homotopy theory

Contents

abstract nonsense	1
elementary concepts	7
the right category	8
cofibrations	9
model structures	13
whitehead theorem	16
lecture notes 14 mar 18 mar Serre fibration long exact sequence (21 march) blakers-massey (26 march) freudenthal theorem (2 april) another application of Blakers-Massey (2 april)	22 24 25
References	28

abstract nonsense

definition.

- An *initial object* in a category \mathcal{C} is an object \varnothing such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source \varnothing and target x.
- For *C* any category, its *arrow category* Arr(*C*) is the category such that
 - an object a of Arr(C) is a morphism $a : a_0 \rightarrow a_1$ of C,
 - a morphism $f: a \to b$ of $Arr(\mathcal{C})$ is a commutative square



in C,

– composition in $Arr(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

$$Arr(C) := Funct(I, C) = [I, C] = C^{I}$$

for I the intervale category $\{0 \rightarrow 1\}$.

• An *equalizer* is a limit

$$eq \xrightarrow{e} X \xrightarrow{f \atop q} Y$$

over a parallel pair of morphisms f and g. This means that for $f: X \to Y$ and $g: X \to Y$ in a category C, their equalizer, if it exists, is

- an object $eq(f, g) \in C$,
- a morphism $eq(f, g) \rightarrow X$
- such that
 - * pulled back to eq(f, g) both morphisms become equal:

$$eq(f,g) \, \longrightarrow \, X \, \stackrel{f}{\longrightarrow} \, Y \quad \ = \quad [\ eq(f,g) \, \longrightarrow \, X \, \stackrel{g}{\longrightarrow} \, Y$$

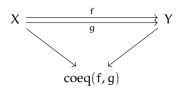
* and eq(f, g) is the universal object with this property.

The dual concept is that of coequalizer.

• The concept of coequalizer in a general category is the generalization of the construction where out of two functions f and g between sets X and Y one forms the set Y/ \sim of equivalence classes induced by the equivalence relation $f(x) \sim g(y)$. This means the quotient function $p: Y \to Y/\sim$ satisfies

$$\mathfrak{p} \circ \mathfrak{f} = \mathfrak{p} \circ \mathfrak{q}$$
.

In some category \mathcal{C} , the *coequalizer* coeq(f,g) of two parallel morphisms f and g between two objects X and Y, if it exists, is the colimit under the diagram formed by these two morphisms

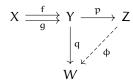


Equivalently, in a category C a diagram

$$X \xrightarrow{f \atop g} Y \xrightarrow{p} Z$$

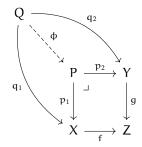
is called a *coequalizer* diagram if

- 1. $p \circ f = p \circ g$,
- 2. p is universal for this property: if $q: Y \to W$ is a morphism of \mathcal{C} such that $q \circ f = q \circ g$, then there is a unique morphism $\phi: Z \to W$ such that $\phi \circ p = q$

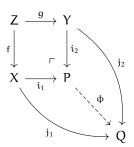


The coequalizer in C is equivalently an equializer in the opposite category C^{op} .

• A *pullback* of the morphisms f and g consists of an object P and two morphisms $p_1: P \to X$ and $p_2: P \to Y$ satisfying the following universal property:



• A *pushout* of the morphisms f and g consists of an object P and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:



remark. Other names for the pushout are *cofibered product of* X *and* Y (especially in algebraic categories when i_1 and i_2 are monomorphisms), or *free product of* X *and* Y with Z *amalgamated sum*, or more simply an *amalgamation* or *amalgam of* X *and* Y.

remark. If coproducts exist in some category, then the pushout

$$Z \xrightarrow{g} Y$$

$$\downarrow f \qquad \qquad \downarrow i_2$$

$$X \xrightarrow{i_1} X \coprod_Z Y$$

is equivalently the coequalizer

$$X \xrightarrow[i_2 \circ q]{} X \coprod Y \longrightarrow X \coprod_Z Y$$

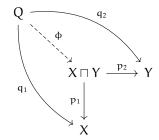
of the two morphisms induced by f and g into the coproduct of X with Y.

example (wiki).

– If X, Y and Z are sets and f, g are functions, the pushout of f and g is the disjoint union of X and Y where elements sharing a common preimage in Z are identified, i.e. $P = (X \coprod Y) / \sim$ where \sim is the finest equivalence relation such that $f(z) \sim g(z)$ for all $z \in Z$.

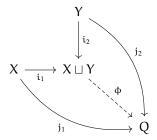
In particular, if X and Y are subsets of some larger set W and Z is their intersection, with f and g the inclusion maps of Z into X and Y, then the pusout can be canonically identified with the union $X \cup Y \subseteq W$.

- The construcion of *adjunction spaces* is an example of pushouts in Top. More precisely, if Z is a subspace of Y and $g:Z\to Y$ is the inclusion map, we can glue Y to another space X along Z using an *attaching map* $f:Z\to X$. The result is the *adjunction space* $X\cup_f Y$ which is just the pushout of f and g. More generally, all identification spaces may be regarded as pushouts in this way. See ?? .
- A *product* of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:



• A *coproduct* of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$,

 $i_2: Y \to X \sqcup Y$ satisfying the following universal property:



remark. More generally, for S any set and $F: S \to C$ a collection of objects in C indexed by S, their *coproduct* is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

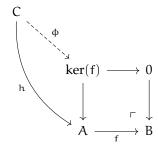
$$F(s) \to \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from F(s).

The *kernel* of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object 0 and pullbacks, the *kernel* ker f of a morphism f: A → B is the pullback ker(f) → A along f of the unique morphism 0 → B

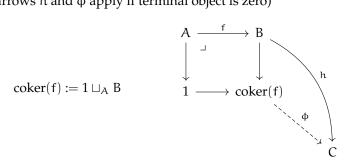
More explicitly, this characterizes the object ker(f) as *the* object (unique up to isomorphism) that satisfies the following universal property:

for every object C and every morphism $h:C\to A$ such that $f\circ h=0$ is the zero morphism, there is a unique morphism $\varphi:C\to \ker(f)$ such that $h=p\circ \varphi.$



• In a category with a terminal object 1, the *cokernel* of a morphism $f : A \to B$ is the

pushout (arrows h and ϕ apply if terminal object is zero)



In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object coker(f) with the following universal property:

for every object C and every morphism $h: B \to C$ such that $h \circ f = 0$ is the zero morphism, there is a unique morphism $\varphi : coker(f) \to C$ such that $h = \varphi \circ i$.

• A morphism $f: X \to Y$ is a *monomorphism* if for every object Z and every pair of morphisms $g_1, g_2: Z \to X$ then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

$$Z \xrightarrow{g_1} X \xrightarrow{f \circ g_2} Y$$

Equivalently, f is a monomorphism if for every Z the hom-functor $\mathsf{Hom}(\mathsf{Z},-)$ takes it to an injective function

$$\text{Hom}(Z,X) \stackrel{f_*}{\longrightarrow} \text{Hom}(Z,Y).$$

Being a monomorphism in a category C means equivalently that it is an epimorphism in the opposite category C^{op} .

• A morphism $f: X \to Y$ is a *epimorphism* if for every object Z and every pair of morphisms $g_1, g_2: Y \to Z$ then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

$$X \xrightarrow{f} Y \xrightarrow{g_1 \otimes f} Z$$

Equivalently, f is a epimorphism if for every Z the hom-functor Hom(-, Z) takes it to an injective function

$$Hom(Y,Z) \stackrel{f^*}{\longrightarrow} Hom(X,Z).$$

Being a monomorphism in a category \mathcal{C} means equivalently that it is an monomorphism in the opposite category \mathcal{C}^{op} .

elementary concepts

definition.

Let X and Y be topological spaces and f, g : X → Y continuous maps. An *homotopy* from f to g is a continuous map

$$H: X \times [0,1] \rightarrow Y$$
, $(x,t) \mapsto H(x,t) = H_t(x)$

) such that f(x) = H(x,0) and g(x) = H(x,1) for all $x \in X$. We denote this situation by $f \simeq g$. The homotopy relation \simeq is an equivalence relation on the set of continuous maps $X \to Y$. A homotopy of maps $H_t : X \to Y$ is called *relative to* $A \subset X$ if $H_t|_A$ is constant.

- Topological spaces and homotopy classes of maps form a quotient category of Top, the *homotopy category* h-Top, where comoposition of homotopy classes is induced by composition of representing maps. If f: X → Y represents an isomorphism in h-Top, then f is called a *homotopy equivalence* or h-*equivalence*. In explicit termins this means f: X → Y is a homotopy equivalence if there exists g: Y → X, a *homotopy inverse of* f, such that gf and fg are both homotopic to the identity. Spaces X and Y are called *homotopy equivalent* or of the same *homotopy type* if there exists a homotopy equivalence X → Y. A space is *contractible* if it is homotopy equivalent to a point. A map f: X → Y is *null homotopic* if it is homotopic to a constant map.
- Let (X, x_0) be a pointed topological space and $s_0 \in S^n$. The elements of the n-th homotopy group are homotopy classes of maps $(S^n, s_0) \to (X, x_0)$. Equivalently, they are homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. (Homotopies are required to preserve the base points, $s_0 \mapsto x_0$ or $\partial I^n \mapsto x_0$.)

Also,

$$\pi_n(X,*) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n/\partial I^n, X]^0$$

where [X, Y] denotes the set of homotopy classes [f] of maps [f] : $X \rightarrow Y$.

proposition 1. $\pi_n(X, x_0)$ is an abelian group for all $n \in \mathbb{N}$.

• Let A be a subspace of X and $x_0 \in A$. The elements of the *relative homotopy group* $\pi_n(X, A, x_0)$ are homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ where J^{n-1} is the union of all but one face of I^n . That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

The elements of such a group are homotopy classes of based maps $D^n \to X$ which carry the boundary S^{n-1} into A. Two maps f, g are called *homotopic relative to* A if they are homotopic by a basepoint-preserving homotopy $F: D_n \times [0,1] \to X$

such that, for each p in S^{n-1} and t in [0,1], the element F(p,t) is in A. Ordinary homotopy groups are recovered for the case in which $A = \{x_0\}$.

remark. This construction is motivated by looking for the kernel of the induced map $i_*: \pi_n(A, x_0) \to \pi_n(X, x_0)$ by the inclusion. This map is in general not injective, and the kernel consists of ?

• For any pair (X, A, x) we have a long exact sequence

$$\pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_{n-1}(X,A,x_0) \xrightarrow{\vartheta} \pi_{n-1}(A,x_0) \longrightarrow \cdots \longrightarrow \pi_0(X,x_0)$$

where i and j are the inclusions $(A,x_0)\hookrightarrow (X,x_0)$ and $(X,x_0,x_0)\hookrightarrow (X,A,x_0)$. The map $\mathfrak d$ comes from restricting maps $(I^n,\mathfrak d I^n,J^{n-1})\to (X,A,x_0)$ to I^{n-1} , or by restricting maps $(D^n,S^{n-1},s_0)\to (X,A,x_0)$. The map, called the *boundary map*, is a homomorphism when n>1.

- A space X with basepoint x_0 is called n-*connected* if $\pi_i(X, x_0) = 0$ for $i \le n$. Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair (X, A) is n-connected if $\pi(X, A, x_0) = 0$ for $i \le n$.
- Two pointed spaces (X, x_0) and (Y, y_0) are n-equivalent if $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$ for all $i \leq n$.

the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on *compactly generated wakly Hausdorff* spaces.
- X is *compactly generated* if, for any subset $C \subset X$, and for all continuous maps $f: K \to X$ from compact Housdorff spaces,

if $f^{-1}(C)$ is closed in K, then C is closed.

claim (What I picked up from the lecture). If X is compactly generated, then X is weakly Hausdorff if the diagonal subset $\Delta_X \subset X \times X$ is k-closed.

From May: The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

proposition 2. If X is compactly generated and $\pi: X \to Y$ is a quotient map, then Y is compactly generated if and only if $(\pi \times \pi)^{-1}(\Delta Y)$ is closed in $X \times X$

The interpretation is that a quotient space of a compactly generated space by a "closed equivalence relation" is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH, Hom(X, Y) is a space with the compact-open topology. This is a compactly generated space, k(Hom(X, Y)).

remark. (Also see wiki on currying)

$$Map(X, Y) := the space of maps X \rightarrow Y.$$

 $Map(X \times Y, Z) \cong Map(X, Map(Y, Z))$
 $Hom(X \times Y, Z) \cong Hom(X, Map(Y, Z))$

In the last line, product is product in CGWH, not in Top.

The functor $- \times Y$ is left adjoint to Map(Y, -).

cofibrations

definition.

• A morphism i has the *left lifting property with respect to a morphism* p and p has the *right lifting property with respect to* i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists φ (I think not necessarily unique) completing the diagram:



(wiki) In mathematics, in particular in homotopy theory, a continuous map between topological spaces i : A → X is a *cofibration* if it has the *homotopy extension* property with respect to all topological spaces S.

That is, i is a cofibration if

- for each topological space S,
- and for any continuous maps $f, f' : A \rightarrow S$
- and $g : X \rightarrow S$ with g ∘ i = f,
- for any homotopy $h: A \times I \rightarrow S$ from f to f',

there is a continuous map $g':X\to S$ and a homotopy $h':X\times I\to S$ from g to g' such that

$$h'(i(a), t) = h(a, t)$$
 for all $a \in A$ and $t \in I$.

• (wiki) Let X be a topological space and let $A \subset X$. We say that the pair (X,A) has the *homotopy extension property* if, given a homotopy $f_{\bullet}: A \to Y^{I}$ and a map $\tilde{f}_{0}: X \to Y$ such that

$$\tilde{f}_0\circ\iota=f_0$$

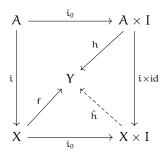
(so $\tilde{\mathbf{f}}$ is the lift of $\mathbf{f}_0: A \to Y$) then there exists an *extension* of \mathbf{f}_{\bullet} to a homotopy $\tilde{\mathbf{f}}_{\bullet}: X \to Y^I$ such that $\tilde{\mathbf{f}}_{\bullet} \circ \iota = \mathbf{f}_{\bullet}$.

That is,



So there's some currying to make usual homotopies $f_{\bullet}: A \times I \to Y$ look like $f_{\bullet}: A \to Y^{I}$. Or, as said in our lectures, !a homotopy $X \times I \to Y$ is the same as a map $X \to Map(I,Y)$ ".

• (May) A map $i:A\to X$ is a *cofibration* if it satisfies the *homotopy extension property (HEP)*. This means that if $h\circ i_0=f\circ i$ in the diagram



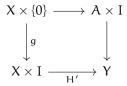
then there exists \tilde{h} that makes the diagram commute.

In traditional topology, one usually means a Hurewicz cofibration. A map i : A →
X between topological spaces is a *Hurewicz cofibration* if it satisfies the homotopy
extension property.

Let's say it one more time: for any $g:X\to Y$ and any homotopy $H:A\times I\to Y$ such that

$$\begin{array}{ccc}
A \times \{0\} & \longrightarrow & A \times I \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & Y
\end{array}$$

there is $H': X \times I \rightarrow Y$,



such that



example. $\partial D^n \to D$ is a Huerwicz cofibration. Why?

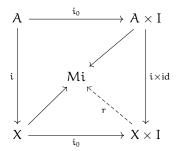
exercise. Prove that an inclusion $f: A \to X$ is a Hurewicz cofibration if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.

definition (Mapping cylinder).

• (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. make the dashed map unique—up to something maybe). Namely, we can let Y in our original test diagram be the *mapping cylinder*

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of i and i_0 . Indeed, suppose that we can construct a map r that makes the following diagram commute



By the universal property of the pushouts, given maps f and h in our original test diagram induce a map $Mi \rightarrow Y$, and its comoposite with r gives a homotopy \tilde{h} that makes the diagram commute. So just saying that Mi is universal.

• (nLab) Given a continuous map $f: X \to Y$ of topological spaces, one can define its

mapping cylinder as a pushout

$$X \xrightarrow{f} Y$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \times I \xrightarrow{(\sigma_0)_*(f)} Cyl(f)$$

in Top, where I = [0,1] and $\sigma : X \to X \times I$ is given by $x \mapsto (x,0)$.

Set theoretically, the mapping cyllinder is usually represented as que quotient space

$$(X \times I \coprod Y) / \sim$$

where \sim is the smallest equivalence relation identifying $(x, 0) \sim f(x)$ for all $x \in X$.

• (wiki) The *mapping cylinder* of a function f between topological spaces X and Y is the quotient

$$M_f = (([0,1] \times X) \coprod Y) / \sim$$

where II denotes disjoint union, and ~ is the equivalence relation generated by

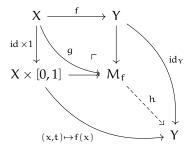
$$(0, x) \sim f(x)$$
 for each $x \in X$.

That is, the mapping cylinder M_f is obtained by gluing one end of $X \times [0,1]$ to Y via the map f. Notice that the "top" of the cylinder $\{1\} \times X$ is homeomorphic to X, while the "bottom" is the space $f(X) \subset Y$.

(Dani) So the mapping cylinder is just a deformation of X to f(X).

• (Homework) Let $f: X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the *mapping cylinder of* f, i.e. the pushout of $X \stackrel{\cong}{\to} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f: X \times Y$.

exercise. Let $g: X \to M_f$ be the map $X \stackrel{\cong}{\to} X \times \{1\} \to M_f$. Let $h: M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y: (x,t) \mapsto f(x)$ and $id_Y: Y \to Y$. Observe that f is the composition of g and h.



In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(colim A_i) \times B$ is canonically homeomorphic with $colim(A_i \times B)$.

- 1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
- 2. Show that $g: X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

- 1. That h is a deformation retract is consequence of the last diagram. Why is it a homotopy equivalence?
- 2. Consider the following lifting problem:

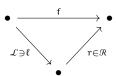


Looks nice but why should the dashed arrow exist...?

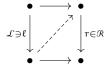
model structures

definition (Riehl). A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{M} is comprised o two clases of morphisms \mathcal{L} and \mathcal{R} so that

1. Every morphism in $\mathcal M$ may be factored as a morphism in $\mathcal L$ followed by a morphism in $\mathcal R\colon$

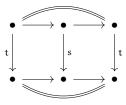


2. The maps in \mathcal{L} have the *left lifting property* with respect to each map in \mathcal{R} and equivalently the maps in \mathcal{R} have the *right lifting property* with respect to each map in \mathcal{L} , that is, any commutative square



admits a diagonal filler as indicated making both triangles commute.

3. The classes $\mathcal L$ and $\mathcal R$ are each closed under retracts in the arrow category: given a commutative diagram



if s is in that class then so is its retract t.

definition (Lecture). A *model structure* on a category \mathcal{A} is a choice of subcategories $\mathcal{W}, \mathcal{C}, \mathcal{F}$ called *weak-equivalences, cofibrations* and *fibrations* with the following properties:

- 1. Given $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$, if either 2 out of 3 among f, g, f \circ g are in W then all of them are.
- 2. $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are both weak factorization systems. $(\mathcal{B}, \mathcal{D})$ is a weak factorization system.
 - (a) Any morphism in $\mathcal I$ can be factored as a morphism in $\mathcal B$ followed by a morphism in $\mathcal D$.
 - (b) Lifts:



Two interesting model category structures on CGWH.

- 1. Hurewicz model structure (Strom).
 - Cofibrations:= Huerwicz cofibrations.
 - Fibrations:= maps $E \rightarrow B$ such that for all spaces X [Photo1].
 - Weak equivalences:= homotopy equivalences.
- 2. Quillen model structure.
 - Cofibrations = retracts of relative cell complexes.

• Weak equivalences: $f: X \to Y$

exercise (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal L$ characterized by the left lifting property against a fixed class of morphisms $\mathcal R$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. (*Coproducts.*) Sergey: Coproduct of morphisms $A_i \to B_i$ in a category C is the obvious morphism $\sqcup A_i \to \sqcup B_i$. (Because in this construction morphisms $A_i \to B_i$ are seen as objects of what's called the arrow category of the category C)

Suppose the maps $\ell_i : A_i \to B_i$ are in \mathcal{L} . Then their coproduct in the arrow category is the obvious map $\coprod A_i \to \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i:\ell_i\to\coprod\ell_i$ such that for any other object $m:A\to B$ in the arrow category and a map $g:\ell\to m$, the following diagram is completed uniquely:

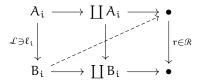
$$\ell_i \xrightarrow{f_i} \coprod \ell_i \xrightarrow{-\exists !} m \qquad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

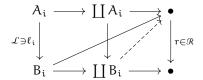
$$\begin{array}{ccc} A_{i} & \xrightarrow{\ell_{i}} & B_{i} \\ f_{i}^{1} & & \downarrow f_{i}^{2} \\ \coprod A_{i} & \xrightarrow{\coprod \ell_{i}} & \coprod B_{i} \\ \exists ! & & \downarrow \exists ! \\ A & \xrightarrow{B} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

Since $\ell_i \in \mathcal{L}$, we have maps



which in turn means we have unique maps



by the universal property of the coproduct $\coprod B_i$.

So, to check that the lower-right triangle commutes, it would be sufficient to show that the map $B_i \to \coprod B_i$ "can be cancelled" since



Likeways, to make sure that the remaining triangle commutes we observe that



Why can we "cancel" the maps $A_i \to \coprod A_i$ and $B_i \to \coprod B_i$?

remark (Plan). Blakers-Massey excision theorem (relies on technical lema, proof from Tom Dieck's book) ⇒ Cellular approximation. Also ⇒ Freudental theorem.

exercise. $X \to M_f \to Y$. Prove $X \to M_f$ is a cofibration.

whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

definition (May).

1. A *CW complex* X is a space X which is the union of an expanding sequence of subspaces X^n such that, inductively, X^0 is a discrete set of points (called *vertices*) and X^{n+1} is the pushout obtained from X^n by attaching disks D^{n+1} along *attaching maps* $j: S^n \to X^n$. Thus X^{n+1} is the quotient space obtained from $X^n \cup (J_{n+1} \times J_n)$

 D^{n+1}) by identifying (j,x) with j(x) for $x \in S^n$, where J_{n+1} is the discrete set of such attaching maps j (see ??). Each resulting map $D^{n+1} \to X$ is called a *cell*. The subspace X^n is called the n-skeleton of X.

$$S^{n} \stackrel{i}{\longleftrightarrow} D^{n+1}$$

$$\downarrow \qquad \qquad \downarrow$$

$$X^{n} \longrightarrow X^{n+1}$$

lemma 3 (HELP). content...

Theorem 4 (Whitehead, May). If X is a CW complex and $e: Y \to Z$ is an n-equivalence, then $e_*: [X, Y] \to [X, Z]$ is a bijection if dim X < n and surjection if dim X = n.

Theorem 5 (Whitehead, May). An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 6 (Whitehead (4.5), Hatcher). If a map $f: X \to Y$ between connected CW complexes induces isomorphisms $f_*: \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12). Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X)=0$ for all $i\leqslant n$, which means that X is contractible by theorem 5.

lecture notes

14 mar

$$(X^{Y})^{Z} \cong Z^{Y \times X}$$

$$g: X' \to X$$

$$Hom(X,Y) \mapsto Hom(X',Y)$$

$$Hom(A,B) \cong Hom(A,B') \text{ natual in } A \Longrightarrow$$

$$Hom(B,B) \cong Hom(B,B') \& Hom(B',B) \cong Hom(B',B')$$

$$\Longrightarrow B \cong B'.$$

- for () commutativity of the hypotesis gives us commutativity of the right-most square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an n-equivalence.
- π_n of loop space is the same as π_{n+1} of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.
- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably *some name* theorem.

18 mar

lemma 7 (Yoneda).

{Natural transformations
$$Hom(-, X) \rightarrow F$$
} \cong $F(X)$

corollary 8.
$$(\text{Hom}(-,X) \to \text{Hom}(-,Y)) \cong \text{Hom}(X,Y).$$

corollary 9. The correspondence $X \mapsto \text{Hom}(-,X)$ is fully faithful, that is, the correspondence $\text{Hom}(X,X') \to \text{Hom}(\text{Hom}(-,X),\text{Hom}(-,X'))$ is injective and bijective. (The right hand side are natural transformations of functors.)

Solution of exercise 1. The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude $X \cong X'$.

lemma 10. Let $E \times_B X$ be the pullback of

$$X \xrightarrow{\simeq} B$$

be such that $E \to B$ is an homotopy fibration and $f: X \to B$ is a homotopy equivalence. Let

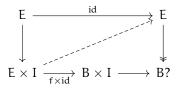
be the pullback. Then $E \times_B X \to E$ is a homotopy equivalence.

Proof. Let $g: B \to X$ be the homotopy inverse of f.

(Step 1) Construct another pullback

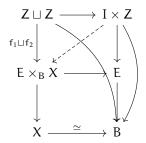
(Step 2) Constuct $E \to E \times_B B$.

Consider



And then $E \to E \times_B B \to E \times_B X \to E$ is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.



corollary 11. B $\stackrel{f}{\rightarrow}$ B is homotopy equivalence and E \rightarrow B is a fibration, in

$$\begin{array}{cccc} E \times_B B & \longrightarrow & E \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B \end{array}$$

 $E \times_B B \to E$ is a homotopy equivalence.

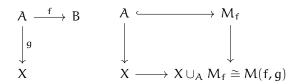
exercise. If fg is an isomorphism and f and g have right inverses, then f and g are isomorphisms.

lemma 12. Let

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow^{g} & \downarrow \\
X & \longrightarrow X \cup_{A} & B
\end{array}$$

be a pushout with $A \to X$ a cofibration. Then the canonical map from the double mapping cylinder $M(f,g) \to X \cup_A B$ is a homotopy equivalence.

remark.



definition.

• The *homotopy pullback* of a diagram



is

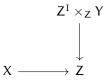
Intuitively, for any $x \in X$ and $y \in Y$ this object has the space of paths connecting x and y.

• The *homotopy fiber* if $f: Y \to Z$ is the pullback of



 $F \subset Z^I \times_Z Y \to Z$, where F is the space of paths starting at x and ending at the same point f(y).

remark. The pullback of



is the motopy pullback of

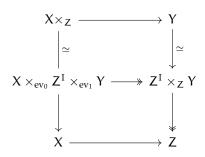


lemma 13. If $X \to Z$ is a fibration then for

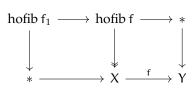


the map from the pullback to the homotopy pullback is a homotopy equivalence.

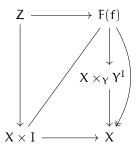
Proof.



Finally,



and



and an exact sequence

$$\Omega^2 \, hofib \, \rightarrow \, \Omega^2 X \, \rightarrow \, \Omega^2 Y \, \rightarrow \, \Omega \, hofib \, f \, \rightarrow \, \Omega X \, \rightarrow \, \Omega Y \, \rightarrow \, hofib \, f \, \rightarrow \, X \, \stackrel{f}{\rightarrow} \, Y$$

lemma 14 (Exactness). $\forall z$, [z hofib f] \rightarrow [Z, X] \rightarrow [Z, Y].

and we get the exact sequence

$$\pi_0(\Omega^2 X) \, \rightarrow \, \pi_0(\Omega^2 Y) \, \rightarrow \, \pi_0(\Omega \, hofib \, f) \, \rightarrow \, \pi_0(\Omega X) \, \rightarrow \, \pi_0(\Omega Y) \, \rightarrow \, \pi_0(hofib \, f) \, \rightarrow \, \pi_0(X) \, \rightarrow \, \pi_0(Y) \, \rightarrow \, \pi_0(\Omega^2 X) \, \rightarrow \, \pi_0(\Omega^2 Y) \, \rightarrow \, \pi_0($$

and then

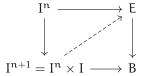
$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewickz fibrations. Let's talk about Serre fibrations. Notice that H. fibration \implies S. fibration. What is the most natural example of a Serre fibration?

proposition 15. Let E be a fiber bundle with fiber F. Then f is a Serre fibration.

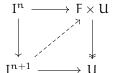
Proof. What sdoes it mean to be a Serre fibration? It means that



So if \mathcal{U} is a covering of B such that $f^{-1}U \cong U \times F$. By Lebesgue lemma, there is a $\delta > 0$ such that for all $x \in I^{n+1}$, the ball $B(x, \delta)$ lies in some $f^{-1}U$ for some U.

Then we subdivide I^{n+1} in smaller cubes of the same size with diameter $<\delta$. So, each the image of each cube lies in some $U\in\mathcal{U}$.

Then



has a lift for every little square because



is always a fibration (think about this) and because pullbacks of fibrations are fibrations:

. Then we may just add up the squares because

and we're done. \Box

proposition 16 (Construction of homotopy long exact sequence from relative homotopy long exact sequence). Let $g: E \to B$ is a Serre fibration. $e \in E$, g(e) = b and $g^{-1} = F$. Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

example. We have shown that $\pi_2(\mathbb{C}P^n) \cong \mathbb{Z}$ using the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ and the fact that $\pi_k(S^n) = 0$ for k < n.

Theorem 17. Let X be a CW-comples, A, B \subset X subcomplexes, C = A \cap B $\neq \emptyset$, so

$$\begin{array}{ccc}
C & \longrightarrow & A \\
\downarrow & & \downarrow \\
B & \longrightarrow & X
\end{array}$$

is a pushout (this happens for inclusions, check it?).

If (A, C) is n-connected and (B, C) is m-connected, then

$$\pi_i(A,C) \to \pi_i(X,B)$$

is an isomorphism for i < m + n and sujerctive for i = m + n.

blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let $f: X \to Y$ be a map. Consider the pullback

$$\begin{array}{c} W(f) & \longrightarrow & Y^I \\ (q,p) & & \downarrow (ev_0,ev_1) \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

where

$$W(f) = \{(x, w) \in X \times Y^{I} | f(x) = w(0) \},$$

 $q(x, w) = x, \quad p(x, w) = w(1).$

Since (ev_0, ev_1) is a fibration, the maps (q, p), q and p are fibrations.

Now suppose f is a pointed map with base points *. Then $W(f) \to W'$ is given the base point $(*, k_*)$.

Let $f : A \hookrightarrow X$ be an inclusion.

definition. By $(I^n, \partial I^n) \to (* \times_{ev_0} X^I \times_{ev_1} A, pt)$ is the same as a map $I^n \times I \to X$ that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \rightarrow *$.
- $I^n \times \{1\} \rightarrow A$.

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow hofib \longrightarrow A \longrightarrow X$$

Theorem 18 (Blakers-Massey 1). Let

$$\begin{array}{ccc}
Q & \xrightarrow{g} & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}$$

be a homotopy pushout, g is m equivalence, f is n-equivalence and m, n $\geqslant 0$. Then $Q \to X \times_P^h$ is (m+n-1)-equivalence.

Theorem 19 (Blakers-Massey 2). P is a CW-complex, X, Y subcomplexes, $X \cap Y = Q \neq \emptyset$ (*strict pushout*)

$$\begin{array}{ccc}
Q & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longmapsto & X
\end{array}$$

Then $\pi_i(Y,\mathbb{Q}) \to \pi_i(P,X)$ is epi for i = m + n and iso for $0 \le i < m + n$.

Theorem 20 (Blakers-Massey 3). $P = X \cup Y$, X and Y are open in P, $X \cap Y = Q \neq \emptyset$.

We proved the third version based on Tom Dieck's proof.

definition.

- A map is a k-equivalence if the induced map on the ith homotopy group is an isomorphism for i < k and an epimorphism for i = k.
- $K_p(W) := \{x \in W : \text{ at least } p \text{ coordinates of } x \text{ are } \}$ the same coordinates of the center of $W\}$

lemma 21. Let W be a cube in \mathbb{R}^d with $\dim W \leq d$. If for all faces W' of ∂W , $f(W') \in A \implies w' \in K_p(W')$, then there is a homotopy $f \simeq g$ rel ∂W such that $g(w) \in A \implies w \in K_p(W)$.

freudenthal theorem (2 april)

definition. The appropriate analogue of the Cartesian product in the category of based spaces is the *smash product* $X \wedge Y$ defined by

$$X \wedge Y = X \times Y/X \vee Y$$
.

Here $X \vee Y$ is viewed as the subspace of $X \times Y$ consisting of those pairs (x, y) such that either x is the basepoint of X or y is the basepoint of Y.

We also have the *suspension of pointed spaces*, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X)/(t, x) \sim (0, y) \sim (1, y) \ \forall y \in X.$$

Then we have

$$Hom_{CGWH_*}(\Sigma X, \Sigma X) \cong Hom_{CGWH_*}(X, \Omega \Sigma X)$$

where $\Sigma X = S^{\wedge}X$ and $\Omega \Sigma X = \text{Map}(S^1, \Sigma X)$. That is, $S^1 \wedge -$ is adjoint to $\text{Map}(S^1, -)$.

So let X be a space. The identity map $id_{\Sigma X}: \Sigma X \to \Sigma X$ then induces a map $X \to \Omega \Sigma X$.

Theorem 22 (Freudenthal). Let X be ℓ -connected space. Then $X \to \Omega \Sigma X$ is a $(2\ell + 1)$ -equivalence, that is,

$$\pi_{i}(X) \to \pi_{i+1}(\Sigma X)$$
,

$$(\ell+1) + (\ell+1) - 1 = 2\ell+1.$$

Proof 1. Using pushouts...

Proof 2. Consider

$$\begin{array}{ccc}
X & \longrightarrow & CX \\
\downarrow & & \downarrow \\
CX & \longrightarrow & \Sigma X
\end{array}$$

Then we use relative homotopy long exact sequence with (X,CX) to get $\pi_i(CX,X)\cong \pi_{i-i}(X)$, which is zero for $0\leqslant i\leqslant \ell+1$. Then use relative homotopy exact sequence for the pair $(\Sigma X,CX)$. then we get that $\pi_i(\Sigma X,CX)=\pi_i(\Sigma X)$. And then if you use it for $(\Sigma X,X)$ and

But it also turns out that $\pi_i(\Sigma X)=\pi_{i-1}(\Omega \Sigma X)$ because

$$\pi_n(\mathsf{Z}) = \mathsf{Hom}_{h\text{-}\mathsf{Top},*}(\mathsf{S}^n,\mathsf{Z}) = \mathsf{Hom}(\mathsf{S}^1 \wedge \mathsf{S}^{n-1},\mathsf{Z}) = \mathsf{Hom}(\mathsf{S}^{n-1},\mathsf{\Omega}\mathsf{Z}) = \pi_{n-1}(\mathsf{\Omega},\mathsf{Z})$$

. And then since $CX \hookrightarrow \Sigma X$ we get an arrow $\pi_i(CX,X) \to \pi_i(\Sigma X,CX)$ which is isomorphism for $0 \leqslant i \leqslant 2\ell+1$ and surjective for $i=2\ell+2$.

So apply Blakers-Massey an ell equalities to get maps fro $\pi_{i-1}(X) \to \pi_{i-1}(\Omega \Sigma X)$ for i as desired. \Box

corollary 23. If X is ℓ -connected, then ΣX is $(\ell + 1)$ -connected for $\ell \geqslant 0$.

$$S^0$$
 $\Sigma S^0 = S^1$ $\Sigma^2 S^0 = S^2$ $\Sigma^3 S^0 = S^3$... $\Sigma^n S^0 = S^n$
-1 0 1 2 ... $(n-1)$

corollary 24. S^n is (n-1)-connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0 = \pi_2(S^3) \to \pi_2(S^2) \stackrel{\cong}{\to} \pi_1(S^1) \to \pi_1(S^3) = 0$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map $S^n \to S^n$. We get a map $CS^n \to CS^n$ (in general, for $f: X \to Y$ we have $(t,x) \mapsto (t,f(x))$ in the cones). We also have $CS^n \to CS^n/S^n = S^{n+1}$.

Now if we take $id: S^n \to S^n$ we shall get $id: S^{n+1} \to S^{n+1}$. Think about this like $\pi_1(S^1) \to \pi_2(S^2)$. Now from Freudenthal we get $\pi_{i-1}(X) \to \pi_i(\Sigma X)$ is surjective because i=0. From Hopf fibration we have $\pi_2(S^2)=\mathbb{Z}$. So we have a surjective map $\mathbb{Z} \to \mathbb{Z}$. So it's an isomorphism and we conclude that id_{S^2} is a generator of $\pi_2(S^2)$.

corollary 25. Since S^n is (n-1)-connected, we have

$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$

is isomorphism for $i \le 2(n-1) = 2n-1$ and epimorphism form i = 2n-1. (We just shift the indices of theorem 22 by one.)

corollary 26. $\pi_n(S^n) = \mathbb{Z}$ with id_{S^n} as generator.

corollary 27. $\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$ is isomorphism for $k \le n-1$ and epimorphism for k = n-1.

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are $\mathbb{Z}/2$. This is what are called the k*th stable homotopy groups of a sphere*. And more in general, we take any space and apply $\Omega\Sigma$ enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \to S^2$$

we get

$$0=\pi_{\mathfrak{i}}(S^1)\to\pi_{\mathfrak{i}}(S^3)\stackrel{\cong}{\to}\pi_{\mathfrak{i}}(S^2)\to\pi_{\mathfrak{i}-1}(S^2)=0$$

So $\pi_3(S^2) \cong \mathbb{Z}$ in case you were wondering.

claim (Serre). $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{finite abelian}$. And $\pi_i(S^k)$ is finite abelian in all other cases.

another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{array}{ccc}
S^{n-1} \xrightarrow{(n-1)\text{-equiv}} D^n \\
\text{0-equiv} & \downarrow \\
X & \longrightarrow X \cup D^n
\end{array}$$

Assume X is connected. We get a map from the vertical arrows

$$\pi_{\mathfrak{i}}(D^{\mathfrak{n}},S^{\mathfrak{n}-1})\,\longrightarrow\,\pi_{\mathfrak{i}}(X\cup D^{\mathfrak{n}},X)$$

which is (n-1)-equivalence since $S^{n-1}\to D^n$ is an (n-1)-equivalence, which is the case since

$$0=\pi_{\mathfrak{i}}(\mathsf{D}^{\mathfrak{n}})\, \longrightarrow \, \pi_{\mathfrak{i}}(\mathsf{D}^{\mathfrak{n}},\mathsf{S}^{\mathfrak{n}-1}) \, \stackrel{\cong}{\longrightarrow} \, \pi_{\mathfrak{i}-1}(\mathsf{S}^{\mathfrak{n}-1}) \, \longrightarrow \, \pi_{\mathfrak{i}-1}(\mathsf{D}^{\mathfrak{n}})=0$$

so
$$\pi_i(D^n, S^{n-1}) = 0$$
 for $i \leq n-1$.

References

- [1] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000 (cit. on p. 17).
- [2] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832 (cit. on pp. 8–11, 16, 17).
- [3] Emily Riehl. *Homotopical categories: from model categories to* (∞,1)-*categories.* 2020. arXiv: 1904.00886 [math.AT] (cit. on pp. 13, 15).