

# homotopy theory

## Contents

<b>abstract nonsense</b>	<b>1</b>
<b>elementary concepts</b>	<b>9</b>
<b>the right category</b>	<b>10</b>
<b>cofibrations</b>	<b>11</b>
<b>fibrations</b>	<b>15</b>
<b>model structures</b>	<b>17</b>
<b>whitehead theorem</b>	<b>19</b>
<b>lecture notes</b>	<b>20</b>
14 mar . . . . .	20
18 mar . . . . .	21
Serre fibration long exact sequence (21 march) . . . . .	24
blakers-massey (26 march) . . . . .	26
freudenthal theorem (2 april) . . . . .	27
another application of Blakers-Massey (2 april) . . . . .	30
<b>References</b>	<b>31</b>

## abstract nonsense

### definition.

- (Limits, [wiki](#).)

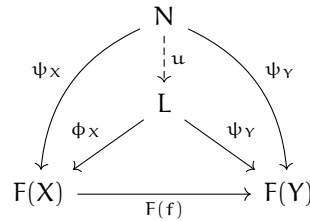
- A *diagram* of shape  $J$  in  $C$  is a functor from  $J$  to  $C$

$$F : J \rightarrow C.$$

The category  $J$  is thought of as an index category, and the diagram  $F$  is thought of as indexing a collection of objects and morphisms in  $C$  patterned on  $J$ .

- Let  $F : J \rightarrow C$  be a diagram of shape  $J$  in a category  $C$ . A *cone* to  $F$  is an object  $N$  in  $C$  together with a family  $\psi_X : N \rightarrow F(X)$  of morphisms indexed by the objects  $X$  of  $J$  (so a cone is an object and a bunch of maps from this object to certain objects that are governed by the diagram), so that for every morphism  $X \rightarrow Y$  in  $J$ , we have  $F(f) \circ \psi_X = \psi_Y$  (I guess this is what nLab meant when he said that everything in sight commutes).

- A **limit** of the diagram  $F : J \rightarrow C$  is a cone  $(L, \phi)$  to  $F$  such that for every cone  $(N, \psi)$  there exists a *unique* morphism  $u : N \rightarrow L$  such that  $\phi_X \circ u = \psi_X$  for all  $X$  in  $J$ .

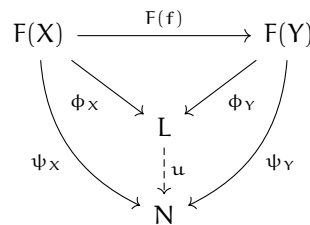


One says that the cone  $(N, \psi)$  factors through the cone  $(L, \phi)$  with the unique factorization  $u$ . The morphism  $u$  is sometimes called the *mediating morphism*.

Limits are also referred to as **universal cones** since they are characterized by a universal property. Limits may also be characterized as terminal objects in the category of cones to  $F$ .

It is possible that a diagram does not have a limit at all. However, if a diagram does have a limit then this limit is essentially unique: it is unique up to a unique isomorphism. For this reason one often speaks of *the* limit of  $F$ .

- **(Colimits, [wiki](#))** The dual notions of limits and cones are colimits and co-cones. Although it is straightforward to obtain the definitions of these by inverting all morphisms in the above definitions, we will explicitly state them here:
  - A **co-cone** of a diagram  $F : J \rightarrow C$  is an object  $N$  of  $C$  together with a family of morphisms  $\psi_X : F(X) \rightarrow N$  (so in the cone we are going from  $N$  and now we're going to  $N$ ) for every object  $X$  of  $J$ , such that for every morphism  $f : X \rightarrow Y$  in  $J$  we have  $\psi_Y \circ F(f) = \psi_X$  *everything in sight commutes*.
  - A **colimit** of a diagram  $F : J \rightarrow C$  is a co-cone  $(L, \phi)$  of  $F$  such that for any other co-cone  $(N, \psi)$  of  $F$  there exists a unique morphism  $u : L \rightarrow N$  such that  $u \circ \phi_X = \psi_X$  for all  $X$  in  $J$ .



Colimits are also referred to as **universal co-cones**. They can be characterized as initial objects in the category of co-cones from  $F$ .

As with limits, if a diagram  $F$  has a colimit then this colimit is unique up to a unique isomorphism.

- An **initial object** in a category  $\mathcal{C}$  is an object  $\emptyset$  such that for any object  $x \in \mathcal{C}$  there is a unique morphism  $\emptyset \rightarrow x$  with source  $\emptyset$  and target  $x$ .
- For  $\mathcal{C}$  any category, its **arrow category**  $\text{Arr}(\mathcal{C})$  is the category such that
  - an object  $a$  of  $\text{Arr}(\mathcal{C})$  is a morphism  $a : a_0 \rightarrow a_1$  of  $\mathcal{C}$ ,
  - a morphism  $f : a \rightarrow b$  of  $\text{Arr}(\mathcal{C})$  is a commutative square

$$\begin{array}{ccc} a_0 & \xrightarrow{f_0} & b_0 \\ a \downarrow & & \downarrow b \\ a_1 & \xrightarrow{f_1} & b_1 \end{array}$$

in  $\mathcal{C}$ ,

- composition in  $\text{Arr}(\mathcal{C})$  is given simply by placing commutative squares side by side to get a commutative oblong.

This is isomorphic to the functor category

$$\text{Arr}(\mathcal{C}) := \text{Funct}(I, \mathcal{C}) = [I, \mathcal{C}] = \mathcal{C}^I$$

for  $I$  the interval category  $\{0 \rightarrow 1\}$ .

- An **equalizer** is a limit

$$\text{eq} \xrightarrow{e} X \rightrightarrows Y$$

over a parallel pair of morphisms  $f$  and  $g$ . This means that for  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  in a category  $\mathcal{C}$ , their equalizer, if it exists, is

- an object  $\text{eq}(f, g) \in \mathcal{C}$ ,
- a morphism  $\text{eq}(f, g) \rightarrow X$
- such that
  - \* pulled back to  $\text{eq}(f, g)$  both morphisms become equal:

$$\text{eq}(f, g) \longrightarrow X \xrightarrow{f} Y = [ \text{eq}(f, g) \longrightarrow X \xrightarrow{g} Y$$

- \* and  $\text{eq}(f, g)$  is the universal object with this property.

The dual concept is that of coequalizer.

- The concept of coequalizer in a general category is the generalization of the construction where out of two functions  $f$  and  $g$  between sets  $X$  and  $Y$  one forms the set  $Y / \sim$  of equivalence classes induced by the equivalence relation  $f(x) \sim g(y)$ . This means the the quotient function  $p : Y \rightarrow Y / \sim$  satisfies

$$p \circ f = p \circ g.$$

In some category  $\mathcal{C}$ , the **coequalizer**  $\text{coeq}(f, g)$  of two parallel morphisms  $f$  and  $g$  between two objects  $X$  and  $Y$ , if it exists, is the colimit under the diagram formed by these two morphisms

$$\begin{array}{ccc} X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y \\ & \searrow \quad \swarrow & \\ & \text{coeq}(f, g) & \end{array}$$

Equivalently, in a category  $\mathcal{C}$  a diagram

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y \xrightarrow{p} Z$$

is called a **coequalizer** diagram if

1.  $p \circ f = p \circ g$ ,
2.  $p$  is universal for this property: if  $q : Y \rightarrow W$  is a morphism of  $\mathcal{C}$  such that  $q \circ f = q \circ g$ , then there is a unique morphism  $\phi : Z \rightarrow W$  such that  $\phi \circ p = q$

$$\begin{array}{ccccc} X & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & Y & \xrightarrow{p} & Z \\ & & \downarrow q & \swarrow \phi & \\ & & W & & \end{array}$$

The coequalizer in  $\mathcal{C}$  is equivalently an equalizer in the opposite category  $\mathcal{C}^{\text{op}}$ .

- A **pullback** of the morphisms  $f$  and  $g$  consists of an object  $P$  and two morphisms  $p_1 : P \rightarrow X$  and  $p_2 : P \rightarrow Y$  satisfying the following universal property:

$$\begin{array}{ccccc} Q & & & & \\ & \searrow \phi & & \swarrow q_2 & \\ & P & \xrightarrow{p_2} & Y & \\ & \downarrow p_1 & \lrcorner & \downarrow g & \\ q_1 \swarrow & X & \xrightarrow{f} & Z & \end{array}$$

- A **pushout** of the morphisms  $f$  and  $g$  consists of an object  $P$  and two morphisms  $i_1 : P \rightarrow X$  and  $i_2 : P \rightarrow Y$  satisfying the following universal property:

$$\begin{array}{ccccc} Z & \xrightarrow{g} & Y & & \\ f \downarrow & & \downarrow i_2 & \searrow j_2 & \\ X & \xrightarrow{i_1} & P & \xrightarrow{\phi} & Q \\ & \searrow j_1 & & \swarrow & \end{array}$$

**remark.** Other names for the pushout are *cofibered product of X and Y* (especially in algebraic categories when  $i_1$  and  $i_2$  are monomorphisms), or *free product of X and Y with Z amalgamated sum*, or more simply an *amalgamation* or *amalgam of X and Y*.

**remark.** If coproducts exist in some category, then the pushout

$$\begin{array}{ccc} Z & \xrightarrow{g} & Y \\ f \downarrow & \lrcorner & \downarrow i_2 \\ X & \xrightarrow{i_1} & X \amalg_Z Y \end{array}$$

is equivalently the coequalizer

$$X \xrightarrow[i_2 \circ g]{i_1 \circ f} X \amalg Y \longrightarrow X \amalg_Z Y$$

of the two morphisms induced by  $f$  and  $g$  into the coproduct of  $X$  with  $Y$ .

**example (wiki).**

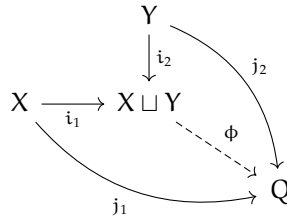
- If  $X$ ,  $Y$  and  $Z$  are sets and  $f, g$  are functions, the pushout of  $f$  and  $g$  is the disjoint union of  $X$  and  $Y$  where elements sharing a common preimage in  $Z$  are identified, i.e.  $P = (X \amalg Y) / \sim$  where  $\sim$  is the finest equivalence relation such that  $f(z) \sim g(z)$  for all  $z \in Z$ .

In particular, if  $X$  and  $Y$  are subsets of some larger set  $W$  and  $Z$  is their intersection, with  $f$  and  $g$  the inclusion maps of  $Z$  into  $X$  and  $Y$ , then the pushout can be canonically identified with the union  $X \cup Y \subseteq W$ .

- The construction of *adjunction spaces* is an example of pushouts in  $\mathbf{Top}$ . More precisely, if  $Z$  is a subspace of  $Y$  and  $g : Z \rightarrow Y$  is the inclusion map, we can glue  $Y$  to another space  $X$  along  $Z$  using an *attaching map*  $f : Z \rightarrow X$ . The result is the *adjunction space*  $X \cup_f Y$  which is just the pushout of  $f$  and  $g$ . More generally, all identification spaces may be regarded as pushouts in this way. See ?? .
- A *product* of  $X$  and  $Y$  is an object  $X \sqcap Y$  and a pair of morphisms  $p_1 : X \sqcap Y \rightarrow X$ ,  $p_2 : X \sqcap Y \rightarrow Y$  satisfying the following universal property:

$$\begin{array}{ccccc} Q & & & q_2 & \\ & \searrow \phi & & \searrow & \\ & & X \sqcap Y & \xrightarrow{p_2} & Y \\ & q_1 \searrow & \downarrow p_1 & & \\ & & X & & \end{array}$$

- A **coproduct** of  $X$  and  $Y$  is an object  $X \sqcup Y$  and a pair of morphisms  $i_1 : X \rightarrow X \sqcup Y$ ,  $i_2 : Y \rightarrow X \sqcup Y$  satisfying the following universal property:



**remark.** More generally, for  $S$  any set and  $F : S \rightarrow \mathcal{C}$  a collection of objects in  $\mathcal{C}$  indexed by  $S$ , their **coproduct** is an object

$$\coprod_{s \in S} F(s)$$

equipped with maps

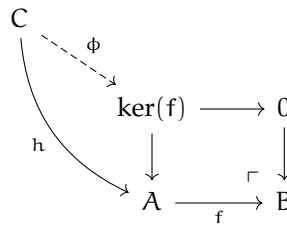
$$F(s) \rightarrow \coprod_{s \in S} F(s)$$

such that this is universal among objects with maps from  $F(s)$ .

- The **kernel** of a morphism is that part of its domain which is sent to zero. Formally, in a category with an initial object  $0$  and pullbacks, the **kernel**  $\ker f$  of a morphism  $f : A \rightarrow B$  is the pullback  $\ker(f) \rightarrow A$  along  $f$  of the unique morphism  $0 \rightarrow B$

More explicitly, this characterizes the object  $\ker(f)$  as *the* object (unique up to isomorphism) that satisfies the following universal property:

for every object  $C$  and every morphism  $h : C \rightarrow A$  such that  $f \circ h = 0$  is the zero morphism, there is a unique morphism  $\phi : C \rightarrow \ker(f)$  such that  $h = p \circ \phi$ .



- In a category with a terminal object  $1$ , the **cokernel** of a morphism  $f : A \rightarrow B$  is the

pushout (arrows  $h$  and  $\phi$  apply if terminal object is zero)

$$\text{coker}(f) := 1 \sqcup_A B$$

In the case when the terminal object is in fact zero object, one can, more explicitly, characterize the object  $\text{coker}(f)$  with the following universal property:

for every object  $C$  and every morphism  $h : B \rightarrow C$  such that  $h \circ f = 0$  is the zero morphism, there is a unique morphism  $\phi : \text{coker}(f) \rightarrow C$  such that  $h = \phi \circ i$ .

- A morphism  $f : X \rightarrow Y$  is a **monomorphism** if for every object  $Z$  and every pair of morphisms  $g_1, g_2 : Z \rightarrow X$  then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2.$$

Equivalently,  $f$  is a monomorphism if for every  $Z$  the hom-functor  $\text{Hom}(Z, -)$  takes it to an injective function

$$\text{Hom}(Z, X) \xleftarrow{f_*} \text{Hom}(Z, Y).$$

Being a monomorphism in a category  $\mathcal{C}$  means equivalently that it is an epimorphism in the opposite category  $\mathcal{C}^{\text{op}}$ .

- A morphism  $f : X \rightarrow Y$  is a **epimorphism** if for every object  $Z$  and every pair of morphisms  $g_1, g_2 : Y \rightarrow Z$  then

$$g_1 \circ f = g_2 \circ f \implies g_1 = g_2.$$

Equivalently,  $f$  is a epimorphism if for every  $Z$  the hom-functor  $\text{Hom}(-, Z)$  takes it to an injective function

$$\text{Hom}(Y, Z) \xleftarrow{f^*} \text{Hom}(X, Z).$$

Being a monomorphism in a category  $\mathcal{C}$  means equivalently that it is an monomorphism in the opposite category  $\mathcal{C}^{\text{op}}$ .

- (Retraction.)

- (wiki) Let  $X$  be a topological space and  $A$  a subspace of  $X$ . Then a continuous map  $r : X \rightarrow A$  is a **retraction** if the restriction of  $r$  to  $A$  is the identity map on  $A$ .
- (nLab) An object  $A$  in a category is called a **retract** of an object  $B$  if there are morphisms  $i : A \rightarrow B$  and  $r : B \rightarrow A$  such that  $r \circ i = \text{id}_A$ . In this case  $r$  is called a **retraction of  $B$  onto  $A$**  and  $i$  is called a **section of  $r$** .

$$\text{id} : A \xrightarrow[\text{section}]{i} B \xrightarrow[\text{retraction}]{r} A$$

Hence a **retraction** of a morphism  $i : A \rightarrow B$  is a left-inverse and a **section** of a morphism  $r : B \rightarrow A$  is a right-inverse.

- (Deformation retract.)

- (nLab) Let  $\mathcal{C}$  be a category equipped with a notion of homotopy between its morphisms. Then a **deformation retraction** of a morphism  $i : A \rightarrow X$  is another morphism  $r : X \rightarrow A$  such that

?

- (wiki) A continuous map  $F : X \times [0, 1] \rightarrow X$  is a **deformation retraction** of a space  $X$  into a subspace  $A$  if, for every  $x$  in  $X$  and  $a$  in  $A$ ,

$$F(x, 0) = x, \quad F(x, 1) \in A \quad \text{and} \quad F(a, 1) = a.$$

In words, a deformation retraction is a homotopy between a retraction and the identity map on  $X$ . The subspace  $A$  is called a **deformation retract** of  $X$ . A deformation retraction is a special case of a homotopy equivalence.

An equivalent definition of deformation retraction is the following. A continuous map  $r : X \rightarrow A$  is a **deformation retraction** if it is a retraction and its composition with the inclusion is homotopic to the identity map on  $X$ . In this formulation, a deformation retraction carries with it a homotopy between the identity map on  $X$  and itself.

- (wiki) If, in the definition of a deformation retraction we add the requirement that

$$F(a, t) = a \quad \forall t \in [0, 1], \forall a \in A,$$

then  $F$  is called a **strong deformation retraction**. In words, a strong deformation retraction leaves points in  $A$  fixed throughout the homotopy.

**example.**  $S^n$  is a strong deformation retract of  $\mathbb{R}^{n+1} \setminus \{0\}$  through  $F(x, t) = (1 - t)x + t \frac{x}{\|x\|}$ .

- (wiki) The inclusion of a closed subspace  $A$  in the space  $X$  is a ?? if and only if  $A$  is a **neighbourhood deformation retract** of  $X$ , meaning that there is a continuous map  $u : X \rightarrow [0, 1]$  with  $A = u^{-1}(0)$  and a homotopy  $H : X \times [0, 1] \rightarrow X$



such that  $H(x, 0) = x$  for all  $x \in X$ ,  $H(a, t) = a$  for all  $a \in A$  and  $t \in [0, 1]$ , and  $H(x, 1) \in A$  if  $u(x) < 1$ .

For example, the inclusion of a subcomplex in a CW complex is a cofibration.

## elementary concepts

### definition.

- Let  $X$  and  $Y$  be topological spaces and  $f, g : X \rightarrow Y$  continuous maps. An *homotopy* from  $f$  to  $g$  is a continuous map

$$H : X \times [0, 1] \rightarrow Y, \quad (x, t) \mapsto H(x, t) = H_t(x)$$

such that  $f(x) = H(x, 0)$  and  $g(x) = H(x, 1)$  for all  $x \in X$ . We denote this situation by  $f \simeq g$ . The homotopy relation  $\simeq$  is an equivalence relation on the set of continuous maps  $X \rightarrow Y$ . A homotopy of maps  $H_t : X \rightarrow Y$  is called *relative to*  $A \subset X$  if  $H_t|_A$  is constant.

- Topological spaces and homotopy classes of maps form a quotient category of  $\text{Top}$ , the *homotopy category*  $\text{h-Top}$ , where composition of homotopy classes is induced by composition of representing maps. If  $f : X \rightarrow Y$  represents an isomorphism in  $\text{h-Top}$ , then  $f$  is called a *homotopy equivalence* or *h-equivalence*. In explicit terms this means  $f : X \rightarrow Y$  is a homotopy equivalence if there exists  $g : Y \rightarrow X$ , a *homotopy inverse of*  $f$ , such that  $gf$  and  $fg$  are both homotopic to the identity. Spaces  $X$  and  $Y$  are called *homotopy equivalent* or of the same *homotopy type* if there exists a homotopy equivalence  $X \rightarrow Y$ . A space is *contractible* if it is homotopy equivalent to a point. A map  $f : X \rightarrow Y$  is *null homotopic* if it is homotopic to a constant map.
- Let  $(X, x_0)$  be a pointed topological space and  $s_0 \in S^n$ . The elements of the *n-th homotopy group* are homotopy classes of maps  $(S^n, s_0) \rightarrow (X, x_0)$ . Equivalently, they are homotopy classes of maps  $(I^n, \partial I^n) \rightarrow (X, x_0)$ . (Homotopies are required to preserve the base points,  $s_0 \mapsto x_0$  or  $\partial I^n \mapsto x_0$ .)

Also,

$$\pi_n(X, *) = [(I^n, \partial I^n), (X, \{*\})] \cong [I^n / \partial I^n, X]^0$$

where  $[X, Y]$  denotes the set of homotopy classes  $[f]$  of maps  $[f] : X \rightarrow Y$ .

**proposition 1.**  $\pi_n(X, x_0)$  is an abelian group for all  $n \in \mathbb{N}$ .

- Let  $A$  be a subspace of  $X$  and  $x_0 \in A$ . The elements of the *relative homotopy group*  $\pi_n(X, A, x_0)$  are homotopy classes of maps  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$  where  $J^{n-1}$  is the union of all but one face of  $I^n$ . That is,

$$\pi_{n+1}(X, A, *) = [(I^{n+1}, \partial I^{n+1}, J^n), (X, A, x_0)].$$

The elements of such a group are homotopy classes of based maps  $D^n \rightarrow X$  which carry the boundary  $S^{n-1}$  into  $A$ . Two maps  $f, g$  are called *homotopic relative to*

$A$  if they are homotopic by a basepoint-preserving homotopy  $F : D_n \times [0, 1] \rightarrow X$  such that, for each  $p$  in  $S^{n-1}$  and  $t$  in  $[0, 1]$ , the element  $F(p, t)$  is in  $A$ . Ordinary homotopy groups are recovered for the case in which  $A = \{x_0\}$ .

**remark.** This construction is motivated by looking for the kernel of the induced map  $i_* : \pi_n(A, x_0) \rightarrow \pi_n(X, x_0)$  by the inclusion. This map is in general not injective, and the kernel consists of ?

- For any pair  $(X, A, x)$  we have a long exact sequence

$$\pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_{n-1}(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \rightarrow \cdots \rightarrow \pi_0(X, x_0)$$

where  $i$  and  $j$  are the inclusions  $(A, x_0) \hookrightarrow (X, x_0)$  and  $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$ . The map  $\partial$  comes from restricting maps  $(I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$  to  $I^{n-1}$ , or by restricting maps  $(D^n, S^{n-1}, s_0) \rightarrow (X, A, x_0)$ . The map, called the **boundary map**, is a homomorphism when  $n > 1$ .

- A space  $X$  with basepoint  $x_0$  is called  **$n$ -connected** if  $\pi_i(X, x_0) = 0$  for  $i \leq n$ . Thus 0-connected means path-connected and 1 connected means simply-connected.
- A pair  $(X, A)$  is  **$n$ -connected** if  $\pi_i(X, A, x_0) = 0$  for  $i \leq n$ .
- Two pointed spaces  $(X, x_0)$  and  $(Y, y_0)$  are  **$n$ -equivalent** if  $\pi_i(X, x_0) \cong \pi_i(Y, y_0)$  for all  $i < n$  and surjective for  $i = n$ .

## the right category

- We don't care so much about Top. We care much more about CGWH, the full subcategory of Top on **compactly generated weakly Hausdorff** spaces.
- $X$  is **compactly generated** if, for any subset  $C \subset X$ , and for all continuous maps  $f : K \rightarrow X$  from compact Hausdorff spaces,

if  $f^{-1}(C)$  is closed in  $K$ , then  $C$  is closed.

**claim** (What I picked up from the lecture). If  $X$  is compactly generated, then  $X$  is weakly Hausdorff if the diagonal subset  $\Delta_X \subset X \times X$  is  **$k$ -closed**.

From [May](#): The ordinary category of spaces allows pathology that obstructs a clean development of the foundations. The homotopy and homology groups of spaces are supported on compact subspaces, and it turns out that if one assumes a separation property that is a little weaker than the Hausdorff property, then one can refine the point-set topology of spaces to eliminate such pathology without changing these invariants.

One major source of point-set level pathology can be passage to quotient spaces. Use of compactly generated topologies alleviates this.

**proposition 2.** If  $X$  is compactly generated and  $\pi : X \rightarrow Y$  is a quotient map, then  $Y$  is compactly generated if and only if  $(\pi \times \pi)^{-1}(\Delta Y)$  is closed in  $X \times X$

The interpretation is that a quotient space of a compactly generated space by a “closed equivalence relation” is compactly generated.

Several other propositions follow in May. Now some other notes from the lectures:

In CGWH,  $\text{Hom}(X, Y)$  is a space with the compact-open topology. This is a compactly generated space,  $\mathbf{k}(\text{Hom}(X, Y))$ .

**remark.** (Also see [wiki on currying](#))

$$\begin{aligned}\text{Map}(X, Y) &:= \text{the space of maps } X \rightarrow Y. \\ \text{Map}(X \times Y, Z) &\cong \text{Map}(X, \text{Map}(Y, Z)) \\ \text{Hom}(X \times Y, Z) &\cong \text{Hom}(X, \text{Map}(Y, Z))\end{aligned}$$

In the last line, product is product in CGWH, not in  $\text{Top}$ .

The functor  $- \times Y$  is left adjoint to  $\text{Map}(Y, -)$ .

## cofibrations

**definition.**

- ([wiki](#)) In mathematics, in particular in homotopy theory, a continuous map between topological spaces  $i : A \rightarrow X$  is a **cofibration** if it has the **homotopy extension property** with respect to all topological spaces  $S$ .

That is,  $i$  is a cofibration if

- for each topological space  $S$ ,
- and for any continuous maps  $f, f' : A \rightarrow S$
- and  $g : X \rightarrow S$  with  $g \circ i = f$ ,
- for any homotopy  $h : A \times I \rightarrow S$  from  $f$  to  $f'$ ,

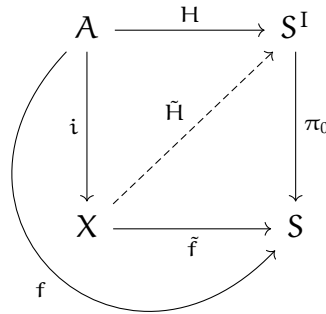
there is a continuous map  $g' : X \rightarrow S$  and a homotopy  $h' : X \times I \rightarrow S$  from  $g$  to  $g'$  such that

$$h'(i(a), t) = h(a, t) \quad \text{for all } a \in A \text{ and } t \in I.$$

- ([wiki](#)) In what follows, let  $I = [0, 1]$  denote the unit interval.

A map  $i : A \rightarrow X$  is a **cofibration** if for any map  $f : A \rightarrow S$  such that there is an extension to  $X$ , meaning there is a map  $\tilde{f} : X \rightarrow S$  such that  $\tilde{f} \circ i = f$ , we can extend a homotopy of maps  $H : A \times I \rightarrow S$  to a homotopy of maps  $\tilde{H} : X \times I \rightarrow S$  where

$$\begin{aligned}H(a, 0) &= f(a) \\ \tilde{H}(x, 0) &= \tilde{f}(x)\end{aligned}$$



- (wiki) Let  $X$  be a topological space and let  $A \subset X$ . We say that the pair  $(X, A)$  has the **homotopy extension property** if, given a homotopy  $f_\bullet : A \rightarrow Y^I$  and a map  $\tilde{f}_0 : X \rightarrow Y$  such that

$$\tilde{f}_0 \circ \iota = f_0$$

(so  $\tilde{f}$  is the lift of  $f_0 : A \rightarrow Y$ ) then there exists an **extension** of  $f_\bullet$  to a homotopy  $\tilde{f}_\bullet : X \rightarrow Y^I$  such that  $\tilde{f}_\bullet \circ \iota = f_\bullet$ .

That is,

$$\begin{array}{ccc} A & \xrightarrow{f_\bullet} & Y^I \\ \downarrow \iota & \nearrow \tilde{f}_\bullet & \downarrow \pi_0 \\ X & \xrightarrow{\tilde{f}_0} & Y \end{array}$$

So there's some **currying** to make usual homotopies  $f_\bullet : A \times I \rightarrow Y$  look like  $f_\bullet : A \rightarrow Y^I$ . Or, as said in our lectures, "a homotopy  $X \times I \rightarrow Y$  is the same as a map  $X \rightarrow \text{Map}(I, Y)$ ".

- (May) A map  $i : A \rightarrow X$  is a **cofibration** if it satisfies the **homotopy extension property (HEP)**. This means that if  $h \circ i_0 = f \circ i$  in the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ \downarrow i & \nearrow h & \downarrow i \times \text{id} \\ X & \xrightarrow{i_0} & X \times I \end{array}$$

$\begin{array}{ccc} & Y & \\ & \nwarrow f & \nearrow \tilde{h} \\ & X & \end{array}$

then there exists  $\tilde{h}$  that makes the diagram commute.

- In traditional topology, one usually means a Hurewicz cofibration. A map  $i : A \rightarrow X$  between topological spaces is a **Hurewicz cofibration** if it satisfies the homotopy extension property.

Let's say it one more time: for any  $g : X \rightarrow Y$  and any homotopy  $H : A \times I \rightarrow Y$  such that

$$\begin{array}{ccc} A \times \{0\} & \longrightarrow & A \times I \\ \downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

there is  $H' : X \times I \rightarrow Y$ ,

$$\begin{array}{ccc} X \times \{0\} & \longrightarrow & A \times I \\ \downarrow g & & \downarrow \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

such that

$$\begin{array}{ccc} A \times I & & \\ \downarrow & \searrow H & \\ X \times I & \xrightarrow{H'} & Y \end{array}$$

**example.**  $\partial D^n \rightarrow D$  is a Hurewicz cofibration. **Why?**

**exercise.** Prove that an inclusion  $f : A \rightarrow X$  is a Hurewicz cofibration if and only if  $A \times I \cup X \times \{0\}$  is a retract of  $X \times I$ .

**definition (Mapping cylinder).**

- (May) Although HEP is expressed in terms of general test diagrams, there is a certain universal test diagram (i.e. **make the dashed map unique—up to something maybe**). Namely, we can let  $Y$  in our original test diagram be the **mapping cylinder**

$$Mi \equiv X \cup_i (A \times I)$$

which is the pushout of  $i$  and  $i_0$ . Indeed, suppose that we can construct a map  $r$  that makes the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{i_0} & A \times I \\ \downarrow i & & \downarrow i \times \text{id} \\ X & \xrightarrow{i_0} & X \times I \end{array} \quad \begin{array}{c} \nearrow \\ Mi \\ \nwarrow \end{array} \quad \begin{array}{c} \nwarrow \\ r \\ \nearrow \end{array}$$

By the universal property of the pushouts, given maps  $f$  and  $h$  in our original test diagram induce a map  $Mi \rightarrow Y$ , and its composite with  $r$  gives a homotopy  $\tilde{h}$  that makes the diagram commute. **So just saying that  $Mi$  is universal.**

- (nLab) Given a continuous map  $f : X \rightarrow Y$  of topological spaces, one can define its *mapping cylinder* as a pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X \times I & \xrightarrow{(\sigma_0)_*(f)} & \text{Cyl}(f) \end{array}$$

in Top, where  $I = [0, 1]$  and  $\sigma : X \rightarrow X \times I$  is given by  $x \mapsto (x, 0)$ .

Set theoretically, the mapping cylinder is usually represented as the quotient space

$$(X \times I \amalg Y) / \sim$$

where  $\sim$  is the smallest equivalence relation identifying  $(x, 0) \sim f(x)$  for all  $x \in X$ .

- (wiki) The *mapping cylinder* of a function  $f$  between topological spaces  $X$  and  $Y$  is the quotient

$$M_f = ([0, 1] \times X \amalg Y) / \sim$$

where  $\amalg$  denotes disjoint union, and  $\sim$  is the equivalence relation generated by

$$(0, x) \sim f(x) \text{ for each } x \in X.$$

That is, the mapping cylinder  $M_f$  is obtained by gluing one end of  $X \times [0, 1]$  to  $Y$  via the map  $f$ . Notice that the “top” of the cylinder  $\{1\} \times X$  is homeomorphic to  $X$ , while the “bottom” is the space  $f(X) \subset Y$ .

(Dani) So the mapping cylinder is just deforming  $X$  to  $Y$  putting  $X$  inside  $Y$  via  $f$ .

- (Homework) Let  $f : X \rightarrow Y$  be a map. Let  $M_f = X \times [0, 1] \cup_f Y$  be the *mapping cylinder of  $f$* , i.e. the pushout of  $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0, 1]$  and of  $f : X \rightarrow Y$ .

**exercise.** Let  $g : X \rightarrow M_f$  be the map  $X \xrightarrow{\cong} X \times \{1\} \rightarrow M_f$ . Let  $h : M_f \rightarrow Y$  be the map that is induced by  $X \times [0, 1] \rightarrow Y : (x, t) \mapsto f(x)$  and  $\text{id}_Y : Y \rightarrow Y$ . Observe that  $f$  is the composition of  $g$  and  $h$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{id} \times 1 \downarrow & \searrow g & \downarrow \text{id}_Y \\ X \times [0, 1] & \xrightarrow{\quad} & M_f \\ & \searrow h & \downarrow \\ & & Y \end{array}$$

$(x, t) \mapsto f(x)$

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e.  $(\text{colim } A_i) \times B$  is canonically homeomorphic with  $\text{colim}(A_i \times B)$ .

1. Show that  $h$  is a deformation retract, and in particular is a homotopy equivalence.
2. Show that  $g : X \rightarrow M_f$  is a cofibration. You may use exercise (a), but the direct proof might be simpler.

## fibrations

- (nLab) A morphism  $i$  has the *left lifting property with respect to a morphism*  $p$  and  $p$  has the *right lifting property with respect to*  $i$  if for each morphisms  $f$  and  $g$ , if the outer square in the following diagram commutes, there exists  $\phi$  (I think not necessarily unique) completing the diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i \downarrow & \nearrow \phi & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

- (nLab) Let  $C$  be a category with products and with interval object  $I$ . A morphism  $E \rightarrow B$  has the homotopy lifting property if it has the right lifting property with respect to all morphisms of the form  $(\text{id}, 0) : Y \rightarrow Y \times I$ .

This means that for all commuting squares

$$\begin{array}{ccc} Y & \xrightarrow{f} & E \\ \downarrow & \nearrow \sigma & \downarrow p \\ Y \times I & \xrightarrow{F} & B \end{array}$$

there exists a morphism  $\sigma : Y \times I \rightarrow E$  such that both triangles in the former diagram commute.

- (Hatcher) A map  $p : E \rightarrow B$  is said to have the *homotopy lifting property* with respect to a space  $X$  if, given a homotopy  $g_t : X \rightarrow B$  and a map  $\tilde{g}_0 : X \rightarrow E$  lifting  $g_0$ , so  $p\tilde{g}_0 = g_0$ , then there exists a homotopy  $\tilde{g}_t : X \rightarrow E$  lifting  $g_t$ .

The *lift extension property for a pair*  $(Z, A)$  asserts that every map  $X \rightarrow B$  has a lift  $Z \rightarrow E$  extending a given lift defined on the subspace  $A \subset Z$ . The case  $(Z, A) = (X \times I, X \times \{0\})$  is the homotopy lifting property.

A *fibration* is a map  $p : E \rightarrow B$  having the homotopy property with respect to all spaces  $X$ .

**Theorem 3 (4.41 Hatcher, Long exact sequence of Serre fibrations, see proposition 17).** Suppose  $p : E \rightarrow B$  has the homotopy lifting property with respect to disks  $D^k$  for all  $k \geq 0$ . Choose basepoints  $b_0 \in B$  and  $x_0 \in F = p^{-1}(b_0)$ . Then the map  $p_* : \pi_n(E, F, x_0) \rightarrow \pi_n(B, b_0)$  is an isomorphism for all  $n \geq 1$ . Hence  $b$  is

path-connected and there is a long exact sequence

$$\cdots \rightarrow \pi_n(F, x_0) \rightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \rightarrow \pi_{n-1}(F, x_0) \rightarrow \cdots \rightarrow \pi_0(E, x_0) \rightarrow 0$$

The map  $p : E \rightarrow B$  is said to have the *homotopy lifting property for a pair*  $(X, A)$  if each homotopy  $f_t : X \rightarrow B$  lifts to a homotopy  $\tilde{g}_t : X \rightarrow E$  starting with a given lift  $\tilde{g}_0$  and extending a given lift  $\tilde{g}_t : A \rightarrow E$ . In other words, the homotopy lifting property for  $(X, A)$  is the lift extension property for  $(X \times I, X \times \{0\} \cup A \times I)$ .

The point is that the homotopy lifting property for disks is equivalent to the homotopy lifting property for all CW pairs  $(X, A)$ . A map  $p : E \rightarrow B$  satisfying the homotopy lifting property for disks is sometimes called a *Serre fibration*.

A *fiber bundle* structure on a space  $E$ , with fiber  $F$ , consists of a projection map  $p : E \rightarrow B$  such that each point  $B$  has a neighbourhood  $U$  for which there is a homeomorphism  $h : p^{-1}(U) \rightarrow U \times F$  making the following diagram commute

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ & \searrow p & \swarrow \\ & U & \end{array}$$

**example.** Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces  $S^n \rightarrow \mathbb{R}P^n$  with fiber  $S^0$ . Over the complex numbers the analog of this is a fiber bundle  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$ . Here  $S^{2n+1}$  is the unit sphere in  $\mathbb{C}^{n+1}$  and  $\mathbb{C}P^n$  is viewed as the quotient space of  $S^{2n+1}$  under the equivalence relation  $(z_0, \dots, z_n) \sim \lambda(z_0, \dots, z_n)$  for  $\lambda \in S^1$ . The projection  $p : S^{2n+1} \rightarrow \mathbb{C}P^n$  sends  $(z_0, \dots, z_n)$  to its equivalence class  $[z_0, \dots, z_n]$ .

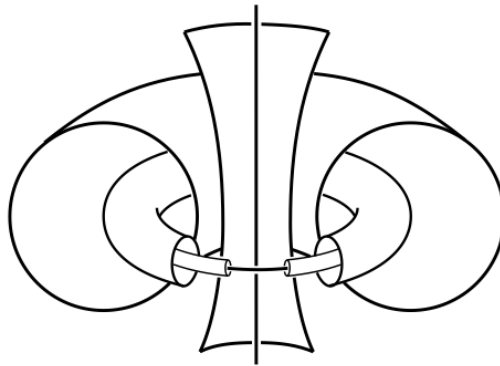
To see that the local triviality condition for fibre bundles is satisfied, ...

The construction of the bundle  $S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$  also works when  $n = \infty$ , so there is a fiber bundle  $S^1 \rightarrow S^\infty \rightarrow \mathbb{C}P^\infty$ .

The case  $n = 1$  is particularly interesting since  $\mathbb{C}P^1 = S^2$  and bundle becomes  $S^1 \rightarrow S^3 \rightarrow S^2$  with fiber, total space, and base all spheres. This is known as the *Hopf bundle*. The projection  $S^3 \rightarrow S^2$  can be taken to be  $(z_0, z_1) \mapsto z_0/z_1 \in \mathbb{C} \cup \{\infty\} = S^2$ . (That is, seeing  $S^2$  as the one-point compactification of  $\mathbb{C}$ .)

In polar coordinates we may see  $S^3$  as the union of several tori. Stereographic projection yields the following figure:





The limiting cases  $T_0$  and  $T_\infty$  correspond to the unit circle in the  $xy$ -plane and the  $z$ -axis under the stereographic projection. Each torus  $T_\rho$  is a union of circle fibers. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. As  $\rho$  goes to 0 or  $\infty$  the fiber circles approach the circles  $T_0$  and  $T_\infty$ , which are also fibers. The figure below shows four tori decomposed into fibers:



How could we visualize the projection onto  $S^2$ ? Could it work to think  $S^2 = \mathbb{C} \cup \infty$  and just do stereographic projection from 3-space to the plane disregarding one point? What would that even mean hehe

Replacing the field  $\mathbb{C}$  by the quaternions  $\mathbb{H}$ , the same constructions yield fiber bundles  $S^3 \rightarrow S^{4n+3} \rightarrow \mathbb{H}P^n$  over quaternionic projective spaces  $\mathbb{H}P^n$ . Here the fiber  $S^3$  is the unit quaternions, and  $S^{4n+3}$  is the unit sphere in  $\mathbb{H}^{n+1}$ . Taking  $n = 1$  gives a second Hopf bundle  $S^3 \rightarrow S^7 \rightarrow S^4 = \mathbb{H}P^1$ .

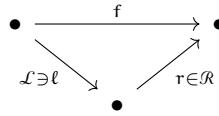
Another Hopf bundle  $S^7 \rightarrow S^{15} \rightarrow S^8 \dots$

## model structures

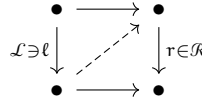
**definition (Riehl).** A *weak factorization system*  $(\mathcal{L}, \mathcal{R})$  on a category  $\mathcal{M}$  is comprised of two classes of morphisms  $\mathcal{L}$  and  $\mathcal{R}$  so that

1. Every morphism in  $\mathcal{M}$  may be factored as a morphism in  $\mathcal{L}$  followed by a mor-

phism in  $\mathcal{R}$ :

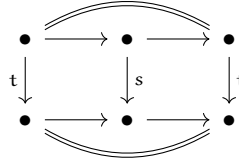


2. The maps in  $\mathcal{L}$  have the *left lifting property* with respect to each map in  $\mathcal{R}$  and equivalently the maps in  $\mathcal{R}$  have the *right lifting property* with respect to each map in  $\mathcal{L}$ , that is, any commutative square



admits a diagonal filler as indicated making both triangles commute.

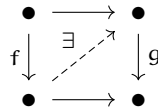
3. The classes  $\mathcal{L}$  and  $\mathcal{R}$  are each closed under retracts in the arrow category: given a commutative diagram



if  $s$  is in that class then so is its retract  $t$ .

**definition (Lecture).** A *model structure* on a category  $\mathcal{A}$  is a choice of subcategories  $\mathcal{W}, \mathcal{C}, \mathcal{F}$  called *weak-equivalences*, *cofibrations* and *fibrations* with the following properties:

- Given  $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ , if either 2 out of 3 among  $f, g, f \circ g$  are in  $\mathcal{W}$  then all of them are.
- $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$  are both weak factorization systems.  $(\mathcal{B}, \mathcal{D})$  is a weak factorization system.
  - Any morphism in  $\mathcal{A}$  can be factored as a morphism in  $\mathcal{B}$  followed by a morphism in  $\mathcal{D}$ .
  - Lifts:



Two interesting model category structures on CGWH.

- Hurewicz model structure (Strom).
  - Cofibrations:= Hurewicz cofibrations.
  - Fibrations:= maps  $E \rightarrow B$  such that for all spaces  $X$  [Photo1].

- Weak equivalences:= homotopy equivalences.

## 2. Quillen model structure.

- Cofibrations = retracts of relative cell complexes.

• (Serre) Fibrations =

$$\begin{array}{ccc} D^n & \xrightarrow{\quad} & E \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ D^n \times I & \longrightarrow & B \end{array}$$

- Weak equivalences:  $f : X \rightarrow Y$

**exercise (3.1.8 from Riehl).** Verify that the class of morphisms  $\mathcal{L}$  characterized by the left lifting property against a fixed class of morphisms  $\mathcal{R}$  is closed under coproducts, closed under retracts, and contains the isomorphisms.

**remark (Plan).** Blakers-Massey excision theorem (relies on technical lemma, proof from Tom Dieck's book)  $\implies$  Cellular approximation. Also  $\implies$  Freudenthal theorem.

**exercise.**  $X \rightarrow M_f \rightarrow Y$ . Prove  $X \rightarrow M_f$  is a cofibration.

## whitehead theorem

We introduce a large class of spaces, called CW complexes, between which a weak equivalence is necessarily a homotopy equivalence. Thus, for such spaces, the homotopy groups are, in a sense, a complete set of invariants. Moreover, we shall see that every space is weakly equivalent to a CW complex.

**definition (May).**

1. A **CW complex**  $X$  is a space  $X$  which is the union of an expanding sequence of subspaces  $X^n$  such that, inductively,  $X^0$  is a discrete set of points (called **vertices**) and  $X^{n+1}$  is the pushout obtained from  $X^n$  by attaching disks  $D^{n+1}$  along **attaching maps**  $j : S^n \rightarrow X^n$ . Thus  $X^{n+1}$  is the quotient space obtained from  $X^n \cup (J_{n+1} \times D^{n+1})$  by identifying  $(j, x)$  with  $j(x)$  for  $x \in S^n$ , where  $J_{n+1}$  is the discrete set of such attaching maps  $j$  (see ??). Each resulting map  $D^{n+1} \rightarrow X$  is called a **cell**. The subspace  $X^n$  is called the **n-skeleton** of  $X$ .

$$\begin{array}{ccc} S^n & \xhookrightarrow{i} & D^{n+1} \\ j \downarrow & \lrcorner & \downarrow \\ X^n & \longrightarrow & X^{n+1} \end{array}$$

**lemma 4 (HELP).** content...

**Theorem 5 (Whitehead, May).** If  $X$  is a CW complex and  $e : Y \rightarrow Z$  is an  $n$ -equivalence, then  $e_* : [X, Y] \rightarrow [X, Z]$  is a bijection if  $\dim X < n$  and surjection if  $\dim X = n$ .

**Theorem 6** (Whitehead, May). An  $n$ -equivalence between CW complexes of dimension less than  $n$  is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

**Theorem 7** (Whitehead (4.5), Hatcher). If a map  $f : X \rightarrow Y$  between connected CW complexes induces isomorphisms  $f_* : \pi_n(X) \rightarrow \pi_n(Y)$  for all  $n$ , then  $f$  is a homotopy equivalence. In case  $f$  is the inclusion of a subcomplex  $X \hookrightarrow Y$ , the conclusion is stronger:  $X$  is a deformation retract of  $Y$ .

**exercise** (Hatcher 4.1.12). Show that an  $n$ -connected,  $n$ -dimensional CW complex is contractible.

*Solution.* Just recall that  $n$ -connectedness means that  $\pi_i(X) = 0$  for all  $i \leq n$ , which means that  $X$  is contractible by theorem 6.  $\square$

## lecture notes

14 mar

$$(X^Y)^Z \cong Z^{Y \times X}$$

$$g : X' \rightarrow X$$

$$\text{Hom}(X, Y) \mapsto \text{Hom}(X', Y)$$

$$\begin{aligned} \text{Hom}(A, B) \cong \text{Hom}(A, B') \text{ natural in } A &\implies \\ \text{Hom}(B, B) \cong \text{Hom}(B, B') \&\text{Hom}(B', B) \cong \text{Hom}(B', B') \\ \implies B \cong B'. \end{aligned}$$

- for (  $\Leftarrow$  ) commutativity of the hypothesis gives us commutativity of the right-most square in the diagram below. In fact, the double square diagram below is a rephrasing of the hypothesis.
- Lemma 2. To build CW complexes
- What we did? Prove the bijection between the homotopic sets given an  $n$ -equivalence.
- $\pi_n$  of loop space is the same as  $\pi_{n+1}$  of original space.
- Then we moved on to homotopic pushouts and pullback. We saw, for instance, that if in a double square diagram each of the squares is a homotopic pushout, then so is the outer square.

- We also looked at those exact sequences on cofibers, spaces of homotopy classes, cohomology and (barely) loop spaces. There was a lemma about this.
- Next time: cofiber of cofiber is homotopy equivalence, then fibers, fibrations and probably \*some name\* theorem.

18 mar

**lemma 8 (Yoneda).**

$$\{\text{Natural transformations } \text{Hom}(-, X) \rightarrow F\} \cong F(X)$$

**corollary 9.**  $(\text{Hom}(-, X) \rightarrow \text{Hom}(-, Y)) \cong \text{Hom}(X, Y)$ .

**corollary 10.** The correspondence  $X \mapsto \text{Hom}(-, X)$  is fully faithful, that is, the correspondence  $\text{Hom}(X, X') \rightarrow \text{Hom}(\text{Hom}(-, X), \text{Hom}(-, X'))$  is injective and bijective. (The right hand side are natural transformations of functors.)

*Solution of exercise 1.* The latter correspondence sends isomorphisms to isomorphisms. Since we are given a natural isomorphism in the problem, we conclude  $X \cong X'$ .  $\square$

**lemma 11.** Let  $E \times_B X$  be the pullback of

$$\begin{array}{ccc} & E & \\ & \downarrow & \\ X & \xrightarrow{\sim} & B \end{array}$$

be such that  $E \rightarrow B$  is an homotopy fibration and  $f : X \rightarrow B$  is a homotopy equivalence. Let

$$\begin{array}{ccccc} E \times_B X & \rightarrow & E & \xrightarrow{\sim} & E \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\sim} & B & & B \end{array}$$

be the pullback. Then  $E \times_B X \rightarrow E$  is a homotopy equivalence.

*Proof.* Let  $g : B \rightarrow X$  be the homotopy inverse of  $f$ .

**(Step 1)** Construct another pullback

$$\begin{array}{ccccc} E \times_B B & \longrightarrow & X \times_B E & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow \\ B & \xrightarrow{g} & X & \xrightarrow{f} & B \end{array}$$

**(Step 2)** Construct  $E \rightarrow E \times_B B$ .

Consider

$$\begin{array}{ccc}
 E & \xrightarrow{\text{id}} & E \\
 \downarrow & \nearrow & \downarrow \\
 E \times I & \xrightarrow{f \times \text{id}} & B \times I \longrightarrow B?
 \end{array}$$

And then  $E \rightarrow E \times_B B \rightarrow E \times_B X \rightarrow E$  is homotopic to the identity.

Constructing the other homotopic inverse is the hard part.

$$\begin{array}{ccc}
 Z \sqcup Z & \longrightarrow & I \times Z \\
 \downarrow f_1 \sqcup f_2 & \nearrow & \downarrow \\
 E \times_B X & \longrightarrow & E \\
 \downarrow & \searrow & \downarrow \\
 X & \xrightarrow{\cong} & B
 \end{array}$$

□

**corollary 12.**  $B \xrightarrow{f} B$  is homotopy equivalence and  $E \rightarrow B$  is a fibration, in

$$\begin{array}{ccc}
 E \times_B B & \longrightarrow & E \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{f} & B
 \end{array}$$

$E \times_B B \rightarrow E$  is a homotopy equivalence.

**exercise.** If  $fg$  is an isomorphism and  $f$  and  $g$  have right inverses, then  $f$  and  $g$  are isomorphisms.

**lemma 13.** Let

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow g & & \downarrow \\
 X & \longrightarrow & X \cup_A B
 \end{array}$$

be a pushout with  $A \rightarrow X$  a cofibration. Then the canonical map from the double mapping cylinder  $M(f, g) \rightarrow X \cup_A B$  is a homotopy equivalence.

**remark.**

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 \downarrow g & & \downarrow \\
 X & & X \cup_A B
 \end{array}
 \quad
 \begin{array}{ccc}
 A & \hookrightarrow & M_f \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & X \cup_A M_f \cong M(f, g)
 \end{array}$$

**definition.**

- The *homotopy pullback* of a diagram

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is

$$\begin{array}{ccccc} X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

Intuitively, for any  $x \in X$  and  $y \in Y$  this object has the space of paths connecting  $x$  and  $y$ .

- The *homotopy fiber* if  $f : Y \rightarrow Z$  is the pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow f \\ \text{pt} & \longrightarrow & Z \end{array}$$

$F \subset Z^I \times_Z Y \rightarrow Z$ , where  $F$  is the space of paths starting at  $x$  and ending at the same point  $f(y)$ .

**remark.** The pullback of

$$\begin{array}{ccc} & & Z^I \times_Z Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is the homotopy pullback of

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

**lemma 14.** If  $X \rightarrow Z$  is a fibration then for

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \twoheadrightarrow & Z \end{array}$$

the map from the pullback to the homotopy pullback is a homotopy equivalence.

*Proof.*

$$\begin{array}{ccc}
 X \times_Z Y & \longrightarrow & Y \\
 \downarrow \simeq & & \downarrow \simeq \\
 X \times_{\text{ev}_0} Z^I \times_{\text{ev}_1} Y & \twoheadrightarrow & Z^I \times_Z Y \\
 \downarrow & & \downarrow \\
 X & \longrightarrow & Z
 \end{array}$$

□

Finally,

$$\begin{array}{ccccc}
 \text{hofib } f_1 & \longrightarrow & \text{hofib } f & \longrightarrow & * \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & X & \xrightarrow{f} & Y
 \end{array}$$

and

$$\begin{array}{ccc}
 Z & \longrightarrow & F(f) \\
 \downarrow & \nearrow & \downarrow \\
 X \times I & \longrightarrow & X
 \end{array}$$

$X \times_Y Y^I$

and an exact sequence

$$\Omega^2 \text{hofib} \rightarrow \Omega^2 X \rightarrow \Omega^2 Y \rightarrow \Omega \text{hofib } f \rightarrow \Omega X \rightarrow \Omega Y \rightarrow \text{hofib } f \rightarrow X \xrightarrow{f} Y$$

**lemma 15 (Exactness).**  $\forall z, [z \text{hofib } f] \rightarrow [Z, X] \rightarrow [Z, Y]$ .

and we get the exact sequence

$$\pi_0(\Omega^2 X) \rightarrow \pi_0(\Omega^2 Y) \rightarrow \pi_0(\Omega \text{hofib } f) \rightarrow \pi_0(\Omega X) \rightarrow \pi_0(\Omega Y) \rightarrow \pi_0(\text{hofib } f) \rightarrow \pi_0(X) \rightarrow \pi_0(Y)$$

and then

$$[S^0, \Omega^2 X] = [\Sigma S^0, \Omega X] = [\Sigma^2 S^0, X] = [S^2, X] = \pi_2(X)$$

### Serre fibration long exact sequence (21 march)

We've been talking a lot about Hurewicz fibrations. Let's talk about Serre fibrations. Notice that H. fibration  $\implies$  S. fibration. What is the most natural example of a Serre fibration?

**proposition 16** (also [Hatcher 4.48](#)). Let  $E$  be a fiber bundle with fiber  $F$ . Then  $f$  is a Serre fibration.



*Proof.* What does it mean to be a Serre fibration? It means that

$$\begin{array}{ccc} I^n & \xrightarrow{\quad} & E \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} = I^n \times I & \longrightarrow & B \end{array}$$

So if  $\mathcal{U}$  is a covering of  $B$  such that  $f^{-1}U \cong U \times F$ . By Lebesgue lemma, there is a  $\delta > 0$  such that for all  $x \in I^{n+1}$ , the ball  $B(x, \delta)$  lies in some  $f^{-1}U$  for some  $U$ .

Then we subdivide  $I^{n+1}$  in smaller cubes of the same size with diameter  $< \delta$ . So, each the image of each cube lies in some  $U \in \mathcal{U}$ .

Then

$$\begin{array}{ccc} I^n & \longrightarrow & F \times U \\ \downarrow & \nearrow & \downarrow \\ I^{n+1} & \longrightarrow & U \end{array}$$

has a lift for every little square because

$$\begin{array}{ccc} X & \longrightarrow & U \\ \downarrow & \nearrow & \downarrow \\ X \times I & \longrightarrow & \text{pt} \end{array}$$

is always a fibration (**think about this**) and because pullbacks of fibrations are fibrations:

$$\begin{array}{ccc} U \times F & \longrightarrow & U \\ \downarrow & & \downarrow \\ F & \longrightarrow & \text{pt} \end{array}$$

. Then we may just add up the squares because

$$\begin{array}{c} D^n \\ \downarrow \\ D^n \times I \end{array}$$

and we're done. □

**proposition 17** (Sere fibration long exact sequence, see theorem 3). Let  $g : E \rightarrow B$  is a Serre fibration.  $e \in E$ ,  $g(e) = b$  and  $g^{-1}b = F$ . Then consider the exact sequence in homotopy of the Serre fibration and the relative homotopy exact sequence. Then there is a long exact sequence (top row):

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(B) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \cong \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \pi_n(F) & \longrightarrow & \pi_n(E) & \longrightarrow & \pi_n(E, F) & \longrightarrow & \pi_{n-1}(F) & \longrightarrow & \pi_{n-1}(E) & \longrightarrow & \cdots \end{array}$$

**example.** We have shown that  $\pi_2(\mathbb{CP}^n) \cong \mathbb{Z}$  using the Hopf fibration  $S^1 \hookrightarrow S^{2n+1} \rightarrow \mathbb{CP}^n$  and the fact that  $\pi_k(S^n) = 0$  for  $k < n$ .

**Theorem 18.** Let  $X$  be a CW-complex,  $A, B \subset X$  subcomplexes,  $C = A \cap B \neq \emptyset$ , so

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & \lrcorner & \downarrow \\ B & \longrightarrow & X \end{array}$$

is a pushout (this happens for inclusions, **check it?**).

If  $(A, C)$  is  $n$ -connected and  $(B, C)$  is  $m$ -connected, then

$$\pi_i(A, C) \rightarrow \pi_i(X, B)$$

is an isomorphism for  $i < m + n$  and surjective for  $i = m + n$ .

### blakers-massey (26 march)

First I show some basic constructions from Tom Dieck (sec. 5.7). Let  $f : X \rightarrow Y$  be a map. Consider the pullback

$$\begin{array}{ccc} W(f) & \longrightarrow & Y^I \\ (q,p) \downarrow & & \downarrow (ev_0, ev_1) \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

where

$$\begin{aligned} W(f) &= \{(x, w) \in X \times Y^I \mid f(x) = w(0)\}, \\ q(x, w) &= x, \quad p(x, w) = w(1). \end{aligned}$$

Since  $(ev_0, ev_1)$  is a fibration, the maps  $(q, p)$ ,  $q$  and  $p$  are fibrations.

Now suppose  $f$  is a pointed map with base points  $*$ . Then  $W(f) \rightarrow W'$  is given the base point  $(*, k_*)$ .

Let  $f : A \hookrightarrow X$  be an inclusion.

**definition.** By  $(I^n, \partial I^n) \rightarrow (* \times_{ev_0} X^I \times_{ev_1} A, pt)$  is the same as a map  $I^n \times I \rightarrow X$  that satisfies:

- $I^n\{0\} \cup \partial I^n \times I \rightarrow *$ .
- $I^n \times \{1\} \rightarrow A$ .

It is fairly straightforward to show that

$$\cdots \longrightarrow \Omega A \longrightarrow \Omega X \longrightarrow \text{hofib} \longrightarrow A \longrightarrow X$$

$$\pi_0(\nearrow) = \begin{array}{ccccccc} \pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_{n-1}(\text{hofib}) & \longrightarrow & \pi_{n-1}(A) \longrightarrow \pi_{n-1}(X) \\ & & & \searrow & \downarrow \cong & \nearrow & \\ & & & & \pi_n(X, A) & & \end{array}$$

**Theorem 19** (Blakers-Massey 1). Let

$$\begin{array}{ccc} Q & \xrightarrow{g} & Y \\ f \downarrow & & \downarrow \\ X & \longrightarrow & P \end{array}$$

be a homotopy pushout,  $g$  is  $m$  equivalence,  $f$  is  $n$ -equivalence and  $m, n \geq 0$ . Then  $Q \rightarrow X \times_P^h Y$  is  $(m + n - 1)$ -equivalence.

**Theorem 20** (Blakers-Massey 2).  $P$  is a CW-complex,  $X, Y$  subcomplexes,  $X \cap Y = Q \neq \emptyset$  (*strict pushout*)

$$\begin{array}{ccc} Q & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ X & \hookrightarrow & P \end{array}$$

Then  $\pi_i(Y, Q) \rightarrow \pi_i(P, X)$  is epi for  $i = m + n$  and iso for  $0 \leq i < m + n$ .

**Theorem 21** (Blakers-Massey 3).  $P = X \cup Y$ ,  $X$  and  $Y$  are open in  $P$ ,  $X \cap Y = Q \neq \emptyset$ .

We proved the third version based on Tom Dieck's proof.

**definition.**

- A map is a *k-equivalence* if the induced map on the  $i$ th homotopy group is an isomorphism for  $i < k$  and an epimorphism for  $i = k$ .
- $K_p(W) := \{x \in W : \text{at least } p \text{ coordinates of } x \text{ are } j \text{ the same coordinates of the center of } W\}$

**lemma 22.** Let  $W$  be a cube in  $\mathbb{R}^d$  with  $\dim W \leq d$ . If for all faces  $W'$  of  $\partial W$ ,  $f(W') \in A \implies w' \in K_p(W')$ , then there is a homotopy  $f \simeq g \text{ rel } \partial W$  such that  $g(w) \in A \implies w \in K_p(W)$ .

**freudenthal theorem (2 april)**

**definition.** The appropriate analogue of the Cartesian product in the category of based spaces is the *smash product*  $X \wedge Y$  defined by

$$X \wedge Y = X \times Y / X \vee Y.$$

Here  $X \vee Y$  is viewed as the subspace of  $X \times Y$  consisting of those pairs  $(x, y)$  such that either  $x$  is the basepoint of  $X$  or  $y$  is the basepoint of  $Y$ .

We also have the *suspension of pointed spaces*, which is like usual suspension but also collapsing the distinguished point, which has become an interval:

$$\Sigma X = (I \times X) / (t, x) \sim (0, y) \sim (1, y) \quad \forall y \in X.$$

Then we have

$$\text{Hom}_{\text{CGWH}_*}(\Sigma X, \Sigma X) \cong \text{Hom}_{\text{CGWH}_*}(X, \Omega \Sigma X)$$

where  $\Sigma X = S^1 \wedge X$  and  $\Omega \Sigma X = \text{Map}(S^1, \Sigma X)$ . That is,  $S^1 \wedge -$  is adjoint to  $\text{Map}(S^1, -)$ .

So let  $X$  be a space. The identity map  $\text{id}_{\Sigma X} : \Sigma X \rightarrow \Sigma X$  then induces a map  $X \rightarrow \Omega \Sigma X$ .

**Theorem 23 (Freudenthal).** Let  $X$  be  $\ell$ -connected space. Then  $X \rightarrow \Omega \Sigma X$  is a  $(2\ell + 1)$ -equivalence, that is,

$$\pi_i(X) \rightarrow \pi_{i+1}(\Sigma X),$$

is a bijection for  $i < 2\ell + 1$  and a surjection for  $i = 2\ell + 1$  (May).

*Proof 1.*

$$\begin{array}{ccc} X & \xrightarrow{(\ell+1)\text{-equiv}} & * \\ & \searrow & \nearrow \\ & \Omega \Sigma X & \\ & \swarrow \quad \searrow & \\ * & \xrightarrow{\quad} & \Sigma X \end{array}$$

$\downarrow (\ell+1)\text{-equiv}$        $\downarrow h_\Gamma$

$\lrcorner_h$

□

*Proof 2.* Consider

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & \Sigma X \end{array}$$

Then we use relative homotopy long exact sequence with  $(X, CX)$  to get  $\pi_i(CX, X) \cong \pi_{i-1}(X)$ , which is zero for  $0 \leq i \leq \ell + 1$ . Then use relative homotopy exact sequence for the pair  $(\Sigma X, CX)$ . then we get that  $\pi_i(\Sigma X, CX) = \pi_i(\Sigma X)$ . And then if you use it for  $(\Sigma X, X)$  and

But it also turns out that  $\pi_i(\Sigma X) = \pi_{i-1}(\Omega \Sigma X)$  because

$$\pi_n(Z) = \text{Hom}_{\text{h-Top},*}(S^n, Z) = \text{Hom}(S^1 \wedge S^{n-1}, Z) = \text{Hom}(S^{n-1}, \Omega Z) = \pi_{n-1}(\Omega, Z)$$

. And then since  $CX \hookrightarrow \Sigma X$  we get an arrow  $\pi_i(CX, X) \rightarrow \pi_i(\Sigma X, CX)$  which is isomorphism for  $0 \leq i \leq 2\ell + 1$  and surjective for  $i = 2\ell + 2$ .

So apply Blakers-Massey an ell equalities to get maps from  $\pi_{i-1}(X) \rightarrow \pi_{i-1}(\Omega \Sigma X)$  for  $i$  as desired. □

**corollary 24.** If  $X$  is  $\ell$ -connected, then  $\Sigma X$  is  $(\ell + 1)$ -connected for  $\ell \geq 0$ .

space	$S^0$	$\Sigma S^0 = S^1$	$\Sigma^2 S^0 = S^2$	$\Sigma^3 S^0 = S^3$	$\dots$	$\Sigma^n S^0 = S^n$
conectedness	-1	0	1	2	$\dots$	$(n-1)$

**corollary 25.**  $S^n$  is  $(n-1)$ -connected.

Back to Hopf fibration:

$$S^1 \hookrightarrow S^3 \rightarrow S^2$$

we get

$$0 = \pi_2(S^3) \rightarrow \pi_2(S^2) \xrightarrow{\cong} \pi_1(S^1) \rightarrow \pi_1(S^3) = 0,$$

so

$$\mathbb{Z} = \pi_2(S^2).$$

Now consider a map  $S^n \rightarrow S^n$ . We get a map  $CS^n \rightarrow CS^n$  (in general, for  $f : X \rightarrow Y$  we have  $(t, x) \mapsto (t, f(x))$  in the cones). We also have  $CS^n \rightarrow CS^n/S^n = S^{n+1}$ .

Now if we take  $\text{id} : S^n \rightarrow S^n$  we shall get  $\text{id} : S^{n+1} \rightarrow S^{n+1}$ . Think about this like  $\pi_1(S^1) \rightarrow \pi_2(S^2)$ . Now from Freudenthal we get  $\pi_{i-1}(X) \rightarrow \pi_i(\Sigma X)$  is surjective because  $i = 0$ . From Hopf fibration we have  $\pi_2(S^2) = \mathbb{Z}$ . So we have a surjective map  $\mathbb{Z} \rightarrow \mathbb{Z}$ . So it's an isomorphism and we conclude that  $\text{id}_{S^2}$  is a generator of  $\pi_2(S^2)$ .

**corollary 26.** Since  $S^n$  is  $(n-1)$ -connected, we have

$$\pi_i(S^n) \rightarrow \pi_{i+1}(S^{n+1})$$

is isomorphism for  $i \leq 2(n-1) = 2n-1$  and epimorphism for  $i = 2n-1$ . (We just shift the indices of theorem 23 by one.)

**corollary 27.**  $\pi_n(S^n) = \mathbb{Z}$  with  $\text{id}_{S^n}$  as generator.

**corollary 28.**  $\pi_{n+k}(S^n) \rightarrow \pi_{n+k+1}(S^{n+1})$  is isomorphism for  $k \leq n-1$  and epimorphism for  $k = n-1$ .

So for example

$$\pi_4(S^3) = \pi_5(S^4) = \pi_6(S^5).$$

And in fact they are  $\mathbb{Z}/2$ . This is what are called the *kth stable homotopy groups of a sphere*. And more in general, we take any space and apply  $\Omega\Sigma$  enough times, and the homotopy will start to stabilize.

Or for example from

$$S^1 \hookrightarrow S^3 \rightarrow S^2$$

we get

$$0 = \pi_i(S^1) \rightarrow \pi_i(S^3) \xrightarrow{\cong} \pi_i(S^2) \rightarrow \pi_{i-1}(S^2) = 0$$

So  $\pi_3(S^2) \cong \mathbb{Z}$  in case you were wondering.

**claim (Serre).**  $\pi_{4n-1}(S^{2n}) \cong \mathbb{Z} \oplus \text{finite abelian}$ . And  $\pi_i(S^k)$  is finite abelian in all other cases.

## another application of Blakers-Massey (2 april)

Glue a disk to a space and what happens to homotopy groups?

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{(n-1)\text{-equiv}} & D^n \\ \downarrow 0\text{-equiv} & \lrcorner & \downarrow \\ X & \longrightarrow & X \cup D^n \end{array}$$

Assume  $X$  is connected. We get a map from the vertical arrows

$$\pi_i(D^n, S^{n-1}) \longrightarrow \pi_i(X \cup D^n, X)$$

which is  $(n-1)$ -equivalence by Blakers-Massey. So, by attaching  $\sqcup D^n$  we can kill  $\pi_{n-1}(X)$ , that is,  $X \cup (\sqcup D^n)$  has trivial  $\pi_{n-1}$ .

Now notice that

$$0 = \pi_i(D^n) \longrightarrow \pi_i(D^n, S^{n-1}) \xrightarrow{\cong} \pi_{i-1}(S^{n-1}) \longrightarrow \pi_{i-1}(D^n) = 0$$

that is,  $\pi_i(D^n, S^{n-1}) = 0$  for  $i \leq n-1$ . This implies that  $\pi_i(X \cup D^n, X) = 0$  for  $i \leq n-1$ .

Also by homotopy long exact sequence,

$$\pi_{n-1}(X) \rightarrow \pi_i(X \cup D^n) \text{ is surjective}$$

$$\pi_i(X) \rightarrow \pi_i(X \cup D^n) \text{ is isomorphism for } i \leq n-2.$$

So what we have thus far is

$$\pi_n(X \cup D^n) \longrightarrow \pi_{n-1}(X) \longrightarrow \pi_{n-1}(X \cup D^n) \longrightarrow 0 = \pi_{n-1}(X \cup D^n)$$

Notice that  $\pi_n(X \cup D^n, X)$  is not ingeneral cyclic (counterexample  $S^1 \cup D^2$  taking universal cover which is real line with spheres on integers, homotopy equivalent to  $\bigvee_{\mathbb{Z}} S^2$  which is not finitely generated).

**proposition 29.** For any CW-complex  $X$ ,  $X^\ell \rightarrow X$  is an  $\ell$ -equivalence.

**remark.** We have used that for  $A \hookrightarrow X$  from long exact sequence of relative homotopy groups we get  $\pi_n(X, A) = 0$ .

**corollary 30.** Let  $i \leq \ell$  and  $g : D^i \rightarrow X$ ,  $g : \partial D^i \rightarrow X^\ell$ . Then there is a homotopy rel  $\partial D^i$  to a map with  $\text{img} \subset X^\ell$ .

**Theorem 31 (Cellular approximation theorem).** Let  $X$  and  $Y$  be CW-complexes and  $\xi : Y \rightarrow X$  be any map. Then  $\xi$  is homotopic to a *cellular map*, that is, a map  $\psi : Y \rightarrow X$ , such that for all  $\ell$ ,  $\psi Y^0 \subset X^\ell$ .

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