algebraic topology exercises

Contents

some definitions

definition.

- An ***initial object*** in a category $\mathcal{C}$ is an object $\varnothing$ such that for any object $x \in \mathcal{C}$ there is a unique morphism $\varnothing \to x$ with source $\varnothing$ and target x.
- A ***pullback*** of the morphisms f and g consists of an object P and two morphisms $p_1 : P \to X$ and $p_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

- A ***pushout*** of the morphisms f and g consists of an object P and two morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] \arrow[ur, phantom, "\ulcorner", very near end] & P \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

- A ***product*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $p_1 : X \sqcap Y \to X$, $p_2 : X \sqcap Y \to Y$ satisfying the following universal property:

$$\begin{tikzcd}
Q \arrow[drr, bend left, "q_2"] \arrow[dr, dashed, "\phi"] \arrow[ddr, bend right, "q_1"'] & & \\
& X \sqcap Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
& X &
\end{tikzcd}$$

- A ***coproduct*** of X and Y is an object $X \sqcup Y$ and a pair of morphisms $i_1 : X \to X \sqcup Y$, $i_2 : Y \to X \sqcup Y$ satisfying the following universal property:

$$\begin{tikzcd}
& Y \arrow[d, "i_2"] \arrow[ddr, bend left, "j_2"] & \\
X \arrow[r, "i_1"'] \arrow[drr, bend right, "j_1"'] & X \sqcup Y \arrow[dr, dashed, "\phi"] & \\
& & Q
\end{tikzcd}$$

  remark. More generally, for S any set and $F : S \to C$ a collection of objects in the category C indexed by S, their ***coproduct*** is an object $\coprod_{s \in S} F(s)$ equipped with maps

$$F(s) \to \coprod_{s \in S} F(s)$$

  such that this is universal among objects with maps from $F(s)$.

- A morphism i has the ***left lifting property with respect to a morphism*** p and p has the ***right lifting property with respect to*** i if for each morphisms f and g, if the outer square in the following diagram commutes, there exists ϕ (I think not necessarily unique) completing the diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "g"'] \arrow[ur, dashed, "\phi"] & Y
\end{tikzcd}$$

- For $\mathcal{C}$ any category, its ***arrow category*** $\mathrm{Arr}(\mathcal{C})$ is the category such that
  - an object a of $\mathrm{Arr}(C)$ is a morphism $a : a_0 \to a_1$ of $\mathcal{C}$,

– a morphism $f : a \to b$ of $\mathrm{Arr}(\mathcal{C})$ is a commutative square

$$\begin{tikzcd} a_0 \arrow[r, "f_0"] \arrow[d, "a"'] & b_0 \arrow[d, "b"] \\ a_1 \arrow[r, "f_1"'] & b_1 \end{tikzcd}$$

in $\mathcal{C}$,

– composition in $\mathrm{Arr}(\mathcal{C})$ is given simply by placing commutative squares side by side to get a commutative oblong.

This is osomorphic to the functor category

$$\mathrm{Arr}(\mathcal{C}) := \mathrm{Funct}(I, C) = [I, C] = C^I$$

for I the interval category $\{0 \to 1\}$.

exercise on mapping cylinder and Hurewicz cofibrations

exercise. Let $f : X \to Y$ be a map. Let $M_f = X \times [0,1] \cup_f Y$ be the ***mapping cylinder of*** f, i.e. the pushout of $X \xrightarrow{\cong} X \times \{0\} \hookrightarrow X \times [0,1]$ and of $f : X \times Y$. Let $g : X \to M_f$ be the map $X \xrightarrow{\cong} X \times \{1\} \to M_f$. Let $h : M_f \to Y$ be the map that is induced by $X \times [0,1] \to Y : (x,t) \mapsto f(x)$ and $\mathrm{id}_Y : Y \to Y$. Observe that f is the composition of g and h.

In both exercises below you might have to use the fact that pushouts are colimits and that colimits commute with products in CGWH, i.e. $(\mathrm{colim}\, A_i) \times B$ is canonically homeomorphic with $\mathrm{colim}(A_i \times B)$.

1. Show that h is a deformation retract, and in particular is a homotopy equivalence.
2. Show that $g : X \to M_f$ is a cofibration. You may use exercise (a), but the direct proof might be simpler.

Solution.

1. We have that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id} \times 1"'] \arrow[dr, bend right, "g"] & Y \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_Y"] & \\ X \times [0,1] \arrow[r] \arrow[r, phantom, shift left=3, "\ulcorner"] \arrow[drr, bend right, "(x,t) \mapsto f(x)"'] & M_f \arrow[dr, dashed, "h"] & \\ & & Y \end{tikzcd}$$

We must show that there is a homotopy between the identity map on M_f and a retraction from M_f to Y. So we want $h : M_f \times [0,1] \to M_f$ such that

$$h(-,0) = \mathrm{id}_{M_f}, \quad \mathrm{img}\, h(-,1) \subset Y \quad \text{and} \quad h(-,1)|_Y = \mathrm{id}_Y$$

The fact that h is a deformation retract is consequence of this diagram. But I still can't see why it must be a homotopy equivalence...

2. Consider the following lifting problem:

$$\begin{tikzcd} X \arrow[r, "H"] \arrow[d, "g"'] & Z^I \arrow[d, "\pi_0"] \\ M_f \arrow[r, "h"'] \arrow[ur, dashed] & Z \end{tikzcd}$$

Looks OK but why should the dashed arrow exist...?

$\square$

path spaces and fibrations

exercise.

a. Show that $\mathrm{Map}(I, Y)$ deformation retracts on $\mathrm{Map}(pt, Y)$. Most likely you'll have to find a correct map $I \times I \to I$. Also show that $\mathrm{Map}(I, Y) \to \mathrm{Map}(pt, Y)$ is a Hurewicz fibration. The key map will be of the form $I \times I \to I \times I$.

Solution.

a. **($\mathrm{Map}(I, Y) \to \mathrm{Map}(pt, Y)$ is a Hurewicz fibration.)** Let A be any space. We must show that for any homotopy H and lift h_0 there exists an homotopy $\tilde{H}$ as in the following diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "h_0"] \arrow[d, hook] & \mathrm{Map}(I, Y) \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & \mathrm{Map}(0, Y) \end{tikzcd}$$

From the isomorphism $\mathrm{Map}(X \times Y, Z) \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ we may rewrite the problem as

$$\begin{tikzcd} (A \times \{0\}) \times I \arrow[dr, "H_0"] & \\ (A \times I) \times I \arrow[r, dashed, "\tilde{H}"] & Y \\ (A \times I) \times \{0\} \arrow[ur, "H"'] & \end{tikzcd}$$

So we define the dashed arrow by

$$(a, s, t) \mapsto \begin{cases} H_0(a, 0, s-t) & \text{when } s - t \geqslant 0 \\ H(a, t-s, 0) & \text{when } s - t \leqslant 0 \end{cases}$$

so that when $s = t$ the functions coincide, when $s = 0$ we get H and when $t = 0$ we get H_0.

($\mathrm{Map}(I, Y)$ deformation retracts on $\mathrm{Map}(pt, Y)$.) We must show there is a homotopy

$$h : \mathrm{Map}(I, Y) \times I \longrightarrow \mathrm{Map}(I, Y)$$

such that

$$h(-, 0) = \mathrm{id}_{\mathrm{Map}(I,Y)}, \qquad h(-, 1) \subset \mathrm{Map}(pt, Y)$$

$$\text{and} \qquad h(-, 1)|_{\mathrm{Map}(pt,Y)} = \mathrm{id}_{\mathrm{Map}(pt,Y)} \,.$$

Consider the map

$$\begin{aligned} I \times I &\to I \\ (s,t) &\mapsto s - st \end{aligned}$$

Our deformation retract may be written like

$$\begin{aligned} h : \mathrm{Map}(I,Y) \times I &\longrightarrow \mathrm{Map}(I,Y) \\ (f(s),t) &\longmapsto f(s-st) \end{aligned}$$

Then for $t = 0$ we have the identity on $\mathrm{Map}(I,Y)$, and when $t = 1$ we have ev_0.

$\square$

exercise on model categories

exercise (3.1.8 from Riehl). Verify that the class of morphisms $\mathcal{L}$ characterized by the left lifting property against a fixed class of morphisms $\mathcal{R}$ is closed under coproducts, closed under retracts, and contains the isomorphisms.

Solution. ***(Coproducts.)*** Comment from Sergey: Coproduct of morphisms $A_i \to B_i$ in a category $\mathcal{C}$ is the obvious morphism $\sqcup A_i \to \sqcup B_i$. (Because in this construction morphisms $A_i \to B_i$ are seen as objects of what's called the arrow category of the category $\mathcal{C}$)

Suppose the maps $\ell_i : A_i \to B_i$ are in $\mathcal{L}$. Then their coproduct in the arrow category is the obvious map $\coprod A_i \to \coprod B_i$.

Explicitly, their coproduct is an arrow $\coprod \ell_i$ and a collection of maps $f_i : \ell_i \to \coprod \ell_i$ such that for any other object $m : A \to B$ in the arrow category and a map $g : \ell \to m$, the following diagram is completed uniquely:

$$\begin{tikzcd} \ell_i \arrow[r, "f_i"] \arrow[rr, bend right, "g"'] & \coprod \ell_i \arrow[r, dashed, "\exists!"] & m \end{tikzcd} \qquad \forall i$$

So we conclude that the source of $\coprod \ell_i$ is $\coprod A_i$ and its target $\coprod B_i$. Indeed, we really looking at

$$\begin{tikzcd} A_i \arrow[r, "\ell_i"] \arrow[d, "f_i^1"'] & B_i \arrow[d, "f_i^2"] \\ \coprod A_i \arrow[r, "\coprod \ell_i"] \arrow[d, dashed, "\exists!"'] & \coprod B_i \arrow[d, dashed, "\exists!"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

Now consider the following lifting problem with respect to a morphism $r \in \mathcal{R}$:

$$\begin{tikzcd} \coprod A_i \arrow[r] \arrow[d, "\coprod \ell_i"'] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

Since $\ell_i \in \mathcal{L}$, we have maps

$$\begin{tikzcd} A_i \arrow[r] \arrow[d, "\mathcal{L} \ni \ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r \in \mathcal{R}"] \\ B_i \arrow[r] \arrow[urr, dashed] & \coprod B_i \arrow[r] & \bullet \end{tikzcd}$$

which in turn means we have a unique map

$$
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}
$$

by the universal property of the coproduct $\coprod B_i$.

To conclude we need to check that the triangles below and above the dashed arrow in the former diagram commute. This follows from the universal property of the coproducts $\coprod A_i$ and $\coprod B_i$ since, in general,

$$
\mathrm{Hom}\left(\coprod X_i, Y\right) \cong \prod \mathrm{Hom}(X_i, Y).
$$

More explicitly, we now that the paths in red in the following diagrams are the same:

$$
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \color{red}\bullet \arrow[d, red, "r\in\mathcal{R}"] \\
\color{red}B_i \arrow[r, red] \arrow[urr] & \color{red}\coprod B_i \arrow[r] \arrow[ur, dashed, red] & \color{red}\bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
A_i \arrow[r] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \coprod A_i \arrow[r] \arrow[d] & \bullet \arrow[d, "r\in\mathcal{R}"] \\
\color{red}B_i \arrow[r, red] \arrow[urr] & \color{red}\coprod B_i \arrow[r, red] \arrow[ur, dashed] & \color{red}\bullet
\end{tikzcd}
$$

and also

$$
\begin{tikzcd}
\color{red}A_i \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \color{red}\coprod A_i \arrow[r] \arrow[d, red] & \color{red}\bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \color{red}\coprod B_i \arrow[r] \arrow[ur, dashed, red] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\color{red}A_i \arrow[r, red] \arrow[d, "\mathcal{L}\ni\ell_i"'] & \color{red}\coprod A_i \arrow[r, red] \arrow[d] & \color{red}\bullet \arrow[d, "r\in\mathcal{R}"] \\
B_i \arrow[r] \arrow[urr] & \coprod B_i \arrow[r] \arrow[ur, dashed] & \bullet
\end{tikzcd}
$$

so the conclusion follows from the former comment.

(Closed under retracts.) Let us at least state what a retract of a morphism g should be in the arrow category. Recall that a retract is just

$$
\begin{tikzcd}
X \arrow[r] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r] & X
\end{tikzcd}
$$

So in the arrow category we get

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[rr, bend left] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet
\end{tikzcd}
$$

$\square$

Hatcher's exercise on Whitehead's theorem

Theorem 1 (Whitehead, May)**.** If X is a CW complex and $e : Y \to Z$ is an n-equivalence, then $e_* : [X, Y] \to [X, Z]$ is a bijection if $\dim X < n$ and surjection if $\dim X = n$.

Theorem 2 (Whitehead, May)**.** An n-equivalence between CW complexes of dimension less than n is a homotopy equivalence. A weak equivalence between CW complexes is a homotopy equivalence.

Theorem 3 (Whitehead (4.5), Hatcher)**.** If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conlusion is stronger: X is a deformation retract of Y.

exercise (Hatcher 4.1.12)**.** Show that an n-connected, n-dimensional CW complex is contractible.

Solution. Just recall that n-connectedness means that $\pi_i(X) = 0$ for all $i \leqslant n$, which means that X is contractible by theorem 2. □

References

[1] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000 (cit. on p. 9).

[2] J.P. May. *A Concise Course in Algebraic Topology*. Chicago Lectures in Mathematics. University of Chicago Press, 1999. ISBN: 9780226511832 (cit. on p. 9).

[3] Emily Riehl. *Homotopical categories: from model categories to $(\infty, 1)$-categories*. 2020. arXiv: 1904.00886 [math.AT] (cit. on p. 7).