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Two new chiral 4-polytopes in $\mathbb{E}^4$

TESINA

QUE PARA OPTAR POR EL GRADO DE:

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A mis queridos tutores Roli e Isabel por transmitirme tanta emoción. A Bris por ser un modelo a seguir. Al equipo de politoperes por todo su cariño. A Vinicio Gómez por estar siempre. A mi familia.

Summary

In the context of skeletal geometric complexes, chiral polytopes are those with maximal rotational symmetry but no reflection symmetry. We show that the natural rotation about the base edge in two of the chiral polyhedra from [2] yields, in each case, a chiral 4-polytope in $\mathbb{E}^4$. Both polytopes are shown to be combinatorially chiral.

Contents

1 Introduction

Despite polygons and polyhedra being basic concepts in mathematics, it is not at all obvious what these words exactly mean. Chiral polytopes arise from the so-called skeletal approach, where polygons and their higher-dimensional analogues are introduced as 1-dimensional complexes, with no need of an enclosed surface or solid.

Such definition allows for symmetric structures otherwise unseen, an example of which are chiral polytopes. While the symmetry group of a regular polyhedron acts transitively on the set of flags, a polyhedron is geometrically chiral when its symmetry group has two orbits on the flags and adjacent flags are in distinct orbits. This captures the idea of maximal rotational symmetry but no reflection symmetry.

Chiral polyhedra first appeared in 2005, when Schulte classified those realisable in $\mathbb{E}^3$, non of which is finite nor its faces are contained in planes (see [13, 14]). The first example of a chiral 4-polytope in $\mathbb{E}^4$, so-called Roli's cube, was constructed in 2014 by Bracho, Hubard and Pellicer [1]. It was later shown that the facets of this polytope belong to a broader family of chiral polyehdra with helical faces in $\mathbb{S}^3$ [2].

In this thesis we show that the natural rotation about the base edge in two such spherical polyhedra successfully yields, in each case, a chiral 4-polytope. Their combinatorial structures are then shown to be chiral as well—something that does not occur for their facets, which are combinatorially regular.

Proofs are given by GAP programs in which the abstract and geometric structures are compared. Geometric realizations are based on the reflection matrices used in [2]. All programs used are available online (see Appendix A).

2 Historical background

We begin with a brief historical discussion on the concept of polytope.

We cannot find a moment in history when triangles and squares came to attract the attention of humans. Later, when mathematics became an established discipline, simple polygons and polyhedra were thoroughly studied: the first proposition in Euclid's elements is the construction of a regular triangle, and the last book is devoted to the study of Platonic Solids [8].

After the greeks, the definition of polygon remained essentialy unchanged for many centuries. The first account of an important difference dates to the XIV century, when an Archbishop of Canterbury investigated star polygons [12]. These are essentially different by being non-convex: their edges intersect in points that are not vertices of the polygon. They may be, however, studied by their symmetry properties just like regular convex polygons.

Regular star polyhedra are the natural generlization of regular star polygons to euclidean 3-space. They take us to the XVII century with Johannes Kepler, who studied the two whose faces are star polygons (pentagrams). Their duals, whose faces are convex but their vertex-figures are not, where studied in 1809 by Louis Poinsot [12].

Higher-dimensional polytopes were first studied in the XIX century, when Ludwig Schläfli found all the regular polytopes whose symmetry groups are generated by reflections in hyperplanes in euclidean spaces [12].

Next in history is Coxeter, who, among many other results, classiffied all discrete euclidean reflection groups. In the 1930's him and Petrie redefined the concept of polytope by letting them have infinite faces, thus finding three more regular polyhedra in $\mathbb{E}^3$ [3]. These have non-planar vertex-figures.

Fourty years later, in 1977, Grünbaum once again reintroduced polytopes, this time permitting the faces to be non-planar [6]. The list of regular polyhedra in $\mathbb{E}^3$ was extended to 18 in the finite case and 48 in total [10]. All but one were classified by Grünbaum, the remaining one and the completeness of the list are due to Dress [4, 5].

The study of the underlying combinatorial structure of polytopes led to the concept of abstract polytope. In 1967 McMullen studied the lattice of faces of a polytope and compared its automorphisms with geometric symmetries [9]. In Grümbaum's [7], the term *polystroma* (stroma=stratum, layer) is defined to mean an abstract partially order structure resembling the face lattice of a polytope.

Chiral polytopes were first studied in this abstract sense, a general theory first given in 1991 by Schulte and Weiss [15]. Back to geometry, skeletal chiral polyhedra in $\mathbb{E}^3$ were studied in [13, 14], and the first example of a skeletal chiral 4-polytope in $\mathbb{E}^4$ is Roli's cube from [1].

3 Skeletal polyhedra and 4-polytopes in $\mathbb{E}^4$

Now we give formal definitions and basic results.

A *skeletal polyhedron* in $\mathbb{E}^4$ consists of *vertices* (points in $\mathbb{E}^4$), *edges* (segments between vertices) and *faces* (cycles on the graph determined by the vertices and edges) such that:

(i) every edge belongs to two faces,

(ii) the graph determined by the vertices and edges is connected,

(iii) every compact subset of $\mathbb{E}^4$ meets finitely many edges, and

(iv) the *vertex-figure*, defined as follows, is a connected graph. For any vertex v, the vertices of the vertex-figure are the neighbours of v and the edges are segments joining any two neighbours that are both in some face.

A *skeletal 4-polytope* in $\mathbb{E}^4$ consits of vertices, edges, faces and *cells* (skeletal polyhedra), such that

(i) every face belongs to two cells,

(ii) the graph determined by the vertices and the edges is connected,

(iii) every compact subset of $\mathbb{E}^4$ meets finitely many edges, and

(iv) the vertex-figure at every vertex is a skeletal polyhedron.

Hereafter we omit the term "skeletal" when the context permits it. If $\mathcal{P}$ is a 4-polytope, the set of vertices, edges, faces and cells is a partial order with respect to inclusion. We say two such elements are *incident* if they are comparable. This poset is called the *abstract polytope associated to* $\mathcal{P}$. As we shall see, 4-polytopes defined as above are realizations of abstract polytopes in the sense of [12] and Section 4.

A *flag* is a 4-tuple of incident vertex, edge, face and cell. Two flags are *adjacent* when they differ by only one element. We call two flags *0-adjacent* if they differ by a vertex, *1-adjacent* if they differ by an edge, *2-adjacent* if they differ by a face and *3-adjacent* if they differ by a cell.

A *symmetry* of $\mathcal{P}$ is an isometry of $\mathbb{E}^4$ that preserves it set-wise. The group of symmetries of $\mathcal{P}$ will be denoted by $G(\mathcal{P})$. We call $\mathcal{P}$ *regular* if $G(\mathcal{P})$ acts transitively on the set of flags, and *chiral* if $G(\mathcal{P})$ induces two orbits on flags so that adjacent flags are on different orbits. In fact, whether $\mathcal{P}$ is regular or chiral, its symmetry group acts transitively on the sets of vertices, edges, faces and cells.

4 Abstract 4-polytopes

The theory of abstract polytopes is an essential tool for our work. In this section we state basic results and constructions upon which our proofs are based. To avoid using the word 'abstract' in every definition, we use the same words as in the previous section to mean different objects. All polytopes are abstract up to Section 4.2, where they are precisely distinguished.

An *abstract* 4*-polytope* is a partial order $\mathcal{P}$ that satisfies the following properties (P1) to (P4). The elements of $\mathcal{P}$ are called *faces* and the maximally ordered subsets (chains) are called *flags*.

(P1) $\mathcal{P}$ contains a minimal face and a greatest face, denoted respectively by F_{-1} and F_4.

(P2) Flags contain 6 elements, including F_{-1} and F_4.

Since every element in a poset is in some chain, we may assign to every face a number from -1 to 4. Such is the *rank function* of $\mathcal{P}$. We say that two flags are *i-adjacent* if they differ only by a face of rank i.

(P3) $\mathcal{P}$ is *strongly flag connected* in the following sense. For any two distinct flags Φ and Ψ there is a sequence

$$\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$$

so that Φ_{i-1} and Φ_i are adjacent and $\Phi \cap \Psi \subset \Phi_i$ for all i.

(P4) If F and G are a $(i-1)$-face and a $(i+1)$-face with $F < G$ and $0 \leq j \leq 2$, then there are exactly two i-faces H such that $F < H < G$.

4.1 Regular and chiral abstract 4-polytopes

Now we study regular 4-polytopes and their automorphism groups. An *automorphism* of $\mathcal{P}$ is a bijection of the polytope that preserves incidence, and we denote the automorphism group by $\Gamma(\mathcal{P})$. $\mathcal{P}$ is called *regular* if $\Gamma(\mathcal{P})$ is transitive on flags.

For a regular polytope $\mathcal{P}$ and some distinguised flag Φ, there exists a unique involutory automorphism ρ_j of $\mathcal{P}$ such that $\Phi\rho_i = \Phi^i$ for $i = 0, 1, 2, 3$ (Prop. 2B4, [12]). These are called the *distinguished generators* of $\Gamma(\mathcal{P})$ and satisfy that

$$(\rho_i\rho_j)^{p_{ij}} = \mathrm{Id} \text{ when } |i-j| \geq 2 \tag{1}$$

for $i, j = 0, 1, 2, 3$ (prop. 2B11 [12]). The numbers $p_i := p_{i-1}p_i$ form the *Schläfli type* $\{p_1, p_2, p_3\}$ of $\mathcal{P}$. The distinguished generators also satisfy

$$\langle \rho_i | i \in I\rangle \cap \langle \rho_j | j \in J\rangle = \langle \rho_i \in I \cap J\rangle \qquad \text{for } I, J \subseteq \{0, 1, 2, 3\} \tag{2}$$

which we call the *intersection property* (prop. 2B10 [12]). Any group Γ generated by the involutions ρ_0 to ρ_3 satisfying eqs. (1) and (2) is called a *C-string group*. We have established that $\Gamma(\mathcal{P})$ is one such group.

Conversely, we may construct a regular abstract 4-polytope from a C-string group Γ. Define $\Gamma_j := \langle \rho_j | i \neq j\rangle$ and $\Gamma_{-1} = \Gamma_4 = \Gamma$, and take the set of i-faces to be the set of right cosets $\Gamma_i\varphi$ for $\varphi \in \Gamma$. Then there is an order relation with respect to which the set of all faces is a regular 4-polytope whose automorphism group is Γ (thm. 2E11, [12]).

Now we revise the analogue construction for chiral polytopes. An abstract 4-polytope $\mathcal{P}$ is called *chiral* if $\Gamma(\mathcal{P})$ induces two orbits on flags and two adjacent flags are in different orbits.

First observe that by defining $\sigma_i := \rho_{i-1}\rho_i$ for the distinguished generators of a regular polytope with bipartite flags, it holds that

$$\begin{gathered}\sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = \mathrm{Id}\\ (\sigma_1\sigma_2)^2 = (\sigma_1\sigma_2\sigma_3)^2 = (\sigma_2\sigma_3)^2 = \mathrm{Id}\end{gathered} \tag{3}$$

for some positive integers p_1, p_2 and p_3 (see [15]).

The group generated by these elements is called the *rotation subgroup* of $\mathcal{P}$, denoted by $\Gamma^+(\mathcal{P})$. For chiral polytopes this will be the whole group: if $\mathcal{P}$ is chiral, $\Gamma(\mathcal{P})$ is generated by three automorphisms σ_1, σ_2 and σ_3 that satisfy eq. (3) (prop. 3, [15]).

To construct a chiral or regular abstract polytope from a group we must also require some sort of intersection property. It will suffice that

$$\begin{gathered}\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \{\mathrm{Id}\} = \langle\sigma_2\rangle \cap \langle\sigma_3\rangle\\ \langle\sigma_1, \sigma_2\rangle \cap \langle\sigma_2,\sigma_3\rangle = \langle\sigma_2\rangle.\end{gathered} \tag{4}$$

Let $\Gamma = \langle\sigma_1, \sigma_2, \sigma_3\rangle$ be a group satisfying both eqs. (3) and (4). Define

$$\begin{gathered}\Gamma^0 = \langle\sigma_2, \sigma_3\rangle, \quad \Gamma^1 = \langle\sigma_3, \sigma_1\sigma_2\rangle, \quad \Gamma^2 = \langle\sigma_1, \sigma_2\sigma_3\rangle, \quad \Gamma^3 = \langle\sigma_1, \sigma_2\rangle,\\ \text{and} \quad \Gamma_{-1} = \Gamma^4 = \Gamma,\end{gathered}$$

and take the i-faces to be the right cosets $\Gamma_i\varphi$ for $\varphi \in \Gamma$. Again, the set of all faces admits a partial order with respect to which it is a chiral or a regular abstract 4-polytope $\mathcal{P}$ such that $\Gamma^+(\mathcal{P}) = \Gamma$ (thm. 1 and lem. 11, [15]).

In this construction we may distinguish chiral from regular abstract 4-polytopes as follows. $\Gamma^+(\mathcal{P})$ is of index 2 in $\Gamma(\mathcal{P})$ if and only if there exists an involutory automorphism $\rho : \Gamma \to \Gamma$ such that

$$\rho(\sigma_1) = \sigma_1^{-1}, \quad \rho(\sigma_2) = \sigma_1^2\sigma_2 \quad \text{and} \quad \rho(\sigma_3) = \sigma_3, \tag{5}$$

in which case $\mathcal{P}$ cannot be chiral (thm. 1, [15]).

4.2 Realizations

Now we review the relationship between abstract and skeletal polytopes as defined in Section 3.

A *realization* of an abstract polytope $\mathcal{P}$ is a map β from the set of abstract 0-faces $\mathcal{P}_0$ into $\mathbb{E}^4$, so that the set $V_0 := \mathcal{P}_0\beta$ is the set of geometric vertices. All other geometric faces are defined by functions from the set of abstract i-faces $\mathcal{P}_i$ to some nested power set of the geometric vertex set: edges are sets of vertices, faces are sets of edges and so on.

Formally, for $i = 1, 2, 3$, β induces a surjection $\beta_i : \mathcal{P}_i \to V_i$, where V_i is thought as the set of geometric i-faces, consisting of the elements $F\beta_i := \{G\beta_{i-1} | G \in \mathcal{P}_{i-1} \text{ and } G \leq F\}$ for $F \in \mathcal{P}_i$ (thm. 5A1, [12]). For example, an abstract edge $E \in \mathcal{P}_1$ is mapped to a set consisting of the two points (there's only two by (P4)) in $\mathbb{E}^4$ whose preimages are abstract vertices smaller than E in $\mathcal{P}$.

Of course, we expect the number of i-faces of the abstract and geometric structures to be the same. A realization is *faithful* if every β_i is a bijection.

We also expect automorphisms to correspond with isometries of $\mathbb{E}^4$. A realization is *symmetric* when every automorphism of $\mathcal{P}$ induces a permutation of V_0, which in turn determines a unique isometry of $\mathbb{E}^4$ (if the vertex set affinely spans $\mathbb{E}^4$). Then these isometries are an euclidean representation of $\Gamma(\mathcal{P})$.

Conversely, given an euclidean representation of the automorphism group $\Gamma(\mathcal{P})$ of an abstract 4-polytope, we obtain a realization by *Wythoff's construction* as follows.

For the regular case, let $\langle R_0, R_1, R_2, R_3\rangle$ be an euclidean representation of the automorphism group of a regular abstract 4-polytope $\mathcal{P}$. Should there be any, define a point $v \in \mathbb{E}^4$ that is fixed by R_1, R_2 and R_3 but not by R_0 as the base vertex and take the orbit of v for the geometric vertex-set of a realization. Further, define the base edge $e = v\langle R_0\rangle$, the base face $f = e\langle R_0, R_1\rangle$ and the base cell $c = f\langle R_0, R_1, R_2\rangle$ for a geometric base flag.

Now let $\langle S_1, S_2, S_3\rangle$ be a representation of the automorphism group, or rotation subgroup, of a chiral or regular abstract 4-polytope, respectively. For a geometric base vertex choose any point that is fixed by S_2 and S_3 but not by S_1. The orbit of this point is the geometric vertex-set of a realization. For a geometric flag we define the base edge as $e = v\langle S_1 S_2\rangle$, or equivalently, as $e = \{v, vS_1^{-1}\}$ since $S_1^{-1} = R_1 R_0$ when $\mathcal{P}$ is regular. Define the base face as $f = e\langle S_1\rangle$ and the base cell as $c = f\langle S_1, S_2\rangle$.

While it is true that realizations of 4-polytopes are mere clouds of points, they are no generalization of skeletal polytopes—the segment between vertices is still taken to be flat. Save this technical difference, it is straightforward to check that a regular or chiral skeletal 4-polytope as defined in Section 3 is a faithful and symmetric realization of the abstract 4-polytope related to it.

It remains only to formally define star polytopes. Following Section 7D from [12], we say a faithfully realized abstract 4-polytope is *classical* if every geometric i-face has dimension i (its affine hull is i-dimensional). For any such geometric regular polytope, we may write its generating hyperplane reflections as

$$R_i = \{x \in \mathbb{E}^4 | \langle x, u_i\rangle = 0\}$$

for some unit vectors u_i. Then $\langle u_i, u_j\rangle = 0$ for $|i - j| \geq 2$, and, since $\langle R_i, R_j\rangle$ is a finite group, there is a rational number p_i such that $\langle u_{i-1}, u_i\rangle = \cos(\pi/p_i)$. We define the *Shläfli type* of this polytope as $\{p_1, p_2, p_3\}$ and call it a *star 4-polytope* if any of the p_i is not an integer. Of course, the definition for polyhedra is analogous.

5 A realization of $\left\{\frac{5}{2}, 3, 5\right\}$

Out first chiral polytope will have the same vertices and edges as the star 4-polytope $\left\{\frac{5}{2}, 3, 5\right\}$. We shall start by constructing $\left\{\frac{5}{2}, 3, 5\right\}$ from a given realization of $\{3, 3, 5\}$. To this end we first express the generating reflections of $\left\{\frac{5}{2}, 3, 5\right\}$ in terms of those of $\{3, 3, 5\}$. Then we prove they generate a C-string group, which in turn yields an abstract polytope. Finally we show it is realized as $\left\{\frac{5}{2}, 3, 5\right\}$.

Our construction is based in the following observations. In the regular star polyhedron $\left\{\frac{5}{2}, 3\right\}$ the edges intersect at points that are not vertices, and the convex hull of these intersections is a regular icosahedron. The vertex-figure of $\{3, 3, 5\}$ is also a regular icosahedron. Further, the vertices of $\left\{\frac{5}{2}, 3, 5\right\}$ must be the same as those of $\{3, 3, 5\}$ ([12], p. 212).

We are thus inclined to look for a copy of $\left\{\frac{5}{2}, 3\right\}$ within $\{3, 3, 5\}$ by taking the vertex-figure icosahedron of some vertex and at each of its faces consider the ajacent tetrahedron that is not contained in the interior of the icosahedron. We may visualize this by stereographically projecting the vertices and taking the convex hull of each tetrahedron as in fig. 1.

Notice that an edge of $\left\{\frac{5}{2}, 3\right\}$ is composed of three edges of $\{3, 3, 5\}$. Choosing a geometric flag for $\left\{\frac{5}{2}, 3, 5\right\}$ ammounts to selecting an incident vertex, edge and face within this figure, which is the base cell. We define the generating reflections of the symmetry group of $\left\{\frac{5}{2}, 3, 5\right\}$ with respect to the base flag as follows. Call the base vertex w_0, the centroid of the base edge w_1, the centroid of the base face w_2 and the centroid of the base cell w_3. The mirror of the reflection P_i is the hyperplane through the origin and the w_j such that $j \neq i$.

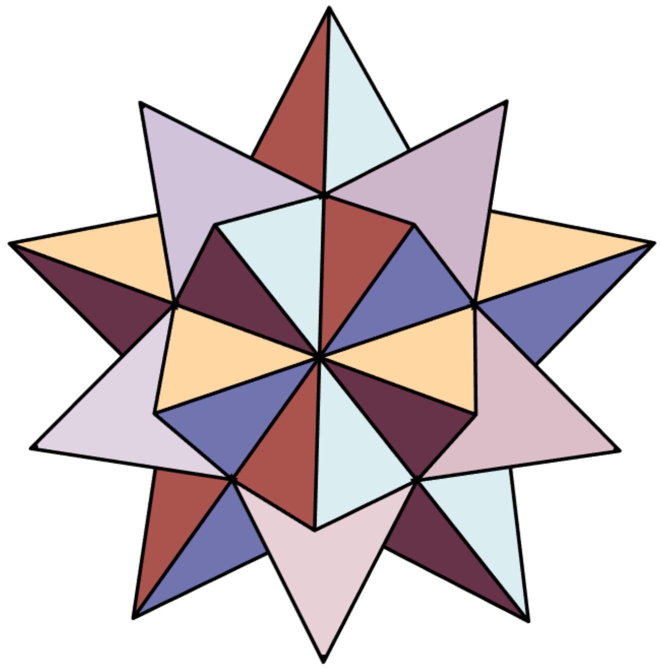


Fig. 1: The base cell of $\left\{\frac{5}{2},3,5\right\}$.

There is a choice of base flag for $\{3,3,5\}$ for which the generating reflections R_0 to R_3 result the following relations, that we shall use as definitions:

$$\begin{aligned} &P_0 = R_0, \qquad P_2 = R_3, \qquad P_3 = R_2 \\ &P_1 = R_1R_2R_3R_2R_1R_0R_1R_2R_3R_2R_1 \end{aligned}$$

To construct a realization of $\left\{\frac{5}{2},3,5\right\}$ we first suppose that R_0 to R_3 are the distinguished generators of the abstract polytope $\{3,3,5\}$. That is, they satisfy the relations

$$R_i^2 = (R_0R_1)^3 = (R_1R_2)^3 = (R_2R_3)^5 = (R_0R_2)^2 = (R_0R_3)^2 = (R_1R_3)^2 = \mathrm{Id}$$

for all i and the intersection property. It has been confirmed that

$$P_i^2 = (P_0P_1)^5 = (P_1P_2)^3 = (P_2P_3)^5 = (P_0P_2)^2 = (P_0P_3)^2 = (P_1P_3)^2 = \mathrm{Id}$$

for all i, and that the intersection property holds (see Appendix A.1). Then the abstract polytope generated by P_0 to P_3 is $\{5,3,5\}$.

Using a particular realization of $\{3,3,5\}$, we may explicitly find the base vertex for $\left\{\frac{5}{2},3,5\right\}$ (see Appendix B). Then Wythoff's construction yields a realization that we expect to be faithful and symmetric. Faithfullness follows by comparing the number of

i-faces of the abstract polytope and the realization (see Appendices A.1 and A.2). Symmetry follows from the fact the vertex-set of both $\{3,3,5\}$ and $\left\{\frac{5}{2},3,5\right\}$ is the same.

Since the R_i are hyperplane reflections, we have a classical polytope as defined in Section 4.2 (see [11]). The only such polytopes in $\mathbb{E}^4$ arising from the abstract regular polytope $\{5,3,5\}$ are the star 4-polytopes $\left\{\frac{5}{2},3,5\right\}$ and $\left\{5,3,\frac{5}{2}\right\}$ (thm. 7D13 [12]). However, the rotation about the edge in this polytope is the reverse of that of $\{3,3,5\}$, so that it cannot be of type $\frac{5}{2}$. This completes the proof that we have constructed $\left\{\frac{5}{2},3,5\right\}$.

6 A chiral 4-polytope from $\left\{\frac{5}{2},3,5\right\}$

To construct a chiral 4-polytope within $\left\{\frac{5}{2},3,5\right\}$, let

$$S_1 = P_0P_1P_3P_2, \qquad S_2 = P_2P_1, \quad \text{and} \quad S_3 = P_3P_2.$$

Then

$$S_1^{12} = S_2^3 = S_3^5 = (S_1S_2)^2 = (S_1S_2S_3)^2 = (S_2S_3)^2 = \mathrm{Id} \tag{6}$$

and eq. (4) holds (see Appendix A.1). It follows that the group generated by S_1 to S_3 yields either a regular or a chiral abstract polytope.

For a realization by Wythoff's construction define the base vertex to be the same as the one that was used for $\left\{\frac{5}{2},3,5\right\}$. Such a choice and the fact that the group $\langle S_1,S_2,S_3\rangle$ a subgroup of $\langle P_0,P_1,P_2,P_3\rangle$ make the realization symmetric. Again, by comparing the number of i-faces in the abstract and geometric polytopes, our realization is seen to be faithful.

The definitions of S_1 and S_2 are as in [2], so that the cells are copies of the chiral polyhedron denoted as $H_1(\left\{5,3,\frac{5}{2}\right\})$. In fact, chirality in our 4-polytope follows from the chirality of the cells, since any symmetry sending a flag to its i-th adjacent is also a symmetry of the cell.

We have shown this structure to have 120 vertices, 720 edges, 300 faces and 50 cells. Every face has 12 vertices and edges arranged in helical fashion as shown in [2] (see fig. 2). In virtue of such arrangement we denote this polytope by $\left\{\frac{12}{1,5},3,5\right\}$.

We now show this structure satisfies *(i)*-*(iv)* in our definition of skeletal 4-polytope. By the realization being faithful and symmetric, conditions *(i)* and *(ii)* follow from (P4) and (P3), respectively. For *(iii)* notice the vertex figures are icosahedra, which follows from Wythoff's construction on any vertex adjacent to the base vertex by the group $\langle S_2,S_3\rangle$. Finally, *(iv)* is immediate from the finiteness of the group. This concludes the proof that we have constructed a chiral 4-polytope.

Further, it was found that there exists no automorphism ρ of the group generated by the S_i that satisfies eq. (5), so that the abstract polytope associated to $\left\{\frac{12}{1,5},3,5\right\}$ is combinatorially chiral (see Appendix A.1).

Fig. 2: Stereographic projection of the base face of $\left\{\frac{12}{1,5}, 3, 5\right\}$. We also show the base face of $\left\{\frac{5}{2}, 3, 5\right\}$

7 A chiral 4-polytope from $\left\{5, 3, \frac{5}{2}\right\}$

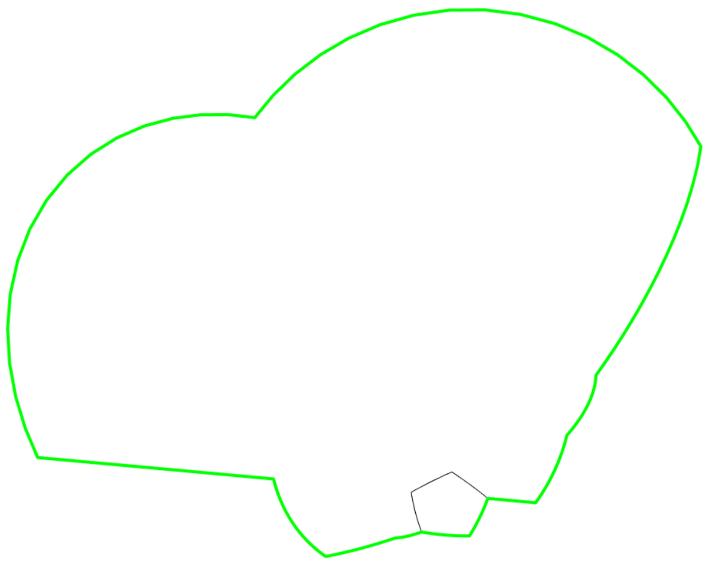
For the dual star polytope $\left\{5, 3, \frac{5}{2}\right\}$ we simply define

$$Q_0 = P_3, \qquad Q_1 = P_2 \qquad Q_2 = P_3, \quad \text{and} \quad Q_3 = P_0$$

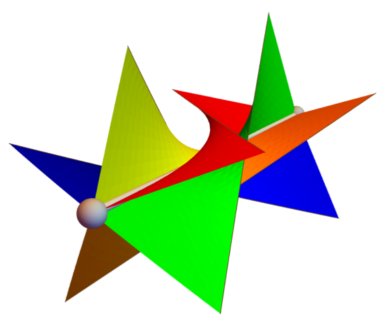
and apply Wythoff's construction on the centroid of the base cell of $\left\{\frac{5}{2}, 3, 5\right\}$. This yields a realization of $\left\{5, 3, \frac{5}{2}\right\}$.

Analogue definitions for the S_i as in the last section also satisfy eq. (6) and eq. (4), so that we have another abstract 4-polytope that is, in fact, the same as the (abstract) one constructed before.

Wythoff's construction on the base vertex then yields a symmetric and faithful realization of a chiral 4-polytope whose cells are copies of $H_0(\left\{5, 3, \frac{5}{2}\right\})$ from [2]. We call it $\left\{\frac{12}{1,5}, 3, \frac{5}{2}\right\}$.



(a) The base face. We also show the base face of $\left\{5, 3, \frac{5}{2}\right\}$



(b) Five faces at the edge. We show part of the surface spanned between the edge and the axis of the twist in every face.

Fig. 3: Stereographic projections of $\left\{\frac{12}{1,5}, 3, \frac{5}{2}\right\}$.

Appendix A

The original code for the programs used in this thesis may be consulted at github.com/dan-gc/tesina.

In the following two sections we show the output of the files `abstract.txt` and `geometric.txt` as executed in GAP, in which the abstract and geometric polytopes associated to $\{\frac{5}{2},3,5\}$ are constructed. The intersection property of $\{\frac{5}{2},3,5\}$ was confirmed in another program named `int-prop.txt`, and the constructions regarding $\{5,3,\frac{5}{2}\}$ are found in analogue programs tagged with the word `dual`.

A.1 abstract.txt

```
-------------------------------
ABSTRACT POLYTOPES
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-------------------------------
STRING RELATIONS OF {5/2,3,5}
|<P0>|=2
|<P1>|=2
|<P2>|=2
|<P3>|=2

|<P0*P1>|=5
|<P1*P2>|=3
|<P2*P3>|=5

|<P0*P2>|=2
|<P0*P3>|=2
|<P1*P3>|=2

Are the groups of {3,3,5} and
   {5/2,3,5} the same? true


-------------------------------
FACE COUNT OF THE THREE POLYTOPES

{3,3,5}

Vertices in the face: 3
Vertices in the cell: 4
Vertices in the polytope: 120
Edges in the face: 3
Edges in the cell: 6
Edges in the polytope: 720
Faces in the cell: 4
Faces in the polytope: 1200
Cells in the polytope: 600


{5/2,3,5}

Vertices in the face: 5
Vertices in the cell: 20
Vertices in the polytope: 120
Edges in the face: 5
Edges in the cell: 30
Edges in the polytope: 720
Faces in the cell: 12
Faces in the polytope: 720
Cells in the polytope: 120


Chiral

Vertices in the cell: 48
Vertices in the polytope: 120
Edges in the cell: 72
Edges in the polytope: 720
Faces in the cell: 12
Faces in the polytope: 300
Cells in the polytope: 50
```

```
-------------------------------
STRING RELATIONS OF CHIRAL

|<S1>|=12
|<S2>|=3
|<S3>|=5

|<S1*S2>|=2
|<S1*S2*S3>|=2
|<S2*S3>|=2


-------------------------------
INTERSECTION PROPERTY OF CHIRAL

<S1>INT<S2>==<1>  true
<S2>INT<S3>==<1>  true
<S1,S2>INT<S2,S3>==<S2>  true
```

```
-------------------------------
COMBINATORIALLY CHIRAL

Is there an automorphism that
   satisfies eq. (3)?

rho(S1)=S1^-1  true
rho(S2)=S1^2*S2  true

Can we extend it to the whole group?
    fail
```

A.2 geometric.txt

```
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GEOMETRIC POLYTOPES
-------------------------------


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STRING RELATIONS OF {5/2,3,5}
|<P0>|=2
|<P1>|=2
|<P2>|=2
|<P3>|=2

|<P0*P1>|=5
|<P1*P2>|=3
|<P2*P3>|=5

|<P0*P2>|=2
|<P0*P3>|=2
|<P1*P3>|=2

Are the groups of {3,3,5} and
   {5/2,3,5} the same? true
```

```
-------------------------------
BASE VERTEX OF {5/2,3,5}

w0P0=w0 false
w0P1=w0 true
w0P2=w0 true
w0P3=w0 true

-------------------------------
FACE COUNT FOR THE THREE POLYTOPES

{3,3,5}

Vertices in the face: 3
Vertices in the cell: 4
Vertices in the polytope: 120
Edges in the face: 3
Edges in the cell: 6
Edges in the polytope: 720
Faces in the cell: 4
Faces in the polytope: 1200
Cells in the polytope: 600
```

```
{5/2,3,5}

Vertices in the face: 5
Vertices in the cell: 20
Vertices in the polytope: 120
Edges in the face: 5
Edges in the cell: 30
Edges in the polytope: 720
Faces in the cell: 12
Faces in the polytope: 720
Cells in the polytope: 120

Chiral

Vertices in the face: 12
Vertices in the cell: 48
Vertices in the polytope: 120
Edges in the face: 12
Edges in the cell: 72
Edges in the polytope: 720
Faces in the cell: 12
Faces in the polytope: 300
Cells in the polytope: 50

-------------------------------
STRING RELATIONS OF CHIRAL
|<S1>|=12
|<S2>|=3
|<S3>|=5

|<S1*S2>|=2
|<S1*S2*S3>|=2
|<S2*S3>|=2
```

```
-------------------------------
STABILIZERS OF CHIRAL

S3 fixes the base vertex and edge?

w0.S3=w0  true
w0.S1^-1.S3=w0.S1^-1  true

Stabilizers within the cell

Stab_<S1,S2>w0=<S2>  true
Stab_<S1,S2>e=<S1*S2>  true
Stab_<S1,S2>f=<S1>  true


Stabilizers within the whole polytope

Stab_<S1,S2,S3>w0=<S2,S3>  true
Stab_<S1,S2,S3>e=<S1*S2,S3>  true
Stab_<S1,S2,S3>f=<S1,S2*S3>  true
Stab_<S1,S2,S3>c=<S1,S2>  true
```

Appendix B

In this appendix we show the coordinates of geometric realizations. Let ϕ be the golden ratio. As in [2], take the base flag of $\{3,3,5\}$ given by the centroids of the base vertex, edge, face and cell by

$$v_0 = (1,0,0,0), \quad v_1 = \left(\phi+2, 1, 0, \frac{1}{\phi}\right), \quad v_2 = \left(\phi, \frac{1}{\phi}, 0, 0\right), \quad v_3 = \left(\phi^2, 1, -\frac{1}{\phi^2}, 0\right),$$

respectively. Then the coordinates of the base vertex of $\{\frac{5}{2}, 3, 5\}$ are

$$w_0 = \left(\frac{\phi}{2}, -\frac{1}{2}, 0, -\frac{1}{2\phi}\right).$$

according to $w_0 = v_0 R_0 R_1 R_2 R_1 R_2 R_3 R_2 R_1 R_3 R_2 R_3 R_2 R_1 R_2 R_3 R_2 R_1$, which was found geometrically (see `52-3-5.nb` for a detailed account). The centroid of the base cell (and hence the base vertex of the dual $\{5, 3, \frac{5}{2}\}$) has coordinates

$$w_3 = \left(\frac{\phi}{4}, \frac{1}{4}, 0, -\frac{1}{4\phi}\right).$$

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