Philosophy of Statistics

Daniel Kenneally

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Science to Statistics

"Statistics is applied philosophy of Science" - *Kempthorne (1976)*

The Scientific Method

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- A **hypothesis** is a supposition or proposed explanation of a phenomenon.
- They observe the world and then they deduce from their observation whether the hypothesis was falsified (proved to be false).

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Example:

- Newton supposes an explanation of how the planets move.
- The planets arrive at their predicted destination.
- He cannot reject his hypothesis.

Definitions

H_0 : The null hypothesis

H_a : The alternative hypothesis

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$\beta = \text{P(false acceptance (Type-II Error))}$

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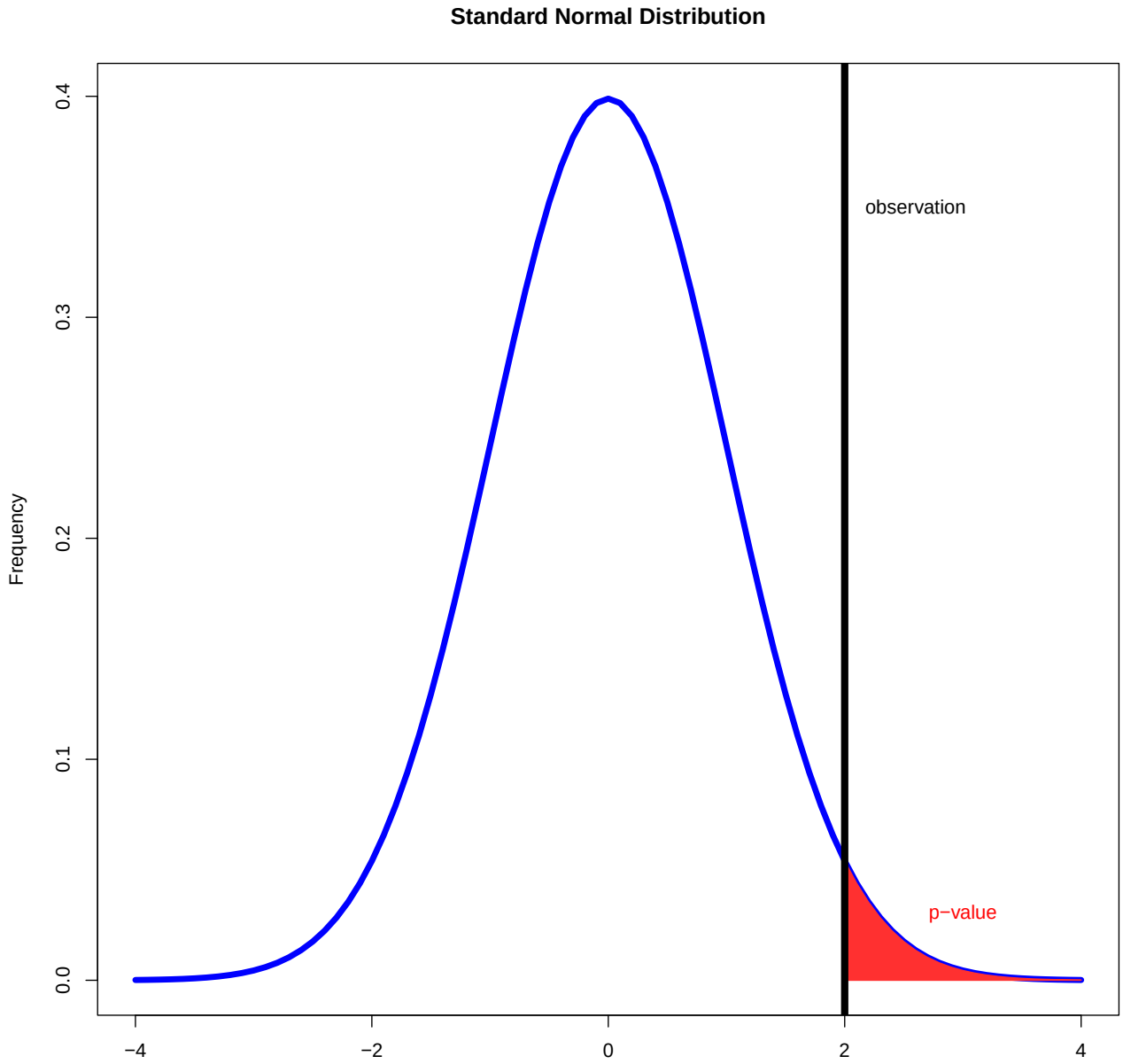
Definition

The **significance level** for the test is used to assess the **p-value** which is defined as follows:

$$\text{p-value} = P(\text{Data} \mid H_0)$$

i.e. the probability that the test statistic be at least as large as the value obtained from the data, given that the null hypothesis H_0 is true.

p-value


Standard Normal Distribution
Frequency
0.0
0.1
0.2
0.3
0.4
−4
−2
0
2
4
observation
p−value

Main Philosophies of Hypothesis Testing

Fisher's Null Hypothesis Testing

1. Set up a statistical null hypothesis.
2. Report the exact level of significance (e.g. p-value $= 0.051$ or p-value $= 0.049$).
3. Do not use a conventional 5% level, and do not talk about accepting or rejecting hypotheses.
4. *Fisher (1955)*[1] "We have the duty of [...] communicating our conclusions, in intelligible form, in recognition of the right of *other* free minds to utilize them in making their own decisions".

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Limits

- Analysis relies on subjective interpretation.

Neyman and Pearson Decision Theory

1. Set up two statistical hypotheses, H_0 and H_a.
2. Decide about α, β, and sample size before the experiment, based on a cost-benefit analysis. These define a rejection region for each hypothesis.
3. If the data falls into the rejection region of H_0, accept H_a; otherwise accept H_0.

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Limits

- Subjective choice of α and β.
- Limited to situations where there is a meaningful cost-benefit trade-off for choosing alpha and beta.
- Limited to situations where you have a disjunction of hypotheses (e.g., either $\mu_1 = 8$ or $\mu_2 = 10$ is true)

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Disagreements:

- For Fisher, Neyman's method is unscientific and only useful in limited cases within industry.
- For Neyman, Fisher's method requires a significance threshold to be useful and does not account for consequences of mistakes.

Current Practices

Current Practices

Gigerenzer (2004)[2]:

1. Set up a statistical null hypothesis.
2. Use 5% as a convention for rejecting the null. If significant, accept your research hypothesis.
3. Report the result as $p < 0.05$, $p < 0.01$, or $p < 0.001$ (whichever comes next to the obtained p-value).
4. Always perform this procedure.

Misusing Statistics

Misusing Statistics

- Statistical metrics are stochastic/random variables.
- Statistical metrics are computations on a sample of observations.
- *Neyman (1957)*[3]: "these observations are random variables, the results of the computations are also random."
- Redraw samples, calculate the metric and take the upper/lower bound.
- The distribution of the maximum/minimum is different to the underlying distribution.

Selection of Best p-value

Selection of Best p-value

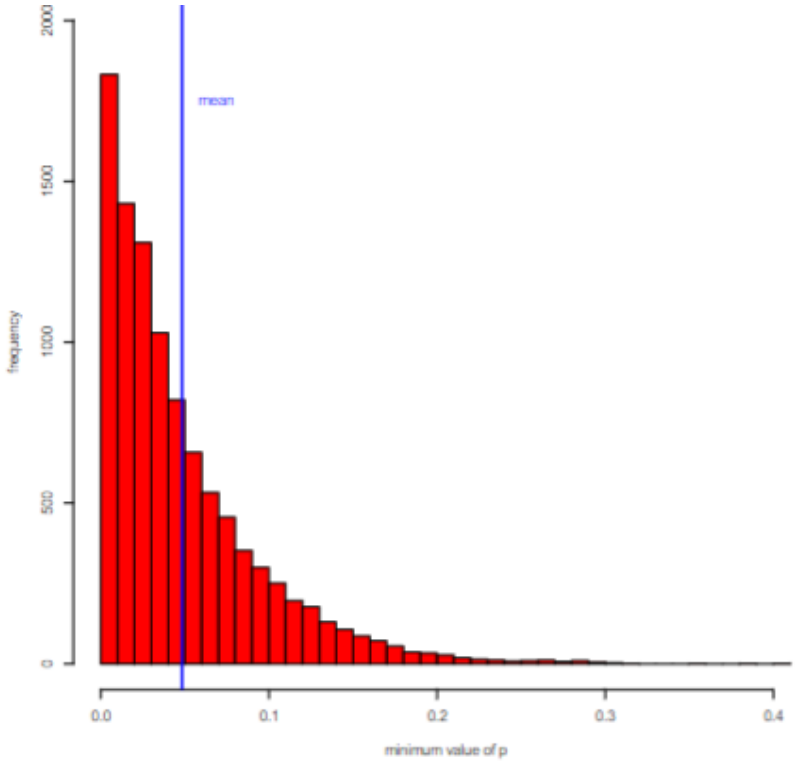
Taleb (2018)[4] illustrates;
Monte Carlo (M = 1000):

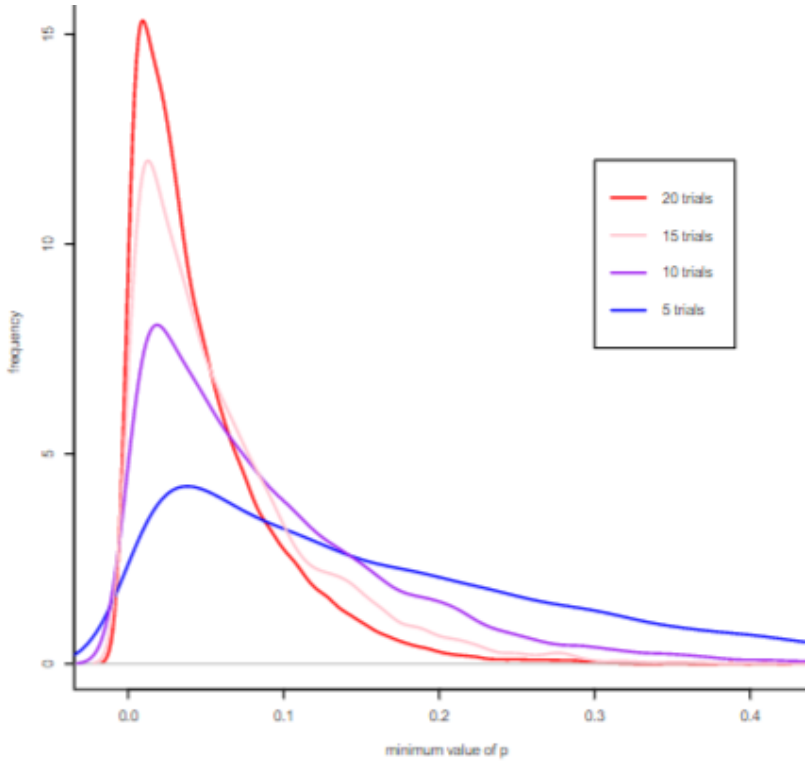
1. A trial consists of selecting two sets (X, Y), of 100 samples from a *N(0,1)* distribution.
2. Apply linear regression:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i, \qquad \epsilon_i \overset{iid}{\sim} N(0,1)$$

3. H_0: $\beta_1 = 0$
4. Complete t trials.
5. Select the trial with the minimum p-value for publication.

Distribution of minimum p-value





Trials	5	10	15	20
Mean	0.1676	0.0909	0.0627	0.0483
Proportion	0.2316	0.4039	0.5365	0.6423

Table 4.1: Mean and proportion of statistically significant p-values

Repeated Sampling

Repeated Sampling

Monte Carlo (M = 1000):

1. Draw two sets (X, Y), of 20 samples from a standard normal distribution, *N(0,1)*.
2. Apply linear regression:

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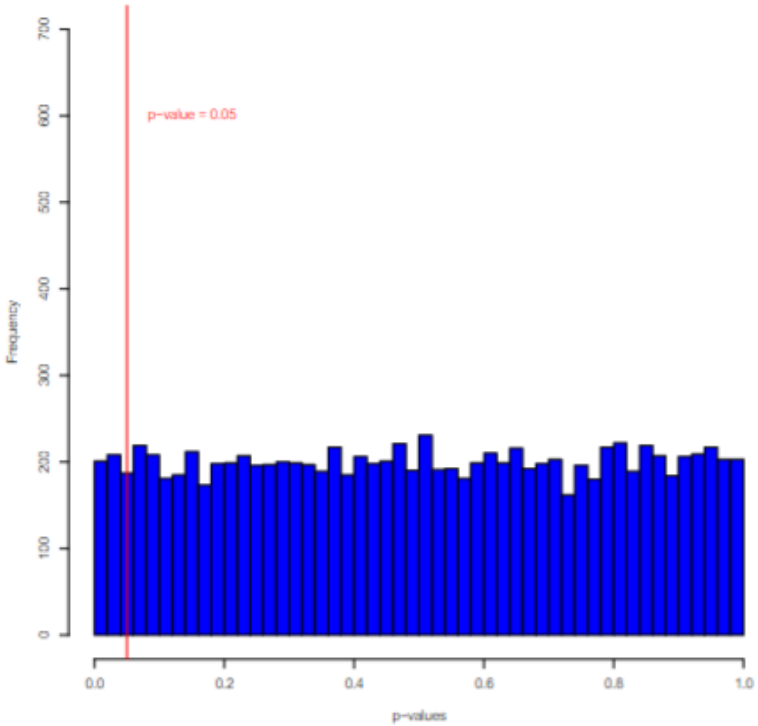
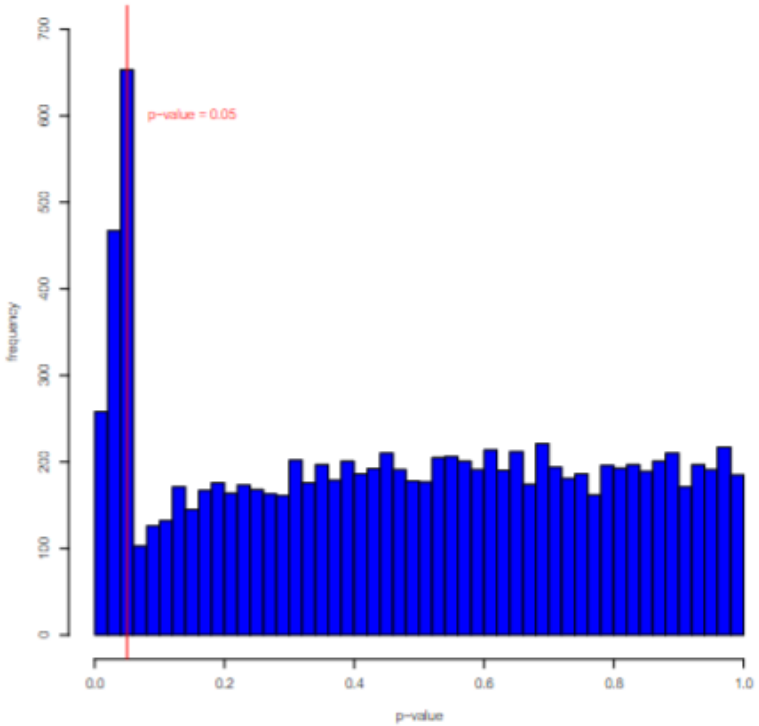
3. H_0: $\beta_1 = 0$
4. If the p-value is statistically significant (<0.05), end the experiment.
5. If the p-value is not statistically significant draw one more sample for X and Y and retest the hypothesis.
6. Repeat step 5 until you achieve a statistically significant result or until your sample size is 40.

Monte Carlo Distribution



(a) Distribution of p-value for fixed sampling size



(b) Distribution of p-value for continued sampling

Experiment:	Mean sample size	p-value < 0.05
Normal	20.0	0.0494
Optimised	38.1	0.1355

Table 4.2: Results for fixed and varying sample sizes

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- A form of survivorship bias occurs.
- Meta-analysis is carried out to account for the largest amount of data.
- But the meta-analysis will only take into account the positive results and so it will be extremely distorted.

Multiple Testing

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Definition

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Example:

- Generate 20 vectors, $X_j's$, each of sample size 100.

$$X_{j,i} \overset{iid}{\sim} N(0,1) \quad \text{with} \quad j = 1, ..., 20$$

- Apply linear regression to each combination of the $X_j's$:

$$X_{j_1} = \beta_0 + \beta_1 X_{j_2} + \epsilon_i, \qquad \epsilon_i \overset{iid}{\sim} N(0,1)$$

- Where $j_1 \neq j_2$
- H_0: $\beta_1 = 0$

Multiple Testing

$$\binom{20}{2} = 190$$

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Method	Significance Level	False Positives
Original	0.05000	13
Bonferroni	0.00026	1
Harmonic mean	0.00530	1
Order of Magnitude	0.00500	1

Table: Adjusted Significance Level

Multiple testing

- These methods only work if we know that we are testing multiple hypotheses.
- Before we test the hypotheses, we look at our data to see if there are any linear relationships between our variables.

Multiple Testing

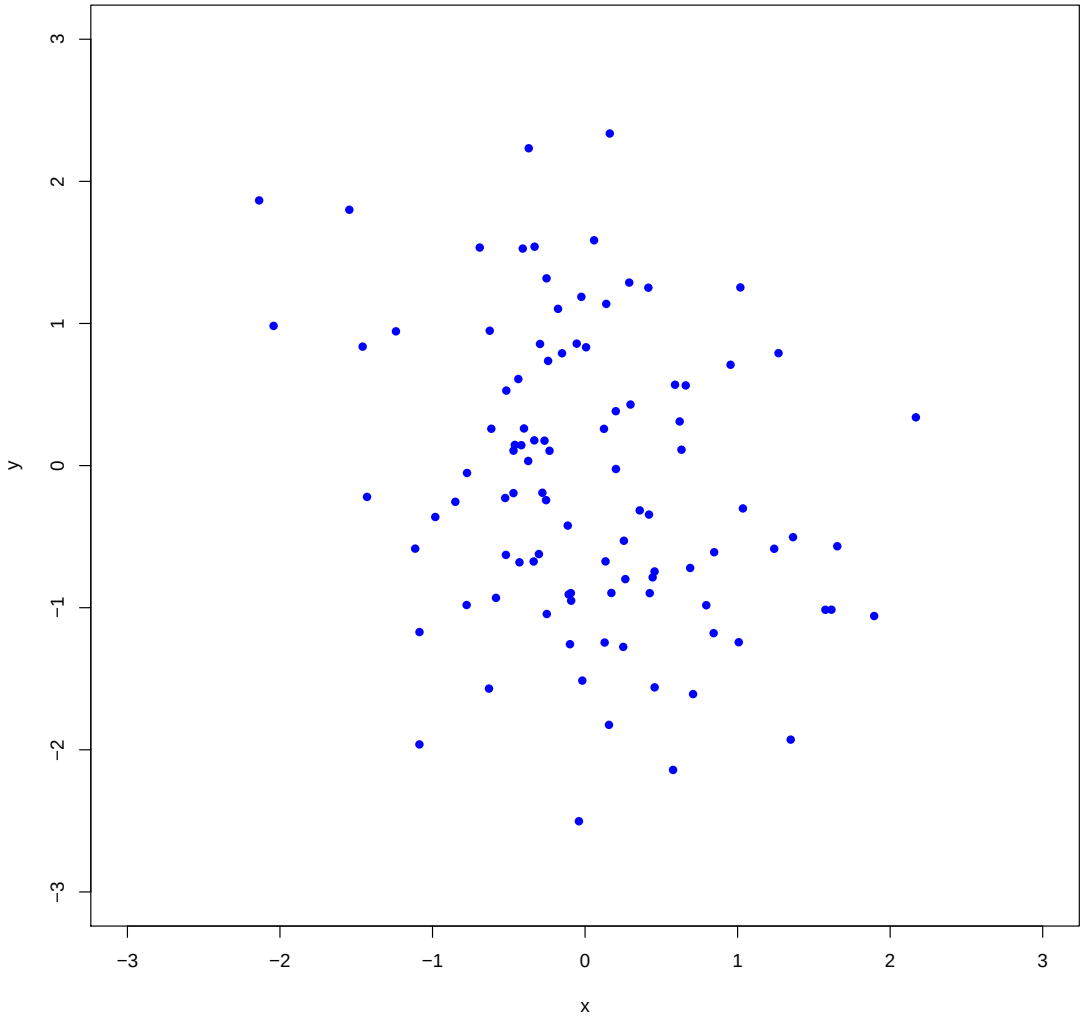

Figure: X_2 and X_{10} plotted against each other

Multiple Testing

- We carry out one hypothesis test.
- Apply linear regression:

$$X_{2,i} = \beta_0 + \beta_1 X_{10,i} + \epsilon_i, \qquad \epsilon_i \overset{iid}{\sim} N(0,1)$$

- H_0: $\beta_1 = 0$

Multiple Testing

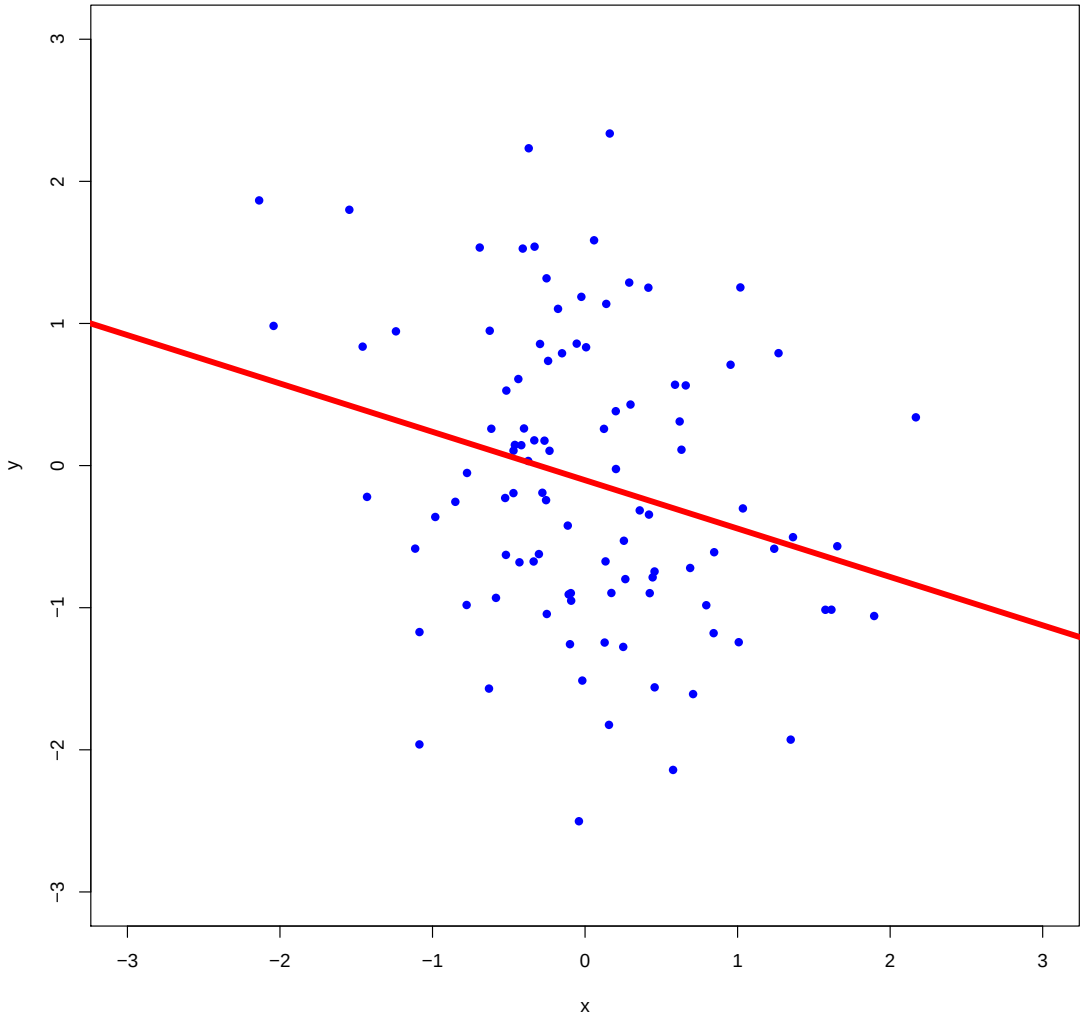

Figure: Model plotted with the data.

Multiple Testing

- We get a statistically significant result (p-value < 0.05).
- By observing our data first, we explicitly carried out only one hypothesis.
- But we implicitly carried out 190 hypothesis testing without adjusting our level of significance.

Curse of Dimensionality

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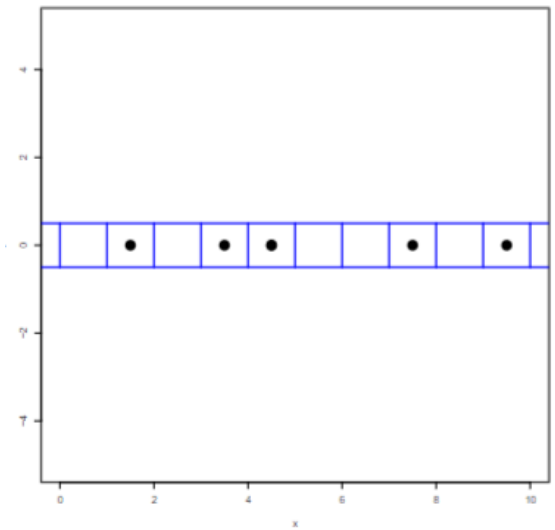
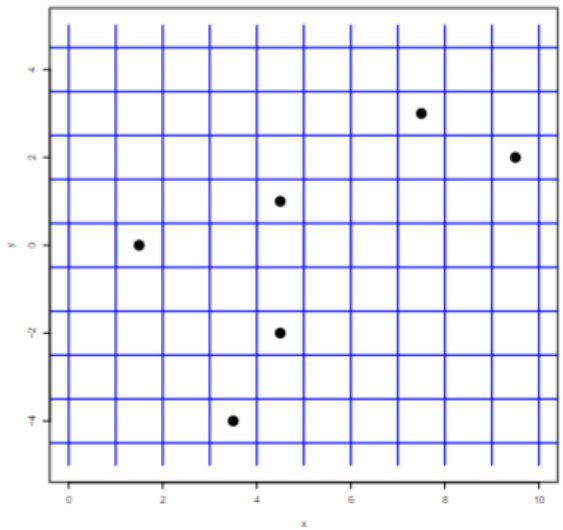
As the number of features or dimensions grows, the amount of data we need to generalise accuracy grows exponentially



(a) One dimensional with 5 data points



(b) Two dimensional with 5 data points

Feature Elimination

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Example:
Create data set $S = (X, Y)$

$$X_{1,i} \in \{1, 2, 3, 4\} \qquad X_{2,i} \overset{iid}{\sim} Pois(6)$$

$$X_{3,i} \overset{iid}{\sim} 5 - exp(2) \qquad X_{j,i} \overset{iid}{\sim} N(0, 1) \quad \text{with} \quad j = 4, ..., 23$$

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. With

$$Y_i = 2\exp(0.15X_{2,i}) + 3.2\log(X_{1,i}) + 4X_{3,i}$$

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$$Y_i = 2\exp(0.15X_{2,i}) + 3.2\log(X_{1,i}) + 4X_{3,i}$$

Create models using different feature elimination methods to deal with these 23 features. We can see the percentage of times these methods selected each feature in 10-fold cross validation.

Feature Elimination

Variable	LASSO	Exhaustive	Backward Selection	Random Forests
X_1	100	100	100	100
X_2	100	100	100	100
X_3	100	100	100	100
X_4	50	40	0	0
X_5	10	40	0	70
X_6	40	30	0	0
X_7	0	0	0	10
X_8	10	10	20	30
X_9	90	90	0	10
X_{10}	20	80	10	20
X_{11}	10	0	0	0
X_{12}	60	100	10	10
X_{13}	0	0	70	10
X_{14}	40	100	0	10
X_{15}	0	0	0	50
X_{16}	70	70	0	50
X_{17}	0	10	20	60
X_{18}	0	0	100	20
X_{19}	60	90	70	0
X_{20}	0	0	0	0
X_{21}	0	0	10	0
X_{22}	80	100	0	0
X_{23}	0	10	90	50

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- **Loss of trust in Science**

Conclusion

Thank you for listening


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