

algebraic topology exercises

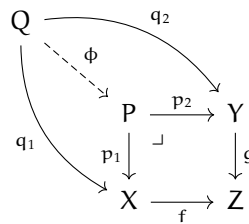
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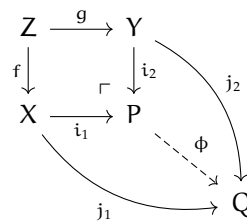
some definitions

definition.

- An *initial object* in a category \mathcal{C} is an object \emptyset such that for any object $x \in \mathcal{C}$ there is a unique morphism $\emptyset \rightarrow x$ with source \emptyset and target x .
- A *pullback* of the morphisms f and g consists of an object P and two morphisms $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ satisfying the following universal property:

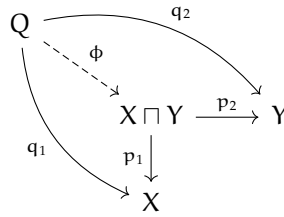


- A *pushout* of the morphisms f and g consists of an object P and two morphisms $i_1 : P \rightarrow X$ and $i_2 : P \rightarrow Y$ satisfying the following universal property:

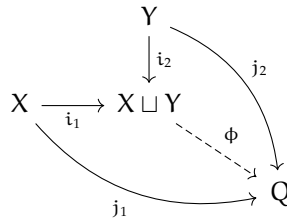


- A *product* of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcup Y \rightarrow X$,

$p_2 : X \sqcap Y \rightarrow Y$ satisfying the following universal property:



- A **coproduct** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \rightarrow X \sqcup Y$, $i_2 : Y \rightarrow X \sqcup Y$ satisfying the following universal property:

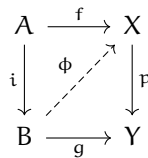


remark. More generally, for S any set and $F : S \rightarrow \mathcal{C}$ a collection of objects in the category \mathcal{C} indexed by S , their **coproduct** is an object $\coprod_{s \in S} F(s)$ equipped with maps

$$F(s) \rightarrow \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from $F(s)$.

- A morphism i has the **left lifting property with respect to a morphism** p and p has the **right lifting property with respect to** i if for each morphisms f and g , if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:



- For \mathcal{C} any category, its **arrow category** $\text{Arr}(\mathcal{C})$ is the category such that
 - an object a of $\text{Arr}(\mathcal{C})$ is a morphism $a : a_0 \rightarrow a_1$ of \mathcal{C} ,

- a morphism $f : a \rightarrow b$ of $\text{Arr}(\mathcal{C})$ is a commutative square

$$\begin{array}{ccc} a_0 & \xrightarrow{f_0} & b_0 \\ a \downarrow & & \downarrow b \\ a_1 & \xrightarrow{f_1} & b_1 \end{array}$$

in \mathcal{C} ,

- composition in $\text{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

$$\text{Arr}(\mathcal{C}) := \text{Funct}(I, \mathcal{C}) = [I, \mathcal{C}] = \mathcal{C}^I$$

for I the interval category $\{0 \rightarrow 1\}$.

exercise on mapping cylinder and Hurewicz cofibrations

exercise. Let $f : X \rightarrow Y$ be a map. Let $M_f = X \times [0, 1] \cup_f Y$ be the *mapping cylinder of f* , i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$ and of $f : X \times Y$. Let $g : X \rightarrow M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \rightarrow M_f$. Let $h : M_f \rightarrow Y$ be the map that is induced by $X \times [0, 1] \rightarrow Y : (x, t) \mapsto f(x)$ and $\text{id}_Y : Y \rightarrow Y$. Observe that f is the composition of g and h .

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\text{colim } A_i) \times B$ is canonically homeomorphic with $\text{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \rightarrow M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

1. We have that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id} \times 1 \downarrow & \searrow g & \downarrow \text{id}_Y \\ X \times [0, 1] & \xrightarrow{\quad} & M_f \\ & \searrow (x, t) \mapsto f(x) & \downarrow h \\ & & Y \end{array}$$

We must show that there is a homotopy between the identity map on M_f and a retraction from M_f to Y . So we want $h : M_f \times [0, 1] \rightarrow M_f$ such that

$$h(-, 0) = \text{id}_{M_f}, \quad \text{img } h(-, 1) \subset Y \quad \text{and} \quad h(-, 1)|_Y = \text{id}_Y$$

The fact that h is a deformation retract is consequence of this diagram. But I still can't see why it must be a homotopy equivalence...

2. Consider the following lifting problem:

$$\begin{array}{ccc} X & \xrightarrow{H} & Y^I \\ g \downarrow & \nearrow \text{dashed} & \downarrow \pi_0 \\ M_f & \xrightarrow{h} & Y \end{array}$$

Looks OK but why should the dashed arrow exist...?

□

exercise on model categories

exercise (3.1.8 from Riehl). Verify that the class of morphisms \mathcal{L} characterized by the left lifting property against a fixed class of morphisms \mathcal{R} is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. (Coproducts.) Comment from Sergey: Coproduct of morphisms $A_i \rightarrow B_i$ in a category \mathcal{C} is the obvious morphism $\coprod A_i \rightarrow \coprod B_i$. (Because in this construction morphisms $A_i \rightarrow B_i$ are seen as objects of what's called the arrow category of the category \mathcal{C})

Suppose the maps $\ell_i : A_i \rightarrow B_i$ are in \mathcal{L} . Then their coproduct in the arrow category is the obvious map $\coprod A_i \rightarrow \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \rightarrow \coprod \ell_i$ such that for any other object $m : A \rightarrow B$ in the arrow category and a map $g : \ell \rightarrow m$, the following diagram is completed uniquely:

$$\begin{array}{ccc} \ell_i & \xrightarrow{f_i} & \coprod \ell_i \xrightarrow{\exists!} m \\ & \searrow g & \uparrow \\ & & \end{array} \quad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{array}{ccc} A_i & \xrightarrow{\ell_i} & B_i \\ f_i^1 \downarrow & & \downarrow f_i^2 \\ \coprod A_i & \xrightarrow{\coprod \ell_i} & \coprod B_i \\ \exists! \downarrow & & \downarrow \exists! \\ A & \xrightarrow{m} & B \end{array}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{array}{ccc} \coprod A_i & \longrightarrow & \bullet \\ \downarrow \coprod \ell_i & & \downarrow r \in \mathcal{R} \\ \coprod B_i & \longrightarrow & \bullet \end{array}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

which in turn means we have a unique map

$$\begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \mathcal{L} \ni \ell_i \downarrow & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array}$$

by the universal property of the coproduct $\coprod B_i$.

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts $\coprod A_i$ and $\coprod B_i$ since, *in general*,

$$\text{Hom}\left(\coprod X_i, Y\right) \cong \prod \text{Hom}(X_i, Y).$$

More explicitly, we now that the paths in red in the following diagrams are the same:

$$\begin{array}{ccc} \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \downarrow \mathcal{L} \ni \ell_i & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} & \text{and} & \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \downarrow \mathcal{L} \ni \ell_i & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} \end{array}$$

and also

$$\begin{array}{ccc} \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \downarrow \mathcal{L} \ni \ell_i & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} & \text{and} & \begin{array}{ccccc} A_i & \longrightarrow & \coprod A_i & \longrightarrow & \bullet \\ \downarrow \mathcal{L} \ni \ell_i & & \downarrow & \nearrow & \downarrow r \in \mathcal{R} \\ B_i & \longrightarrow & \coprod B_i & \longrightarrow & \bullet \end{array} \end{array}$$

so the conclusion follows from the former comment. \square

Hatcher's exercise on Whitehead's theorem

Theorem 1 (Whitehead, May). If X is a CW complex and $e : Y \rightarrow Z$ is an n -equivalence, then $e_* : [X, Y] \rightarrow [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 2 (Whitehead, May). An n -equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 3 (Whitehead (4.5), Hatcher). If a map $f : X \rightarrow Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ for all n , then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is stronger: X is a deformation retract of Y .

exercise (Hatcher 4.1.12). Show that an n -connected, n -dimensional CW complex is contractible.

Solution. Just recall that n -connectedness means that $\pi_i(X) = 0$ for all $i \leq n$, which means that X is contractible by theorem 2. \square

References

- [1] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000 (cit. on p. 6).
- [2] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832 (cit. on p. 6).
- [3] Emily Riehl. *Homotopical categories: from model categories to $(\infty, 1)$ -categories*. 2020. arXiv: 1904.00886 [math.AT] (cit. on p. 4).