notes on mirror symmetry

3 may (Alex)

Main reference for these notes is Gross, sect. 14.

We wish to understand

Theorem 1 (Bogomolov-Tian-Todorov)**.** Any Calabi-Yau manifold has unobstructed deformations.

Definition. An ***almost compelx structure*** is an endomorphism J. . .

Remark. It is a fact by Borel & Serre (1953) that the only spheres which admit an almost complex structure are S^2 and S^6.

Example. All complex manifolds are almost complex manifolds.

Theorem 2. A necessary and sufficient condition for a $2u$-smooth manifold M to admit an almost complex structure is that the group of tangent bundle of M could be reduced to $U(n)$.

Theorem 3 (Newlander-Nirenberg)**.** Let (M, J) be an almost complex manifold. Then, the following are equivalent:

1. (six conditions. . .)

proposition 4. An almost complex structure on a real 2-dimensional manifold is a complex structure.

Proof. By the Newlander-Nirenberg theorem, given a point $p \in U \subset M$ and a vector field $\mathfrak{X}U$, we have that (V, JV) is a frame, and

$$N(V, JV) = [V, JV] + J[V, JV] + J[V, J^2V] - [JV, J^2V] = 0$$

□

Definition. A ***deformation*** of complex analytic space M over a germ (S, s_0) of complex analytic space is a triple $\pi, X, i)$ such that

$$\begin{tikzcd} X \arrow[d, "\pi"'] & M \arrow[l, "\text{embedding}"'] \arrow[d] \\ (S, s_0) & \text{pt} \arrow[l, "s_0"] \end{tikzcd}$$

where M is a compact manifold, $M \simeq \pi^{-1}(s_0)$ and π is proper smooth.

Theorem 5 (Ehresmann)**.** Let $\pi : X \to S$ be a proper family of differentiable manifold. If S is conncected, then all fibres are diffeomorphic.

Theorem 6 (Kodaira)**.** Let X_0 be a compact Kähler manifold. If $X \to S$ is a deformation, then any fibre X_t is again Kähler.

Theorem 7 (Kuranashi)**.**

1. Any compact complex manifold admits a universal deformation.
2. If $\Gamma(X_0, T_{x_0}) = 0$ then it admits a universal deformation.

Lemma. Let J be an almost complex structure sufficiently close to J_0 so that it is represented by a form $\lambda \in A^{0,1}T^{1,0}M$. Then J is integrable if and only if

$$\bar{\partial}\lambda_i + \frac{1}{2}[\lambda_j, \lambda_j] = 0.$$

Theorem 8 (Maurier-Cartan)**.**

$$\bar{\partial}\phi + [\phi, \phi] = 0$$

where

$$\phi = \phi(t) = \sum_{i=1} \phi_i t_i$$

Definition.

- The ***Kodaira-Spencer class*** of a one-parameter deformation J_t of a complex stucture J is induced by a homology class $\phi_1 \in H^1(X, Tx)$.
- The ***Kodaira-Spancer map*** is

$$T_s S \to H^1(X_s, T_{X_s}) = T_{[X_s]} \operatorname{Def}(X_{s_0})$$

References

[1] Mark Gross. "Calabi—Yau Manifolds and Mirror Symmetry". In: *Calabi-Yau Manifolds and Related Geometries: Lectures at a Summer School in Nordfjordeid, Norway, June 2001*. Ed. by Geir Ellingsrud et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003, pp. 69–159. ISBN: 978-3-642-19004-9. DOI: 10.1007/978-3-642-19004-9_2. URL: https://doi.org/10.1007/978-3-642-19004-9_2 (cit. on p. 1).