CONVEXITY ADJUSTMENT FOR LIBOR PAYMENTS WITH DELAY

Abstract. This note gives some formulas for pricing Libor payments with arbitrary delay. Exact results are given for the convexity adjustments assuming log-normally distributed rates in their forward measure.

Key words. convexity adjustments

1. Introduction. Consider an instrument paying the Libor rate L_{ab} at time c. The price of this instrument V_0 is given by an expectation in the c-forward measure

$$V_0 = P_{0,c}\mathbb{E}_c[L_{ab}] \tag{1.1}$$

where we denote $P_{0,c}$ the zero coupon bond price with maturity c at time $t = 0$. Up to the discount factor $P_{0,c}$, this is the expectation of the Libor rate L_{ab} in the c-forward measure.

The particular case $c = b$ is called "natural payment date". The price V_0 for this case is known model independently from no-arbitrage pricing and is given by

$$V_0|_{c=b} = P_{0,b}L_{ab}^{\rm fwd} \tag{1.2}$$

with $L_{ab}^{\rm fwd} = (b-a)^{-1}(P_{0,a}/P_{0,b} - 1)$ the forward Libor for the period (a, b). This does not depend on any rates volatilities.

Any other value of c different from b is an "un-natural payment date" and the price V_0 will depend on rate volatilities and correlations. In the limit of zero volatility (assuming deterministic rates) the price is simply known in terms of the forward rates

$$V_0|_{\rm zero-vol} = P_{0,c}L_{ab}^{\rm fwd} \tag{1.3}$$

For non-zero rates volatilities the price will start to differ from this value. The difference with the deterministic price is called a "convexity adjustment".

2. Convexity adjustment. We will assume $c > b$. The alternative possibility $a \leq c \leq b$ is also possible (in particular $c = a$ corresponds to the so-called Libor payment in arrears), but will be treated separately. We would like to compute exactly the price of the Libor payment with delay (1.1) under the assumption that all rates appearing are log-normally distributed in their respective forward measure. The forward measure for L_{ab} is the b-forward measure, and for L_{bc} it is the c-forward measure.

This is the problem considered in Sec. 13.8.6 of Brigo and Mercurio [1], who treat it using the so-called frozen drift approximation. In this section we give an exact result, and obtain an explicit evaluation using a moment matching approximation.

Under the log-normal assumption, the process for the forward rate $L_t(bc)$ in c-forward measure can be written as a drift-less diffusion

$$\frac{dL_t(bc)}{L_t(bc)} = \sigma_2 dW(t)\,. \tag{2.1}$$

The initial condition is $L_0(bc) = L_{bc}^{\rm fwd}$. This equation is solved immediately as

$$L_t(bc) = L_{bc}^{\rm fwd}\exp\Big(\sigma_2 W(t) - \frac{1}{2}\sigma_2^2 t\Big)\,. \tag{2.2}$$

The process for $L_t(ab)$ in the same measure (c-forward measure) has a drift and can be written as

$$\frac{dL_t(ab)}{L_t(ab)} = \sigma_1 dW(t) - \sigma_1\sigma_2\rho\frac{L_t(bc)\tau_{bc}}{1+L_t(bc)\tau_{bc}}dt\,, \tag{2.3}$$

with initial condition $L_0(ab) = L_{ab}^{\rm fwd}$.

We are interested in the solution of (2.3) at time $t = a$, which gives the rate L_{ab} setting at time a. This can be written explicitly as

$$L_{ab} = L_{ab}^{\rm fwd}\exp\Big(\sigma_1 W(a) - \frac{1}{2}\sigma_1^2 a - \rho\sigma_1\sigma_2\int_0^a \frac{ds}{1+(L_{bc}^{\rm fwd}\tau_{bc})^{-1}e^{-\sigma_2 W(s)+\frac{1}{2}\sigma_2^2 s}}\Big) \tag{2.4}$$

We are interested in the expectation of this stochastic variable.

Brigo and Mercurio approximate the rate $L_t(bc)$ in the drift term with the forward rate $L_{bc}^{\rm fwd}$ (the so-called frozen drift approximation). Then the equation (2.3) is easily solved. The average of L_{ab} in the c-forward measure is given in this approximation by

$$\mathbb{E}_c[L_{ab}]_{\rm f.d.} = L_{ab}^{\rm fwd}\exp\Big(-\sigma_1\sigma_2\rho a\frac{L_{bc}^{\rm fwd}\tau_{bc}}{1+L_{bc}^{\rm fwd}\tau_{bc}}\Big) \tag{2.5}$$

Can we do better than the frozen drift approximation? In the next section we compute the expectation of the stochastic variable in (2.4) using a moment matching approximation, and compare it with the frozen drift approximation.

3. Moment matching approximation. The validity of the frozen drift approximation is restricted to very small volatilities, much smaller than those encountered in the caplet markets. We give next a more precise approximation based on moment matching.

The main assumption is that L_{bc} is log-normally distributed in the b-forward measure, with an average value $\tilde{L}_{bc}$ which is determined by moment matching with the exact distribution. We write this as

$$\mathbb{P}_b : L_{ab} = L_{ab}^{\rm fwd}e^{\sigma_1\sqrt{a}X-\frac{1}{2}\sigma_1^2 a} \tag{3.1}$$

$$\mathbb{P}_b : L_{bc} = \tilde{L}_{bc}e^{\sigma_2\sqrt{b}Y-\frac{1}{2}\sigma_2^2 b} \tag{3.2}$$

Here X, Y are normally distributed random variables with mean zero and variance 1, and correlated with correlation ρ.

The calculation of the expectation appearing in Eq. (1.1) proceeds in two steps.

Step 1. Determine $\tilde{L}_{bc}$. This is done exactly as follows

$$\tilde{L}_{bc} = \mathbb{E}_b[L_{bc}] = \frac{P_{0c}}{P_{0b}}\mathbb{E}_c[L_{bc}(1+L_{bc}\tau_{bc})] = L_{bc}^{\rm fwd}\frac{1+L_{bc}^{\rm fwd}\tau_{bc}e^{\sigma_2^2 b}}{1+L_{bc}^{\rm fwd}\tau_{bc}}\,. \tag{3.3}$$

Step 2. Compute the required expectation.

$$\mathbb{E}_c[L_{ab}] = \frac{P_{0b}}{P_{0c}}\mathbb{E}_b\Big[L_{ab}\frac{1}{1+L_{bc}\tau_{bc}}\Big] \tag{3.4}$$

$$= \frac{P_{0b}}{P_{0c}}\int dXdY n(X,Y;\rho)\frac{L_{ab}^{\rm fwd}e^{\sigma_1\sqrt{a}X-\frac{1}{2}\sigma_1^a a}}{1+\tilde{L}_{bc}\tau_{bc}e^{\sigma_2\sqrt{b}Y-\frac{1}{2}\sigma_2^2 b}} \tag{3.5}$$

$$= \frac{P_{0b}}{P_{0c}}\int_{-\infty}^{\infty}\frac{dY}{\sqrt{2\pi}}e^{-\frac{1}{2}Y^2}\frac{L_{ab}^{\rm fwd}}{1+\tilde{L}_{bc}\tau_{bc}e^{\rho\sigma_1\sigma_2\sqrt{ab}}e^{\sigma_2\sqrt{b}Y-\frac{1}{2}\sigma_2^2 b}} \tag{3.6}$$

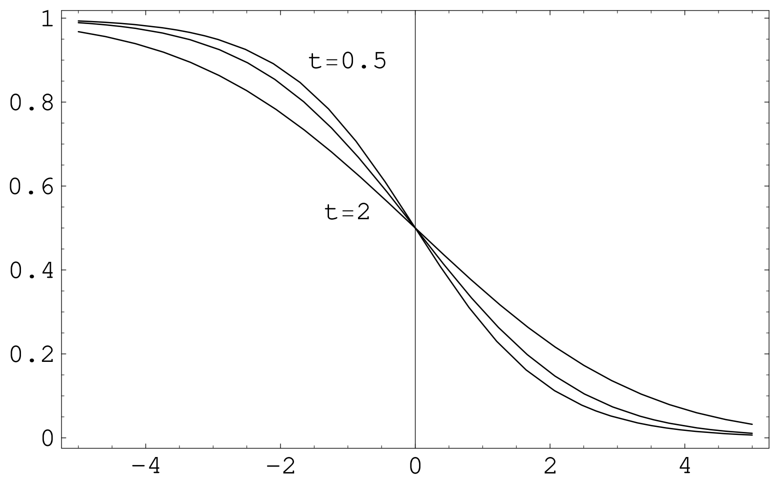

FIG. 3.1. *Plot of the logistic-normal integral $\varphi(z;t)$ with $t = 0.5, 1, 2$.*

This integral appearing here is related to the so-called logistic-normal integral, defined as

$$\varphi(z;t) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-z)^2} \frac{1}{1+e^x} \,. \tag{3.7}$$

This integral cannot be evaluated in closed form, but has many exact properties which can help with its numerical evaluation, see [2]. Typical plots of this function for $t = 0.5, 1, 2$ are shown in Fig. 3.1.

The expectation to be computed is expressed in terms of this integral as

$$\mathbb{E}_c[L_{ab}] = \frac{P_{0b}}{P_{0c}} L_{ab}^{\rm fwd} \varphi(\log(\tilde{L}_{bc}\tau_{bc}) + \rho\sigma_1\sigma_2\sqrt{ab} - \frac{1}{2}\sigma_2^2 b, \sigma_2\sqrt{b}) \,. \tag{3.8}$$

For very small negative argument $z \to -\infty$ (far in the left wing) the logistic-normal integral can be approximated as

$$\varphi(z,t) \simeq 1 - e^{z+\frac{1}{2}t} \tag{3.9}$$

Substituting this into (3.8) one finds the following approximation for the convexity adjustment of Libor payments with delay

$$\begin{aligned} \mathbb{E}_c[L_{ab}] &= (1 + L_{bc}^{\rm fwd}\tau_{bc}) L_{ab}^{\rm fwd} \Big(1 - (\tilde{L}_{bc}\tau_{bc}) e^{\rho\sigma_1\sigma_2\sqrt{ab}}\Big) \\ &= L_{ab}^{\rm fwd} \Big(1 + L_{bc}^{\rm fwd}\tau_{bc} - L_{bc}^{\rm fwd}\tau_{bc}(1 + L_{bc}^{\rm fwd}\tau_{bc} e^{\sigma_2^2 b}) e^{\rho\sigma_1\sigma_2\sqrt{ab}}\Big) \\ &= L_{ab}^{\rm fwd} \Big(1 - L_{bc}^{\rm fwd}\tau_{bc}(e^{\rho\sigma_1\sigma_2\sqrt{ab}} - 1) - (L_{bc}^{\rm fwd}\tau_{bc})^2 e^{\rho\sigma_1\sigma_2\sqrt{ab}+\sigma_2^2 b}\Big) \end{aligned} \tag{3.10}$$

We used here $\frac{P_{0b}}{P_{0c}} = 1 + L_{bc}^{\rm fwd}\tau_{bc}$. In practice it is sufficient to keep only the term linear in $L_{bc}^{\rm fwd}\tau_{bc}$.

From this result we note the following qualitative properties of the convexity adjustment:

1. The convexity adjustment vanishes when $b = c$ (natural payment time), as expected, because of the linear factors of τ_{bc}.

2. The convexity adjusted rate $\mathbb{E}_c[L_{ab}]$ is less than the forward rate $L_{ab}^{\rm fwd}$ if the correlation is positive $\rho > 0$. This is always the case in practice. In the unlikely case of a negative correlation $\rho < 0$, the convexity adjustment will be larger than 1.

We can compare the result (3.10) with the frozen drift approximation of Brigo and Mercurio (2.5). Expanding the exponential factor appearing in their result to first order, gives an expression similar to the first term in (3.10) (after expanding also the exponential to linear order), but with the replacement $\sqrt{ab} \to a$.

4. Yet another approximation. Another possible approximation is to assume log-normality in the c-forward measure for both rates L_{ab} and L_{bc}.

$$\mathbb{P}_c : L_{ab} = \tilde{L}_{ab} e^{\sigma_1 \sqrt{a} X - \frac12 \sigma_1^2 a} \tag{4.1}$$

$$\mathbb{P}_c : L_{bc} = L_{bc}^{\rm fwd} e^{\sigma_2 \sqrt{b} Y - \frac12 \sigma_2^2 b}\,. \tag{4.2}$$

As before, X, Y are normally distributed random variables with mean zero and variance 1, and correlated with correlation ρ.

Here $\tilde{L}_{ab}$ is just the expectation we would like to find, as we have $\mathbb{E}_c[L_{ab}] = \tilde{L}_{ab}$. This can be determined from the condition that the expectation of L_{ac} in c-forward measure is just the forward rate $L_{ac}^{\rm fwd}$. This rate is given by

$$1 + L_{ac}\tau_{ac} = (1 + L_{ab}\tau_{ab})(1 + L_{bc}\tau_{bc}) \tag{4.3}$$

Taking expectations of both sides in the c-forward measure we get

$$1 + L_{ac}^{\rm fwd}\tau_{ac} = 1 + \tilde{L}_{ab}\tau_{ab} + L_{bc}^{\rm fwd}\tau_{bc} + (\tilde{L}_{ab}\tau_{ab})(L_{bc}^{\rm fwd}\tau_{bc}) e^{\sigma_1\sigma_2\rho\sqrt{ab}} \tag{4.4}$$

This is solved for $\tilde{L}_{ab}$ with the result

$$\tilde{L}_{ab}\tau_{ab} = \frac{L_{ac}^{\rm fwd}\tau_{ac} - L_{bc}^{\rm fwd}\tau_{bc}}{1 + L_{bc}^{\rm fwd}\tau_{bc} e^{\rho\sigma_1\sigma_2\sqrt{ab}}} \tag{4.5}$$

The forward rates satisfy the multiplicative relation

$$1 + L_{ac}^{\rm fwd}\tau_{ac} = (1 + L_{ab}^{\rm fwd}\tau_{ab})(1 + L_{bc}^{\rm fwd}\tau_{bc}) \tag{4.6}$$

Using this relation, the result (4.5) can be simplified and we obtain the final result

$$\mathbb{E}_c[L_{ab}] = \tilde{L}_{ab} = L_{ab}^{\rm fwd} \frac{1 + L_{bc}^{\rm fwd}\tau_{bc}}{1 + L_{bc}^{\rm fwd}\tau_{bc} e^{\rho\sigma_1\sigma_2\sqrt{ab}}}\,. \tag{4.7}$$

This agrees with the approximation of the previous section (3.10) upon expanding in powers of $L_{bc}^{\rm fwd}\tau_{bc}$ and keeping only the linear order term. One possible advantage of this approximation is that there is no need for the evaluation of the logistic-normal integral.

REFERENCES

[1] D. Brigo and F. Mercurio, *Interest rates models - Theory and practice: With smile, inflation and credit*, 2nd edition, Springer Finance, Springer New York, 2006

[2] D. Pirjol, The logistic-normal integral and its generalizations, *Journal of Computational and Applied Mathematics* 237(1), 460-469, 2013