Daniel González Casanova Azuela
github.com/danimalabares/riemann-surfaces

Home Assignment 3: Lie groups

Definition. A ***Lie group*** is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G ***acts on a manifold*** M if the group action is given by the smooth map $G \times M \to M$.

Exercise 3.1. Prove that $SL(n, \mathbb{R})$ is a Lie group. Prove that it is connected.

Proof. Recall that $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ of matrices with determinant 1, so it is the preimage of $\{1\}$ under the smooth function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. In fact, 1 is a regular value of det because det is surjective and of constant rank $\equiv 1$, making $SL(n, \mathbb{R})$ a submanifold. (Of course, $GL(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{2n} = M(n, \mathbb{R})$ because it is an open subset, namely, the preimage of $\mathbb{R}\backslash 0$ under the continuous function det.)

Moreover, we may think of det as a group homomorphism from $GL(n, \mathbb{R})$ to the multiplicative group $\mathbb{R}\backslash 0$, so that $SL(n, \mathbb{R}) = \ker \det$, making it a subgroup. The restriction of the group operations from $GL(n, \mathbb{R})$ are smooth, making $SL(n, \mathbb{R})$ a Lie group. □

Exercise 3.2. Prove that the special unitary group $SU(n)$ acts transitively on the projective space $\mathbb{C}P^{n-1}$. Find the stabilizer $St_x(SU(n))$ of a point $x \in \mathbb{C}P^{n-1}$. Prove that it is connected, or find a counterexample.

Proof.

($SU(n)$ acts transitevly on $\mathbb{C}P^{n-1}$.) Any point in $\mathbb{C}P^{n-1}$ has two representants in the set of points of $\mathbb{C}^n$ of norm 1. Indeed, suppose $x = z_1 : \ldots : z_n$ is a point of $\mathbb{C}P^{n-1}$. Since not all coordinates are zero, we may normalize dividing by $\sqrt{z_1^2 + \ldots + z_n^2}$. But of course the point $(-z_1, \ldots, -z_n) \in \mathbb{C}^n$ is also a representant of x that has norm 1.

Anyway, a matrix $U \in SU(n)$ not only will preserve the set of points of norm 1, but act transitively on them. This follows from Gram-Schmidt orthogonalization process for Hermitian forms and from Hadamard's inequality. The latter says that the determinant of a matrix equals the product of the column vectors if they are orthogonal.

(Find $St_x(SU(n))$.) By the arguments above it can only be the cyclic group of two elements.

(Is $SU(n)$ connected?) □

Definition. Let W be an n-dimensional complex vector space equipped with a complex-linear non-degenerate quadratic form s. Consider the ***complex orthogonal group*** $O(n, \mathbb{C})$ of all matrices $A \in GL(W)$ preserving s. A subspace $V \subset W$ is called ***isotropic*** if $s|_W = 0$. It is called ***maximally isotropic***, or ***Lagrangian***, if $\dim V = [n/2]$.

Exercise 3.3. Prove that $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ is a Lie group which has index 2 in $O(n, \mathbb{C})$. Prove that it is connected.

Proof. content... $\square$