Home Assignment 1: holomorphic functions

Exercise 1.1. Let f be a holomorphic function on a disk Δ. Prove that the zero set Z_f of f is discrete in Δ.

Proof. Suppose there exists $z_0 \in Z_f \cap Z_f' \cap \Delta$, a zero of f in Δ that is also an accumulation point of Z_f. Then there is a sequence $(z_k) \subset Z_f$ that converges to z_0. So $f \equiv 0$ in $(z_k) \cup \{z_0\}$, and by the identity principle $f \equiv 0$ in Δ. □

Exercise 1.2. Let $P(t)$ be a polynomial. Prove that $f(z) = z - \frac{P(z)}{P'(z)}$ is holomorphic in a neighbourhood of any α which is a root of $P(t)$. Prove that $f(\alpha) = \alpha$ and $|f'(\alpha)| < 1$.

Proof. We have that

$$P(z) = Q(z)(z-\alpha)^k$$

where $Q(\alpha) \neq 0$. So

$$\begin{aligned} P'(z) &= Q'(z)(z-\alpha)^k + kQ(z)(k-\alpha)^{k-1} \\ &= (z-\alpha)^{k-1}(Q'(z)(z-\alpha) + kQ(z)), \end{aligned}$$

and then the quotient is

$$\frac{P(z)}{P'(z)} = (z-\alpha)\frac{Q(z)}{Q'(z)(z-\alpha) + kQ(z)}$$

which is well-defined in a neighbourhood of α where there are no other roots of $P(z)$ (we may use exercise 1.1). Thus f is holomorphic near α.

It is clear that $f(\alpha) = \alpha$.

I couldn't really prove that $|f'(\alpha)| < 1$.

Here's some ideas:

Notice that

$$f'(z) = 1 - \frac{P'(z)^2 - P''(z)P(z)}{P'(z)^2} = \frac{P''(z)P(z)}{P'(z)^2}.$$

Now suppose that

$$P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n.$$

Then

$$\begin{aligned} P'(z) &= a_1 + 2a_2 z + 3a_3 z^2 \ldots + n a_n z^{n-1} \\ P''(z) &= 2a_2 + 3 \cdot 2a_3 z + \ldots + n \cdot (n-1) a_n z^{n-2} \end{aligned}$$

and then …? □

Exercise 1.3. Let $f(z)$ be a non-constant holomorphic function on a disk Δ. Prove that there exist $t, s \in]0,1]$ such that the function $f(tz) - f(sz)$ has no zeros when $|z| = 1$.

Proof. By contradiction, suppose that for all $t, s \in]0,1]$ there is z of norm 1 such that $f(tz) = f(sz)$. For $s = 1$ and $t_n = 1/n$ with $n \in \mathbb{N}$, there exists z_n of norm 1 such that $f(z_n/n) = f(z_n)$. Denoting $z_0 = \lim_n z_n$, we get by continuity that $f(0) = f(z_0)$.

By a similar argument but now varying s in $]0,1]$, we obtain numbers $z_{0,s}$ of norm s such that $f(0) = f(z_{0,s})$. But then $f \equiv f(0)$ in a infinite set within a compact set, which must have an accumulation point, so f is constant.

□

Exercise 1.4. Prove that if f is holomorphic on the unit disk Δ and continuous on $\partial\Delta$, then $\frac{1}{\pi}\int_\Delta f = f(0)$.

Proof. We know that the real and imaginary parts of f are harmonic functions. By the mean value property

$$f(0) = u(0) + \sqrt{-1}v(0) = \frac{1}{\pi}\left(\int_\Delta u + \sqrt{-1}\int_\Delta v\right),$$

where $f = u + \sqrt{-1}v$, using that the area of the unit disk is π. □

Exercise 1.5. Prove that any holomorphic map from $\mathbb{C}\backslash 0$ to a disk Δ is constant.

Solution. By Riemann's criterion of removable singularities, 0 is a removable singularity of f since f is bounded. Thus there is a holomorphic function extending f to all of $\mathbb{C}$, which must be constant by Liouville's theorem, making f constant as well. □

Exercise 1.6. Prove that any holomorphic function on a disk Δ which is continuous on its boundary $\partial\Delta$ and takes real values on $\partial\Delta$ is constant, or find a counterexample.

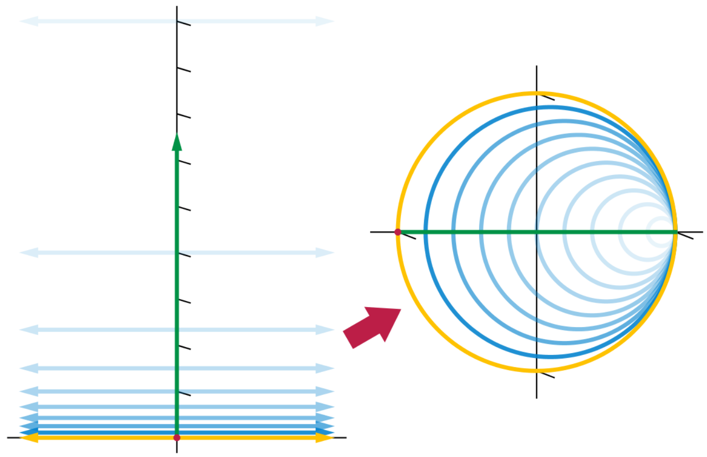
Solution. If we had f holomorphic on an open neighbourhood of Δ, then we can borrow this reasoning from StackExchange:

> $\mathrm{Im}(f)$ is harmonic, and 0 on $\partial\Delta$. If it were not identically 0, it would have a maximum or minimum in Δ, and that is impossible.

This would mean that f is a real-valued function, so its derivative must be zero.

But in fact, f can be holomorphically extended by Schwarz reflection principle, which states that any function that is holomorphic on the upper half plane and real-valued on the real axis is holomorphically extendable to the whole plane. So we only need to precompose f with some holomorphic function taking the upper half-plane to our given disk (and then translate and resize).

The Cayley transform $z \mapsto \frac{z-i}{z+i}$ has such an action with respect to the unit disk:



$\square$