

Home Assignment 3: Lie groups

Definition. A *Lie group* is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G *acts on a manifold* M if the group action is given by the smooth map $G \times M \rightarrow M$.

Exercise 3.1. Prove that $SL(n, \mathbb{R})$ is a Lie group. Prove that it is connected.

Proof.

($SL(n, \mathbb{R})$ is a Lie group.) Recall that $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ of matrices with determinant 1, so it is the preimage of $\{1\}$ under the smooth function $\det : GL(n, \mathbb{R}) \rightarrow \mathbb{R}$. In fact, 1 is a regular value of \det because \det is surjective and of constant rank $\equiv 1$, making $SL(n, \mathbb{R})$ a submanifold. (Of course, $GL(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{2n} = M(n, \mathbb{R})$ because it is an open subset, namely, the preimage of $\mathbb{R} \setminus \{0\}$ under the continuous function \det .)

Moreover, we may think of \det as a group homomorphism from $GL(n, \mathbb{R})$ to the multiplicative group $\mathbb{R} \setminus \{0\}$, so that $SL(n, \mathbb{R}) = \ker \det$, making it a subgroup. The restriction of the group operations from $GL(n, \mathbb{R})$ are smooth, making $SL(n, \mathbb{R})$ a Lie group.

($SL(n, \mathbb{R})$ is connected.) □

Exercise 3.2. Prove that the special unitary group $SU(n)$ acts transitively on the projective space \mathbb{CP}^{n-1} . Find the stabilizer $St_x(SU(n))$ of a point $x \in \mathbb{CP}^{n-1}$. Prove that it is connected, or find a counterexample.

Proof.

($SU(n)$ acts transitively on \mathbb{CP}^{n-1} .) Any point in \mathbb{CP}^{n-1} has two representants in the set of points of \mathbb{C}^n of norm 1. Indeed, suppose $x = z_1 : \dots : z_n$ is a point of \mathbb{CP}^{n-1} . Since not all coordinates are zero, we may normalize dividing by $\sqrt{z_1^2 + \dots + z_n^2}$. But of course the point $(-z_1, \dots, -z_n) \in \mathbb{C}^n$ is also a representant of x that has norm 1.

Anyway, a matrix $U \in SU(n)$ not only will preserve the set of points of norm 1, but act transitively on them. This follows from Gram-Schmidt orthogonalization process and from [Hadamard's inequality](#). The latter says that the determinant of a matrix equals the product of the column vectors if they are orthogonal.

(Find $St_x(SU(n))$.) By the arguments above it can only be the cyclic group of two elements.

(Is $SU(n)$ connected?) □

Definition. Let W be an n -dimensional complex vector space equipped with a complex-linear non-degenerate quadratic form s . Consider the *complex orthogonal group* $O(n, \mathbb{C})$ of all matrices $A \in GL(W)$ preserving s . A subspace $V \subset W$ is called *isotropic* if $s|_V = 0$. It is called *maximally isotropic*, or *Lagrangian*, if $\dim V = \lceil n/2 \rceil$.

Exercise 3.3. Prove that $SO(n, \mathbb{C}) := O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ is a Lie group which has index 2 in $O(n, \mathbb{C})$. Prove that it is connected.

Proof. content...

□