Daniel González Casanova Azuela
github.com/danimalabares/riemann-surfaces

# Home Assignment 3: Lie groups

**Definition.** A ***Lie group*** is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group $G$ ***acts on a manifold*** $M$ if the group action is given by the smooth map $G \times M \to M$.

**Exercise 3.1.** Prove that $SL(n, \mathbb{R})$ is a Lie group. Prove that it is connected.

*Proof.*

**($SL(n, \mathbb{R})$ is a Lie group.)** (Idea from Lee) Since $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ of matrices with determinant 1, it is the preimage of $\{1\}$ under the smooth function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. In fact, 1 is a regular value of det because det is surjective and of constant rank $\equiv 1$, making $SL(n, \mathbb{R})$ a submanifold. (Of course, $GL(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{2n} = M(n, \mathbb{R})$ because it is an open subset, namely, the preimage of $\mathbb{R}\backslash 0$ under the continuous function det.)

Moreover, we may think of det as a group homomorphism from $GL(n, \mathbb{R})$ to the multiplicative group $\mathbb{R}\backslash 0$, so that $SL(n, \mathbb{R}) = \ker \det$, making it a subgroup. The restriction of the group operations from $GL(n, \mathbb{R})$ are smooth, making $SL(n, \mathbb{R})$ a Lie group.

**($SL(n, \mathbb{R})$ is connected.)** (Idea from StackExchange) We have seen in our lectures that for any vector space $V$ we have $SL(V) = e^{\mathrm{End}_0(V)}$ where $\mathrm{End}_0(V)$ denotes the space of matrices with trace 0. We may show that $SL(n, \mathbb{R})$ is path-connected by taking any matrix $e^X \in SL(n, \mathbb{R})$ with $X \in \mathrm{End}_0(\mathbb{R}^n)$ and connecting it to the identity element by the path $e^{tX}$. □

**Exercise 3.2.** Prove that the special unitary group $SU(n)$ acts transitively on the projective space $\mathbb{C}P^{n-1}$. Find the stabilizer $\mathrm{St}_x(SU(n))$ of a point $x \in \mathbb{C}P^{n-1}$. Prove that it is connected, or find a counterexample.

*Proof.*

**($SU(n)$ acts transitevly on $\mathbb{C}P^{n-1}$.)** Any point in $\mathbb{C}P^{n-1}$ has a whole circle of representants in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Indeed, suppose $x = z_1 : \ldots : z_n$ is a point of $\mathbb{C}P^{n-1}$. Since not all coordinates are zero, we may normalize dividing by $\sqrt{z_1^2 + \ldots + z_n^2}$, or perhaps by $h(x, x)$. Then for every $\lambda \in S^1$, the point $\lambda(z_1, \ldots, z_n) \in \mathbb{C}^n$ is also a representant of $x$ in $S^{2n-1}$.

Anyway, a matrix $U \in SU(n)$ not only will preserve $S^{2n-1}$, but will act transitively on it. This follows from Gram-Schmidt orthogonalization process and from Hadamard's inequality. The latter says that the determinant of a matrix equals the product of the column vectors if they are orthogonal.

Explicitly, we proceed as follows. Any point on the sphere may be extended to an othonormal basis, which is equivalent to a matrix in $SU(n)$ by Hadamard's inequality. Given any two points on the sphere, the composition of their correponding matrices (using the inverse of one of them) takes one point to another. Transitivity on $S^{2n-1}$ implies transitivity on $\mathbb{C}P^n$.

**(Find $St_x(SU(n))$.)** Consider any representant $z \in S^{2n-1}$ of $x \in \mathbb{C}P^{n-1}$. Elements in $SU(n)$ that fix $z$ will of course fix $x$. And for each of them, multiplying by $\lambda \in S^1$ will yield another operator that fixes $z$ as well.

Now the elements in $SU(n)$ that fix $z$ can be identified with $SU(n-1)$. This can be easily seen for the particular case of a simple vector such as $v = (1, 0, \dots, 0)$ noticing that the unitary matrices that fix it must have $v$ as a vector in their first column and row, leaving the remaining $n-1$ entries corresponding to a matrix in $SU(n-1)$. Up to a change of coordinates, this is true for any point in $S^{2n-1}$.

We conclude that $St_x(SU(n)) = SU(n-1) \otimes S^1$.

**($SU(n)$ is connected.)** I have found two proofs:

**(Proof from Hall, prop. 1.13)** Any matrix in $U \in SU(n)$ can be diagonalized with eigenvalues of norm 1. This just means that $U = U_1 D U_1^{-1}$ for $U_1 \in SU(n)$ and $D$ diagonal. Now consider

$$D(t) = \begin{pmatrix} e^{\sqrt{-1}(1-t)\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{\sqrt{-1}(1-t)\theta_n} \end{pmatrix}$$

Then the path $U(t) = U_1 D(t) U_1^{-1}$ connects $U$ and $\mathrm{Id}$ as $t$ moves from 1 to 0. To make sure the path stays within $SL(n)$ we redefine the last entry to be the inverse of the product of the first $n-1$ entries. This proves that $SU(n)$ is connected.

**(Proof using homogeneous spaces.)** We have seen in our lectures that a transitive action of a Lie group $G$ on a manifold $M$ yields $M = G/St_x(G)$. Then we have that $\mathbb{C}P^{n-1} = SU(n)/(SU(n-1) \otimes S^1)$.

Another way to put this is in the form of a fiber bundle

$$SU(n-1) \otimes S^1 \longrightarrow SU(n) \longrightarrow \mathbb{C}P^{n-1}$$

which in turn yields

$$\pi_0(SU(n-1) \otimes S^1) \longrightarrow \pi_0(SU(n)) \longrightarrow 0.$$

The final ingredient was found in Piccione and Tausk, example 3.2.25: do induction on $n$. Since $SU(1) = 0$, then $SU(n-1) \otimes S^1$ is connected for all $n > 1$ and so is $SU(n)$. $\square$

**Definition.** Let $W$ be an $n$-dimensional complex vector space equipped with a complex-linear non-degenerate quadratic form $s$. Consider the ***complex orthogonal group*** $O(n, \mathbb{C})$ of all matrices $A \in GL(W)$ preserving $s$. A subspace $V \subset W$ is called ***isotropic*** if $s|_V = 0$. It is called ***maximally isotropic***, or ***Lagrangian***, if $\dim V = [n/2]$.

**Exercise 3.3.** Prove that $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ is a Lie group which has index 2 in $O(n, \mathbb{C})$. Prove that it is connected.

*Proof.* We know that $SO(n, \mathbb{C})$ is a Lie group from our lectures. Now the function $\det : O(n, \mathbb{C}) \to \{\pm 1\} = \mathbb{Z}/2$ is a surjective group homomorphism with kernel $SO(n, \mathbb{C})$, meaning $[O(n, \mathbb{C}) : SO(n, \mathbb{C})] = |\mathbb{Z}/2| = 2$.

This yields a fibration

$$SO(n, \mathbb{C}) \longrightarrow O(n, \mathbb{C}) \longrightarrow \{\pm 1\}$$

yielding

$$\pi_0(SO(n, \mathbb{C})) \longrightarrow \pi_0(O(n, \mathbb{C})) \longrightarrow \pi_0(\{\pm 1\})$$

and since both $O(n, \mathbb{C})$ and $\{\pm 1\}$ have two connected components, we have the arrow on the right is an isomorphism and thus $SO(n, \mathbb{C}) = 0$ by exactness of the sequence.

**(Alternative approach following Hall)** The complex-linear quadratic form $s$ is associated to a symmetric bilinear form, which means we can diagonalize any matrix in $SO(n, \mathbb{C})$ with eigenvalues of norm 1 and construct a path to the identity matrix just as in the previous exercise. □

**Exercise 3.4.** Let $X$ be the space of all maximally isotropic subspaces in $W$ (the ***maximally isotropic Grassmanian***).

(a) Prove that $O(n, \mathbb{C})$ acts on $X$ transitively.

(b) Prove that $X$ is disconnected for $n = 2, 3$.

(c) Prove that it is connected for $n \geqslant 4$, or find a counterexample.

*Proof.*

(a) The idea here is to construc a Hermitian metric as a sum of the symmetric and a skew-symmetric form as follows:

$$h = g - \sqrt{-1}\omega.$$

This should be possible given the natural complex structure on $V$ and the bilinear form associated to $s$.

I am unsure wether $s$ is associated to a symmetric or a skew-symmetric bilinear form: while I initially though it would be just a symmetric form, it appears that Grassmanian Lagrangian subspaces are usually defined as those where the skew-symmetric form $\omega$ vanishes.

In what follows I suppose $s$ is associated to a symmetric bilinear form $g$.

Now consider two Lagragian Grassmanians $L_1, L_2 \in X$ and fix orthonormal bases $(b_j)_{j=1}^n$ and $(b'_j)_{j=1}^n$ with respect to $\omega$. Since $s$ vanishes in both spaces, the restriction of $h$ to either of them becomes $\sqrt{-1}\omega$ and thus these bases are $h$-orthonormal. Then we use transitivity of hermitian forms on bases.

(b, c) If the previous exercise is correct, the stabilizer $\mathrm{St}_A$ of every maximally isotropic subspace $V \in X$ is the group of matrices that preserve the skew-symmetric form $\omega$, that is, the symplectic group $\mathrm{Sp}([n/2], \mathbb{C})$. We have a fiber bundle

$$\mathrm{Sp}([n/2], \mathbb{C}) \longrightarrow \mathrm{O}(n, \mathbb{C}) \longrightarrow X(n)$$

where $X(n)$ is just $X$ for dimension $n$. This gives

$$\pi_0(\mathrm{Sp}([n/2], \mathbb{C}) \longrightarrow \pi_0(\mathrm{O}(n, \mathbb{C})) \longrightarrow \pi_0(X(n)) \longrightarrow 0.$$

Since the symplectic group is connected for all $n$ (Piccione and Tausk example 3.2.25), we get $\pi_0(X(n)) = \pi_0(\mathrm{O}(n, \mathbb{C}))$ for all $n$, meaning $X(n)$ has two connected components for all $n$.

If we associate the quadratic form $s$ to a skew-symmetric form, we get the analogous construction for $U(n)$ instead of $\mathrm{Sp}([n/2], \mathbb{C})$ and the result still holds.

□

# References


[1] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Cham: Springer International Publishing, 2015, pp. 371–405. ISBN: 978-3-319-13467-3. DOI: 10.1007/978-3-319-13467-3_13. URL: https://doi.org/10.1007/978-3-319-13467-3_13.

[2] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000.

[3] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. 2013.

[4] P. Piccione and D. V. Tausk. *On the geometry of Grassmannians and the symplectic group: the Maslov index and its applications*. Citeseer, 2000.