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Home Assignment 7: Coverings

Definition. A ***covering*** is a continuos map $\pi : M_1 \to M$ such that for each $x \in M_1$ there exists a neighbourhood $U \subset M$ such that $\pi^{-1}(U) = U \times S$ where S is a fixed set with discrete topology and $\pi : \pi^{-1} = U \times S \to M$ is a projection. A ***morphism of coverings*** $\pi_1 : M_1 \to M, \pi_2 : M_2 \to M$ is a map $f : M_1 \to M_2$ which commutes with the projections to M. A covering $M_1 \to M$ is called ***universal covering*** if M_1, M are connected and M_1 is simply connected.

Definition. A continuos map $\pi : M_1 \to M$ is called ***proper*** if for any compact $K \subset M$ the preimage $\pi^{-1}(K)$ is compact.

Exercise 7.1. Let $\pi : M \to M_1$ be a smooth map of n-manifolds with differential non-degenerate everywhere. Assume that π is proper. Prove that π is a covering.

Proof. Since $d\pi$ is non-degenerate everywhere, $\pi^{-1}(p)$ is a submanifold of M for every $p \in M_1$. But also every point $p \in M$ must have a compact neighbourhood, whose preimage under π is also compact. This makes the preimage manifold $\pi^{-1}(p)$ compact, but not necessarily discrete.

It appears that a fiber bundle is proper if fibers are compact, giving a large family of counter-examples such as the Hopf fibration. □

Definition. Let $\pi_1 : M_1 \to M$, $\pi_2 : M_2 \to M$ be continuous maps. ***Fibered product*** $M_1 \times M_2$ is the subset of $M_1 \times M_2$ defined as $M_1 \times_M M_2 := \{(x, y) \in M_1 \times M_2 | \pi_1(x) = \pi_2(y)\}$ with induced topology.

Exercise 7.2. Let $\pi_1 : M_1 \to M, \pi_2 : M_2 \to M$ be coverings. Prove that $M_1 \times_M M_2 \to M$ is also a covering.

Proof. Every point in M has two open sets such that their preimages under π_1 and π_2, respectively, are a union of disjoint open sets. The intersection of these two open sets in M is also an open set in M, whose preimage under both projections is a disjoint union of open sets. □

Recall the followings results that may be found in **hatcher**:

Proposition (Homotopy lifting property)**.** Given a covering space $M_1 \to M$, a homotopy $f_t : Y \to X$ and a map $\tilde{f} : Y \to \tilde{X}$ lifting f_0, there exists a unique homotopy $\tilde{f}_t : Y \to \tilde{X}$ lifting f_t.

Proposition (Lifting criterion)**.** Suppose given a covering space $\pi : (M_1, m_1) \to (M, m)$ and a map $f : (X, x) \to (M, m)$ with X path connected and locally path-connected. Then a lift $(\tilde{X}, x) \to (M_1, m_1)$ of f exists iff $f_*(\pi_1(X, x)) \subset \pi_*(\pi_1(M_1, m))$.

Proposition (Uniqueness of lifts). Given a covering space $\pi : M_1 \to M$ and a map $f : X \to M$, if two lifts $\tilde{f}_1, \tilde{f}_2 : X \to M_1$ agree at one point of X and X is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree throughout X.

Exercise 7.3. Let $\pi_1 : M_1 \to M$, $\pi_2 : M_2 \to M$ be universal coverings. Prove that $M_1 \times_M M_2 \to M$ is a union of $\pi_1(M)$ disconnected copies of M_1. Deduce that the universal covering is unique up to an isomorphism of coverings.

Proof of uniqueness of universal covering. Universal covering is a covering of every other covering by Exercise 7.5. Then we have coverings $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_1$. Now notice that $f_2 f_1$ is a lift of p_1 to M_1 as in the next diagram

$$\begin{tikzcd}
 & & M_1 \arrow[dd, "\pi_1"] \\
 & M_2 \arrow[ur, "f_2"] & \\
M_1 \arrow[ur, "f_1"] \arrow[rr, "\pi_1"'] & & M
\end{tikzcd}$$

This follows from simply conectedness of M_1 and the lifting criterion. Notice the identity map of M_2 is also a lift of π_1. Recall that two lifts are the same if they agree in one point. The coverings must be pointed for these arguments to work, and indeed $f_2 f_1$ must fix the basepoint of M_1. This shows that $f_2 f_1 = \mathrm{id}_{M_1}$. A similar argument shows that $f_1 f_2 = \mathrm{id}_{M_2}$. $\square$

Exercise 7.4. Let $\pi : M_1 \to M$ be the universal covering. Prove that the group $\mathrm{Aut}_M(M_1)$ of its automorphisms (in the category of coverings) is isomorphic to $\pi_1(M)$.

Proof. Take $[\gamma] \in \pi_1(M, m)$ and consider the lift of γ to a path $\tilde{\gamma}$ in M_1 that joins two elements in $\pi^{-1}(m)$, say m_1 and m_2. Once again we may use the lifting property to find a covering automorphism:

$$\begin{tikzcd}
 & (M_1, m_2) \arrow[d, "\pi_1"] \\
(M_1, m_1) \arrow[ur, dashed] \arrow[r, "\pi_1"'] & (M, m)
\end{tikzcd}$$

This map is independent of the choice of γ by the homotopy lifting criterion. By uniqueness of lifts, the covering automorphism is completely determined by mapping m_1 to m_2. Thus any two elements of $\pi_1(M)$ whose lifts join m_1 and m_2 induce the same automorphism. Two distinct homotopy classes of loops cannot join the same points since M_1 is simply connected and an homotopy between the lifted paths would project to a homotopy between the loops.

This map is also surjective, since any automorphism of covers sending m_1 to m_2 corresponds to some loop whose lift joins these points. It also a group homomorphism since concatenation corresponds to composition, and an isomorphism since the construction of the inverse map is analogous. ☐

Exercise 7.5. Let $\pi : M_1 \to M$ be a connected covering, and $u : M_u \to M$ be the universal covering. Prove that $M_u \times_M M_1$ is a disconnected sum of several copies of M_u. Prove that the map $u : M_u \to M$ can be factorized through $\pi : M_1 \to M$, with u equal to a composition $M_u \xrightarrow{\phi} M_1 \xrightarrow{\pi_1} M$.

Proof. (From StackExchange) By the lifting criterion, since M_u is simply connected we have a map

$$\begin{tikzcd} & M_1 \arrow[d, "\pi_1"] \\ M_u \arrow[ur, dashed, "\tilde{\pi}"] \arrow[r, "\pi_u"'] & M \end{tikzcd}$$

We claim $\tilde{\pi}$ is a covering.

($\tilde{\pi}$ is surjective.) Let $m_1 \in M_1$. Choose any path γ in M_1 from any other point $m_0 \in M_1$ to m_1. Project such path to M, then lift to M_u to obtain a path $\tilde{\gamma}$. Then project to M_1 via $\tilde{\pi}$:

$$\begin{tikzcd} & & M_u \arrow[dl, "\tilde{\pi}"] \arrow[d, "\pi_u"] \\ I \arrow[urr, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & M_1 \arrow[r, "\pi_1"'] & M \end{tikzcd}$$

By uniqueness of path lifting, endpoint of $\tilde{\pi} \circ \tilde{\gamma}$ coincides with the endpoint of γ, meaning there is a point in M_u mapping to m_1.

($\tilde{\pi}$ is covering) Let $m_1 \in M_1$. Project to M to obtain regular neighbourhoods U_u and U_1 of $\pi_1(m_1)$ for π_u and π_1. (***Regular*** means as in definition of covering.) Let V be the connected component of m_1 in $\pi_1^{-1}(\pi_1(m_1))$. Let $\pi_u^{-1}(U_u) = \coprod U_\alpha$. Then $\tilde{\pi}^{-1}(V)$ is contained in $\coprod U_\alpha$ and mapped homeomorphically to each U_α by commutativty of the diagram. ☐

Exercise 7.6. Let $\pi : M_1 \to M$ be a connected covering. Prove that π induces a group monomorphism $\pi_1(M_1) \to \pi_1(M)$. Prove that the isomorphism classes of connected coverings $\pi : M_1 \to M$ are in bijective correspondence with subgroups of $\pi_1(M)$.

Proof. **(Induced map is monomorphism.)** Let $[\gamma]$ be an element of the kernel of the induced map. This means that there is a homotopy between $\pi \circ \gamma$ and the constant loop in M. By the homotopy lifting property of covering spaces, this homotopy lifts to a homotopy of loops between $[\gamma]$ and the constant loop in M_1.

(Bijective correspondence) Consider the correspondence

$$\text{covering} \quad p : M_1 \to M \qquad \to \qquad p_*(\pi_1(M_1)) \subset \pi_1(M)$$

Claim. Two connected coverings $p_1 : M_1 \to M$ and $p_2 : M_2 \to M$ are isomorphic iff $p_{1*}(\pi_1(M_1) = p_{2*}(\pi_1(M_2))$.

Proof. Suppose $f : M_1 \to M_2$ is a covering isomorphism. Then $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$, which means $p_{1*} = p_{2*} f_*$ and $p_{2*} = p_{1*} f_*^{-1}$. Since f is an isomorphism it induces isomorphism on fundamental groups, yielding the result.

The converse follows by the lifting criterion and uniquencess of lifts. □

Claim. For every subgroup $H \subset \pi_1(M)$ there is a covering space M_H such that $\pi_*(\pi_1(X_H)) = H$.

Sketch of proof. This proof uses the construction of the universal covering as the space of paths in M starting at a given point. Define an equivalence relation ~ between two such paths it they end at the same point and their concatenation is in H. Define $M_H = M_1/\sim$. □

□

Exercise 7.7. Let $x \in M$ be a point in a connected manifold M. Consider a functor Φ from the category $\mathcal{C}_M$ of coverings of M to the category of sets mapping a covering $\pi : M_1 \to M$ to $\pi^{-1}(x)$. Prove that the fundamental group $\pi_1(M, x)$ naturally acts on the set $\pi^{-1}(x)$. Prove that Φ defines an equivalence of categories from $\mathcal{C}_M$ to the category of sets with an action of the group $\pi_1(M, x)$.

Proof. Fix a covering $\pi : M_1 \to M$, a point in $x_0 \in \pi^{-1}(x)$ and an element $[\gamma] \in \pi(M, x)$. Any represenant of $[\gamma]$ lifts uniquely to a path starting in x_0, yielding an action on $\pi^{-1}(x)$. Trivial loop fixes every point in $\pi^{-1}(x)$ and concatenation of loops is composition, yielding a group action.

This shows how any covering space corresponds to a set with a group action of $\pi_1(M, x)$. For the converse I wish to show that a set with an action of $\pi_1(M, x)$ corresponds to a subgroup of $\pi_1(M, x)$ and use Exercie 7.6. □