notes on mirror symmetry

Contents

3 may (Alex) **1**

May 10 **3**

1 Sergey: preliminaries . . . 3
2 Bruno: more on deformation . . . 4
3 Griffiths transversality (Victor) . . . 5
4 Hodge Theory for Calabi-Yau (afternoon) . . . 6
4.1 Preliminaries (Sergey) . . . 6
4.2 Pseudoholomorphic curves (Victor) . . . 6

May 17 **8**

1 Pseudoholomorphic curves cont. (Victor) . . . 8
2 Dirichlet energy functional (Alex) . . . 8

May 24 (Alex) **9**

June 7 Frobenius manifolds (Sergey) **10**

1 Introduction . . . 10
2 Milnor ring and Milnor number . . . 11
3 Versal deformation of isolated singularities . . . 12
4 Frobenius algebra . . . 12

3 may (Alex)

We wish to understand

Theorem (Bogomolov-Tian-Todorov)**.** Any Calabi-Yau manifold has unobstructed deformations.

Definition. An ***almost compelx structure*** is an endomorphism J. . .

Remark. It is a fact by Borel & Serre (1953) that the only spheres which admit an almost complex structure are S^2 and S^6.

Example. All complex manifolds are almost complex manifolds.

Theorem. A necessary and sufficient condition for a $2u$-smooth manifold M to admit an almost complex structure is that the group of tangent bundle of M could be reduced to $U(n)$.

Theorem (Newlander-Nirenberg)**.** Let (M, J) be an almost complex manifold. Then, the following are equivalent:

1. (six conditions. . .)

Proposition. An almost complex structure on a real 2-dimensional manifold is a complex structure.

Proof. By the Newlander-Nirenberg theorem, given a point $p \in U \subset M$ and a vector field $\mathfrak{X}U$, we have that (V, JV) is a frame, and

$$N(V, JV) = [V, JV] + J[V, JV] + J[V, J^2V] - [JV, J^2V] = 0$$

$\square$

Definition. A ***deformation*** of complex analytic space M over a germ (S, s_0) of complex analytic space is a triple $\pi, X, i)$ such that

$$\begin{tikzcd} X \arrow[d, "\pi"'] & M \arrow[l, "\text{embedding}"'] \arrow[d] \\ (S, s_0) & pt \arrow[l, "s_0"] \end{tikzcd}$$

where M is a compact manifold, $M \simeq \pi^{-1}(s_0)$ and π is proper smooth.

Theorem (Ehresmann)**.** Let $\pi : X \to S$ be a proper family of differentiable manifold. If S is conncected, then all fibres are diffeomorphic.

Theorem (Kodaira)**.** Let X_0 be a compact Kähler manifold. If $X \to S$ is a deformation, then any fibre X_t is again Kähler.

Theorem (Kuranashi)**.**

1. Any compact complex manifold admits a universal deformation.
2. If $\Gamma(X_0, T_{x_0}) = 0$ then it admits a universal deformation.

Lemma. Let J be an almost complex structure sufficiently close to J_0 so that it is represented by a form $\lambda \in A^{0,1}T^{1,0}M$. Then J is integrable if and only if

$$\bar{\partial}\lambda_i + \frac{1}{2}[\lambda_j, \lambda_j] = 0.$$

Theorem (Maurer-Cartan)**.**

$$\bar{\partial}\phi + [\phi, \phi] = 0$$

where

$$\phi = \phi(t) = \sum_{i=1} \phi_i t_i$$

Definition.

- The ***Kodaira-Spencer class*** of a one-parameter deformation J_t of a complex stucture J is induced by a homology class $\phi_1 \in H^1(X, Tx)$.
- The ***Kodaira-Spencer map*** is

$$T_sS \to H^1(X_s, T_{X_s}) = T_{[X_s]} \operatorname{Def}(X_{s_0})$$

May 10

1 Sergey: preliminaries

We work in the category of schemes over $\mathbb{C}$.

Definition. A morphism $f : X \to Y$ is ***projective*** if

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "i"'] & & Y \\ & \mathbb{P}^n_Y = Y \times \mathbb{P}^n \arrow[ur] & \end{tikzcd}$$

where i is a closed embedding and in fact $Y \times \mathbb{P}^n = \operatorname{Spec} \mathbb{C}$.

Definition. A ***Hilbert function*** for a given $Z \hookrightarrow \mathbb{P}^N$ is

$$h_Z(n) = \chi(Z, \mathcal{O}_Z(n))$$

Definition (Found later in [?], p. 273)**.** A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ if ***flat*** if the stalk $\mathcal{O}_{X,x}$ [...]

Claim (Criterium for flatness of projective morphisms)**.** A projective morhphism is flat if and only if $h_{X_t}(n)$ is constant as a function of t?

$Y \to \mathbb{Q}[n]$.

Example (non-flat projective morphism (blowup), which is also a non-submersion)**.** Let's find some $f : X \to Y$ projective but not flat. Suppose X, Y are smooth and connected.

A closed embedding.

We tried

$$\begin{tikzcd} & \mathbb{C}^2 \times \mathbb{P}^1 \arrow[dr] & \\ X = \operatorname{Bl}_0 \mathbb{C} \arrow[ur, hook] \arrow[rr, "\pi"] & & Y = \mathbb{C}^2 \end{tikzcd}$$

but (I think) its differential is not surjective due to the tangent space of the exceptional divisor.

Definition (Wiki)**.** In algebraic geometry, a morphism $f : X \to S$ between schemes is said to be ***smooth*** if

1. it is locally of finite presentation.
2. it is flat, and
3. for every geometric point $\bar{s} \to S$ the fiber $X_{\bar{s}} = X \times_X \bar{s}$.

2 Bruno: more on deformation

Definition (of smooth submersion). A map whose differential is surjective.

Definition ([?]). A ***deformation*** X consists of a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and S are connected complex spaces, and an isomorphism $X \cong \mathcal{X}_0$, where $0 \in S$ is a distinguished point. We call $\mathcal{X} \to S$ a ***family of complex manifolds***.

In order to define the deformation space Def(X) suppose X is Kähler with $H^0(X, \mathcal{T}_X) = 0$. Then there exists a universal deformation:

Definition ([?]). A deformation $X \to (S, 0)$ of X is called ***universal*** if any other deformation $X' \to (S', 0')$ is isomorphic to the pullback under a uniquely determined morphism $\varphi : S' \to S$ with $\varphi(0') = 0$.

$$\begin{array}{ccc} \mathcal{X}_S & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi_S}\downarrow & & \downarrow \\ S & \xrightarrow[\exists!]{} & \mathrm{Def}_S(X) \end{array}$$

Definition. The ***Teichmüller space*** of X is

$$\mathrm{Teich}(X) = \frac{\text{complex structures on } M}{\mathrm{Diff}_0}$$

and it is such that

$$\mathcal{T}_X \,\mathrm{Teich}(X) = H^1(X, \mathcal{T}_I X^{1,0})$$

Remark (The Misha Verbitsky way). Let $X = (M, I)$ and $\bar{\partial} : C^\infty(M) \to \Omega^1(M, \mathbb{C})$ and remember that

- $\mathrm{img}\,\bar{\partial} = \Omega^{0,1}_{(I)}(M)$
- $\bar{\partial}^2 = 0$.

Take a solution of the Maurer-Carten equation:

$$\bar{\partial}\gamma + [\gamma, \gamma] = 0$$

where $\gamma \in T^{1,0} \otimes \Omega^{0,1}$. Then we do

$$(\bar{\partial} + \gamma)(\bar{\partial} f + \gamma f) = \bar{\partial}(\gamma f) + \gamma\bar{\partial} f + \gamma(\gamma f)$$
$$\bar{\partial}_{\text{new}} f = \bar{\partial} f + \gamma f.$$
$$\ldots ?$$

Now take $s \in T^{1,0} \otimes \Omega^{0,1}$ such that

$$\bar{\partial} s + [s, s] = 0$$

and consider also its cohomology class $[s] \in H^1(T^{1,0})$. We have the ***Kodaira-Spencer map***

$$\begin{aligned} KS : T_{s_0}S &\to H^1(T^{1,0}) \cong T_X \operatorname{Def} X \\ s &\mapsto [s] \end{aligned}$$

which is useful because de ***deformation space*** of X is

$$(\operatorname{Def} X, 0) = \frac{\text{solutions to Maurer-Cartan}}{\mathrm{Diff}_0}$$

Ok, but what is the bracket? Answer: take the usual vector field conmutator on vector fields and the wedge product on differential forms. This makes $(\mathcal{T}_X^{1,0} \otimes \Omega_X^{0,\bullet}, [,], \bar{\partial})$ into a ***differential graded Lie algebra (DGLA)***.

So suppose

$$s = \sum_{n \geqslant 1} t^m s_m$$

and we wish to find

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_n = \sum_{i+j=n-1} [s_i, s_j]$$

The right-hand-side equation says s_n is $\bar{\partial}$-exact.

Now since our objective is to understand Bogomolov-Tian-Todorov, we are interested in what ***unobsturctedness*** is. It means that

$$\bar{\partial} s_1 = 0 \qquad \text{and} \qquad \bar{\partial} s_2 = [s_1, s_2]$$

Also recall that

Definition. Two manifolds $M_1, M_2 \subseteq \mathbb{C}^n$ define the same ***germ*** at $0 \in \mathbb{C}^n$ if there is an open set $U \subseteq \mathbb{C}^n$ containing 0 such that

$$M_1 \cap U = M_\cap U.$$

and then...

Theorem (Bogomolov-Tian-Todorov)**.** content...

3 Griffiths transversality (Victor)

Claim. Let X be a complex manifold. For a 1-parameter family of complex structures (X, J_t) and forms $\alpha_t \in \Omega^{p,q}(X, J_T)$ we have

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_t \in \Omega^{p+1,q-1}(X) \oplus \Omega^{p,q}(X) \oplus \Omega^{p-1,q+1}(X).$$

Proof. content... □

4 Hodge Theory for Calabi-Yau (afternoon)

4.1 Preliminaries (Sergey)

Let's first recall that

Definition. The ***Hodge star*** operator is

$$* : H_{\bar{\partial}}^{p,q} \to H_{\bar{\partial}}^{n-q,n-p}$$

Proposition. For any complex manifold,

$$\begin{aligned} H_{\bar{\partial}}^{p,q} \times H_{\bar{\partial}}^{n-p,n-q} &\to H_{\bar{\partial}}^{n,n} \cong \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto \int_{[X]} \alpha \wedge \beta := (\alpha, \beta) \end{aligned}$$

is bilinear and non-degenerate.

Proof. $\forall \alpha \exists \beta = *\bar{\alpha}$ such that $(\alpha, \beta) \neq 0$ so

$$\begin{aligned} 0 < \|\alpha\|^2 &= \int \alpha \wedge *\bar{\alpha} \\ \langle \alpha, \beta \rangle &= \int \alpha \wedge *\bar{\beta} \end{aligned}$$

where $\langle -, - \rangle$ is the induced metric by some hermitian/riemannian metric on X? □

Theorem (Serre duality)**.** For any complex manifold,

$$\begin{aligned} H_{\bar{\partial}}^{p,q} \times H_{\bar{\partial}}^{n-p,n-q} &\to H_{\bar{\partial}}^{n,n} \cong \mathbb{C} \\ ([\alpha],[\beta]) &\mapsto \int_{[X]} \alpha \wedge \beta \end{aligned}$$

is a perfect pairing. That is

$$H^{p,q} \cong (H^{n-p,n-q})^*$$

And we also have

Theorem (Hodge)**.** For Kähler manifolds

$$H^{p,q} \cong \overline{H^{q,p}}$$

4.2 Pseudoholomorphic curves (Victor)

We follow [**?**], lecture 3.

Remark. For every Calabi-Yau manifold X,

$$H^{p,0} = H^{n,n-p} = H_{\bar{\partial}}^{n-p}(X, \Omega_X^n) = H_{\bar{\partial}}^{n-p}(X, \Omega_X) = H^{0,n-p} = H^{n-p,0}$$

So we have some symmetry:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & a & & 0 & \\
1 & & b & & b & & 1 \\
 & 0 & & a & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & b & & 0 & \\
1 & & a & & a & & 1 \\
 & 0 & & b & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Definition. Let (X^{2n}, ω) be a symplectic manifold, J a compatible almost-complex structure, $\omega(\cdot, J\cdot)$ the associated Riemannian metric. Furthermore, let (Σ, j) be a Riemann surface of genus g and $z_1, \ldots, z_k$ marked points.

There is a well-defined moduli space of $\mathcal{M}_{g,k} = \{(\Sigma, z_1, \ldots, z_k)\}$ which is a complex manifold of dimension $3k - 3 + k$.

$u : \Sigma \to X$ is a J-***holomorphic (or pseudoholomorphic) map*** if

$$J \circ du = du \circ j$$

that is,

$$\bar{\partial}_J u = \frac{1}{2}(du + J du j) = 0. \tag{.1}$$

For $\beta \in H_2(X, \mathbb{Z})$, we obtain an associated space

$$M_{g,k}(X, J, \beta) = \{(\Sigma, j, z_1, \ldots, z_k, u : \Sigma \to X | u_*[\Sigma] = \beta, \bar{\partial}_J u = 0\} / \sim$$

where $\sim$ is the equivalence given by ϕ below:

$$
\begin{tikzcd}
\Sigma, z_1, \ldots, z_k \arrow[r, "u"] \arrow[d, "\phi"', "\cong"] & X \\
\Sigma', z_1', \ldots, z_k' \arrow[ur, "u'"'] &
\end{tikzcd}
$$

Question. Where does the object in eq. (.1) live? The differential of any map of complex manifolds can be decomposed in ∂ and $\bar{\partial}$. The operator $\bar{\partial}_J u$ is an element of $\Omega^{0,1}(\Sigma, u^*TX) = \Gamma(\Sigma, \Omega^{0,1}(\Sigma) \otimes u^*TX)$.

Remark. See wiki for interpretation of this definition as a map satisfying the Cauchy-Riemann equations.

Remark. See What is... a pseudoholomorphic curve? for another friendly explanation:

> A pseudoholomorphic curve is just the natural modification of the notion of a holomorphic curve to the case when the ambient manifold is almost-complex.

May 17

1 Pseudoholomorphic curves cont. (Victor)

We continue to read [**?**], lecture 3.

Definition. We say that $u : \Sigma \to X$ is ***simple*** if there exists $z \in \Sigma$ such that $du(z) \neq 0$ and $u^{-1}(u(z)) = z$.

Which roughly means that the function is not generically one to one on its image.

Example. The function

$$\begin{aligned} u : \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x : y] &\mapsto [x^2 : y^2 : 0] \end{aligned}$$

is not simple. Indeed, near a point $[x : y] \in \mathbb{P}^1$ with $x \neq 0$, the differential of u may be expressed in coordinates as the linear map $du = \begin{pmatrix} 2 & 0 \end{pmatrix} \neq 0$; however $u^{-1}([x^2 : y^2 : 0]) = \{[x : y], [-x : y]\}$. The case of $y \neq 0$ is analogous. We also see there are no singular points, so u cannot be simple.

Then we define

$$D_{\bar\partial} : W^{r+1,p}(\Sigma, u^*TX) \times T\mathcal{M}_{g,k} \to W^{r,p}(\Sigma, \Omega^{0,1}_{\Sigma} \otimes U^*TX)$$

by

$$D_{\bar\partial}(\nu, j') = \bar\partial\nu + \frac{1}{2}(\nabla_\nu J)du \cdot j + \frac{1}{2}J \cdot du \cdot j'$$

where $W^{r,p}$ is a completion of $C^\infty(-)$ of (?) to $L^{r,p}$ norm defined by $\|f\|_{r,p} = \left(\sum_{i=0}^{r} \int |f^{(i)}(t)|^p \, dt\right)^{1/p}$.

$D_{\bar\partial}$ is Fredholm, (meaning the dimensions of its kernel and cokernel are finite), with index (the difference of such numbers)

$$\operatorname{index}_{\mathbb{R}} D_{\bar\partial} := 2d = 2\langle c_1(TX), \beta\rangle + n(2 - 2g) + (6g - 6 + 2k).$$

We may interpret this equation as differentiation of the Cauchy-Riemann equations.

2 Dirichlet energy functional (Alex)

We follow [**?**], sec. 2.2

Consider a map

$$u : (\Sigma, j) \to (X, \omega, J, g)$$

and define the ***energy functional***

$$\varepsilon = \int_{\Sigma} |du|_g^2 \operatorname{Vol}_g$$

Now, we may take local isothermic coordinates where the metric is expressed as

$$g = \lambda(x, y)(dx^2 + dy^2)$$

giving

$$\begin{aligned} du &= \partial_x u \otimes dx + \partial_y u \otimes dy \\ |du|^2 &= |\partial_x u|^2 \lambda^{-2} + |\partial_y u|^2 \lambda^{/2} \\ \mathrm{Vol}_{\Sigma g'} &= \lambda^2 dx \wedge dy \end{aligned}$$

Then

$$\varepsilon(u) = \int_\Sigma |\partial_x u|_g^2 + |\partial_y u|_g^2 dx \wedge dy.$$

The following equality shows that the energy functional attains its minimum on pseudoholomorphic maps (in virtue of eq. (1)).

Claim. For every smooth map $u : \Sigma \to X$,

$$\varepsilon(u) = \int_\Sigma 2|\bar{\partial}_J|^2 \,\mathrm{Vol} + \int_\Sigma u^*\omega$$

Proof. content... □

May 24 (Alex)

We start with two short questions from last session.

Question.

- What exactly is $\Omega^1(\Sigma, E)$ where E is a vector bundle? It is the space of sections of the bundle $T^*M\Sigma \otimes E$.
- Let $u : (\Sigma, j) \to (X, J)$. Is du is an element of $\Omega^1(\Sigma, u^*TM)$? Yes, notice that du is an element of $\mathrm{Hom}(T\Sigma, TX)$. Forget about all of TX and consider only its image under u. There is an isomorphism $T\Sigma^* \otimes u^*TX \cong \mathrm{Hom}(T\Sigma, u^*TX)$.

Remark. There is a bundle $\mathcal{E} \to \mathcal{B}$ where $\mathcal{B} = C^\infty(\Sigma, M)$ and the fibers are $\mathcal{E}_u = \Omega^{0,1}(\Sigma, u^*TM)$. For a map $u : (\Sigma, j) \to (X, J)$, the nonlinear operator

$$u \mapsto (u, \bar{\partial}_J(u))$$

is a sections of this bundle whose zero-set is the space of J-holomorphic curves.

Then we concluded the proof of the final claim of the last session:

Lemma (2.2.1, [?]). Let ω be a nondegenerate 2-form on a smooth manifold M. If J is ω-tame then every J-homolomorphic curve $u : \Sigma \to M$ satisfies the energy identity

$$E(u) = \int_\Sigma u^*\omega.$$

If J is ω-compatible then every smooth map $u : \Sigma \to M$ satisfies

$$E(u) = \int_\Sigma |\bar{\partial}_J(u)|_J^2 \operatorname{Vol}_\Sigma + \int_\Sigma u^*\omega.$$

Proof. We may take local isothermic coordinates where the metric is expressed as

$$g = \lambda(x,y)(dx^2 + dy^2)$$

giving

$$\begin{aligned} du &= \partial_x u \otimes dx + \partial_y u \otimes dy \\ \implies |du|^2 &= |\partial_x u|^2 \lambda^{-2} + |\partial_y u|^2 \lambda^{/2} \\ \operatorname{Vol}_{\Sigma g'} &= \lambda^2 dx \wedge dy \end{aligned}$$

Then

$$\varepsilon(u) = \int_\Sigma |\partial_x u|_g^2 + |\partial_y u|_g^2 dx \wedge dy.$$

□

June 7 Frobenius manifolds (Sergey)

1 Introduction

Dubrovin in 1991 formulated the notion of ***Frobenius Manifold*** in the context of the ***WDVV equation*** in singularity theory. In late 1970s or early 1980s, Kyoji Saito formulated the notion of ***flat structure***, or ***Saito (pre-)structure***. In 1962, when May (?) was 24, there was a lot of activity in Paris. Not far from then in Japan, Saito was looking for a PhD, and eventually became a student of Brieskorn (though he initially intended to work with Grauert).

Saito started studiyng quotient singularities, and then we continued to all isolated singularitied.

Brieskorn sphere is given by

$$\begin{cases} x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5 = 0 \\ \sum |x_i|^2 = \Sigma \end{cases}$$

in $\mathbb{C}^5$.

More generally we can study links of ***isolated singularities***. Let $f \in \mathbb{C}[[x_1, \ldots, x_n]]$ and define $H = \{f(x) = 0\}$.

What is $T_0^* H$? Well,

$$T_0^* H = \frac{\mathfrak{m}}{\mathfrak{m}_0}$$

where $\mathfrak{m}_o = \langle x_1, \ldots, x_n \rangle$ is a maximal ideal.

The important thing is $f(0) = 0 \iff f \in$ $\mathtt{mathfrakm}_0$. Also, $0 \in H_{\text{sing}} \iff f \in \mathfrak{m}_0^2$.

Definition. Given $\mathbb{C}^* = V$, ie. a $\mathbb{Z}$-grading of V, then $f \in S^\bullet V^*$ is called ***quasi-homogeneous*** of degree D and weight $(w_1, \ldots, w_n)$ if

$$\lambda^* f = \lambda^D \cdot f$$

that is,

$$f(\lambda^{w_1} x_1, \lambda^{w_2} x_2, \ldots, \lambda^{w_n} x_n) = \lambda^D f(x_1, \ldots, x_n)$$

Proposition (Euler). The ***Euler vector field*** is $\sum x_i \frac{\partial}{\partial x_i}$. So,

Proof. $\lambda \to 1 + \varepsilon \ldots$ $\square$

Another way of saying that f is quasi-homogeneous is that f is eigenvector of E_w with eigenvalue D.

Exercise. Homogeneous singularity is isolated iff $Z(f) \subset \mathbb{P}(V)$ is smooth.

If you have weights, you have ***weighted projective space***: $\mathbb{P}(w_1, \ldots, w_n) = \operatorname{Proj} k[x_1, \ldots, x_n]$ for x_i of degree w_i.

Definition. w is ***well-formed*** if $\forall i$, $\operatorname{grd}(w_1, w_2, \ldots, \hat{w}_i, \ldots, w_n) = 1$.

Look for some text containing "singular locus of weighted projective space".

Example. $\mathbb{P}(1, \ldots, 1, d)$. So we have $\mathbb{P}(1, 1, d) \overset{\mathcal{O}(d)}{\hookrightarrow} \mathbb{P}^{d+1}$, which is the Vernoese embedding. Secretely $\mathbb{P}(1, 1, d)$ is a projective cone over $\nu_d(\mathbb{P}^1$ with singularities its vertex.

We can resolve this singularity by blowing up: $Y \overset{\mathcal{L}}{\hookrightarrow} \mathbb{P}^N$, $\mathrm{Bl}_p \mathbb{P}$.

2 Milnor ring and Milnor number

f isolated singularities iff

$$\dim \mathbb{C}[[x]] / \left(\frac{\partial f}{\partial x_i}, f \right) < \infty$$

Define the latter ring to be the ***Milnor ring***:

$$\mu_{ev} = \mathbb{C}[[x]] / \left(\frac{\partial f}{\partial x_i}, f \right)$$

There's also the ***Milnor fibration***, and the ***Milnor map***:

$$\begin{tikzcd} S_\varepsilon \arrow[r] \arrow[dr, "\text{Milnor map}"'] & \mathbb{C}^* \arrow[d, "\arg"] \\ & U(1) \cong S^1 \end{tikzcd}$$

Theorem (Milnor). $\mu_{alg} = \mu_{tor}$

3 Versal deformation of isolated singularities

Let $\lambda \in \mathbb{C}^n$,

$$f_\lambda = f + \lambda_1 g_1 + \lambda_2 g_2 + \ldots + \lambda_\mu g_\mu$$

Theorem (Saito). On versal deformation spaces of isolated singularities there exists a rich structure called ***flat structure***, a ***primitive form***, and some higher pairings, and so on and so on.

So, Dubrrovin defined Frobenius manifolds in 1991, and what is it?

Definition. Take a manifold either real or complex, and introduce some geometric structure as follows: a metric g, a 3-tensor C, such that $\forall m \in M$, $T_m M$ has the structure of a Frobenius algebra.

4 Frobenius algebra

You have A and a pairing $A \otimes A \to A$. So the 3-tensor is

$$g(a \cdot b, c) = g(a, b \cdot c)$$

You can also write this as $g(e, a) = \tau(a)$ for neutral element e. Notice this is the same as 2d TQFT. Also a connection

$$\nabla = \nabla^g + K$$

and we can also consider a 1-parametric family of connections

$$\nabla^{(\alpha)} = \nabla^g + \alpha K$$

with $\nabla^{(\alpha)}$ flat for all α. So this is a pencil of metrics.

> For every isolated singularity, its space of versal deformations has a Frobenius manifolds structure.