notes on mirror symmetry

3 may (Alex)

We wish to understand

Theorem (Bogomolov-Tian-Todorov)**.** Any Calabi-Yau manifold has unobstructed deformations.

Definition. An ***almost compelx structure*** is an endomorphism $J\dots$

Remark. It is a fact by Borel & Serre (1953) that the only spheres which admit an almost complex structure are S^2 and S^6.

Example. All complex manifolds are almost complex manifolds.

Theorem. A necessary and sufficient condition for a $2u$-smooth manifold M to admit an almost complex structure is that the group of tangent bundle of M could be reduced to $U(n)$.

Theorem (Newlander-Nirenberg)**.** Let (M, J) be an almost complex manifold. Then, the following are equivalent:

1. (six conditions...)

proposition. An almost complex structure on a real 2-dimensional manifold is a complex structure.

Proof. By the Newlander-Nirenberg theorem, given a point $p \in U \subset M$ and a vector field $\mathfrak{X}U$, we have that (V, JV) is a frame, and

$$N(V, JV) = [V, JV] + J[V, JV] + J[V, J^2V] - [JV, J^2V] = 0$$

□

Definition. A ***deformation*** of complex analytic space M over a germ (S, s_0) of complex analytic space is a triple $\pi, X, i)$ such that

$$\begin{tikzcd} X \arrow[d, "\pi"'] & M \arrow[l, "\text{embedding}"'] \arrow[d] \\ (S, s_0) & pt \arrow[l, "s_0"] \end{tikzcd}$$

where M is a compact manifold, $M \simeq \pi^{-1}(s_0)$ and π is proper smooth.

Theorem (Ehresmann)**.** Let $\pi : X \to S$ be a proper family of differentiable manifold. If S is conncected, then all fibres are diffeomorphic.

Theorem (Kodaira)**.** Let X_0 be a compact Kähler manifold. If $X \to S$ is a deformation, then any fibre X_t is again Kähler.

Theorem (Kuranashi)**.**

1. Any compact complex manifold admits a universal deformation.
2. If $\Gamma(X_0, T_{x_0}) = 0$ then it admits a universal deformation.

Lemma. Let J be an almost complex structure sufficiently close to J_0 so that it is represented by a form $\lambda \in A^{0,1}T^{1,0}M$. Then J is integrable if and only if

$$\bar{\partial}\lambda_i + \frac{1}{2}[\lambda_j, \lambda_j] = 0.$$

Theorem (Maurer-Cartan)**.**

$$\bar{\partial}\phi + [\phi, \phi] = 0$$

where

$$\phi = \phi(t) = \sum_{i=1} \phi_i t_i$$

Definition.

- The ***Kodaira-Spencer class*** of a one-parameter deformation J_t of a complex stucture J is induced by a homology class $\phi_1 \in H^1(X, Tx)$.
- The ***Kodaira-Spancer map*** is

$$T_s S \to H^1(X_s, T_{X_s}) = T_{[X_s]} \operatorname{Def}(X_{s_0})$$

10 May

1 Sergey: preliminaries

We work in the category of schemes over $\mathbb{C}$.

Definition. A morphism $f : X \to Y$ is ***projective*** if

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & \mathbb{P}^n_Y = Y \times \mathbb{P}^n \arrow[ur] & \end{tikzcd}$$

where i is a closed embedding and in fact $Y \times \mathbb{P}^n = \operatorname{Spec} \mathbb{C}$.

Definition. A ***Hilbert function*** for a given $Z \hookrightarrow \mathbb{P}^N$ is

$$h_Z(n) = \chi(Z, \mathcal{O}_Z(n))$$

Definition (Found later in [?], p. 273)**.** A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ if ***flat*** if the stalk $\mathcal{O}_{X,x}$ [...]

Claim (Criterium for flatness of projective morphisms)**.** A projective morhphism is flat if and only if $h_{X_t}(n)$ is constant as a function of t?

$Y \to \mathbb{Q}[n]$.

Example (non-flat projective morphism (blowup), which is also a non-submersion)**.** Let's find some $f : X \to Y$ projective but not flat. Suppose X, Y are smooth and connected.

A closed embedding.

We tried

$$\begin{tikzcd} & \mathbb{C}^2 \times \mathbb{P}^1 \arrow[dr] & \\ X = \mathrm{Bl}_0 \mathbb{C} \arrow[ur, hook] \arrow[rr, "\pi"] & & Y = \mathbb{C}^2 \end{tikzcd}$$

but (I think) its differential is not surjective due to the tangent space of the exceptional divisor.

Definition (Wiki)**.** In algebraic geometry, a morphism $f : X \to S$ between schemes is said to be ***smooth*** if

1. it is locally of finite presentation.
2. it is flat, and
3. for every geometric point $\bar{s} \to S$ the fiber $X_{\bar{s}} = X \times_X \bar{s}$.

2 Bruno: more on deformation

Definition (of smooth submersion)**.** A map whose differential is surjective.

Definition ([?])**.** A ***deformation*** X consists of a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and S are connected complex spaces, and an isomorphism $X \cong \mathcal{X}_0$, where $0 \in S$ is a distinguished point. We call $\mathcal{X} \to S$ a ***family of complex manifolds***.

In order to define the deformation space $\mathrm{Def}(X)$ suppose X is Kähler with $H^0(X, \mathcal{T}_X) = 0$. Then there exists a universal deformation:

Definition ([?])**.** A deformation $X \to (S, 0)$ of X is called ***universal*** if any other deformation $X' \to (S', 0')$ is isomorphic to the pullback under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

$$\begin{tikzcd} \mathcal{X}_S \arrow[r] \arrow[d, "\pi_S"'] & \mathcal{X} \arrow[d] \\ S \arrow[r, "\exists!"'] & \mathrm{Def}_S(X) \end{tikzcd}$$

Definition. The ***Teichmüller space*** of X is

$$\mathrm{Teich}(X) = \frac{\text{complex structures on } M}{\mathrm{Diff}_0}$$

and it is such that

$$\mathcal{T}_X \,\mathrm{Teich}(X) = H^1(X, \mathcal{T}_I X^{1,0})$$

Remark (The Misha Verbitsky way). Let $X = (M, I)$ and $\bar{\partial} : C^\infty(M) \to \Omega^1(M, \mathbb{C})$ and remember that

- $\operatorname{img} \bar{\partial} = \Omega^{0,1}_{(I)}(M)$
- $\bar{\partial}^2 = 0$.

Take a solution of the Maurer-Carten equation:

$$\bar{\partial}\gamma + [\gamma, \gamma] = 0$$

where $\gamma \in T^{1,0} \otimes \Omega^{0,1}$. Then we do

$$\begin{gathered}(\bar{\partial} + \gamma)(\bar{\partial} f + \gamma f) = \bar{\partial}(\gamma f) + \gamma \bar{\partial} f + \gamma(\gamma f) \\ \bar{\partial}_{\mathrm{new}} f = \bar{\partial} f + \gamma f. \\ \ldots ?\end{gathered}$$

Now take $s \in T^{1,0} \otimes \Omega^{0,1}$ such that

$$\bar{\partial} s + [s, s] = 0$$

and consider also its cohomology class $[s] \in H^1(T^{1,0})$. We have the ***Kodaira-Spencer map***

$$\begin{aligned} \mathrm{KS} : T_{s_0} S &\to H^1(T^{1,0}) \cong T_X \,\mathrm{Def}\, X \\ s &\mapsto [s] \end{aligned}$$

which is useful because de ***deformation space*** of X is

$$(\mathrm{Def}\, X, 0) = \frac{\text{solutions to Maurer-Cartan}}{\mathrm{Diff}_0}$$

Ok, but what is the bracket? Answer: take the usual vector field conmutator on vector fields and the wedge product on differential forms. This makes $(\mathcal{T}^{1,0}_X \otimes \Omega^{0,\bullet}_X, [,], \bar{\partial})$ into a ***differential graded Lie algebra (DGLA)***.

So suppose

$$s = \sum_{n \geqslant 1} t^m s_m$$

and we wish to find

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_n = \sum_{i+j=n-1} [s_i, s_j]$$

The right-hand-side equation says s_n is $\bar{\partial}$-exact.

Now since our objective is to understand Bogomolov-Tian-Todorov, we are interested in what ***unobsturctedness*** is. It means that

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_2 = [s_1, s_2]$$

Also recall that

Definition. Two manifolds $M_1, M_2 \subseteq \mathbb{C}^n$ define the same ***germ*** at $0 \in \mathbb{C}^n$ if there is an open set $U \subseteq \mathbb{C}^n$ containing 0 such that

$$M_1 \cap U = M_\cap U.$$

and then...

Theorem (Bogomolov-Tian-Todorov). content...

3 Griffiths transversality (Victor)

Claim. Let X be a complex manifold. For a 1-parameter family of complex structures (X, J_t) and forms $\alpha_t \in \Omega^{p,q}(X, J_T)$ we have

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_t \in \Omega^{p+1,q-1}(X) \oplus \Omega^{p,q}(X) \oplus \Omega^{p-1,q+1}(X).$$

Proof. content... □

4 Hodge Theory for Calabi-Yau (afternoon)

4.1 Preliminaries (Sergey)

Let's first recall that

Definition. The ***Hodge star*** operator is

$$* : H^{p,q}_{\bar{\partial}} \to H^{n-q,n-p}_{\bar{\partial}}$$

proposition. For any complex manifold,

$$\begin{aligned} H^{p,q}_{\bar{\partial}} \times H^{n-p,n-q}_{\bar{\partial}} &\to H^{n,n}_{\bar{\partial}} \cong \mathbb{C} \\ ([\alpha], [\beta]) &\mapsto \int_{[X]} \alpha \wedge \beta := (\alpha, \beta) \end{aligned}$$

is bilinear and non-degenerate.

Proof. $\forall \alpha \exists \beta = *\bar{\alpha}$ such that $(\alpha, \beta) \neq 0$ so

$$0 < \|\alpha\|^2 = \int \alpha \wedge *\bar{\alpha}$$
$$\langle \alpha, \beta \rangle = \int \alpha \wedge *\bar{\beta}$$

where $\langle -, - \rangle$ is the induced metric by some hermitian/riemannian metric on X? □

Theorem (Serre duality)**.** For any complex manifold,

$$H_{\bar{\partial}}^{p,q} \times H_{\bar{\partial}}^{n-p,n-q} \to H_{\bar{\partial}}^{n,n} \cong \mathbb{C}$$
$$([\alpha], [\beta]) \mapsto \int_{[X]} \alpha \wedge \beta$$

is a perfect pairing. That is

$$H^{p,q} \cong (H^{n-p,n-q})^*$$

And we also have

Theorem (Hodge)**.** For Kähler manifolds

$$H^{p,q} \cong \overline{H^{q,p}}$$

4.2 Pseudoholomorphic curves (Victor)

We follow [?], lecture 3.

Remark. For every Calabi-Yau manifold X,

$$H^{p,0} = H^{n,n-p} = H_{\bar{\partial}}^{n-p}(X, \Omega_X^n) = H_{\bar{\partial}}^{n-p}(X, \Omega_X) = H^{0,n-p} = H^{n-p,0}$$

So we have some symmetry:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & a & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & a & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & b & & 0 & \\ 1 & & a & & a & & 1 \\ & 0 & & b & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Definition. Let (X^{2n}, ω) be a symplectic manifold, J a compatible almost-complex structure, $\omega(\cdot, J\cdot)$ the associated Riemannian metric. Furthermore, let (Σ, j) be a Riemann surface of genus g and $z_1, \dots, z_k$ marked points.

There is a well-defined moduli space of $\mathcal{M}_{g,k} = \{(\Sigma, z_1, \dots, z_k)\}$ which is a complex manifold of dimension $3k - 3 + k$.

$u : \Sigma \to X$ is a J-***holomorphic (or pseudoholomorphic) map*** if

$$J \circ du = du \circ j$$

that is,

$$\bar{\partial}_J u = \frac{1}{2}(du + J du j) = 0. \tag{.1}$$

For $\beta \in H_2(X, \mathbb{Z})$, we obtain an associated space

$$M_{g,k}(X, J, \beta) = \{(\Sigma, j, z_1, \dots, z_k, u : \Sigma \to X | u_*[\Sigma] = \beta, \bar{\partial}_J u = 0\} / \sim$$

where $\sim$ is the equivalence given by ϕ below:

$$\begin{tikzcd} \Sigma, z_1, \dots, z_k \arrow[r, "u"] \arrow[d, "\phi"', "\cong"] & X \\ \Sigma', z_1', \dots, z_k' \arrow[ur, "u'"'] & \end{tikzcd}$$

Question. Where does the object in eq. (.1) live? The differential of any map of complex manifolds can be decomposed in ∂ and $\bar{\partial}$. The operator $\bar{\partial}_J u$ is an element of $\Omega^{0,1}(\Sigma, u \otimes TX) = \Gamma(\Sigma, \Omega^{0,1}(\Sigma) \otimes u^* TX)$.

Remark. See wiki for interpretation of this definition as a map satisfying the Cauchy-Riemann equations.

Remark. See What is... a pseudoholomorphic curve? for another friendly explanation:

> A pseudoholomorphic curve is just the natural modification of the notion of a holomorphic curve to the case when the ambient manifold is almost-complex.

May 17

1 Pseudoholomorphic curves cont. (Victor)

We continue to read [?].

Definition. We say that $u : \Sigma \to X$ is ***simple*** if there exists $z \in \Sigma$ such that $du(z) \neq 0$ and $u^{-1}(u(z)) = z$.

Which roughly means that the function is not generically one to one on its image.

Example. The function

$$\begin{aligned} u : \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x : y] &\mapsto [x^2 : y^2 : 0] \end{aligned}$$

is not simple. Indeed, near a point $[x : y] \in \mathbb{P}^1$ with $x \neq 0$, the differential of u may be expressed in coordinates as the linear map $du = \begin{pmatrix} 2 & 0 \end{pmatrix} \neq 0$; however $u^{-1}([x^2 : y^2 : 0]) = \{[x : y], [-x : y]\}$. The case of $y \neq 0$ is analogous. We also see there are no singular points, so u cannot be simple.

Then we define

$$D_{\bar{\partial}} : W^{r+1,p}(\Sigma, u^*TX) \times T\mathcal{M}_{g,k} \to W^{r,p}(\Sigma, \Omega^{0,1}_{\Sigma} \otimes U^*TX)$$

by

$$D_{\bar{\partial}}(\nu, j') = \bar{\partial}\nu + \frac{1}{2}(\nabla_\nu J)du \cdot j + \frac{1}{2}J \cdot du \cdot j'$$

where $W^{r,p}$ is a completion of $C^\infty(-)$ of (?) to $L^{r,p}$ norm defined by $\|f\|_{r,p} = \left(\sum_{i=0}^{r} \int |f^{(i)}(t)|^p \, dt\right)^{1/p}$.

$D_{\bar{\partial}}$ is Fredholm, (meaning the dimensions of its kernel and cokernel are finite), with index (the difference of such numbers)

$$\mathrm{index}_{\mathbb{R}} D_{\bar{\partial}} := 2d = 2\langle c_1(TX), \beta\rangle + n(2 - 2g) + (6g - 6 + 2k).$$

We may interpret this equation as differentiation of the Cauchy-Riemann equations.

2 Dirichlet energy (Alex)

Consider a map

$$u : (\Sigma, j) \to (X, \omega, J, g)$$

and define the ***energy functional***

$$\varepsilon = \int_\Sigma |du|_g^2 \, \mathrm{Vol}_g$$

Now, we may take local isothermic coordinates where the metric is expressed as

$$g = \lambda(x, y)(dx^2 + dy^2)$$

giving

$$\begin{aligned} du &= \partial_x u \otimes dx + \partial_y u \otimes dy \\ |du|^2 &= |\partial_x u|^2 \lambda^{-2} + |\partial_y u|^2 \lambda^{/2} \\ \mathrm{Vol}_{\Sigma g'} &= \lambda^2 dx \wedge dy \end{aligned}$$

Then

$$\varepsilon(u) = \int_\Sigma |\partial_x u|_g^2 + |\partial_y u|_g^2 dx \wedge dy.$$

The following equality shows that the energy functional attains its minimum on pseudoholomorphic maps (in virtue of eq. (1)).

Claim. For a pseudoholomorphic map,

$$\varepsilon(u) = \int_{\Sigma} 2|\bar{\partial}_J|^2 \operatorname{Vol} + \int_{\Sigma} u^* \omega$$

Proof. content... ☐