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github.com/danimalabares/riemann-surfaces

Home Assignment 2: quadratic forms

Theorem (Sylvester's law of inertia). If A is the symmetric matrix that defines a quadratic form and S is any invertible matrix such that $D = SAS^T$ is diagonal, then the number of negative elements in the diagonal of D is always the same, for all such S; and the same goes for the number of positive elements.

Definition. Let the number of +1s be denoted by n_+ and the number of -1s by n_-. The pair (n_+, n_-) is called the ***signature*** of A.

Exercise 2.1. Let q be a quadratic form of signature $(1,1)$ on $\mathbb{R}^2$ with integer coefficients. Prove that there is always a non-trivial rational pair $v = (a,b) \in \mathbb{R}^2$ such that $q(v) = 0$, or find a counter example.

Idea of proof. Let $q(x,y) = ax^2 + 2bxy + cy^2$ with $a, b, c \in \mathbb{Z}$. For the associated matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ there exists a nonsingular matrix S such that $S^{-1}AS$ is either of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for every vector $v \in \mathbb{R}^2$ with coordinates (x,y) in the new basis,

$$\tilde{q}(v) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ -y \end{pmatrix} = x^2 - y^2$$

or

$$\tilde{q}(v) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -x & y \end{pmatrix} = -x^2 + y^2.$$

So the solutions of $\tilde{q}(v) = 0$ are the lines $x = y$ and $x = -y$. When returning to the original basis, these lines are mapped to two other lines through the origin. Should these new lines have irrational slope, there would be no rational points in them. A necessary condition for this to happen is that S^{-1} maps $(1,1)$ to a scalar multiple of $(1, \alpha)$ for some irrational number α. But then at least one of the entries of S would be irrational, and this might mean that A cannot be an integer matrix. □

Proof by Lada.

$$x^2 - 2y^2$$

□

Definition. The group $O(p,q)$ is the group of linear isometries of the $(p+q)$-dimensional vector space with scalar product of signature (p,q), and $SO(p,q) \subset O(p,q)$ is the group of isometries preserving the orientation. We use the notation $SO^+(p,q)$ for the connected component of $SO(p,q)$.

Exercise 2.2. Prove that $O(1,1)$ has 4 connected components, and $SO(1,1)$ has 2 connected components.

Exercise 2.3. Prove that $O(p,q)$ has 4 connected components, when $p, q > 0$, and $SO(1,1)$ has 2 connected components. **Hint:** use the previous exercise.

Idea of proof of exercises 2.2 and 2.3.

(Step 1) First we make sure that $O(n)$ was two connected components for all n. Any orthonormal base is mapped to another orthonormal base, so that the columns of any matrix $A \in O(n)$ are orthonormal. Since the (i,j)-entry of the product of AA^T, where T denotes transpose, is the scalar product of the i-th row with the j-th column of A, we have that $AA^T = I$. It follows that

$$1 = \det(I) = \det(AA^T)\det(A)\det(A^T) = \det(A)^2$$

so

$$\det(A) = \pm 1.$$

Then $O(n)$ has two connected components because $\det : O(n) \to \mathbb{C}$ is a continuous function and $O(n)$ is the preimage of two small open neighbourhoods of 1 and -1.

(Step 2) (This is exercise 1.6.1 in B. Hall, *Lie Groups, Lie, Algebras, and Representations: an Elementary Introduction*, Second Edition.)

For the scalar product of signature (p,q)

$$q(x,y) = x_1y_2 + \ldots + x_py_p - x_{p+1}y_{p1} - \ldots - x_{p+q}y_{p+q},$$

a matrix A is in $O(p,q)$ if and only if

$$GA^TG = A^{-1}$$

where

$$G = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and I_k denotes de $k \times k$ identity matrix.

(Step 3) We borrow an answer from StackExchange:

> Since $O(p,q)$ as characterized is closed under transposes, the group $O(p,q) \cap O(p,q) = O(p) \times O(q)$ is a maximal compact subgroup of $O(p,q)$. Therefore, the number of connected components of $O(p,q)$ is the same as that of $O(p) \times O(q)$, and this is $2 \times 2 = 4$.

Where the equality $O(p,q) \cap O(p,q) = O(p) \times O(q)$ holds since

> the intersection preserves the quadratic forms with plus and minus signs being a subgroup of $O(p,q)$ and preserves the quadratic form with only pluses being a subgroup of $O(p+q)$. Hence it preserves the sum and difference of the quadratic forms and their null spaces, Thus it preserves $\mathbb{R}^p$ and $\mathbb{R}^q$, hence it lies in $O(p) \times O(q)$.

(Step 4) It remains to show that the result holds for a general scalar product of signature (p, q).

$\square$

Proof of exercise 2.2 by Misha. Choose two vectors a, b such that

$$q(Ua, Ub) = q(a, b) = 1$$

and

$$\begin{array}{ll} q(a) = b & q(a) = -a \\ q(b) = a & q(b) = -b \end{array}$$

for $U \in SO(1, 1)$. The point is that given the choise, all $\lambda \in \mathbb{R}^{>0}$ will do, so that the connected component is a line. The four choices are just signs and orientations. $\square$

Idea for exercise 2.3 by Vanya. Consider the fibration

$$O(p-1, q) \longrightarrow O(p, q) \longrightarrow B = S^{q-1}$$

since hyperboloid retracts to a sphere. We obtain

$$\pi_1(S^{q-1}) \longrightarrow \pi_0(O(p-1, q)) \longrightarrow \pi_0(S^{q-1})$$

$\square$

Exercise 2.4. Let q be a quadratic form of signature $(1, 2)$ on $\mathbb{R}^3$ with integral coefficients. Prove that there is always a non-trivial rational triple $u = (a, b, c) \in \mathbb{R}^3$ such that $q(u) = 0$, or find a counterexample.

Proof by Vanya.

$$q(a, b, c) = 3a^2 - b^2 - c^2$$

$\square$

Definition. Let $V = \mathbb{R}^3$ be a vector space with quadratic form q of signature $(1, 2)$. A line (= 1-dimensional vector subspace) ℓ in V is called ***positive*** if $q(x, x) > 0$ for some $x \in \ell$, ***negative*** if $q(x, x) < 0$ for some $x \in \ell$ and ***isotropic*** if $q(x, x) = 0$ for all $x \in \ell$. Let $\alpha \in SO^+$ be a non-trivial element. It is called ***elliptic*** if it preserves a positive line $\ell \in V$, ***hyperbolic*** if it preserves a negative line, and ***parabolic*** if all lines preserved by α are isotropic.

Exercise 2.5. Let q be quadratic form of signature $(1, 2)$ on $\mathbb{R}^3$ with integral coefficients, $h \in SO^+(1, 2)$ a hyperbolic isometry with integral coefficients, and $P_h(t)$ its characteristic polynomial. Prove that $P_h(t)$ has precisely 1 rational root.

Idea. Think of $SO^+(1,2)$ as the set of orientation-preserving isometries of $\mathbb{R}^{1,2}$ that fix one of the sheets of some hyperboloid given by $q(x,x) = \pm 1$. We may identify $SO^+(1,2)$ with $PGL(2,\mathbb{C}) \cong \mathrm{Aut}(\mathbb{C}P^1)$. This might simply h to a 2×2 matrix, but I still couldn't find a rational root. $\square$

Exercise 2.6. Let $f : \partial\Delta \to \mathbb{C}$ be a continuous function. Prove that f can be extended to a holomorphic function on Δ or find a counterexample.

Proof. A counter example is the function $f(z) = \frac{1}{z}$. Suppose that f can be extended to a holomorphic function on Δ and define $g(z) = zf(z)$. Then

$$0 = g(0) = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{g(z)}{z} dz = \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{dz}{z} = 1.$$

$\square$