Home Assignment 3: Lie groups

Definition. A ***Lie group*** is a smooth manifold equipped with a group structure such that the group operations are smooth. Lie group G ***acts on a manifold*** M if the group action is given by the smooth map $G \times M \to M$.

Exercise 3.1. Prove that $SL(n, \mathbb{R})$ is a Lie group. Prove that it is connected.

Proof.

($SL(n, \mathbb{R})$ is a Lie group.) (Idea from Lee) Since $SL(n, \mathbb{R})$ is the subgroup of $GL(n, \mathbb{R})$ of matrices with determinant 1, it is the preimage of $\{1\}$ under the smooth function $\det : GL(n, \mathbb{R}) \to \mathbb{R}$. In fact, 1 is a regular value of det because det is surjective and of constant rank $\equiv 1$, making $SL(n, \mathbb{R})$ a submanifold. (Of course, $GL(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{2n} = M(n, \mathbb{R})$ because it is an open subset, namely, the preimage of $\mathbb{R}\backslash 0$ under the continuous function det.)

Moreover, we may think of det as a group homomorphism from $GL(n, \mathbb{R})$ to the multiplicative group $\mathbb{R}\backslash 0$, so that $SL(n, \mathbb{R}) = \ker \det$, making it a subgroup. The restriction of the group operations from $GL(n, \mathbb{R})$ are smooth, making $SL(n, \mathbb{R})$ a Lie group.

($SL(n, \mathbb{R})$ is connected.) (Idea from StackExchange) We have seen in our lectures that for any vector space V we have $SL(V) = e^{\mathrm{End}_0(V)}$ where $\mathrm{End}_0(V)$ denotes the space of matrices with trace 0. We may show that $SL(n, \mathbb{R})$ is path-connected by taking any matrix $e^X \in SL(n, \mathbb{R})$ with $X \in \mathrm{End}_0(\mathbb{R}^n)$ and connecting it to the identity element by the path e^{tX}. □

Exercise 3.2. Prove that the special unitary group $SU(n)$ acts transitively on the projective space $\mathbb{C}P^{n-1}$. Find the stabilizer $St_x(SU(n))$ of a point $x \in \mathbb{C}P^{n-1}$. Prove that it is connected, or find a counterexample.

Proof.

($SU(n)$ acts transitevly on $\mathbb{C}P^{n-1}$.) Any point in $\mathbb{C}P^{n-1}$ has a whole circle of representants in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. Indeed, suppose $x = z_1 : \ldots : z_n$ is a point of $\mathbb{C}P^{n-1}$. Since not all coordinates are zero, we may normalize dividing by $\sqrt{z_1^2 + \ldots + z_n^2}$, or perhaps by $h(x, x)$. Then for every $\lambda \in S^1$, the point $\lambda(z_1, \ldots, z_n) \in \mathbb{C}^n$ is also a representant of x in S^{2n-1}.

Anyway, a matrix $U \in SU(n)$ not only will preserve S^{2n-1}, but will act transitively on it. This follows from Gram-Schmidt orthogonalization process and from Hadamard's inequality. The latter says that the determinant of a matrix equals the product of the column vectors if they are orthogonal.

Explicitly, we proceed as follows. Any point on the sphere may be extended to an othonormal basis, which is equivalent to a matrix in $\mathrm{SU}(n)$ by Hadamard's inequality. Given any two points on the sphere, the composition of their correponding matrices (using the inverse of one of them) takes one point to another. Transitivity on S^{2n-1} implies transitivity on $\mathbb{C}P^n$.

(Find $\mathrm{St}_x(\mathrm{SU}(n))$**.)** Consider any representant $z \in S^{2n-1}$ of $x \in \mathbb{C}P^{n-1}$. Elements in $\mathrm{SU}(n)$ that fix z will of course fix x. And for each of them, multiplying by $\lambda \in S^1$ will yield another operator that fixes z as well.

Now the elements in $\mathrm{SU}(n)$ that fix z can be identified with $\mathrm{SU}(n-1)$. This can be easily seen for the particular case of a simple vector such as $v = (1, 0, \dots, 0)$ noticing that the unitary matrices that fix it must have v as a vector in their first column and row, leaving the remaining $n-1$ entries corresponding to a matrix in $\mathrm{SU}(n-1)$. Up to a change of coordinates, this is true for any point in S^{2n-1}.

We conclude that $\mathrm{St}_x(\mathrm{SU}(n)) = \mathrm{SU}(n-1) \otimes S^1$.

($\mathrm{SU}(n)$ **is connected.)** (Proof from Hall, prop. 1.13) Any matrix in $U \in \mathrm{SU}(n)$ can be diagonalized with eigenvalues of norm 1. This just means that $U = U_1 D U_1^{-1}$ for $U_1 \in \mathrm{SU}(n)$ and D diagonal. Now consider

$$D(t) = \begin{pmatrix} e^{\sqrt{-1}(1-t)\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{\sqrt{-1}(1-t)\theta_n} \end{pmatrix}$$

Then the path $U(t) = U_1 D(t) U_1^{-1}$ connects U and Id as t moves from 1 to 0. To make sure the path stays within $\mathrm{SL}(n)$ we redefine the last entry to be the inverse of the product of the first $n-1$ entries. This proves that $\mathrm{SU}(n)$ is connected.

(Proof using theory of homogeneous spaces.) We have seen in our lectures that a transitive action of a Lie group G on a manifold M yields $M = G/\mathrm{St}_x(G)$. If my stabilizer is correct, then we have that $\mathbb{C}P^{n-1} = \mathrm{SU}(n)/(\mathrm{SU}(n-1) \otimes S^1)$. It certainly would be nice to conclude this reasoning since its probably the intention of the exercise... haha!)

□

Definition. Let W be an n-dimensional complex vector space equipped with a complex-linear non-degenerate quadratic form s. Consider the ***complex orthogonal group*** $\mathrm{O}(n, \mathbb{C})$ of all matrices $A \in \mathrm{GL}(W)$ preserving s. A subspace $V \subset W$ is called ***isotropic*** if $s|_V = 0$. It is called ***maximally isotropic***, or ***Lagrangian***, if $\dim V = [n/2]$.

Exercise 3.3. Prove that $\mathrm{SO}(n, \mathbb{C}) := \mathrm{O}(n, \mathbb{C}) \cap \mathrm{SL}(n, \mathbb{C})$ is a Lie group which has index 2 in $\mathrm{O}(n, \mathbb{C})$. Prove that it is connected.

Proof. We know that $\mathrm{SO}(n, \mathbb{C})$ is a Lie group from our lectures. Now the function $\det : \mathrm{O}(n, \mathbb{C}) \to \{\pm 1\} = \mathbb{Z}/2$ is a surjective group homomorphism with kernel $\mathrm{SO}(n, \mathbb{C})$, meaning $[\mathrm{O}(n, \mathbb{C}) : \mathrm{SO}(n, \mathbb{C})] = |\mathbb{Z}/2| = 2$.

Any complex-linear quadratic form is associated to a hermitian form, which in turn means we can diagonalize any matrix in $SO(n, \mathbb{C})$ and construct a path to the identity matrix just as in the previous exercise. Or maybe use some homogeneous spaces/quotient again? $\square$

Exercise 3.4. Let X be the space of all maximally isotropic subspaces in W (the ***maximally isotropic Grassmanian***).

(a) Prove that $O(n, \mathbb{C})$ acts on X transitively.

(b) Prove that X is disconnected for $n = 2, 3$.

(c) Prove that it is connected for $n \geqslant 4$, or find a counterexample.

Proof.

(a) Coro. 1.4.27 in here might help

(b) **(First attempt)** An interesting example (Hatcher, 4.53) of a fiber bundle is $S^1 \to S^{2n+1} \to \mathbb{C}P^n$, given by the map that sends a point to its equivalence class and, as noted in exercise 3.2, has fiber S^1. Surprisingly, this is the case $n = 1$ for the more general fiber bundle $U(n) \to V_n(\mathbb{C}^k) \to G_n(C^k)$ where $V_n(\mathbb{C}^k)$ is the space of n-tuples of orthonormal vectors in $\mathbb{C}^k$ and $G_n(C^k)$ is the space of n-subspaces in $\mathbb{C}^k$. The arrow in the right assigns to a n-tuple the vector space it generates, so the fiber is given by all the orthonormal n-tuples that generate any given linear space, which is precisely the group of $U(n)$ by the Gram-Schmidt process.

Unfortunately, our space X cannot correspond to an n-tuple of s-orthonormal vectors since every n-tuple that generates a given vector space V in X will vanish under s.

(Second attempt) In search for an alternative approach, I have found in Wikipedia that the Lagrangian Grassmanian of a symplectic vector space is identified with $U(n)/O(n)$.

Given the transitive action from item (a), it's enough to find the stabilizer of the action to obtain a quotient $O(n, \mathbb{C})/St_x = X$. Then prove such a quotient is connected for the required n.

$\square$

References

[1] Brian Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Cham: Springer International Publishing, 2015, pp. 371–405. ISBN: 978-3-319-13467-3. DOI: 10.1007/978-3-319-13467-3_13. URL: https://doi.org/10.1007/978-3-319-13467-3_13.

[2] A. Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000.

[3] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. 2013.