Daniel González Casanova Azuela
github.com/danimalabares/riemann-surfaces

Home Assignment 6: Isometries of $\mathbb{H}^2$

Definition. ***Order*** of $A \in GL(n)$ is the smallest positive integer such that $A^k = \mathrm{Id}$.

Exercise 6.1. Let A be an element of finite order k in $GL(2, \mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Solution. (**From StackExchange.**) First notice that A is diagonalizable. This follows from the fact that a matrix is diagonalizable if and only if its minimal polynomial factors completely into distinct linear factors. The minimal polynomial m_A of A must divide any polynomial that annihilates A, so $m_A | X^k - 1$. Also, $X^k - 1$ factors into k distinct linear factors given by the k-th roots of unity, and so does m_A.

Let λ_1 and λ_2 be the eigenvalues of A, so that $\lambda_1^k = 1$ and $\overline{\lambda}_2 = \lambda_1$. Then there exists $\theta \in \mathbb{R}$ such that $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{-i\theta}$. Since $A \in GL(2, \mathbb{Z})$ and the trace is invariant under change of coordinates, $\operatorname{Tr} A = \lambda_1 + \lambda_2 = 2\cos\theta$ is an integer. Therefore, we have only 5 possibilities:

$$\begin{aligned}
\cos\theta = 1 &\implies A^2 = \mathrm{Id} \\
\cos\theta = -1/2 &\implies A^3 = \mathrm{Id} \\
\cos\theta = 0 &\implies A^4 = \mathrm{Id} \\
\cos\theta = 1/2 &\implies A^6 = \mathrm{Id} \\
\cos\theta = 1 &\implies A = \mathrm{Id}
\end{aligned}$$

□

Exercise 6.2. Let A be an element of finite order k in $SL(3, \mathbb{Z})$. Prove that $k = 2, 3, 4, 6$.

Solution. As in the former case, A is diagonalizable. Again the trace of A is an integer, and the eigenvalues satisfy $\overline{\lambda}_2 = \lambda_1$ and $\lambda_3 \in \mathbb{R}$. Moreover, its trace must be an integer. We have:

$$\operatorname{Tr}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 2\cos\theta + \lambda_3 \in \mathbb{Z},$$

where θ is the real number such that $\lambda_1 = e^{i\theta}$.

We also know that the determinant of A, which is 1, must be the product of its eigenvalues:

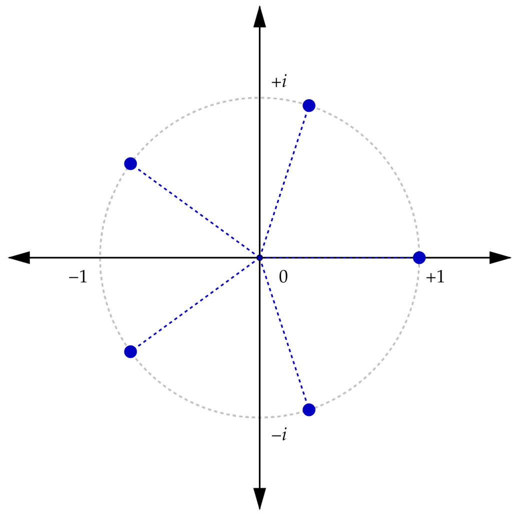
$$1 = \det A = \lambda_1\lambda_2\lambda_3 = |\lambda_1|^2\lambda_3 = \lambda_3$$

since, as in the previous exercise, $\lambda_1^k = 1$. Finally, $2\cos\theta$ is again an integer and we conclude in analgous manner.

□

Exercise 6.3. Find an element of order 5 in $GL(4, \mathbb{Z})$, or prove that it does not exist.

Solution. Suppose A is an element of order 5 in $GL(4, \mathbb{Z})$. Once again, A must be diagonalizable and its eigenvalues are two pairs of complex conjugates. In fact, since we want $A^5 = \mathrm{Id}$, these eigenvalues must be the following two conjugate pairs of 5th roots of unity:



Up to a change of coordinates, if $\lambda_1 = e^{i\theta}$ and $\lambda_2 = e^{i\phi}$,

$$\theta = \frac{2\pi\sqrt{-1}}{5} \qquad \text{and} \qquad \phi = \frac{4\pi\sqrt{-1}}{5}.$$

However, the trace of A must be an integer, that is

$$\operatorname{Tr} A = 2(\cos\theta + \cos\phi) \in \mathbb{Z},$$

which is not the case. $\square$

Remark. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature $(1, 2)$, $V^+ := \{v \in V | q(v) > 0\}$, and $\mathbb{P}V^+$ its projectivisation. Then $\mathbb{P}V^+ = SO^+(1, 2)/SO(1)$, giving $\mathbb{P}V^+ = \mathbb{H}^2$; this is one of the standard models of a hyperbolic plane. The ***absolute*** is projectivization of the set of all isotropic lines; it is identified with the boundary of $\mathbb{P}V^+$ in $\mathbb{P}V$.

Definition. Let $\ell \subset V$ be a line, that is, a 1-dimensional subspace. The property $q(x, x) < 0$ for a non-zero $x \in \ell$ is written as $q(\ell, \ell) < 0$. A line ℓ with $q(\ell, \ell) < 0$ is called ***negative line***, a line with $q(\ell, \ell) > 0$ is called ***positive line***.

Remark. Negative lines bijectively correspond to geodesics in $\mathbb{P}V^+ = \mathbb{H}^2$ (Lecture 8): an orthogonal complement to a negative line is a 2-dimensional plane $\ell^\perp$, its projectivization intersected with $\mathbb{P}V^+ = \mathbb{H}^2$ is a geodesic.

Exercise 6.4. Let γ_1, γ_2 be geodesics on a hyperbolic plane, and ℓ_1, ℓ_2 the corresponding negative lines.

a. Prove that ℓ_1 is orthogonal to ℓ_2 if and only if γ_1 is orthogonal to γ_2.

b. Prove γ_1 intersects γ_2 if and only if the 2-plane $\langle \ell_1, \ell_2 \rangle$ generated by ℓ_1, ℓ_2 has signature (0,2).

c. Prove that γ_1 and γ_2 passes through the same point on the absolute if and only if the 2-plane generated by ℓ_1, ℓ_2 has degenerate scalar product.

Attempt of solution.

a.

Question. What is the definition of two hyperbolic geodesics being ***orthogonal***?

Two planes in Euclidean space are naturally defined to be orthogonal if their normal vectors are orthogonal, but this would make the question tautological.

Question. Can we actually have two orthogonal negative lines?

We find in Alekseevskij, Vinberg, and Solodovnikov that

> "A subspace $U \subseteq \mathbb{R}^{n,1}$ is said to be *elliptic* (respectively, *parabolic*, *hyperbolic*) if the restriction of the scalar product in $\mathbb{R}^{n,1}$ to U is positive definite (respectively, positive semi-definite and degenerate, indefinite).

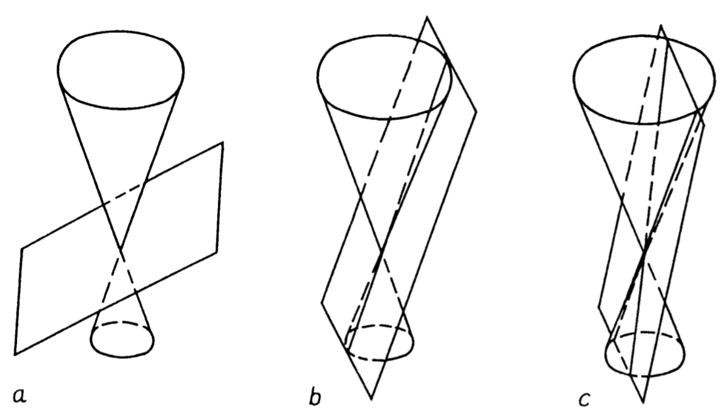


Figure from Alekseevskij, Vinberg, and Solodovnikov

> [...] The orthogonal complement $U^\perp$ to an elliptic (respectively, parabolic, hyperbolic) subspace U is a hyperbolic (respectively, parabolic, elliptic) subspace."

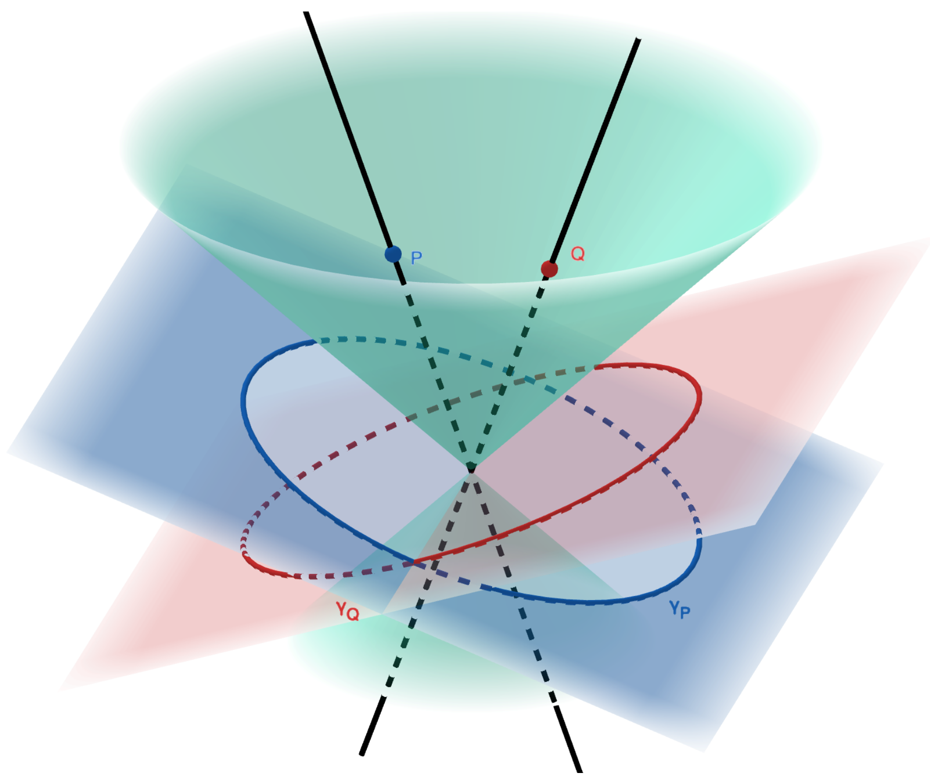
which makes me think that given a negative (hyperbolic) line, any line orthogonal to it will be positive (elliptic), which is confusing.

b.

Question. Two negative (hyperbolic) lines generate a plane that intersects the cone given by $q(x,x) = 0$. (Such a plane is a hyperbolic subspace according to Alekseevskij, Vinberg, and Solodovnikov.)

I struggle to see how it could be possible that such a plane can have different signature depending wether the two geodesics intersect.

Two geodesics in $\mathbb{P}V^+$ do not intersect in $\mathbb{H}^2$ when their intersection is a negative point in $\mathbb{P}V$, that is, a point "inside" the cone. I have represented the situation when two geodesics do intersect:



Two negative lines determined by points P and Q, their corresponding orthogonal planes and the intersection of the planes with a sphere centered about the origin. (See interactive figure online.)

c. This situation corresponds to the case when the planes corresponding to the geodesics are tangent to the cone. Scalar product in any of this planes is degenerate since each of them contains an isotropic line. The plane $\langle \ell_1, \ell_2 \rangle$, a hyperbolic subspace, contains two isotropic lines and is also degenerate.

$\square$

Remark. Review of definition of angle between geodesics is necessary to address d. and e. (and to correctly answer a. b. and c.).

References

[1] D. V. Alekseevskij, E. B. Vinberg, and A. S. Solodovnikov. "Geometry of Spaces of Constant Curvature". In: *Geometry II: Spaces of Constant Curvature*. Ed. by E. B. Vinberg. Berlin, Heidelberg: Springer Berlin Heidelberg, 1993, pp. 1–138. ISBN: 978-3-662-02901-5. DOI: 10.1007/978-3-662-02901-5_1. URL: https://doi.org/10.1007/978-3-662-02901-5_1.