Using Lasso regularization in MAM

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I encourage reading [2] (and [1]) before diving into this technical note.

Instead of dealing with $\|.\|_0$ of the barycenter's probability, one can solve a $\|.\|_1$ regularized barycenter problem to increase the sparsity of the method. The problem writes

$$\min_{\pi} f(\pi) + g(\pi) \tag{1}$$

where

$$f(\pi) := \sum_{m=1}^{M} \sum_{r=1}^{R} \sum_{s=1}^{S^{(m)}} c_{rs}^{(m)} \pi_{rs}^{(m)} + \mathbf{i}_{\Pi^{(m)}}(\pi^{(m)}) \tag{2}$$

$$g(\pi) := \mathbf{i}_{\mathcal{B}}(\pi) + \rho \sum_{r=1}^{R} \left| \sum_{s=1}^{S^{(1)}} \pi_{rs}^{(1)} \right| \tag{3}$$

Now, Prox_g consists in solving

$$\min_{\pi^{(1)},\dots,\pi^{(M)}} \frac{1}{2} \sum_{r=1}^{R} \left[\rho \left| \sum_{s=1}^{S^{(1)}} \pi_{rs}^{(1)} \right| + \sum_{m=1}^{M} \sum_{s=1}^{S^{(m)}} (\pi_{rs}^{(m)} - \theta_{rs}^{(m)})^2 \right] \tag{4}$$

under the constraints

$$\sum_{s=1}^{S^{(m)}} \pi_{rs}^{(m)} = \sum_{s=1}^{S^{(m+1)}} \pi_{rs}^{(m+1)}, \quad r = 1, \dots, R \ ; \quad m = 1, \dots, M-1 \tag{5}$$

The problem is still r-separable, thus for all r, one gets the Lagrangian

$$L_r[\theta](y,\lambda) := \rho \left| \sum_{s=1}^{S^{(1)}} y_s^{(1)} \right| + \sum_{m=1}^{M} \left[\sum_{s=1}^{S^{(m)}} \frac{1}{2} \left(y_s^{(m)} - \theta_{rs}^{(m)} \right)^2 \right] + \sum_{m=1}^{M-1} \left[\lambda^{(m)} \left(\sum_{s=1}^{S^{(m)}} y_s^{(m)} - \sum_{s=1}^{S^{(m+1)}} y_s^{(m+1)} \right) \right] \tag{6}$$

First, assume that $\left| \sum_{s=1}^{S^{(1)}} y_s^{(1)} \right| > 0$, then eq. (6) writes

$$L_r[\theta](y,\lambda) := \rho \sum_{s=1}^{S^{(1)}} y_s^{(1)} + \sum_{m=1}^{M} \left[\sum_{s=1}^{S^{(m)}} \frac{1}{2} \left(y_s^{(m)} - \theta_{rs}^{(m)} \right)^2 \right] + \sum_{m=1}^{M-1} \left[\lambda^{(m)} \left(\sum_{s=1}^{S^{(m)}} y_s^{(m)} - \sum_{s=1}^{S^{(m+1)}} y_s^{(m+1)} \right) \right] \tag{7}$$

and the stationary condition of the dual problem writes

$$\left\{ \begin{array}{llll} \pi_{rs}^{(1)} - \theta_{rs}^{(1)} + \rho & + \ \lambda^{(1)} & = 0 & s = 1, \dots, S^{(1)} \\ \pi_{rs}^{(2)} - \theta_{rs}^{(2)} & + \ \lambda^{(2)} - \lambda^{(1)} & = 0 & s = 1, \dots, S^{(2)} \\ & \ \vdots & & \\ \pi_{rs}^{(M-1)} - \theta_{rs}^{(M-1)} & + \ \lambda^{(M-1)} - \lambda^{(M-2)} & = 0 & s = 1, \dots, S^{(M-1)} \\ \pi_{rs}^{(M)} - \theta_{rs}^{(M)} & - \ \lambda^{(M-1)} & = 0 & s = 1, \dots, S^{(M)}. \end{array} \right. \tag{8}$$

define

$$\tilde{p}_r^{(m)} := \begin{cases} \sum_{s=1}^{S^{(m)}} \left(\theta_{rs}^{(m)} - \rho\right) & \text{if } m = 1 \\ \sum_{s=1}^{S^{(m)}} \theta_{rs}^{(m)} & \text{otherwise} \end{cases} \tag{9}$$

we have

$$p_r := \left(\sum_{m=1}^{M} \frac{1}{S^{(m)}}\right)^{-1} \sum_{m=1}^{M} \left(\frac{\tilde{p}_r^{(m)}}{S^{(m)}}\right) = \left(\sum_{m=1}^{M} \frac{1}{S^{(m)}}\right)^{-1} \left[\left(\sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}}\right) - \rho\right] \tag{10}$$

Thus $p_r > 0 \Leftrightarrow \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} > \rho$.
Suppose now $p_r < 0$ then eq. (6) writes

$$L_r[\theta](y, \lambda) := -\rho \sum_{s=1}^{S^{(1)}} y_s^{(1)} + \sum_{m=1}^{M} \left[\sum_{s=1}^{S^{(m)}} \frac{1}{2} \left(y_s^{(m)} - \theta_{rs}^{(m)}\right)^2\right] + \sum_{m=1}^{M-1} \left[\lambda^{(m)} \left(\sum_{s=1}^{S^{(m)}} y_s^{(m)} - \sum_{s=1}^{S^{(m+1)}} y_s^{(m+1)}\right)\right] \tag{11}$$

and the stationary condition of the dual problem writes

$$\begin{cases} \pi_{rs}^{(1)} - \theta_{rs}^{(1)} - \rho & + & \lambda^{(1)} & = 0 & s = 1, \ldots, S^{(1)} \\ \pi_{rs}^{(2)} - \theta_{rs}^{(2)} & + & \lambda^{(2)} - \lambda^{(1)} & = 0 & s = 1, \ldots, S^{(2)} \\ & \vdots & & & \\ \pi_{rs}^{(M-1)} - \theta_{rs}^{(M-1)} & + & \lambda^{(M-1)} - \lambda^{(M-2)} & = 0 & s = 1, \ldots, S^{(M-1)} \\ \pi_{rs}^{(M)} - \theta_{rs}^{(M)} & - & \lambda^{(M-1)} & = 0 & s = 1, \ldots, S^{(M)}. \end{cases} \tag{12}$$

Then, using the same reasoning than before, we have

$$p_r := \left(\sum_{m=1}^{M} \frac{1}{S^{(m)}}\right)^{-1} \left[\left(\sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}}\right) + \rho\right] \tag{13}$$

Thus $p_r < 0 \Leftrightarrow \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} < -\rho$.
Finally if $\sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \in [-\rho; \rho]$, the stationarity conditions writes

$$\begin{cases} \pi_{rs}^{(1)} - \theta_{rs}^{(1)} + [-\rho; \rho] & + & \lambda^{(1)} & \ni 0 & s = 1, \ldots, S^{(1)} \\ \pi_{rs}^{(2)} - \theta_{rs}^{(2)} & + & \lambda^{(2)} - \lambda^{(1)} & = 0 & s = 1, \ldots, S^{(2)} \\ & \vdots & & & \\ \pi_{rs}^{(M-1)} - \theta_{rs}^{(M-1)} & + & \lambda^{(M-1)} - \lambda^{(M-2)} & = 0 & s = 1, \ldots, S^{(M-1)} \\ \pi_{rs}^{(M)} - \theta_{rs}^{(M)} & - & \lambda^{(M-1)} & = 0 & s = 1, \ldots, S^{(M)}. \\ & & \sum_s^{S(1)} \pi_{rs}^{(1)} & = 0 & \end{cases} \tag{14}$$

Using the fact that $\sum_s \pi_{rs}^{(m)} = p_r = 0, \forall m$ and summing over s the last before last line of the previous equation yields

$$\lambda^{(M-1)} = -\frac{p_r^{(M)}}{S^{(M)}} \tag{15}$$

and then

$$\lambda^{(m-1)} = \lambda^{(m)} - \frac{p_r^{(m)}}{S^{(m)}} = -\sum_{m'=m}^{M} \frac{p_r^{(m')}}{S^{(m')}} \tag{16}$$

thus

$$\lambda^{(1)} = -\sum_{m=2}^{M} \frac{p_r^{(m)}}{S^{(m)}} \tag{17}$$

Now let us check that the first line of eq. (14) holds. To do so let us sum over s

$$-p_r^{(1)} - \rho S^{(1)} - S^{(1)} \sum_{m=2}^{M} \frac{p_r^{(m)}}{S^{(m)}} \leq 0 \leq -p_r^{(1)} + \rho S^{(1)} - S^{(1)} \sum_{m=2}^{M} \frac{p_r^{(m)}}{S^{(m)}} \tag{18}$$

$$-\rho S^{(1)} - S^{(1)} \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \leq 0 \leq \rho S^{(1)} - S^{(1)} \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \tag{19}$$

$$-\rho - \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \leq 0 \leq \rho - \sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \tag{20}$$

which is equivalent to $\sum_{m=1}^{M} \frac{p_r^{(m)}}{S^{(m)}} \in [-\rho; \rho]$ and proves the result

References

[1] Daniel Mimouni, Welington de Oliveira, and Gregorio M Sempere. On the computation of constrained wasserstein barycenters.

[2] Daniel Mimouni, Paul Malisani, Jiamin Zhu, and Welington de Oliveira. Computing wasserstein barycenter via operator splitting: the method of averaged marginals. *arXiv preprint arXiv:2309.05315*, 2023.