two parallel morphisms f and g between two objects X and Y, if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{tikzcd} X \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr] & & Y \arrow[dl] \\ & \mathrm{coeq}(f,g) & \end{tikzcd}$$

Equivalently, in a category $\mathcal{C}$ a diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] & Z \end{tikzcd}$$

is called a ***coequalizer*** diagram if

1. $p \circ f = p \circ g$,
2. p is universal for this property: if $q : Y \to W$ is a morphism of $\mathcal{C}$ such that $q \circ f = q \circ g$, then there is a unique morphism $\phi : Z \to W$ such that $\phi \circ p = q$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "p"] \arrow[d, "q"] & Z \arrow[dl, dashed, "\phi"] \\ & W & \end{tikzcd}$$

The coequalizer in $\mathcal{C}$ is equivalently an equializer in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A ***pullback*** of the morphisms f and g consists of an object P and two morphisms

$p_1 : P \to X$ and $p_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

- A ***pushout*** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

remark. Other names for the pushout are ***cofibered product of*** X ***and*** Y (especially in algebraic categories when i_1 and i_2 are monomorphisms), or ***free product of*** X ***and*** Y with Z ***amalgamated sum***, or more simply an ***amalgamation*** or ***amalgam of*** X ***and*** Y.

remark. If coproducts exist in some category, then the pushout

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \\
X \arrow[r, "i_1"'] & X \coprod_Z Y \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is equivalently the coequalizer

$$X \overset{i_1 \circ f}{\underset{i_2 \circ g}{\rightrightarrows}} X \coprod Y \longrightarrow X \coprod_Z Y$$

of the two morphisms induced by f and g into the coproduct of X with Y.

example (wiki).

- If X, Y and Z are sets and f, g are functions, the pushout of f and g is the disjoint union of X and Y where elements sharing a common preimage in Z are identified, i.e. $P = (X \coprod Y) / \sim$ where $\sim$ is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

In particular, if X and Y are subsets of some larger set W and Z is their intersection, with f and g the inclusion maps of Z into X and Y, then the pusout can be canonically identified with the union $X \cup Y \subseteq W$.

- The construcion of ***adjunction spaces*** is an example of pushouts in Top. More precisely, if Z is a subspace of Y and $g : Z \to Y$ is the inclusion map, we can glue Y to another space X along Z using an ***attaching map*** $f : Z \to X$. The result is the ***adjunction space*** $X \cup_f Y$ which is just the pushout of f and g. More generally, all identification spaces may be regarded as pushouts in this way. See **??** .

- A ***product*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \sqcap Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}$$

- A ***coproduct*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$, $i_2 : Y \to X \sqcup Y$ satisfying the following universal property:

$$\begin{tikzcd}
& Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & X \sqcup Y \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

remark. More generally, for S any set and $F : S \to \mathcal{C}$ a collection of objects in $\mathcal{C}$ indexed by S, their ***coproduct*** is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

- The ***kernel*** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the ***kernel*** $\ker f$ of a morphism $f : A \to B$ is the pullback $\ker(f) \to A$ along f of the unique morphism $0 \to B$

More explicitly, this characterizes the object $\ker(f)$ as *the* object (unique up to isomorphism) that satisfies the following universal property:

> for every object C and every morphism $h : C \to A$ such that $f \circ h = 0$ is the zero morphism, there is a unique morphism $\phi : C \to \ker(f)$ such that $h = p \circ \phi$.

$$
\begin{tikzcd}
C \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "h"'] & & \\
& \ker(f) \arrow[r] \arrow[d] & 0 \arrow[d] \\
& A \arrow[r, "f"'] \arrow[ur, phantom, "\ulcorner" very near end] & B
\end{tikzcd}
$$

- In a category with a terminal object 1, the ***cokernel*** of a morphism $f : A \to B$ is the pushout (arrows h and ϕ apply if terminal object is zero)

$$
\mathrm{coker}(f) := 1 \sqcup_A B
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & B \arrow[d] \arrow[ddr, bend left, "h"] & \\
1 \arrow[r] & \mathrm{coker}(f) \arrow[dr, dashed, "\phi"] & \\
& & C
\end{tikzcd}
$$

  In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object $\mathrm{coker}(f)$ with the following universal property:

  > for every object C and every morphism $h : B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\phi : \mathrm{coker}(f) \to C$ such that $h = \phi \circ i$.

- A morphism $f : X \to Y$ is a ***monomorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Z \to X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

$$
\begin{tikzcd}
Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] \arrow[rr, bend left, "f \circ g_1"] \arrow[rr, bend right, "f \circ g_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}
$$

  Equivalently, f is a monomorphism if for every Z the hom-functor $\mathrm{Hom}(Z, -)$ takes it to an injective function

$$\mathrm{Hom}(Z, X) \xhookrightarrow{f_*} \mathrm{Hom}(Z, Y).$$

  Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- A morphism $f : X \to Y$ is a ***epimorphism*** if for every object Z and every pair of morphisms $g_1, g_2 : Y \to Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g_1 \circ f"] \arrow[rr, bend right, "g_2 \circ f"'] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

Equivalently, f is a epimorphism if for every Z the hom-functor $\mathrm{Hom}(-, Z)$ takes it to an injective function

$$\mathrm{Hom}(Y, Z) \xhookrightarrow{f^*} \mathrm{Hom}(X, Z).$$

Being a monomorphism in a category $\mathcal{C}$ means equivalently that it is an monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.

- (Retraction.)

  - (wiki) Let X be a topological space and A a subspace of X. Then a continuous map $r : X \to A$ is a ***retraction*** if the restriction of r to A is the identity map on A.

  - (nLab) An object A in a category is called a ***retract*** of an object B if there are morphisms $i : A \to B$ and $r : B \to A$ such that $r \circ i = \mathrm{id}_A$. In this case r is called a ***retraction of*** B ***onto*** A and i is called a ***section of*** r.

  $$\mathrm{id} : A \xrightarrow[\text{section}]{i} B \xrightarrow[\text{retraction}]{r} A$$

  Hence a ***retraction*** of a morphism $i : A \to B$ is a left-inverse and a ***section*** of a morphism $r : B \to A$ is a right-inverse.

- (Deformation retract.)

  - (nLab) Let $\mathcal{C}$ be a category equipped with a notion of homotopy between its morphisms. Then a ***deformation retraction*** of a morphism $i : A \to X$ is another morphism $r : X \to A$ such that

  ?

  - (wiki) A continuous map $F : X \times [0,1] \to X$ is a ***deformation retraction*** of a space X into a subspace A if, for every x in X and a in A,

  $$F(x, 0) = x, \qquad F(x, 1) \in A \quad \text{and} \quad F(a, 1) = a.$$

  In words, a deformation retraction is a homotopy between a retraction and the identity map on X. The subspace A is called a ***deformation retract*** of X. A deformation retraction is a special case of a homotopy equivalence.

An equivalent definition of deformation retraction is the following. A continuous map $r : X \to A$ is a ***deformation retraction*** if it is a retraction and its compositition with the inclusion is homotopic to the identity map on X.In this formulation, a deformation retraction carries with it a homotopy between the identity map on X and itself.

– (wiki) If, in the definition of a deformation retraction we add the requirement that

$$F(a, t) = a \qquad \forall t \in [0, 1], \ \forall a \in A,$$

then F is called a ***strong deformation retraction***. In words, a strong deformation retraction leaves points in A fixed throughout the homotopy.

example. S^n is a strong deformation retract of $\mathbb{R}^{n+1}\backslash 0$ through $F(x, t) = (1 - t)x + t\frac{x}{\|x\|}$.

– (wiki) The inclusion of a closed subspace A in the space X is a **??** if and only if A is a ***neighbourhood deformation retract*** of X, meaning that there is a continuous map $u : X \to [0, 1]$ with $A = u^{-1}(0)$ and a homotopy $H : X \times [0, 1] \to X$ such that $H(x, 0) = x$ for all $x \in X$, $H(a, t) = a$ for all $a \in A$ and $t \in [0, 1]$, and $H(x, 1) \in A$ if $u(x) < 1$.

For example, the inclusion of a subcomplex in a CW complex is a cofibration.

elementary concepts

definition.

- Let X and Y be topological spaces and $f, g : X \to Y$ continuous maps. An ***homotopy*** from f to g is a continuous map

$$H : X \times [0, 1] \to Y, \qquad (x, t) \mapsto H(x, t) = H_t(x)$$

) such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation $\simeq$ is an equivalence relation on the set of continuous maps $X \to Y$. A homotopy of maps $H_t : X \to Y$ is called ***relative to*** $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top, the ***homotopy category*** h-Top, where comoposition of homotopy classes is induced by composition of representing maps. If $f : X \to Y$ represents an isomorphism in h-Top, then f is called a ***homotopy equivalence*** or h-***equivalence***. In explicit termins this means $f : X \to Y$ is a homotopy equivalence if there exists $g : Y \to X$, a ***homotopy inverse of*** f, such that gf and fg are both homotopic to the identity. Spaces X and Y are called ***homotopy equivalent*** or of the same ***homotopy type*** if there exists a homotopy equivalence $X \to Y$. A space is ***contractible*** if it is homotopy equivalent to a point. A map $f : X \to Y$ is ***null homotopic*** if it is homotopic to a constant map.

- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the n-***th homotopy group*** are homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

  Also,
  $$\pi_n(X, *) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n/\partial I^n, X]^0$$
  where $[X, Y]$ denotes the set of homotopy classes $[f]$ of maps $[f] : X \to Y$.

  proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

- Let A be a subspace of X and $x_0 \in A$. The elements of the ***relative homotopy group*** $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n. That is,
  $$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

  The elements of such a group are homotopy classes of based maps $D^n \to X$ which carry the boundary S^{n-1} into A. Two maps f, g are called ***homotopic relative to*** A if they are homotopic by a basepoint-preserving homotopy $F : D_n \times [0, 1] \to X$ such that, for each p in S^{n-1} and t in $[0, 1]$, the element $F(p, t)$ is in A. Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

  remark. This construction is motivated by looking for the kernel of the induced map $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair (X, A, x) we have a long exact sequence
  $$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \cdots \longrightarrow \pi_0(X, x_0)$$
  where i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$. The map, called the ***boundary map***, is a homomorphism when $n > 1$.

- A space X with basepoint x_0 is called n-***connected*** if $\pi_i(X, x_0) = 0$ for $i \leqslant n$. Thus 0-connected means path-connected and 1 connected means simply-connected.

- A pair (X, A) is n-***connected*** if $\pi(X, A, x_0) = 0$ for $i \leqslant n$.

- Two pointed spaces (X, x_0) and (Y, y_0) are n-***equivalent*** if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i < n$ and surjective for $i = n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on ***compactly generated wakly Hausdorff*** spaces.

- X is ***compactly generated*** if, for any subset $C \subset X$, and for all continuous maps $f : K \to X$ from compact Housdorff spaces,

$$\text{if } f^{-1}(C) \text{is closed in } K, \text{ then } C \text{ is closed.}$$

claim (What I picked up from the lecture). If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is k-closed.

From May: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi : X \to Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH, $\text{Hom}(X, Y)$ is a space with the compact-open topology. This is a compactly generated space, $\mathbf{k}(\text{Hom}(X, Y))$.

remark. (Also see wiki on currying)

$$\begin{aligned} \text{Map}(X, Y) &:= \text{ the space of maps } X \to Y. \\ \text{Map}(X \times Y, Z) &\cong \text{Map}(X, \text{Map}(Y, Z)) \\ \text{Hom}(X \times Y, Z) &\cong \text{Hom}(X, \text{Map}(Y, Z)) \end{aligned}$$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to $\text{Map}(Y, -)$.

cofibrations

definition.

- (wiki) In mathematics, in particular in homotopy theory, a continuous map between topological spaces $i : A \to X$ is a ***cofibration*** if it has the ***homotopy extension property*** with respect to all topological spaces S.

  That is, i is a cofibration if

- for each topological space S,
- and for any continuous maps $f, f' : A \to S$
- and $g : X \to S$ with $g \circ i = f$,
- for any homotopy $h : A \times I \to S$ from f to f',

there is a continuous map $g' : X \to S$ and a homotopy $h' : X \times I \to S$ from g to g' such that
$$h'(i(a), t) = h(a, t) \qquad \text{for all } a \in A \text{ and } t \in I.$$

- (wiki) In what follows, let $I = [0, 1]$ denote the unit interval.

  A map $i : A \to X$ is a ***cofibration*** if for any map $f : A \to S$ such that there is an extension to X, meaning there is a map $\tilde{f} : X \to S$ such that $\tilde{f} \circ i = f$, we can extend a homotopy of maps $H : A \times I \to S$ to a homotopy of maps $\tilde{H} : X \times I \to S$ where

$$H(a, 0) = f(a)$$
$$\tilde{H}(x, 0) = \tilde{f}(x)$$

$$\begin{tikzcd}
A \arrow[r, "H"] \arrow[d, "i"'] \arrow[dr, bend right=90, "f"'] & S^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}"'] \arrow[ur, dashed, "\tilde{H}"] & S
\end{tikzcd}$$

- (wiki) Let X be a topological space and let $A \subset X$. We say that the pair (X, A) has the ***homotopy extension property*** if, given a homotopy $f_\bullet : A \to Y^I$ and a map $\tilde{f}_0 : X \to Y$ such that
$$\tilde{f}_0 \circ \iota = f_0$$
(so $\tilde{f}$ is the lift of $f_0 : A \to Y$) then there exists an ***extension*** of $f_\bullet$ to a homotopy $\tilde{f}_\bullet : X \to Y^I$ such that $\tilde{f}_\bullet \circ \iota = f_\bullet$.

  That is,

$$\begin{tikzcd}
A \arrow[r, "f_\bullet"] \arrow[d, "\iota"'] & Y^I \arrow[d, "\pi_0"] \\
X \arrow[r, "\tilde{f}_0"'] \arrow[ur, dashed, "\tilde{f}_\bullet"] & Y
\end{tikzcd}$$

So there's some currying to make usual homotopies $f_\bullet : A \times I \to Y$ look like $f_\bullet : A \to Y^I$. Or, as said in our lectures, "a homotopy $X \times I \to Y$ is the same as a map $X \to \mathrm{Map}(I, Y)$".

- (May) A map $i : A \to X$ is a ***cofibration*** if it satisfies the ***homotopy extension property (HEP)***. This means that if $h \circ i_0 = f \circ i$ in the diagram

$$\begin{tikzcd} A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl, "h"'] \arrow[dd, "i \times id"] \\ & Y & \\ X \arrow[ur, "f"] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "\tilde{h}"] \end{tikzcd}$$

  then there exists $\tilde{h}$ that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map $i : A \to X$ between topological spaces is a ***Hurewicz cofibration*** if it satisfies the homotopy extension property.

  Let's say it one more time: for any $g : X \to Y$ and any homotopy $H : A \times I \to Y$ such that

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & A \times I \arrow[d] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

  there is $H' : X \times I \to Y$,

$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d, "g"] & A \times I \arrow[d] \\ X \times I \arrow[r, "H'"'] & Y \end{tikzcd}$$

  such that

$$\begin{tikzcd} A \times I \arrow[d] \arrow[dr, "H"] & \\ X \times I \arrow[r, "H'"'] & Y \end{tikzcd}$$

example. $\partial D^n \to D$ is a Huerwicz cofibration. Why?

exercise. Prove that an inclusion $f : A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

definition (Mapping cylinder)**.**

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. make the dashed map unique—up to something maybe). Namely, we can let Y in our original test diagram be the ***mapping cylinder***

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of i and i_0. Indeed, suppose that we can construct a map r that makes the following diagram commute

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd, "i"'] & & A \times I \arrow[dl] \arrow[dd, "i \times id"] \\
& Mi & \\
X \arrow[ur] \arrow[rr, "i_0"'] & & X \times I \arrow[ul, dashed, "r"]
\end{tikzcd}$$

By the universal property of the pushouts, given maps f and h in our original test diagram induce a map $Mi \to Y$, and its comoposite with r gives a homotopy $\tilde{h}$ that makes the diagram commute. So just saying that Mi is universal.

- (nLab) Given a continuous map $f : X \to Y$ of topological spaces, one can define its ***mapping cylinder*** as a pushout

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X \times I \arrow[r, "(\sigma_0)_*(f)"'] & Cyl(f)
\end{tikzcd}$$

in Top, where $I = [0, 1]$ and $\sigma : X \to X \times I$ is given by $x \mapsto (x, 0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$\left(X \times I \coprod Y\right) \Big/ \sim$$

where $\sim$ is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

- (wiki) The ***mapping cylinder*** of a function f between topological spaces X and Y is the quotient

$$M_f = (([0, 1] \times X) \amalg Y)/ \sim$$

where $\amalg$ denotes disjoint union, and $\sim$ is the equivalence relation generated by

$$(0, x) \sim f(x) \text{ for each } x \in X.$$

That is, the mapping cylinder M_f is obtained by gluing one end of $X \times [0, 1]$ to Y via the map f. Notice that the "top" of the cylinder $\{1\} \times X$ is homeomorphic to X, while the "bottom" is the space $f(X) \subset Y$.

(Dani) So the mapping cylinder is just deforming X to Y putting X inside Y via f.

- (Homework) Let $f : X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f : X \times Y$.

exercise. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\
X \times [0,1] \arrow[r] \arrow[drr, bend right, "(x,t)\mapsto f(x)"'] & M_f \arrow[dr, dashed, "h"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & Y
\end{tikzcd}$$

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\mathrm{colim}\, A_i) \times B$ is canonically homeomorphic with $\mathrm{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

exercise. $X \to M_f \to Y$. Prove $X \to M_f$ is a cofibration.

fibrations

- (nLab) A morphism i has the ***left lifting property with respect to a morphism*** p and p has the ***right lifting property with respect to*** i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\phi"] & Y
\end{tikzcd}$$

- (nLab) Let C be a category with products and with interval object I. A morphism $E \to B$ has the ***homotopy lifting property*** if it has the right lifting property with respect to all morphisms of the form $(\mathrm{id}, 0) : Y \to Y \times I$.

This means that for all commuting squares

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\sigma"] & B \end{tikzcd}$$

there exists a morphism $\sigma : Y \times I \to E$ such that both triangles in the former diagram commute.

A ***fibration*** (also called ***Hurewicz fibration***) is a mapping $p : E \to B$ satisfying the homotopy lifting property for all spaces X.

- (Hatcher) A map $p : E \to B$ is said to have the ***homotopy lifting property*** with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, so $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

  The ***lift extension property for a pair*** (Z, A) asserts that every map $X \to B$ has a lift $Z \to E$ extending a given lift defined on the subspace $A \subset Z$. The case $(Z, A) = (X \times I, X \times \{0\})$ is the homotopy lifting property.

  A ***fibration*** is a map $p : E \to B$ having the homotopy property with respect to all spaces X.

Theorem 3 (4.41 Hatcher, Long exact sequence of Serre fibrations, see proposition 18)**.** Suppose $p : E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geqslant 0$. Choose basepoints $b_0\, nB$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_* : \pi_n(E, F, x_0) \to \pi_0(B, b_0)$ is an isomorphism for all $n \geqslant 1$. Hence b is path-connected and there is a long exact sequence

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \overset{p_*}{\to} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

The map $p : E \to B$ is said to have the ***homotopy lifting property for a pair*** (X, A) if each homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$. In other words, the homotopy lifting property for (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

The point is that the homotopy lifting property for disjs is equivalent to the homotopy lifting property for all CW pairs (X, A). A map $p : E \to B$ satisfying the homotopy lifting property for disks is sometimes called a ***Serre fibration***.

A ***fiber bundle*** structure on a space E, with fiber F, consists of a projection map $p : E \to B$ such that each point B has a neighbourhood U for which there is a homeomorphism $h : p^{-1}(U) \to U \times F$ making the following diagram commute

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

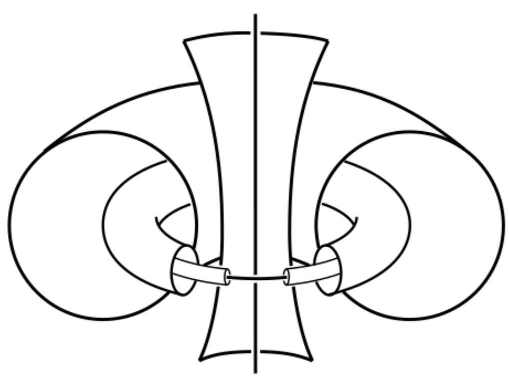
example. Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces $S^n \to \mathbb{R}P^n$ with fiber S^0. Over the complex numbers the analog of this is a fiber bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. Here S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}P^n$ is viewed as the quotient space of S^{2n+1} under the equivalence relation $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$ for $\lambda \in S^1$. The projection $p : S^{2n+1} \to \mathbb{C}P^n$ sends $(z_0, \dots, z_n)$ to its equivalence class $[z_0, \dots, z_n]$.

To see that the local triviality condition for fibre bundles is satisfied, ...

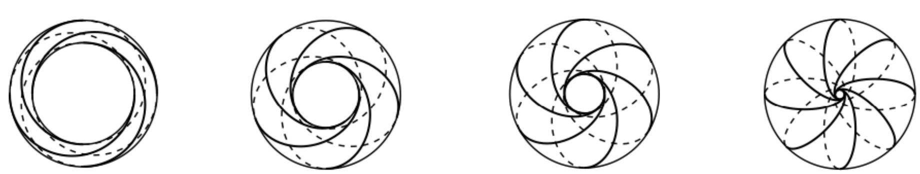
The constructino of the bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$ also works when $n = \infty$, so there is a fiber bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$.

The case $n = 1$ is particularly interesting since $\mathbb{C}P^1 = S^2$ and bundle becomes $S^1 \to S^3 \to S^2$ with fiber, total space, and base all speres. This is known as the ***Hopf bundle***. The projection $S^3 \to S^2$ can be taken to be $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$. (That is, seeing S^2 as the one-point compactification of $\mathbb{C}$.)

In polar coordinates we may see S^3 as the union of several tori. Stereorgraphic projection yields the following figure:



The limiting cases T_0 and T_∞ correspond to the unit circle in the xy-plane and the z-axis under the stereographic projection. Each torus T_ρ is aunion of circle fibers. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. As ρ goes to 0 or ∞ the fiber circles approach the circles T_0 and T_∞, which are also fibers. The figure below shows four tori decomposed into fibers:



How could we visualize the projection onto S^2? Could it work to think $S^2 = \mathbb{C} \cup \infty$ and just do stereographic projection from 3-space to the plane disregarding one point? What would that even mean hehe

Replacing the field $\mathbb{C}$ by the quaternions $\mathbb{H}$, the same constructions yield fiber bundles $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ over quaternionic projective spaces $\mathbb{H}P^n$. Here the fiber S^3 is the unit quaternions, and S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1}$. Taking $n = 1$ gives a second Hopf bundle $S^3 \to S^7 \to S^4 = \mathbb{H}P^1$.

Another Hopf bundle $S^7 \to S^{15} \to S^8 \dots$

model structures

definition (Riehl). A ***weak factorization system*** $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{M}$ is comprised o two clases of morphisms $\mathcal{L}$ and $\mathcal{R}$ so that

1. Every morphism in $\mathcal{M}$ may be factored as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$:

$$\begin{tikzcd} \bullet \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni\ell"'] & & \bullet \\ & \bullet \arrow[ur, "r\in\mathcal{R}"'] & \end{tikzcd}$$

2. The maps in $\mathcal{L}$ have the ***left lifting property*** with respect to each map in $\mathcal{R}$ and equivalently the maps in $\mathcal{R}$ have the ***right lifting property*** with respect to each map in $\mathcal{L}$, that is, any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\mathcal{L}\ni\ell"'] & \bullet \arrow[d, "r\in\mathcal{R}"] \\ \bullet \arrow[r] \arrow[ur, dashed] & \bullet \end{tikzcd}$$

   admits a diagonal filler as indicated making both triangles commute. When this lift is unique, we say the factorization system is ***orthogonal***.

3. The classes $\mathcal{L}$ and $\mathcal{R}$ are each closed under retracts in the arrow category: given a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "t"'] \arrow[rr, bend left, equal] & \bullet \arrow[r] \arrow[d, "s"] & \bullet \arrow[d, "t"] \\ \bullet \arrow[r] \arrow[rr, bend right, equal] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

   if s is in that class then so is its retract t.

exercise (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

definition. Given a contravariant functor $\mathcal{F} : \mathcal{C}^{op} \to \text{Sets}$ there is a corresponding category (***of elements of*** $\mathcal{F}$) that lies over $\mathcal{C}$, that is,

$$\text{el}\,\mathcal{F} \to \mathcal{C}$$

given by

Objects: pairs (C, X) where $C \in \operatorname{Obj} \mathcal{C}$ and $X \in \mathcal{F}(X)$.

Morphisms: $f : (C, X) \to (C', C')$ are morphisms $f : C \to C'$ such that $\mathcal{F}(f)(X') = X$.

remark. We can use the Yoneda embedding to view $\mathcal{C}$ as a subcategory of $\operatorname{Psh}(\mathcal{C})$,

$$\mathcal{C} \hookrightarrow \operatorname{Psh}(\mathcal{C})$$

And also $\mathcal{F} \in \operatorname{Psh}(\mathcal{C})$. In fact, the element category is just the slice category:

$$\operatorname{el} \mathcal{F} \cong \mathcal{C}/\mathcal{F}.$$

question. Given $\mathcal{D} \to \mathcal{C}$ is it isomorphic to $\operatorname{el} \mathcal{F} \to \mathcal{C}$?

definition. $G : \mathcal{D} \to \mathcal{C}$ is a ***discrete fibration*** if for any $d \in \mathcal{D}$ and any $f : C \to G(d)$ there exists a unique lift from f' of f to $f \in \mathcal{D}$ such that the target ot f' is is d. That is,

$$\begin{tikzcd}
\bullet \arrow[r, dashed, "\exists! f'"] \arrow[d, "G"'] & d \arrow[d, "G"] \\
C \arrow[r, "f"'] & G(d)
\end{tikzcd}$$

remark. Given a discrete fibration we may construct a functor $\mathcal{F} : \mathcal{C}^{op} \to \text{Sets}$ simply by defining $\mathcal{F}(C) = G^{-1}(C)$ and if $C \to C' \cdots \to d$.

definition (Lecture)**.** A ***model structure*** on a category $\mathcal{A}$ is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called ***weak-equivalences***, ***cofibrations*** and ***fibrations*** with the following properties:

0. All (finite) small limits an colimits.

1. **(2 of 3)** Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among $f, g, f \circ g$ are in $\mathcal{W}$ then all of them are.

2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system. That is,

   (a) Any morphism in $\mathcal{A}$ can be factored as a morphism in $\mathcal{B}$ followed by a morphism in $\mathcal{D}$.

   (b) Lifts:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\
\bullet \arrow[r] \arrow[ur, dashed, "\exists"] & \bullet
\end{tikzcd}$$

   (c′) Notice that the aciom of retracts is not necessary. $r' \in \mathcal{R} \iff$ it satisfies (b) for all $\ell \in \mathcal{L}$.

definition.

- X is ***fibrant*** if $X \twoheadrightarrow \mathrm{pt}$.
- X is ***cofibrant*** if $X \rightarrowtail\!\!\twoheadrightarrow X$
- X is ***bifibrant*** if $0 \rightarrowtail X \twoheadrightarrow \mathrm{pt}$

examples (Two interesting model category structures on CGWH)**.**

1. ***Hurewicz model structure*** (Strom).
   - Cofibrations:= Huerwicz cofibrations.
   - Fibrations:= maps $E \to B$ such that for all spaces X [Photo1].
   - Weak equivalences:= homotopy equivalences.
2. ***Quillen model structure***. Defined on Top.
   - Cofibrations = retracts of relative cell complexes.
   - Fibrations = Serre Fibrations: $$\begin{tikzcd} D^n \arrow[r] \arrow[d] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$
   - Weak equivalences: $f : X \to Y$

   Also, we have
   - Fibrant: all of Obj Top.
   - Cofibrant: $\exists$\{CW complexes\}.

definition. Given a category $\mathcal{C}$ and a class of morphisms $W \subset \operatorname{Mor} \mathcal{C}$, its ***localization*** is a category $\mathcal{C}[W^{-1}]$ such that there is a functor $\operatorname{Loc}_W \mathcal{C} \to \mathcal{C}[W^{-1}]$ that sends weak equivalences to isomorphisms. Also, its satisfies the universal property that for every $F : \mathcal{C} \to \mathcal{D}$ such that $F(X) \subset \mathrm{Iso}$, the following diagram commutes

$$\begin{tikzcd} & \mathcal{C}[W^{-1}] \arrow[dr, dashed, "\exists! G"] & \\ \mathcal{C} \arrow[ur, "\operatorname{Loc}_W"] \arrow[rr, "F"] & & \mathcal{D} \end{tikzcd}$$

Theorem 4. Let $\mathcal{C}$ and (C, W, F) be a model category and $\mathcal{C}[W^{-1}] \cong \operatorname{Ho} \mathcal{C}$ where the latter is given by

- $\operatorname{Ob} \operatorname{Ho} \mathcal{E} = \{$fibrant-cofibrant-bifibrant objects of $\mathcal{E}\}$.
- $\operatorname{Mor} \operatorname{Ho} \mathcal{E} = \operatorname{Mor}_{\mathcal{E}}(X, Y)/\text{homotopy}$.

Let's say what homotopy means

definition. Given two maps

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

- We say $f \underset{\text{left}}{\sim} g$ if for the ***cylinder*** $\mathrm{Cyl}(X)$ defined by

$$\begin{tikzcd} X \sqcup X \arrow[rr] \arrow[dr, tail, "\text{cofibr.}"'] & & X \\ & \mathrm{Cyl}(X) \arrow[ur, two heads, "\text{trivial fib.}"'] & \end{tikzcd}$$

  we have that

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr] & & Y \times Y \\ & \mathrm{Cyl}(X) \arrow[ur, dashed, "\exists H"'] & \end{tikzcd}$$

- We say $f \underset{\text{right}}{\sim} g$ if

$$\begin{tikzcd} X \sqcup X \arrow[rr, "(f,g)"] \arrow[dr, dashed, "\exists H"'] & & Y \times Y \\ & \mathrm{Path}(Y) \arrow[ur] & \end{tikzcd}$$

claim. Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, if X is cofibrant and Y is fibrant, then $f \underset{\text{left}}{\sim} g \iff f \underset{\text{right}}{\sim} g$ and $\sim$ is an equivalence relation.

whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A ***CW complex*** X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called ***vertices***) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along ***attaching maps*** $j : S^n \to X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times D^{n+1})$ by identifying (j, x) with $j(x)$ for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j (see **??**). Each resulting map $D^{n+1} \to X$ is called a ***cell***. The

subspace X^n is called the n-***skeleton*** of X.

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\ulcorner"] & D^{n+1} \arrow[d] \\ X^n \arrow[r] & X^{n+1} \end{tikzcd}$$

lemma 5 (HELP)**.** content...

Theorem 6 (Whitehead, May)**.** If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 7 (Whitehead, May)**.** An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 8 (Whitehead (4.5), Hatcher)**.** If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12)**.** Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 7. $\square$

lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \to X$$

$$\mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(X', Y)$$

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, B') \text{ natual in } A \implies$$
$$\mathrm{Hom}(B, B) \cong \mathrm{Hom}(B, B') \& \mathrm{Hom}(B', B) \cong \mathrm{Hom}(B', B')$$
$$\implies B \cong B'.$$

- for ($\Longleftarrow$) commutativity of the hypotesis gives us commutativiuty of the rightmost square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an n-equivalence.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 9 (Yoneda).

$$\{\text{Natural transformations } \operatorname{Hom}(-,X) \to F\} \cong F(X)$$

corollary 10. $(\operatorname{Hom}(-,X) \to \operatorname{Hom}(-,Y)) \cong \operatorname{Hom}(X,Y)$.

corollary 11. The correspondence $X \mapsto \operatorname{Hom}(-,X)$ is fully faithful, that is, the correspondence $\operatorname{Hom}(X,X') \to \operatorname{Hom}(\operatorname{Hom}(-,X),\operatorname{Hom}(-,X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$. □

lemma 12. Let $E \times_B X$ be the pullback of

$$\begin{tikzcd} & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be such that $E \to B$ is an homotopy fibration and $f : X \to B$ is a homotopy equivalence. Let

$$\begin{tikzcd} E \times_B X \to E \arrow[r, dash, "\simeq"] \arrow[d, two heads] & E \arrow[d, two heads] \\ X \arrow[r, "\simeq"] & B \end{tikzcd}$$

be the pullback. Then $E \times_B X \to E$ is a homotopy equivalence.

Proof. Let $g : B \to X$ be the homotopy inverse of f.

(Step 1) Construct another pullback

$$\begin{tikzcd}
E \times_B B \arrow[r] \arrow[d, two heads] & X \times_B E \arrow[r] \arrow[d, two heads] & E \arrow[d, two heads] \\
B \arrow[r, "g"'] & X \arrow[r, "f"'] & B
\end{tikzcd}$$

(Step 2) Constuct $E \to E \times_B B$.

Consider

$$\begin{tikzcd}
E \arrow[rr, "id"] \arrow[d] & & E \arrow[d, two heads] \\
E \times I \arrow[r, "f \times id"'] \arrow[urr, dashed] & B \times I \arrow[r] & B?
\end{tikzcd}$$

And then $E \to E \times_B B \to E \times_B X \to E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{tikzcd}
Z \sqcup Z \arrow[r] \arrow[d, "f_1 \sqcup f_2"'] \arrow[ddr, bend left] & I \times Z \arrow[d] \arrow[dl, dashed] \arrow[dd, bend left] \\
E \times_B X \arrow[r] \arrow[d] & E \arrow[d] \\
X \arrow[r, "\simeq"] & B
\end{tikzcd}$$

$\square$

corollary 13. $B \xrightarrow{f} B$ is homotopy equivalence and $E \to B$ is a fibration, in

$$\begin{tikzcd}
E \times_B B \arrow[r] \arrow[d] & E \arrow[d] \\
B \arrow[r, "f"'] & B
\end{tikzcd}$$

$E \times_B B \to E$ is a homotopy equivalence.

exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 14. Let

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, tail, "g"] & B \arrow[d] \\
X \arrow[r] & X \cup_A B
\end{tikzcd}$$

be a pushout with $A \to X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f,g) \to X \cup_A B$ is a homotopy equivalence.

remark.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \\ X & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, hook] \arrow[d] & M_f \arrow[d] \\ X \arrow[r] & X \cup_A M_f \cong M(f,g) \end{tikzcd}$$

definition.

- The ***homotopy pullback*** of a diagram

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  is

$$\begin{tikzcd} X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y.

- The ***homotopy fiber*** if $f : Y \to Z$ is the pullback of

$$\begin{tikzcd} & Y \arrow[d, "f"] \\ pt \arrow[r] & Z \end{tikzcd}$$

  $F \subset Z^I \times_Z Y \to Z$, where F is the space of paths starting at x and ending at the same point $f(y)$.

remark. The pullback of

$$\begin{tikzcd} & Z^I \times_Z Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

is the homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

lemma 15. If $X \to Z$ is a fibration then for

$$\begin{tikzcd} & Y \arrow[d] \\ X \arrow[r, two heads] & Z \end{tikzcd}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

Proof.

$$\begin{tikzcd} X \times_Z \arrow[rr] \arrow[d, dash, "\simeq"] & & Y \arrow[d, "\simeq"] \\ X \times_{ev_0} Z^I \times_{ev_1} Y \arrow[rr, two heads] \arrow[d] & & Z^I \times_Z Y \arrow[d, two heads] \\ X \arrow[rr] & & Z \end{tikzcd}$$

□

Finally,

$$\begin{tikzcd} \text{hofib}\, f_1 \arrow[r] \arrow[d] & \text{hofib}\, f \arrow[r] \arrow[d, two heads] & * \arrow[d] \\ * \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

and

$$\begin{tikzcd} Z \arrow[r] \arrow[dd] & F(f) \arrow[d] \arrow[dd, bend left, two heads] \arrow[ddl, dash] \\ & X \times_Y Y^I \arrow[d] \\ X \times I \arrow[r] & X \end{tikzcd}$$

and an exact sequence

$$\Omega^2 \text{hofib} \to \Omega^2 X \to \Omega^2 Y \to \Omega\, \text{hofib}\, f \to \Omega X \to \Omega Y \to \text{hofib}\, f \to X \xrightarrow{f} Y$$

lemma 16 (Exactness)**.** $\forall z$, $[z\, \text{hofib}\, f] \to [Z, X] \to [Z, Y]$.

and we get the exact sequence

$$\pi_0(\Omega^2 X) \to \pi_0(\Omega^2 Y) \to \pi_0(\Omega\, \text{hofib}\, f) \to \pi_0(\Omega X) \to \pi_0(\Omega Y) \to \pi_0(\text{hofib}\, f) \to \pi_0(X) \to \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewickz fibrations. Let's talk about Serre fibrations. Notice that H. fibration $\implies$ S. fibration. What is the most natural example of a Serre fibration?

proposition 17 (also Hatcher 4.48)**.** Let E be a fiber bundle with fiber F. Then f is a Serre fibration.

Proof. What sdoes it mean to be a Serre fibration? It means that

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & E \arrow[d] \\ I^{n+1} = I^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

So if $\mathcal{U}$ is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U.

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $< \delta$. So, each the image of each cube lies in some $U \in \mathcal{U}$.

Then

$$\begin{tikzcd} I^n \arrow[r] \arrow[d] & F \times U \arrow[d, two heads] \\ I^{n+1} \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

has a lift for every little square because

$$\begin{tikzcd} X \arrow[r] \arrow[d] & U \arrow[d] \\ X \times I \arrow[r] \arrow[ur, dashed] & pt \end{tikzcd}$$

is always a fibration (think about this) and because pullbacks of fibrations are fibrations:

$$\begin{tikzcd} U \times F \arrow[r, two heads] \arrow[d, two heads] & U \arrow[d, two heads] \\ F \arrow[r, two heads] & pt \end{tikzcd}$$

. Then we may just add up the squares because

$$\begin{tikzcd} D^n \arrow[d] \\ D^n \times I \end{tikzcd}$$

and we're done. $\square$

proposition 18 (Sere fibration long exact sequence, see theorem 3)**.** Let $g : E \to B$ is a Serre fibration. $e \in E$, $g(e) = b$ and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(F) \arrow[r] & \pi_n(E) \arrow[r] & \pi_n(B) \arrow[r] & \pi_{n-1}(F) \arrow[r] & \pi_{n-1}(E) \arrow[r] & \cdots \\
\cdots \arrow[r] & \pi_n(F) \arrow[r] \arrow[u, "="] & \pi_n(E) \arrow[r] \arrow[u, "="] & \pi_n(E,F) \arrow[r] \arrow[u, dashed, "\cong"] & \pi_{n-1}(F) \arrow[r] \arrow[u, "="] & \pi_{n-1}(E) \arrow[r] \arrow[u, "="] & \cdots
\end{tikzcd}$$

example. We have shown that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ and the fact that $\pi_k(S^n) = 0$ for $k < n$.

Theorem 19. Let X be a CW-comples, $A, B \subset X$ subcomplexes, $C = A \cap B \neq \varnothing$, so

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout (this happens for inclusions, check it?).

If (A, C) is n-connected and (B, C) is m-connected, then

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an isomorphism for $i < m + n$ and sujerctive for $i = m + n$.

blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f : X \to Y$ be a map. Consider the pullback

$$\begin{tikzcd}
W(f) \arrow[r] \arrow[d, "{(q,p)}"'] & Y^I \arrow[d, "{(ev_0, ev_1)}"] \\
X \times Y \arrow[r, "f \times id"'] & Y \times Y
\end{tikzcd}$$

where

$$W(f) = \{(x, w) \in X \times Y^I | f(x) = w(0)\},$$
$$q(x, w) = x, \quad p(x, w) = w(1).$$

Since (ev_0, ev_1) is a fibration, the maps (q, p), q and p are fibrations.

Now suppose f is a pointed map with base points $*$. Then $W(f) \to W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \to (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \to X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \to *$.
- $I^n \times \{1\} \to A$.

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow \text{hofib} \longrightarrow A \longrightarrow X$$

$$\pi_0(\nearrow) = \qquad \begin{tikzcd} \pi_n(A) \arrow[r] & \pi_n(X) \arrow[r] \arrow[dr] & \pi_{n-1}(\text{hofib}) \arrow[r] \arrow[d, "\cong"] & \pi_{n-1}(A) \arrow[r] & \pi_{n-1}(X) \\ & & \pi_n(X, A) \arrow[ur] & & \end{tikzcd}$$

Theorem 20 (Blakers-Massey 1)**.** Let

$$\begin{tikzcd} Q \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

be a homotopy pushout, g is m equivalence, f is n-equivalence and $m, n \geqslant 0$. Then $Q \to X \times^h_P$ is $(m + n - 1)$-equivalence.

Theorem 21 (Blakers-Massey 2)**.** P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \varnothing$ (***strict pushout***)

$$\begin{tikzcd} Q \arrow[r, tail] \arrow[d, tail] & Y \arrow[d, tail] \\ X \arrow[r, tail] \arrow[ur, phantom, "\lrcorner"] & X \end{tikzcd}$$

Then $\pi_i(Y, \mathbb{Q}) \to \pi_i(P, X)$ is epi for $i = m + n$ and iso for $0 \leqslant i < m + n$.

Theorem 22 (Blakers-Massey 3)**.** $P = X \cup Y$, X and Y are open in P, $X \cap Y = Q \neq \varnothing$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a k-***equivalence*** if the induced map on the ith homotopy group is an isomorphism for $i < k$ and an epimorphism for $i = k$.
- $K_p(W) := \{x \in W :$ at least p coordinates of x are ¡ the same coordinates of the center of $W\}$

lemma 23. Let W be a cube in $\mathbb{R}^d$ with $\dim W \leqslant d$. If for all faces W' of ∂W, $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g \text{ rel } \partial W$ such that $g(w) \in A \implies w \in K_p(W)$.

freudenthal theorem (2 april)

definition. The appropriate analogue of the Cartesian product in the category of based spaces is the ***smash product*** $X \wedge Y$ defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y.

We also have the ***suspension of pointed spaces***, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X)/(t, x) \sim (0, y) \sim (1, y) \ \forall y \in X.$$

Then we have

$$\mathrm{Hom}_{\mathrm{CGWH}_*}(\Sigma X, \Sigma X) \cong \mathrm{Hom}_{\mathrm{CGWH}_*}(X, \Omega\Sigma X)$$

where $\Sigma X = S^{\wedge} X$ and $\Omega\Sigma X = \mathrm{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\mathrm{Map}(S^1, -)$.

So let X be a space. The identity map $\mathrm{id}_{\Sigma X} : \Sigma X \to \Sigma X$ then induces a map $X \to \Omega\Sigma X$.

Theorem 24 (Freudenthal)**.** Let X be ℓ-connected space. Then $X \to \Omega\Sigma X$ is a $(2\ell + 1)$-equivalence, that is,

$$\pi_i(X) \to \pi_{i+1}(\Sigma X),$$

is a bijection for $i < 2\ell + 1$ and a surjection for $i = 2\ell + 1$ (May).

Proof 1.

$$\begin{tikzcd}
X \arrow[rr, "(\ell+1)\text{-equiv}"] \arrow[dr] \arrow[dd, "(\ell+1)\text{-equiv}"'] & & * \arrow[dd] \\
& \Omega\Sigma X \arrow[ur] \arrow[dl] \arrow[d, phantom, "\lrcorner h"] & \\
* \arrow[rr] & {} & \Sigma X \arrow[ul, phantom, "h \ulcorner"]
\end{tikzcd}$$

□

Proof 2. Consider

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X
\end{tikzcd}$$

Then we use relative homotopy long exact sequence with (X, CX) to get $\pi_i(CX, X) \cong \pi_{i-i}(X)$, which is zero for $0 \leqslant i \leqslant \ell + 1$. Then use relative homotopy exact sequence for the pair $(\Sigma X, CX)$. then we get that $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$. And then if you use it for $(\Sigma X, X)$ and

But it also turns out that $\pi_i(\Sigma X) = \pi_{i-1}(\Omega\Sigma X)$ because

$$\pi_n(Z) = \mathrm{Hom}_{\text{h-Top},*}(S^n, Z) = \mathrm{Hom}(S^1 \wedge S^{n-1}, Z) = \mathrm{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX, X) \to \pi_i(\Sigma X, CX)$ which is isomorphism for $0 \leqslant i \leqslant 2\ell + 1$ and surjective for $i = 2\ell + 2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \to \pi_{i-1}(\Omega\Sigma X)$ for i as desired. □

corollary 25. If X is ℓ-connected, then ΣX is $(\ell+1)$-connected for $\ell \geqslant 0$.

space	S^0	$\Sigma S^0 = S^1$	$\Sigma^2 S^0 = S^2$	$\Sigma^3 S^0 = S^3$	$\cdots$	$\Sigma^n S^0 = S^n$
conectedness	-1	0	1	2	$\cdots$	$(n-1)$

corollary 26. S^n is $(n-1)$-connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_2(S^3) \to \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \to \pi_1(S^3) = 0,$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map $S^n \to S^n$. We get a map $CS^n \to CS^n$ (in general, for $f : X \to Y$ we have $(t, x) \mapsto (t, f(x))$ in the cones). We also have $CS^n \to CS^n/S^n = S^{n+1}$.

Now if we take $\mathrm{id} : S^n \to S^n$ we shall get $\mathrm{id} : S^{n+1} \to S^{n+1}$. Think about this like $\pi_1(S^1) \to \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective because $i = 0$. From Hopf fibration we have $\pi_2(S^2) = \mathbb{Z}$. So we have a surjective map $\mathbb{Z} \to \mathbb{Z}$. So it's an isomorphism and we conclude that id_{S^2} is a generator of $\pi_2(S^2)$.

corollary 27. Since S^n is $(n-1)$-connected, we have

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is isomorphism for $i \leqslant 2(n-1) = 2n-1$ and epimorphism form $i = 2n-1$. (We just shift the indices of theorem 24 by one.)

corollary 28. $\pi_n(S^n) = \mathbb{Z}$ with id_{S^n} as generator.

corollary 29. $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \leqslant n-1$ and epimorphism for $k = n-1$.

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are $\mathbb{Z}/2$. This is what are called the k***th stable homotopy groups of a sphere***. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_i(S^1) \to \pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2) \to \pi_{i-1}(S^2) = 0$$

So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

claim (Serre). $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus$finite abelian. And $\pi_i(S^k)$ is finite abelian in all other cases.

another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{tikzcd}
S^{n-1} \arrow[r, "(n-1)\text{-equiv}"] \arrow[d, "0\text{-equiv}"'] & D^n \arrow[d] \\
X \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \cup D^n
\end{tikzcd}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is $(n-1)$-equivalence by Blakers-Massey. So, by attaching $\sqcup D^n$ we can kill $\pi_{n-1}(X)$, that is, $X \cup (\sqcup D^n)$ has trivial π_{n-1}.

Now notice that

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

that is, $\pi_i(D^n, S^{n-1}) = 0$ for $i \leqslant n-1$. This implies that $\pi_i(X \cup D^n, X) = 0$ for $i \leqslant n-1$.

Also by homotopy long exact sequence,

$$\pi_{n-1}(X) \to \pi_i(X \cup D^n) \text{ is sujrective}$$

$$\pi_i(X) \to \pi_i(X \cup D^n) \text{ is isomorphism for } i \leqslant n-2.$$

So what we have thus far is

$$\pi_n(X \cup D^n) \longrightarrow \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup D^n) \longrightarrow 0 = \pi_{n-1}(X \cup D^n)$$

Notice that $\pi_n(X \cup D^n, X)$ is not ingeneral cyclic (counterexample $S^1 \cup D^2$ taking unieversal cover which is real line with spheres on integers, homotopy equivalent to $\bigvee_{\mathbb{Z}} S^2$ which is not finitely generated).

So basically attaching a disk via f will kill $[f]$ inside $\pi_n(X)$ this is called ***killing*** an element of the homotopy group.

proposition 30. For any CW-complex X, $X^\ell \to X$ is an ℓ-equivalence.

remark. We have used that for $A \hookrightarrow X$ from long exact sequence of relative homotopy groups we get $\pi_n(X, A) = 0$.

corollary 31. Let $i \leqslant \ell$ and $g : D^i \to X$, $g : \partial D^i \to X^\ell$. Then there is a homotopy rel ∂D^i to a map with img $\subset X^\ell$.

Theorem 32 (Cellular approximation theorem)**.** Let X and Y be CW-complexes and $\xi : Y \to X$ be any map. Then ξ is homotopic to a ***cellular map***, that is, a map $\psi : Y \to X$, such that for all ℓ, $\psi Y^0 \subset X^\ell$.

We also have

proposition 33. Let $n \geqslant 2$. Then

$$\pi_n\left(\bigvee_{k\in I} S^n\right) = \bigoplus_{k\in I} \imath_n(S^n) = \bigoplus_{k\in I} \mathbb{Z} = Z^{\oplus I}$$

proposition 34. First notice that for finite I,

$$X^n = X^{n+1} = \bigvee_{k\in I} S^n$$

by looking carefully. Then

$$\pi_n(X, X^{n+1}) = 0 = \pi_{n+1}(X, X^{n+1})$$

so

$$\bigoplus_{k\in I} \mathbb{Z} = \prod_{k\in I} \pi_n(S^n) = \pi_n(X) = \pi_n(X^{n+1}) = \pi_n(X^n) = \pi_n\left(\bigvee_{k\in I} S^n\right)$$

and for the infinite case it also works, using finite compactness of the CW complex.

postnikov tower and CW-approximation, 9 abril

- Let X be a space. Then there is a CW-complex Y and a weak homotopy equivalence from $Y \to X$.
- Let $A \to X$ be a map of spaces. Then it can be factored as $A \to Y \to X$ where $A \hookrightarrow Y$ is a relative CW-complex, and $Y \to X$ is a weak homotopy equivalence.

  remark. Notice that the second item is the first one with $A = \varnothing$. Then, the second case is a Serre cofibration since it is a construction involving the cofibration $S^{n-1} \hookrightarrow D^n$ (this is a cofibration by definition).

- Let A be a space. Then there is a space $\tau_{\leqslant n}A$ such that $A \hookrightarrow \tau_{\leqslant n}A$; $\tau_{\leqslant n}A$ is obtained by adding cells of dim $\geqslant n+2$. $A \hookrightarrow \tau_{n \leqslant n}A$ is $(n+1)$-equivalence and

$$\pi_k(\tau_{\leqslant n}A) = 0 \qquad k > n.$$

Moreover, $A \to \tau_{\leqslant n}A$ is unique among morphisms in $\mathrm{Ho}(CGWH)$ from A into spaces with $\pi_k = 0$ for $k > n$.

This is called a ***Postnikow tower*** and it looks like this:

$$\begin{tikzcd}
& \cdots \arrow[d] \\
& \tau_{\leqslant 2}A \arrow[d] \\
& \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

The idea is that $\tau_{\leqslant n}A$ is obtained from A by killing elements of dimension greater than n, that is, by

- attaching $n+2$ cells that kill all $\pi_{n+1}(A)$,
- attaching $n+3$ cells that kill all $\pi_{n+2}(A)$,
- attaching $n+3\ldots$
- attaching $n+n\ldots$

So consider the space X that is obtained from A after attaching cells of dimension $\geqslant n+2$, so we have a map $A \to X$. Consider also a space Y with $\pi_k(Y) = 0$ for $k > n$. Then for any map $A \to Y$ there is a map $X \to Y$ that extends $A \to Y$. This accounts for a bijection

$$[X, Y] \cong [A, Y].$$

In class we struggled a bit to prove surjectivity, finally using an argument related to the pair $(X \times I, X \times \partial I \cup A \times I)$.

> The point is that the spaces in the Postnikov tower are like the original space but with trivial homotopy groups for $k \geqslant n$.

question. What is the limit of the Posnikov tower?

$$\begin{tikzcd}
& \lim \\
& \cdots \arrow[d] \\
& \tau_{\leqslant 2}A \arrow[d] \\
& \tau_{\leqslant 1}A \arrow[d] \\
A \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuuur] & \tau_{\leqslant 0}A
\end{tikzcd}$$

- Let $A \to X$ be a map (of CW-complexes (or spaces?)). Then

$$\begin{tikzcd}
& \cdots \arrow[d] & \\
& Y_2 \arrow[d] \arrow[ddr, bend left] & \\
& Y_1 \arrow[d] \arrow[dr] & \\
A \arrow[r] \arrow[ur] \arrow[uur, bend left] \arrow[rr, bend right] & Y_0 \arrow[r] & X
\end{tikzcd}$$

Proof pending

- We also have the ***Whitehead tower***, obtained from the homotopy fiber

$$\operatorname{hofib} f_n \longrightarrow A \xrightarrow{f_n} \tau_{\leqslant n-1}A$$

which yields

$$\cdots \longrightarrow \pi_{k+1}(A) \xrightarrow{\cong} \pi_{k+1}(\tau_{\leqslant n}A) \longrightarrow \pi_k(\operatorname{hofib}) \longrightarrow \pi_k(A) \longrightarrow \pi_k(\tau_{\leqslant n}A) \longrightarrow \cdots$$

so

$k \leqslant n-1$	$k = n$	$k \geqslant n+1$
$\pi_k(\operatorname{hofib} f_n) = 0$	$\pi_n(\operatorname{hofib} f_n) = 0$	$\pi_k(A) = \pi_k(\operatorname{hofib} f_n)$

- Now there's a natural way to construct the following diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend right] & \tau_{\leqslant n}A \arrow[r] & \tau_{\leqslant n-1}A
\end{tikzcd}$$

which yields the bundle

$$\text{hofib} \longrightarrow \tau_{\leqslant n}A \longrightarrow \tau_{\leqslant n-1}A$$

and in this case we get

$k \neq n$	$k = n$
$\pi_k(\text{hofib}) = 0$	$\pi_n(\text{hofib}) = \pi_n(A)$

and this is what we call a $K(\pi, n)$-***space*** (all homotopy groups are trivial but the nth.)

Moore space, $K(\pi, n)$ and Hurewicz theorem, 11&16 apr

Theorem 35 (Uniqueness of CW-approximations)**.** Recall that a CW-approximation of X is a map $f : Z \to X$ and a CW-complex Z that is a weak homotopy equivalence (induces isomorphisms in all homotopy groups).

We have that

$$\begin{tikzcd} Y_1 \arrow[dr, "\text{CW ap.}"] & \\ & X \\ Y_2 \arrow[uu, dashed, "\exists!"] \arrow[ur, "\text{CW ap.}"'] & \end{tikzcd}$$

up to homotopy equivalences

lemma 36 (Compression)**.** If the relative homotopy groups of a pair (Y, B) is zero for $n = \dim e$ for every cell $e \in X \backslash A$ then any map $(X, A) \to (Y, B)$ is homotopic rel A to $(X, A) \to (B, B)$ (so intuitively we can collapse Y).

Proof. With fibrations (photo) □

proposition 37. Let $f : X \to Y$ be an n-equivalence (in Hatcher stated as weak equivalence but argument is the same). Then f induces an n-***equivalence in homology*** $H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$ (an isomorphism for $i < n$ and surjection for $i = n$).

Proof. photo □

corollary 38. If $f : X \to Y$ is a weak equivalence, then f induces an isomorphism in $H_*(-, G)$ and $H^*(-, G)$.

Proof. Universal coefficients. □

definition. Let π be an abelian group. Take

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \pi$$

a ***free resolution***, i.e. $F_1 = Z^{\oplus J}$ and $F_0 = Z^{\oplus I}$ are free abelian groups and $\pi = F_0/F_1$. Let's take the corresponding maps

$$\begin{aligned} \textstyle\bigvee_{j\in S} S^n \longrightarrow \textstyle\bigvee_{i\in I} S^n &\longrightarrow \operatorname{hocofib} f \\ x_j \longmapsto \textstyle\sum a_i y_i & \end{aligned}$$

where a_i is the degree of $S^n \to S^n$. Recall that the homotopy cofiber $\operatorname{hocofib} f$ is the mapping cone of f. It is the ***cone of pointed spaces***. We call this space the ***Moore space*** $M(\pi, n)$ and it is such that

$$\tilde{H}_i(M(\pi, n)) = \begin{cases} 0, & i \neq n \\ \pi, & i = n \end{cases}$$

What do we get in homology? Exactly the sequence of free groups above. So, $H_n(\operatorname{hocofib} f) = \pi$. What do we get in homotopy? Might be π as well. Let's prove something stronger:

Theorem 39. Let Y be such that $\pi_i(Y) = 0$ for $i > n$ and $\pi_0(Y) = 0$. Then

$$[\operatorname{hocofib} f, Y] \to \operatorname{Hom}(\pi_n(\operatorname{hocofib} f), \pi_n(Y))$$

is a bijection.

Proof. Photo

Take

$$\textstyle\bigvee_I S^n \longrightarrow \bigvee_J S^m \longrightarrow \operatorname{hocofib} f$$

Now apply $[-, Y]$. We get

$$[\textstyle\bigvee_I S^1] \longleftarrow [\bigvee_J S^1] \longleftarrow [\operatorname{hocofib} f, Y] \longleftarrow 0$$

$\square$

lemma 40. If (X, A) is r-connected, A s-connected for all $r, s \geqslant 0$, then the map

$$\pi_i(X, A) \to \pi_i(X/A)$$

induced by the quotient map $X \to X/A$ is an $(r + s + 1)$ equivalence.

Proof. If (X, A) is r-connected. $\square$

After the lemma we proved that $\pi_n(C_f \cong G$.

Theorem 41. Now consider a Moore space, kill all homotopy groups to get $\tau_{\leqslant n}(M)$. It is a $K(\pi, n)$ space with cells in dim $\geqslant n$, obtained from hofib f by attaching cells of dim $\geqslant n+2$. Then

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \tau_{\leqslant n} M \arrow[dl, dashed, "\exists! \text{ up to homotopy}"] \\ & Y & \end{tikzcd}$$

$$[\text{hocofib } f, y] \cong [\tau_{\leqslant n}(\text{hocofib } f), Y] = \text{Hom}(\pi, \pi_n(Y)).$$

If $\pi_n(Y) = \pi$, then there is a weak equivalence

$$\tau_{\leqslant n}(\text{hocofib } f) \to Y.$$

Theorem 42 (Hurewicz)**.** Let X be an $(n-1)$-connected space for $n \geqslant 2$. Then

$$\tilde{H}_i(X) = \begin{cases} 0, & i < n \\ \pi_n(X), & i = n \end{cases}$$

Proof. Photo.

Idea is to construct a Moore space that is a piece of the CW approximation. Why a piece? Write and understand why this worked! □

Theorem 43 (Relative Hurewicz theorem)**.** Let (X, A) be n-connected, A be 1-connected, $n \geqslant 2$. Then

$$H_i(X, A) = \begin{cases} 0, & i < n \\ \pi_n(X, A), & i = n \end{cases}$$

Proof. Take a CW approximation of (X, A),

$$\begin{tikzcd} & Y \arrow[dr, "\cong"] & \\ B \arrow[ur] \arrow[r, "\cong"'] & A \arrow[r] & X \end{tikzcd}$$

So the approximation is (B, Y).

Then we have

$$\begin{aligned} \pi_i(Y, B) &= \pi_i(Y/B), \quad i \leqslant n \\ H_i(Y, B) &= \tilde{H}_i(Y/B), \quad \forall i \end{aligned}$$

and first line implies that $\pi_1(Y/B) = 0$ for $i < n$. But then we are done, right? □

Representability of the functor $H^n(-,G)$

remark. Recall the ***adjoint relation***

$$\langle \Sigma X, K\rangle = \langle X, \Omega K\rangle$$

where ΣX is the ***reduced suspension*** of a space X, ΩK is the ***loop space*** of another space K and the brackets mean homotopy classes of basepointed maps. Choosing $X = S^0$ and $K = M$, the left-hand side becomes $\pi_1(M, m)$ and the right-hand side becomes the path components of $\mathrm{Map}((S^1, 0), (M, m))$.

Now let's do some other interesting remarks.

definition. Let $\mathcal{C}$ be a category. Then $\mathrm{Psh}(\mathcal{C})$ is the category of ***presheaves*** of $\mathcal{C}$ i.e. the category of functors $\mathcal{C}^{op} \to \mathrm{Sets}$ and natural transformations. For any object A in $\mathcal{C}$ there is a presheaf $\mathrm{Hom}_{\mathcal{C}}(-, A)$. A presheaf $\mathcal{F}$ that is isomorphic to $\mathrm{Hom}_{\mathcal{C}}(-, A)$ for some A is called ***representable***.

For example, $\mathrm{Hom}_{CW}(X, K(\pi, n)) \cong H^n(X, \pi)$, that is, H^n is representable.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(H_{n-1}(K(G,n),G)) \arrow[r] \arrow[d, equal] & H^n(K(G,n),G) \arrow[r, "\cong"] & \mathrm{Hom}(H_n(K(G,n)),G) \arrow[r] \arrow[d, equal] & 0 \\
& 0 & & 0 &
\end{tikzcd}$$

There is a special element in $\mathbb{H}^n(K(G,N),G)$, the preimage if id_G.

claim.

$$[X, K(G, n) \cong \tilde{H}^n(X, G)]$$

where on the left we have based CW-complexes.

lemma 44 (Yoneda). Let $\mathcal{F}$ be a presheaf, A be an object in $\mathcal{C}$. Then

$$\mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(-, A), \mathcal{F}) \cong \mathcal{F}(A)$$

naturally in A.

Proof. For $f : C \to A$, we have this commutative diagram:

$$\begin{tikzcd}
\mathrm{id}_A \arrow[r, mapsto] & f \\
\mathrm{Hom}(A,A) \arrow[r] \arrow[d, "\eta_A"'] & \mathrm{Hom}(C,A) \arrow[d, "\eta_C"] \\
\mathcal{F}(A) \arrow[r] & \mathcal{F}(C) \\
\eta_A(\mathrm{id}_A) \arrow[r, mapsto] & \eta_C(f)
\end{tikzcd}$$

So, natural transformations $\eta : \mathrm{Hom}(-, A) \to \mathcal{F}$ are determined by $\eta_A(\mathrm{id}_A)$. And that's it because then the map $\eta \mapsto \eta_A(\mathrm{id}_A)$ is what we are looking for. It remains to write why it is injective, surjective and natural. $\square$

representability theory

If a functor $F : \mathcal{C}^{op} \to \mathrm{Sets}$ is representable, then it sends colimits to limits and sends weak colimits to weak colimits.

Theorem 45. If $F : \mathrm{Ho}\,\mathrm{CW}_* \to \mathrm{Sets}$ sends

1. coproducts (=wedges) to products,
2. $\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \cup_A C \end{array}$ to weak pullback, where $B \cup_A C$ is a CW complex, B and C are CW complexes, $A = B \cap C$,

   then F is representable.

   (By B with isomorphism $F \cong [-, B]$ given by some $X \in F(B)$.)

definition. A CW-complex B together with a choice of $\gamma \in F(B)$ is a ***spherical classifying space*** of F if

$$\begin{aligned} \gamma_* : [S^n, B] &\to F(S^n) \\ f &\mapsto f^*(\gamma) \end{aligned}$$

is an isomorphism for $n > 0$ (because for $n = 0$ S^n is not connected).

proposition 46. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F, then B_1 and B_2 are homotopy equivalent via the map that sends γ_1 to γ_2.

proposition 47. If (B_1, γ_1) and (B_2, γ_2) are two classifying spaces for F, then $g : B_1 \to B_2$ is such that $g^*/(\gamma_2) = \gamma_1$, then f is a homotopy equivalence.

May 2: after representation theorem

Cohomology operations

$$\mathrm{Nat}(H^n(-, G_1), H^n(-, G_2)) \leftrightarrow [K(G_1, n), K(G_2, n)]$$

example.

1. $x \mapsto x \smile x$, $H^n \to H^{2n}$. This is a natural transformation between the H functors.
2. The short exact sequence

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

yields

$$\cdots \longrightarrow H^n(X,\mathbb{Z}/p^2) \longrightarrow H^n(X,\mathbb{Z}/p) \longrightarrow H^{n+1}(X,\mathbb{Z}/p) \longrightarrow H^{n+1}(X,\mathbb{Z}/p^2) \longrightarrow \cdots$$

which is natural in X, yielding the ***Bockstein homomorphism***

$$H^n(-,\mathbb{Z}/p \to H^{n+1}(-,\mathbb{Z}/p)$$

spectral sequences

1. ***Exact couple***: $\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$

   It looks like this is just notation for an exact sequence that repeats over and over.

   example. Take

$$0 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0$$

   and take $C_\bullet$ torsion free chain complex. You get

$$0 \longrightarrow C_\bullet \xrightarrow{\times p} C_\bullet \longrightarrow C_\bullet \otimes \mathbb{Z}/p \longrightarrow 0$$

   which yields a long exact sequence

$$\begin{tikzcd} H_n \arrow[rrr] & & & H_n \arrow[dl] \\ & H_{n-1} \arrow[r] & \ldots & \\ & & H_n \arrow[ul] & \end{tikzcd}$$

   Or better yet (notation)

$$\begin{tikzcd} H_*(C) \arrow[rr] & & H_*(C) \arrow[dl] \\ & H_*(C,\mathbb{Z}_p) \arrow[ul] & \end{tikzcd}$$

2. ***Derived couple of exact couple***

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

Where $A' = \operatorname{img} i$,

And the exact couple yields a homology $E^1 = H(A,)d$ with differential $d = jk$,

$i' = i|_{A'}, j'(ia) := [j(a)], k'([e]) = k(e)$.

We have checked that

- j' and k' are well-defined. i' is well-defined automatically.
- that the image of k' is in fact in A'
- $j'i' = 0, k'j' = 0, i'k' = 0$.
- that $j'(ai) = 0 \implies ia = i'a', k'[e] = 0 \implies [e] = j'(a')$ and that $i'(a') = 0 \implies a' = k[e]$.

So, we have shown that each exact couple gives a derived couple.

3.

$$\begin{tikzcd}
\text{Filtration on something} \arrow[r] & \text{filtration on chain complex} \arrow[r] \arrow[dr] & \text{exact couple} \arrow[d] \\
& & \text{spectral seq}
\end{tikzcd}$$

A ***filtration*** on an abelian group/R-module/chain complex/... C is

$$\ldots \subseteq F_n C \subseteq F_{n+1} C \subseteq C, \qquad n \in \mathbb{Z}$$

and there is an associated graded $\operatorname{gr} F_\bullet C := \bigoplus_{n\in\mathbb{Z}} F_n C / F_{n-1} C$.

We hope to recover C from $\operatorname{gr} F_\bullet$.

Problem 1. If $\bigcap_{n\in\mathbb{Z}} F_n C \neq 0$ then the map $\bigcap_{n\in\mathbb{Z}} F_n C$ looses some information. We may solve this by asking that

1. $F_n C = 0$ for $n < 0$
2. $C \to \lim C/F_n C$ is isomorphism.

Problem 2. If $\bigcup_{n\in\mathbb{Z}} F_n C \neq C$ that would be very bad. Then we should ask that $C = \bigcup_{n\in\mathbb{Z}} F_n C$ which is $C = \operatorname{colim} F_n C$.

example. We discussed the cases of $\bigoplus_{n\in\mathbb{N}} \mathbb{Z}$ and $\prod_{n\in\mathbb{N}} \mathbb{Z}$. We found that even though $\bigcap_{n\in\mathbb{Z}} F_n C = 0$ in both cases, we could not recover the desired information (?).

Now, to each Serre fibration corresponds a spectral sequence.

following sss.pdf

X is a filtered chain complex

$$\cdots \subseteq X_{n-1} \subseteq X_n \subseteq X_{n+1} \subseteq \cdots$$

We automatically get

$$\begin{tikzcd}
H_{p+q}(X_p) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_p/X_{p-1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p-1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p-1}/X_{p-2}) \arrow[r] \arrow[d] & \cdots \\
H_{p+q}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+1}/X_p) \arrow[r] \arrow[d] & H_{p+q-1}(X_p) \arrow[r, "j"'] \arrow[d, "i"'] & H_{p+q-1}(X_p/X_{p-1}) \arrow[r, "k"'] \arrow[d] & \cdots \\
H_{p+q}(X_{p+2}) \arrow[r] \arrow[d, "i"'] & H_{p+q}(X_{p+2}/X_{p+1}) \arrow[r] \arrow[d] & H_{p+q-1}(X_{p+1}) \arrow[r] \arrow[d, "i"'] & H_{p+q-1}(X_{p+1}/X_p) \arrow[r] \arrow[d] & \cdots \\
\cdots & \cdots & \cdots & \cdots &
\end{tikzcd}$$

Notice that the red arrows are the induced exact sequence of

$$\begin{tikzcd}
0 \arrow[r] & X_{p-1} \arrow[r] \arrow[d] & X_p \arrow[r] \arrow[d] & X_p/X_{p-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_p \arrow[r] & X_{p+1} \arrow[r] & X_{p+1}/X_p \arrow[r] & 0
\end{tikzcd}$$

Now consider

$$A = \bigoplus_{p+q} H_{p+q}(X_p)$$

and simply call $A_{p,1} = H_{p+q}(X_p)$. Also take

$$E = \bigoplus_{p,q} H_{p+q}(X_p/X_{p-1})$$

and $E_{p,q} := H_{p+q}(X_p/X_{p-1})$.

We have

$$\begin{tikzcd}
A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

where

map	**bidigree**
$i : A_{p,q} \to A_{p+1,q-1}$	$(1,-1)$
$j : A_{p,q} \to E_{p,q}$	$(0,0)$
$k : E_{p,q} \to A_{p-1,q}$	$(-1,0)$

Now consider the derived couple of this exact couple several times:

$$\begin{tikzcd} A^2 \arrow[rr, "i_2"] & & A^2 \arrow[dl, "j_2"] \\ & E^2 \arrow[ul, "k_2"] & \end{tikzcd} \qquad \begin{tikzcd} A^3 \arrow[rr, "i_3"] & & A^3 \arrow[dl, "j_3"] \\ & E^3 \arrow[ul, "k_3"] & \end{tikzcd}$$

where, going back to definitions

$$E^2 = \ker d_1 / \operatorname{img} d_1 \qquad E^3 = \ker d_2 / \operatorname{img} d_2$$

and so on.

And then think about the bidigrees of the other maps. Well they are

map	**bidigree**
$i_k = i_{k-1}\vert_{\operatorname{img} i_{k-1}}$	$(1, -1)$
j_n	$(-(n-1), n-1)$
k_n	$(-1, 0)$
$d = j_n k_n$	$(-n, n-1)$

(we thought about this).

Here are some reasonable assumtions:

1. $X_p = 0$ for $p < 0$.
2. $\bigcup_p X_p = X$.
3. $H_{p+q}(X_p/X_{p-1}) = 0$ for $q < 0$.

remark. It will happen that for very large r, $E^r_{p,q} = E^r_{p,q} = \cdots = E^\infty_{p,q}$.

Also it will happen that

$$E^\infty_{p,q} \cong \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

and we will know what

$$\bigoplus_{\substack{p+q \text{ fixed} \\ p \in \mathbb{N}}} \operatorname{img} H_{p+q}(X_p) / \operatorname{img} H_{p+q}(X_{p-1})$$

is, and understand

$$H_* \qquad \ldots \subseteq \operatorname{img} H_{p+q}(X_p) \subseteq \operatorname{img} H_{p+q}(X_{p+1}) \subseteq \ldots \subseteq H_{p+q}(X),$$

which is induced by

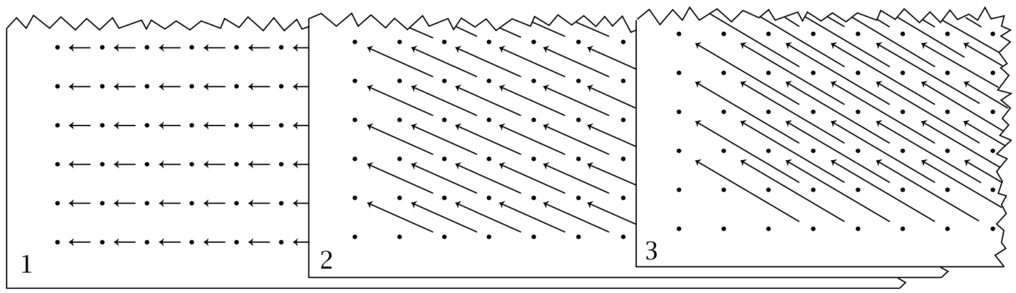
$$X \longrightarrow X_{p+1} \longrightarrow X$$

in our hopes to understand $H_{p+q}(X)$ which is of course just the homology of X.

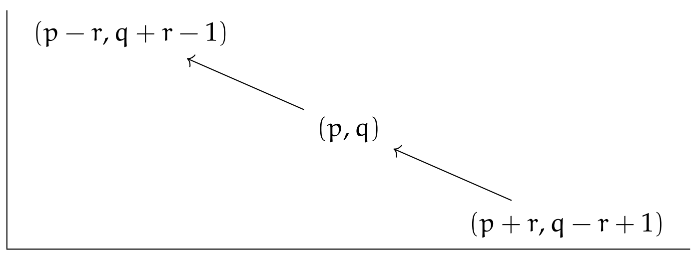
Now let's put the E's in a diagram like

$$\begin{array}{|lll}
\cdots & & \\
E^1_{0,1} & E^1_{1,1} & E^1_{2,1} \\
E^1_{0,0} & E^1_{1,0} & E^1_{2,0} \\
\hline
\end{array}$$

and if arrows are the differential d with bidegree $(-n, n-1)$, we already have Hatcher's picture



And something interesting will happen when r is very big. What? This:



that for a fixed (p, q) there is a large r such that if $p < r$ and $q + 1 < r$ we shall have

$$E^r_{p,q} = E^{r+1}_{p,q} := E^\infty_{p,q}.$$

Serre spectral sequence

Now take a Serre fibration $F \longrightarrow E \longrightarrow B$ with

$$\pi_0(F) = 0 \quad \text{and} \quad \pi_1(B) = 0$$

and it turns out that

$$E^2_{p,q} = H_p(B, H_q(H)) \implies H_{p,q}(E).$$

example. For $S^1 \to S^3 \to S^2$ we have found that the first two pages are

$$\begin{tikzcd} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \arrow[ul, "\cong"', "d"] \end{tikzcd} \qquad\qquad \begin{array}{cc} \mathbb{Z}/\operatorname{img} d_2 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z}/\operatorname{img} d_2 \end{array}$$

$$E_2 \qquad\qquad\qquad\qquad E_3$$

so $E^3 = E^\infty$.

example. We have also done $S^3 \to S^7 \to S^4$ and $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. See **sssguide**.

proofs

Now we will prove that there exists a spectral sequence and that it converges where it converges.

serre spectral sequence

Consider either of

$$\pi_0(B) = 0 \qquad\qquad \text{or} \qquad\qquad \begin{cases} \pi_1(B) = 0 \\ \text{or } \pi_1(B) \text{ acts trivially on } H_q(F) \\ \text{or take } H_* \text{ with local coefficients} \end{cases}$$

Theorem 48. There is spectral sequence E^ν that converges to $H_*(E)$ and such that

$$E^2_{p,q} = H_p(B, H_q(F, G)).$$

Proof. The construction of the spectral sequence is not complicated. Start with E^1. Then Do E^2 with

$$\begin{tikzcd} E^k \times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ D^k \arrow[r] & B \end{tikzcd}$$

∎

Consider the following conditions on the big staircase diagram:

a. In each A column almost all maps are isomorphisms.

b. In each column E almost each entry is 0.

c. $E^1_{p,q} = 0$ for $p < 0$ and $q < 0$.

d. $X_p = 0$ for $p < 0$ and $H_n(X_p) \to H_n(X_{p+1})$ is isomorphism for $p \gg 0$.

And also

- $A_{-\infty,p+q} := A_{p,q}$ for $p \ll 0$.

- $A_{+\infty,p+q} := A_{p,q}$ for $p \gg 0$.

And then

e1. $A_{-\infty,p+q} = 0$.

e2. $A_{+\infty,p+q} = 0$.

claim. b $\implies E^r_{p,q}$ stabilizes for fixed panel q, so $E^\infty_{p,q}$ makes sense.

Now let's check that indices are the way they are in

$$\begin{array}{ccccccccccccc} E^r_{p+r-1,q-r+2} & \xrightarrow{k} & A^r_{p+r-2,q-r+2} & \xrightarrow{i} & A^r_{p+r-1,q-r+1} & \xrightarrow{j} & E^r_{p,q} & \xrightarrow{k} & A^r_{p-1,q} & \xrightarrow{i} & A^r_{p,q-1} & \xrightarrow{j} & E^r_{p-r+1,q-r-2} \\ \| & & & & & & & & \| & & \| & & \| \\ 0 & & & & & & & & \underset{\text{if e1}}{0} & & \underset{\text{if e1}}{0} & & 0 \end{array}$$

Ok, after some other considerations we have concluded that

> $E^\infty_{p,q}$ also makes sense and it is a piece of p-graded associative algebra graded of $A_{+\infty,p+q}$ with

$$\begin{aligned} E^r_{p,q} &= A^r_{p+q+r,q-r+q}/iA^r_{p+r-2,q-1+2} \\ &= i^{r-1}(A^1_{p,q})/i^r(A^1_{p-1,q+1}) \\ &= F_pA_{+\infty,p+q}/F_{p-1}A_{+\infty,p+q} \end{aligned}$$

Something on 14 may

In cover spaces we have homeomorphisms between the fibers. In hurewicz fibrations this may not be true, but we still can have homotopy equivalences. So consider a Hurewicz fibration $f : E \to B$ and a path $I \to B$. Then we have:

$$\begin{tikzcd} F_a \arrow[r, hook] \arrow[d, "i_0"] & E \arrow[d, two heads, "f"] \\ F_a \times I \arrow[ur] \arrow[r, "\pi"'] & I \arrow[r, "p"] & B \end{tikzcd}$$

And we claim that there is an homotopy equivalence

$$F_a \cong F_a \times \{1\} \to F_b = f^{-1}b$$

To see why, consider two homotopic paths $p_1, p_2 : I \to B$. Construct the following diagram:

We have (or will?) established:

proposition 49. For any Hurewikz fibration $f : E \to B$ there is "an action of $\pi_1(B)$ on F_b up to homotopy"

$$\begin{aligned} \Pi_1(B) &\to \mathrm{Ho\,Top} \\ b &\mapsto F_b \end{aligned}$$

We have an action

$$\pi_1(B) \curvearrowright H_*(F_b)$$

Why do need this? Consider the following case:

$$\begin{tikzcd} f_i^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, two heads] \\ X \arrow[r, "f_i"'] & B \end{tikzcd}$$

for $f_1, f_2 : X \to B$.

serre spectral sequence for cohomology

Suppose

$$F \hookrightarrow X \longrightarrow B$$

Is a Serre fibration with $\pi_0(B) = 0$ and that the action described above $\pi_1(B) \curvearrowright H^*(F; G)$ is trivial.

Then there is a spectral sequence $\{E_r^{p,q}, d_r\}$ such that

a. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+q}$; $\quad E_{r+1}^{p,q} = \ker d_r / \operatorname{img} d_r$.

b. $E_\infty^{p,q} \cong F_p^{p+q}/F_{p+1}^{p+q}$; $\quad 0 \subset F_{p+q}^{p+q} \subset \ldots \subset F_0^{p+q} = H^{p+q}(X, G)$.

c. $E_2^{p,q} = H^p(B, H^q(F, G))$.

d. $E_2^{p,q} \times E_2^{s,t} \to E_2^{p+s,q+t}$, which is given by $(-1)^{qs}$.

$$H^p(B, H^q(F, R)) \times H^s(B, H^t(F, R)) \overset{\smile}{\to} H^{p+s}(B, H(F, R))$$

now supposing that $G = R$ is a ring.

e. This

$$F_p^m \times F_s^n \overset{\smile}{\to} F_{p+s}^{m+n}$$

which induces

$$F_p^m / F_{p+1}^m \times F_s^n / F_{s+1}^n \to F_{p+s}^{m+n} / F_{p+s+1}^{m+n}$$

and is

$$F_\infty^{p,m-p} \times E_\infty^{s,n-s} \to E_\infty^{p+s,m+n-p-s}$$

example. We have computed the cohomology of $\mathbb{C}P^n$ using Serre spectral sequence.

example. Consider the fibration

$$\Omega S^3 \longrightarrow \underset{\substack{\simeq \\ \mathrm{pt}}}{PS^3} \longrightarrow S^3$$

We use the following

claim. On $E^3, E^4, \ldots$ are equal when the differentials on the second page are 0.

Proof. It's because

$$E_r^{p,q} = 0 \implies E_r^{p,q} = 0$$

We have $E_2^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, so we have $E_3^{p,q} = 0$ for $q \geqslant 2$ or $q < 0$, and finally $d_r = 0$ for $r \geqslant 3$. More finally,

$$d_r : E^{p,q} \to E^{p+q} \to E^{p+r,q-r+1}$$

for $-r+1 \geqslant -2$. $\square$

Notice that

$$\begin{array}{|cccc}
\cdots & & & \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
0 & 0 & 0 & 0 \\
\mathbb{Z} & 0 & 0 & \mathbb{Z} \\
\hline
\end{array}$$

We have also

References

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