

notes on mirror symmetry

3 may (Alex)

Main reference for these notes is [Gross](#), sect. 14.

We wish to understand

Theorem 1 (Bogomolov-Tian-Todorov). Any Calabi-Yau manifold has unobstructed deformations.

Definition. An *almost complex structure* is an endomorphism $J \dots$

Remark. It is a fact by Borel & Serre (1953) that the only spheres which admit an almost complex structure are S^2 and S^6 .

Example. All complex manifolds are almost complex manifolds.

Theorem 2. A necessary and sufficient condition for a $2n$ -smooth manifold M to admit an almost complex structure is that the group of tangent bundle of M could be reduced to $U(n)$.

Theorem 3 (Newlander-Nirenberg). Let (M, J) be an almost complex manifold. Then, the following are equivalent:

1. (six conditions...)

proposition 4. An almost complex structure on a real 2-dimensional manifold is a complex structure.

Proof. By the Newlander-Nirenberg theorem, given a point $p \in U \subset M$ and a vector field X on U , we have that (V, JV) is a frame, and

$$N(V, JV) = [V, JV] + J[V, JV] + J[V, J^2V] - [JV, J^2V] = 0$$

□

Definition. A *deformation* of complex analytic space M over a germ (S, s_0) of complex analytic space is a triple (π, X, i) such that

$$\begin{array}{ccc} X & \xleftarrow{\text{embedding}} & M \\ \pi \downarrow & & \downarrow \\ (S, s_0) & \xleftarrow{s_0} & \text{pt} \end{array}$$

where M is a compact manifold, $M \simeq \pi^{-1}(s_0)$ and π is proper smooth.

Theorem 5 (Ehresmann). Let $\pi : X \rightarrow S$ be a proper family of differentiable manifold. If S is connected, then all fibres are diffeomorphic.

Theorem 6 (Kodaira). Let X_0 be a compact Kähler manifold. If $X \rightarrow S$ is a deformation, then any fibre X_t is again Kähler.

Theorem 7 (Kuranashi).

1. Any compact complex manifold admits a universal deformation.
2. If $\Gamma(X_0, T_{X_0}) = 0$ then it admits a universal deformation.

Lemma. Let J be an almost complex structure sufficiently close to J_0 so that it is represented by a form $\lambda \in \Lambda^{0,1}T^{1,0}M$. Then J is integrable if and only if

$$\bar{\partial}\lambda_i + \frac{1}{2}[\lambda_j, \lambda_j] = 0.$$

Theorem 8 (Maurer-Cartan).

$$\bar{\partial}\phi + [\phi, \phi] = 0$$

where

$$\phi = \phi(t) = \sum_{i=1} \phi_i t_i$$

Definition.

- The *Kodaira-Spencer class* of a one-parameter deformation J_t of a complex structure J is induced by a homology class $\phi_1 \in H^1(X, T_X)$.
- The *Kodaira-Spencer map* is

$$T_s S \rightarrow H^1(X_s, T_{X_s}) = T_{[X_s]} \text{Def}(X_{s_0})$$

References

- [1] Mark Gross. “Calabi—Yau Manifolds and Mirror Symmetry”. In: *Calabi-Yau Manifolds and Related Geometries: Lectures at a Summer School in Nordfjordeid, Norway, June 2001*. Ed. by Geir Ellingsrud et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003, pp. 69–159. ISBN: 978-3-642-19004-9. DOI: [10.1007/978-3-642-19004-9_2](https://doi.org/10.1007/978-3-642-19004-9_2). URL: https://doi.org/10.1007/978-3-642-19004-9_2 (cit. on p. 1).