# notes on mirror symmetry

## 3 may (Alex)

We wish to understand

**Theorem** (Bogomolov-Tian-Todorov)**.** Any Calabi-Yau manifold has unobstructed deformations.

**Definition.** An ***almost compelx structure*** is an endomorphism J...

**Remark.** It is a fact by Borel & Serre (1953) that the only spheres which admit an almost complex structure are $S^2$ and $S^6$.

**Example.** All complex manifolds are almost complex manifolds.

**Theorem.** A necessary and sufficient condition for a 2u-smooth manifold M to admit an almost complex structure is that the group of tangent bundle of M could be reduced to $U(n)$.

**Theorem** (Newlander-Nirenberg)**.** Let $(M, J)$ be an almost complex manifold. Then, the following are equivalent:

1. (six conditions...)

**proposition.** An almost complex structure on a real 2-dimensional manifold is a complex structure.

*Proof.* By the Newlander-Nirenberg theorem, given a point $p \in U \subset M$ and a vector field $\mathfrak{X}U$, we have that $(V, JV)$ is a frame, and

$$N(V, JV) = [V, JV] + J[V, JV] + J[V, J^2V] - [JV, J^2V] = 0$$

$\square$

**Definition.** A ***deformation*** of complex analytic space M over a germ $(S, s_0)$ of complex analytic space is a triple $\pi, X, i)$ such that

$$
\begin{array}{ccc}
X & \xleftarrow{\text{embedding}} & M \\
{\scriptstyle \pi}\downarrow & & \downarrow \\
(S, s_0) & \xleftarrow{\ s_0\ } & \text{pt}
\end{array}
$$

where M is a compact manifold, $M \simeq \pi^{-1}(s_0)$ and $\pi$ is proper smooth.

**Theorem** (Ehresmann)**.** Let $\pi : X \to S$ be a proper family of differentiable manifold. If S is conncected, then all fibres are diffeomorphic.

**Theorem** (Kodaira). Let $X_0$ be a compact Kähler manifold. If $X \to S$ is a deformation, then any fibre $X_t$ is again Kähler.

**Theorem** (Kuranashi).

1. Any compact complex manifold admits a universal deformation.

2. If $\Gamma(X_0, T_{x_0}) = 0$ then it admits a universal deformation.

**Lemma.** Let $J$ be an almost complex structure sufficiently close to $J_0$ so that it is represented by a form $\lambda \in A^{0,1}T^{1,0}M$. Then $J$ is integrable if and only if

$$\bar{\partial}\lambda_i + \frac{1}{2}[\lambda_j, \lambda_j] = 0.$$

**Theorem** (Maurer-Cartan).
$$\bar{\partial}\phi + [\phi, \phi] = 0$$

where

$$\phi = \phi(t) = \sum_{i=1} \phi_i t_i$$

**Definition.**

- The **Kodaira-Spencer class** of a one-parameter deformation $J_t$ of a complex stucture $J$ is induced by a homology class $\phi_1 \in H^1(X, Tx)$.
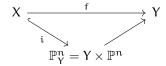
- The **Kodaira-Spancer map** is

$$T_s S \to H^1(X_s, T_{X_s}) = T_{[X_s]} \operatorname{Def}(X_{s_0})$$

# 10 May

## 1 Sergey: preliminaries

We work in the category of schemes over $\mathbb{C}$.

**Definition.** A morphism $f : X \to Y$ is **projective** if



where $i$ is a closed embedding and in fact $Y \times \mathbb{P}^n = \operatorname{Spec}\mathbb{C}$.

**Definition.** A **Hilbert function** for a given $Z \hookrightarrow \mathbb{P}^N$ is

$$h_Z(n) = \chi(Z, \mathcal{O}_Z(n))$$

**Definition** (Found later in Huybrechts, p. 273)**.** A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ if **flat** if the stalk $\mathcal{O}_{X,x}$ [...]

**Claim** (Criterium for flatness of projective morphisms)**.** A projective morhphism is flat if and only if $h_{X_t}(n)$ is constant as a function of t?

$Y \to \mathbb{Q}[n]$.

**Example** (non-flat projective morphism (blowup), which is also a non-submersion)**.** Let's find some $f : X \to Y$ projective but not flat. Suppose $X, Y$ are smooth and connected.

A closed embedding.

We tried

$$
\begin{array}{ccc}
 & \mathbb{C}^2 \times \mathbb{P}^1 & \\
\nearrow & & \searrow \\
X = \mathrm{Bl}_0\, \mathbb{C} \xrightarrow{\quad \pi \quad} & & Y = \mathbb{C}^2
\end{array}
$$

but (I think) its differential is not surjective due to the tangent space of the exceptional divisor.

**Definition** (Wiki)**.** In algebraic geometry, a morphism $f : X \to S$ between schemes is said to be **smooth** if

1. it is locally of finite presentation.

2. it is flat, and

3. for every geometric point $\bar{s} \to S$ the fiber $X_{\bar{s}} = X \times_X \bar{s}$.

## 2  Bruno: more on deformation

**Definition** (of smooth submersion)**.** A map whose differential is surjective.

**Definition** (Gross)**.** A **deformation** X consists of a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and $S$ are connected complex spaces, and an isomorphism $X \cong \mathcal{X}_0$, where $0 \in S$ is a distinguished point. We call $\mathcal{X} \to S$ a **family of complex manifolds**.

In order to define the deformation space $\mathrm{Def}(X)$ suppose X is Kähler with $H^0(X, \mathcal{T}_X) = 0$. Then there exists a universal deformation:

**Definition** (Gross)**.** A deformation $X \to (S, 0)$ of X is called **universal** if any other deformation $X' \to (S', 0')$ is isomorphic to the pullback under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

$$
\begin{array}{ccc}
\mathcal{X}_S & \longrightarrow & \mathcal{X} \\
\pi_S \downarrow & & \downarrow \\
S & \xrightarrow{\ \exists! \ } & \mathrm{Def}_S(X)
\end{array}
$$

**Definition.** The *Teichmüller space* of $X$ is

$$\text{Teich}(X) = \frac{\text{complex structures on } M}{\text{Diff}_0}$$

and it is such that

$$\mathcal{T}_X \text{Teich}(X) = H^1(X, \mathcal{T}_I X^{1,0})$$

**Remark** (The Misha Verbitsky way)**.** Let $X = (M, I)$ and $\bar\partial : C^\infty(M) \to \Omega^1(M, \mathbb{C})$ and remember that

- $\text{img}\,\bar\partial = \Omega^{0,1}_{(I)}(M)$

- $\bar\partial^2 = 0$.

Take a solution of the Maurer-Carten equation:

$$\bar\partial\gamma + [\gamma, \gamma] = 0$$

where $\gamma \in T^{1,0} \otimes \Omega^{0,1}$. Then we do

$$(\bar\partial + \gamma)(\bar\partial f + \gamma f) = \bar\partial(\gamma f) + \gamma\bar\partial f + \gamma(\gamma f)$$
$$\bar\partial_{\text{new}} f = \bar\partial f + \gamma f.$$
$$\dots?$$

Now take $s \in T^{1,0} \otimes \Omega^{0,1}$ such that

$$\bar\partial s + [s, s] = 0$$

and consider also its cohomology class $[s] \in H^1(T^{1,0})$. We have the *Kodaira-Spencer map*

$$KS : T_{s_0} S \to H^1(T^{1,0}) \cong T_X \text{Def}\,X$$
$$s \mapsto [s]$$

which is useful because de *deformation space* of $X$ is

$$(\text{Def}\,X, 0) = \frac{\text{solutions to Maurer-Cartan}}{\text{Diff}_0}$$

Ok, but what is the bracket? Answer: take the usual vector field conmutator on vector fields and the wedge product on differential forms. This makes $(\mathcal{T}_X^{1,0} \otimes \Omega_X^{0,\bullet}, [,], \bar\partial)$ into a *differential graded Lie algebra (DGLA)*.

So suppose

$$s = \sum_{n \geqslant 1} t^m s_m$$

and we wish to find

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_n = \sum_{i+j=n-1} [s_i, s_j]$$

The right-hand-side equation says $s_n$ is $\bar{\partial}$-exact.

Now since our objective is to understand Bogomolov-Tian-Todorov, we are interested in what **unobsturctedness** is. It means that

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_2 = [s_1, s_2]$$

Also recall that

**Definition.** Two manifolds $M_1, M_2 \subseteq \mathbb{C}^n$ define the same **germ** at $0 \in \mathbb{C}^n$ if there is an open set $U \subseteq \mathbb{C}^n$ containing $0$ such that

$$M_1 \cap U = M_\cap U.$$

and then...

**Theorem** (Bogomolov-Tian-Todorov)**.** content...

## 3 Griffiths transversality

**Claim.** Let $X$ be a complex manifold. For a 1-parameter family of complex structures $(X, J_t)$ and forms $\alpha_t \in \Omega^{p,q}(X, J_T)$ we have

$$\frac{d}{dt}\Big|_{t=0} \alpha_t \in \Omega^{p+1, q-1}(X) \oplus \Omega^{p,q}(X) \oplus \Omega^{p-1, q+1}(X).$$

*Proof.* content... $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$