Poncelet Invariants:
an Experimental Promenade

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Chapter 1

Introduction

Chapter 2

Poncelet Preliminaries

2.1 Poncelet's Great Theorem

Consider a pair of nested ellipses $\mathcal{D} \subset \mathcal{C}$ as shown in Figure 2.1. Let $P_0 \in \mathcal{C}$ and draw a tangent line L_1 to $\mathcal{D}$ and passing through P_0. Call P_1 the second intersection of L_1 with $\mathcal{C}$. From P_1 draw the the second tangent line L_2 to $\mathcal{D}$. Clearly this process can be iterated to order n. The sequence of points $\{P_0, P_1, P_2, \ldots, P_n, \ldots\}$ will be called the *Poncelet orbit*.

When $P_n = P_0$ the Poncelet orbit is called periodic and the polygon $\mathcal{P}_n$ with vertices $\{P_0, \ldots, P_{n-1}, P_n\}$ will be called an $n-$gon. So, we obtain a polygon interscribed in the pair of ellipses $\{\mathcal{D}, \mathcal{C}\}$.

Theorem 1. *Consider a pair of nested ellipses $\{\mathcal{D}, \mathcal{C}\}$ as shown in Figure 2.1. If there is a $n-gon$ interscribed between the pair $\mathcal{D}$ and $\mathcal{C}$, then for every $Q_0 \in \mathcal{C}$ there is an $n-$gon interscribed between $\mathcal{D}$ and $\mathcal{C}$ having Q_0 as one of its vertices.*

The classical Poncelet theorem is about a natural generalization of billiards. Consider a pair of nested ellipses Γ (outer) and γ (inner) oriented by the external normal vector. Consider a point $p \in \gamma$ and ℓ_p the tangent line to γ at p. Let p_1 and p_2 the intersection of ℓ_p with Γ. The Poncelet map is induced by the correspondence $P_p : \Gamma \to \Gamma$, $P_p(p_1) = p_2$. Through p_2 consider the other tangent line to γ and let p_3 the intersection of this line with Γ. Therefore, fixed a orientation we have a well defined map $T : \Gamma \to \Gamma$ with $T(p_1) = p_2$, $T(p_2) = p_3$, etc. Changing the orientation is defined the inverse T^{-1}. See Figure 2.1.

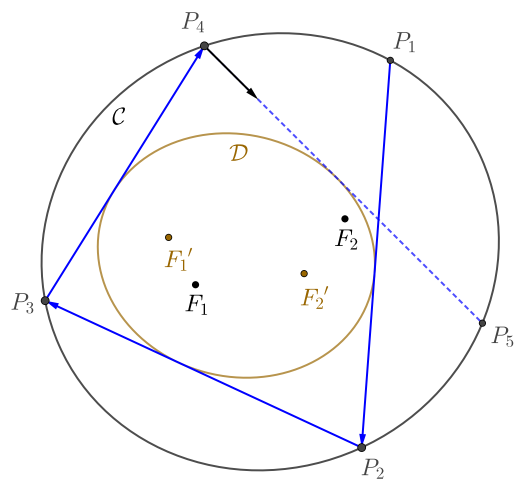

Figure 2.1: Poncelet map associated to the pair of ellipses $\{\mathcal{C}, \mathcal{D}\}$.

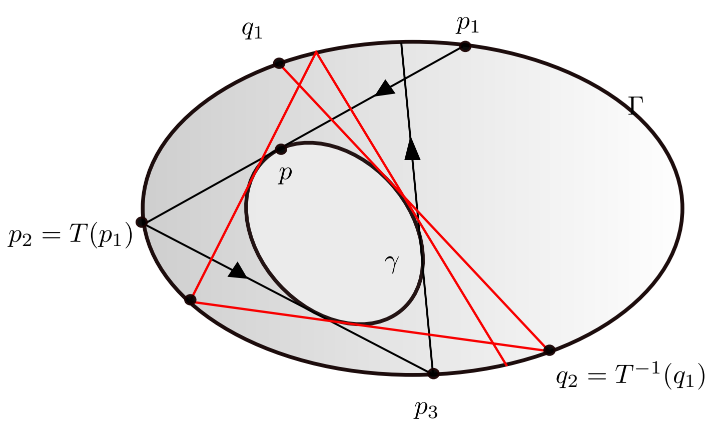

Figure 2.2: Poncelet map $T : \Gamma \to \Gamma$ with $T(p_1) = p_2$ and $T^{-1} : \Gamma \to \Gamma$ with $T^{-1}(q_1) = q_2$ associated to the pair of nested ellipses $\gamma \subset \Gamma$.

The orbit of a point $p_1 \in \Gamma$ is the polygon defined by the vertices $T^k(p_1) = p_k$. A orbit of a point p_1 is called N-periodic when $T^N(p_1) = p_1$. The N-gon is inscribed in Γ and circumscribed about γ. The Poncelet theorem is following.

Theorem 2 (Poncelet's Closure Theorem)**.** *Consider a pair of nested ellipses $\gamma \subset \Gamma$. If $T:\Gamma \to \Gamma$ has a periodic orbit for some p_1 then all orbits are periodic.*

For a historical and proofs of this theorem see **barth1996**, **centina2016a**, **centina2016b**, Chasles (1843), **drag2014** and references therein. See also **berger2010**, **cima2010**,**cieslak2016**, Darboux (1917, Livre III, Chapitres II, III), **drag_milena2011**, **gla2016**, **hahu2015**, **hahu2017**, **leb1921**, **mirman2012**, *Traité sur les propriétés projectives des figures* (1822) and **previ1999**.

The proof presented below is based in **shoe1983**. Consider an ellipse and a circle given by

$$\mathcal{E}_{a,b}:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \mathcal{C}_r:\ x^2+y^2=r^2,\ 0<r<b<a.$$

For the reduction of a general pair of nested ellipses to the pair above and a proof of Poncelet's Porism theorem see **bry**.

Let $P=(a\cos\varphi, b\sin\varphi)$ and $P'=(a\cos\varphi, a\sin\varphi)$ as shown in **??**.

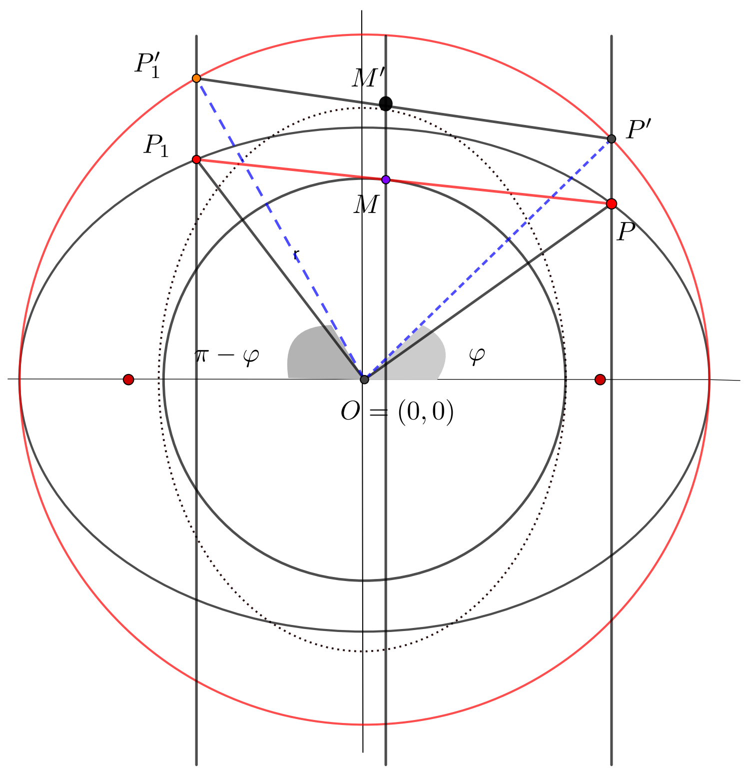

Figure 2.3: Poncelet maps $T:\mathcal{E}_{a,b}\to\mathcal{E}_{a,b}$ with $T(P)=P_1$ and $T:\mathcal{C}_a\to\mathcal{C}_a$ with $T'(P')=P_1'$ associated to the pairs $\{\mathcal{E}_{a,b},\mathcal{C}_r\}$ and $\{\mathcal{C}_a,\mathcal{E}_{r,ra/b}\}$.

Consider the tangent line PP_1 to $\mathcal{C}_r$ at the point M, and let

$$\angle AOP' = \varphi, \ \ \angle AOP_1' = \varphi_1.$$

Proposition 1. *In the above conditions it follows that:*

$$\left(\frac{d\varphi_1}{d\varphi}\right)^2 = \frac{1-k^2\sin^2\varphi_1}{1-k^2\sin^2\varphi} \tag{2.1}$$

Proof. Let $A(x,y) = (x, \frac{b}{a}y)$ be the affine transformation sending the circle $\mathcal{C}_a$ of radius a to the ellipse $\mathcal{E}_{a,b}$. Define the points $M' = T^{-1}M$ and $P_1' = T^{-1}(P_1)$. The envelope of the family of lines $P'P_1'$ is an ellipse $\mathcal{E}_{r,ar/b}$ of semiaxes r and $\frac{ar}{b} > r$ wich is tangent to M_1'. See **??**.

The line $P'M'P_1'$ intersects the circle $\mathcal{C}_a$ in equal angles, and it follows that

$$\frac{d\varphi_1}{d\varphi} = \frac{|M'P_1'|}{|M'P'|} = \frac{|MP_1|}{|MP|}.$$

Now observe that

$$\begin{aligned}|MP|^2 =& |OP|^2 - |OM|^2 = a^2\cos^2\varphi + b^2\sin^2\varphi - r^2\\ &= a^2 - r^2 - (a^2-b^2)\sin^2\varphi\\ &= (a^2-r^2)\left(1 - \frac{a^2-b^2}{a^2-r^2}\sin^2\varphi\right)\\ &= (a^2-r^2)(1-k^2\sin^2\varphi), \ \ k^2 = \frac{a^2-b^2}{a^2-r^2} < 1.\\ |MP_1|^2 =& |OP_1|^2 - |OM_1|^2 = (a^2-r^2)\left(1-k^2\sin^2\varphi_1\right)\end{aligned}$$

□

Proposition 2. *Let*

$$J(\varphi,\varphi_1) = \int_{\varphi}^{\varphi_1} \frac{dx}{\sqrt{1-k^2\sin^2 x}}.$$

Then $J(\varphi,\varphi_1) = cte$ is independent of the initial position φ.

Proof. Follows directly from **??** and Fundamental Theorem of Calculus. □

Remark 1. *By the construction above it follows that $A \circ T' = T \circ A$ and therefore the two billiard maps t and T' are conjugated by the affine map A.*

Theorem 3 (Poncelet)**.** *Consider the nested pair of an ellipse $\mathcal{E}_{a,b}$ and a circle $\mathcal{C}_r$. Consider a polygonal orbit $\mathcal{P}_n = P_1P_2\ldots P_n$ inscribed in $\mathcal{E}_{a,b}$ and circumscribed about $\mathcal{C}_r$. If after one revolution along $\mathcal{E}_{a,b}$ the polygon $\mathcal{P}_n$ is closed, then all orbits will be closed.*

Proof. Denote $P_k = (a\cos\varphi_k, b\sin\varphi_k)$ with $\varphi_0 = \varphi$. As $J(\varphi,\varphi_1) = \omega =$ cte, it follows that for a n-periodic orbit with turning number N we have $J(\varphi,\varphi+2\pi N) = n\omega$. □

2.2 Cayley's Condition

Let C, D be $3x \times 3$ matrices defining quadratic forms which cut out a general pair of conics in the projective plane. Let:

$$f(t) = \sqrt{\det(tC + D)} = \sum_{i=0}^{\infty} a_i t^i = a_0 + a_1 t + a_2 t^2 + \cdots$$

be the Taylor series of a branch of the square root of the cubic $\det(D + tC)$. Then there exists a Poncelet polygon, inscribed in C and circumscribed about D, with vertex count dividing n if and only if the determinant of an associated $[\frac{n-1}{2}]$-dimensional Hankel matrix defined below vanishes.

$$\begin{vmatrix} a_2 & a_3 & \ldots & a_{p+1} \\ a_3 & a_4 & \ldots & a_{p+2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p+1} & a_{p+2} & \ldots & a_{2p} \end{vmatrix} = 0, n = 2p+1, \text{or} : \begin{vmatrix} a_3 & a_4 & \ldots & a_{p+1} \\ a_4 & a_5 & \ldots & a_{p+2} \\ \ldots & \ldots & \ldots & \ldots \\ a_{p+1} & a_{p+2} & \ldots & a_{2p-1} \end{vmatrix} = 0, \ n = 2p.$$

2.3 Darboux theorem

Theorem 4 (Darboux (1917))**.** *Let $S \subset \mathbb{P}_2$ be a curve of degree $n-1$. If there is a complete $n-$gon tangent to a smooth conic C and inscribed into S, then there are infinitely many of them.*

Proof. See Darboux (1917, Chapitre I, page 248). □

Example 1. *Consider the cubic curve and the circle defined by*

$$C_3(x, y) = -4x^2 - 5y^2 + 3x(y^2 - 1) + 9 = 0, \ C(x, y) = x^2 + y^2 - 1 = 0.$$

There is a porism of 4-periodic orbits (quadrilaterals) inscribed in the compact component of $C_3(x, y) = 0$ and tangents to the circle $C(x, y) = 0$. See **??**. *In fact, the square with vertices $(\pm 1, \pm 1)$ is inscribed in $C_3(x, y) = 0$.*

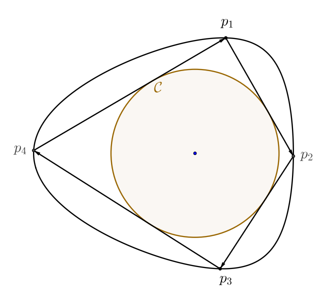

Figure 2.4: Porism of 4-periodic orbits associated to $C_3(x, y) = 0$ and the unit circle.

Consider a pair of conics (ellipses) defined by two quadratic forms $q_1(x,y,z) = \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - z^2 = 0$ and $q_2(x,y,z) = \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - z^2 = 0$ in projective coordinates $[x : y : z]$.

Let $f(t) = \sqrt{\mathrm{d}et(q_1 + tq_2)}$ where

$$q_i = \begin{pmatrix} \frac{1}{a_i^2} & 0 & 0 \\ 0 & \frac{1}{b_i^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2.4 Jacobi's Proof of Poncelet's Theorem

Here we review Jacobi's functions and derive some basic properties. For more on this subject see ...

Let $0 < k < 1$ and consider the elliptic integral

$$u = F(\varphi, k) = \int_0^\varphi \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

The inverse of F will be denoted by $\varphi = \mathrm{am}(u, k)$ and is called the amplitude Jacobi function.

The functions

$$\begin{aligned} \mathrm{cn}(u,k) &= \mathrm{JacobiCN}(u,k) = \cos(\mathrm{am}(u,k)) \\ \mathrm{sn}(u,k) &= \mathrm{JacobiSN}(u,k) = \sin(\mathrm{am}(u,k)) \\ \mathrm{dn}(u,k) &= \sqrt{1 - k^2 \mathrm{sn}^2(u,k)} \end{aligned}$$

are called the Jacobi's elliptic functions. For k fixed they will denoted simply by $\mathrm{cn}(u)$ and $\mathrm{sn}(u)$. From definition basic properties are:

$$\mathrm{cn}(0) = 1,\ \mathrm{sn}(0) = 0,\ \mathrm{dn}(0) = 1;$$
$$\mathrm{cn}(K) = 0,\ \mathrm{sn}(K) = 1,\ \mathrm{dn}(K) = \sqrt{1 - k^2} = k$$
$$\mathrm{cn}(2K) = -1,\ \mathrm{sn}(2K) = 0,\ \mathrm{dn}(2K) = 1.$$

Also,

$$\begin{aligned} \mathrm{sn}^2(u) + \mathrm{cn}^2(u) &= 1, \\ \mathrm{dn}^2(u) + k^2\mathrm{sn}^2(u) &= 1, \\ \mathrm{sn}'(u) &= \mathrm{cn}(u)\mathrm{dn}(u), \\ \mathrm{cn}'(u) &= -\mathrm{sn}(u)\mathrm{dn}(u), \\ \mathrm{dn}'(u) &= -k^2\mathrm{sn}(u)\mathrm{cn}(u). \end{aligned}$$

$$
\begin{aligned}
\mathrm{cn}(u+v) &= \frac{\mathrm{cn}(u)\mathrm{cn}(v) - \mathrm{sn}(u)\mathrm{sn}(v)\mathrm{dn}(u)\mathrm{dn}(v)}{\Delta(u,v)} \\
\mathrm{sn}(u+v) &= \frac{\mathrm{sn}(u)\mathrm{cn}(v)\mathrm{dn}(v) + \mathrm{sn}(v)\mathrm{cn}(u)\mathrm{dn}(u)\mathrm{dn}(u)}{\Delta(u,v)} \\
\mathrm{dn}(u+v) &= \frac{\mathrm{dn}(u)\mathrm{dn}(v) - k^2\mathrm{sn}(u)\mathrm{sn}(v)\mathrm{cn}(u)\mathrm{cn}(v)}{\Delta(u,v)} \\
\Delta(u,v) &= 1 - k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)
\end{aligned}
$$

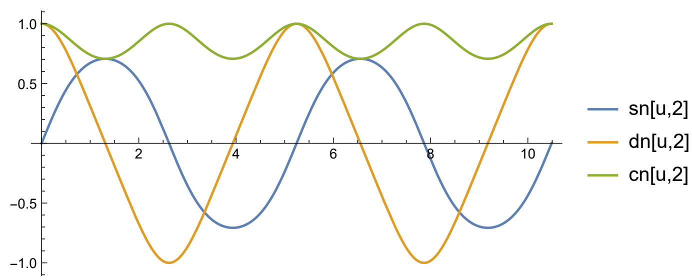

Figure 2.5: Jacobi elliptic functions

Below we recall some facts about three of Jacobi's elliptic functions extended to the complex plane $sn(z,k) = \sin(\mathrm{am}(z,k))$, $cn(z,k) = \cos(\mathrm{am}(z,k))$ and $dn(z,k) = \sqrt{1-k^2sn^2(z,k)}$, where $z \in \mathbb{C}$, and $0 < k < 1$ is the elliptic modulus.

These functions have two independent periods and also have simple poles at the same points. In fact:

$$
\begin{aligned}
\mathrm{sn}(u+4K) &= \mathrm{sn}(u+2iK') = \mathrm{sn}(u) \\
\mathrm{cn}(u+4K) &= \mathrm{cn}(u+2K+2iK') = \mathrm{cn}(u) \\
\mathrm{dn}(u+2K) &= \mathrm{dn}(u+4iK') = \mathrm{dn}(u) \\
K' = K(k'),&\quad k' = \sqrt{1-k^2}
\end{aligned}
$$

The poles of these three functions, which are simple, occur at the points

$$2mK + i(2n+1)K', \quad m, n \in \mathbb{Z}$$

They also display a certain symmetry around the poles. Namely, if z_p is a pole of $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$, then, for every $w \in \mathbb{C}$, we have Armitage and Eberlein (2006, Chapter 2):

$$
\begin{aligned}
\mathrm{sn}(z_p+w) &= -\,\mathrm{sn}(z_p-w) \\
\mathrm{cn}(z_p+w) &= -\,\mathrm{cn}(z_p-w) \\
\mathrm{dn}(z_p+w) &= -\,\mathrm{dn}(z_p-w)
\end{aligned} \tag{2.2}
$$

Proposition 3. *A billiard orbit P_n $(n = 1, \dots, N)$ of period N is parametrized by*

$$P_n = \left[a\,\mathrm{sn}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right), b\,\mathrm{cn}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right)\right]$$

where

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - (\frac{c}{a})^2 \sin^2 x}} dx$$

and $0 \leq u \leq 4K$.

2.5 Griffith and Harris' Proof

to do

2.6 Exercises

Exercise 1. *Show that a pair of ellipses $x^2/A^2 + y^2/B^2 = 1$ and $x^2/a^2 + y^2/b^2 = 1$ with semiaxes (A, B) and (a, b) ($A > a,\ B > b$) has a porism of pentagons (5-periodic orbits) then*

$$\frac{a^3}{A^3} + \frac{b^3}{B^3} + \left(\frac{a}{A} + \frac{b}{B}\right)^2 = 1 + \left(\frac{a}{A} + \frac{b}{B}\right)\left(1 + \frac{ab}{AB}\right)$$

Exercise 2. *Consider a quartic curve $q(x, y) = x^4 + y^4 - 1 = 0$ and a family of circles $\mathcal{C}_r :\ x^2 + y^2 - r^2 = 0$.*
i) Determine r such that there is 1d-family of triangles inscribed in the quartic $x^4 + y^4 = 1$ and sides tangent to the circle $x^2 + y^2 = r^2$. Analyze properties of this family of triangles. See **??**.
ii) Show that the square with vertices $[\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}]$ is inscribed in $q(x, y) = 0$ and its sides are tangent to the circle $x^2 + y^2 = \frac{\sqrt{2}}{2}$. In this case show that there is no porism of Poncelet associated to the algebraic curves. See **??**.

Exercise 3.

Exercise 4.

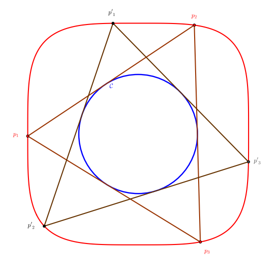

Figure 2.6: Porism of triangular orbits in a pair of a quartic and a circle.

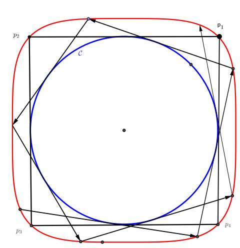

Figure 2.7: Non existence of a porism of quadrilateral orbits in a pair of a quartic and a circle.

Chapter 3

Loci of Triangle Centers over N=3 Poncelet Families

3.1 Introduction

Reader used to projective geometry may find it redundant to list so many Poncelet family cases. However, locus phenomena are euclidean.

History of the Results

Early videos 2011 com Jair Koiller Reznik (2011d) and Reznik (2011b), proof by complexification Romaskevich (2014), proof by Affine Curvature Garcia (2019), circumcenter Fierobe (2021). Centers of Mass of Poncelet Polygon Schwartz and Sergei Tabachnikov (2016a), Circumcenter of Mass Sergei Tabachnikov and Tsukerman (2014).

3.2 Some N=3 Poncelet Families

3.2.1 Concentric, Axis-Parallel Pair

General Case

- Cayley's condition $a_c/a + b_c/b = 1$
- Vertex Parametrization
- Figure

Below we introduce a few special cases.

Confocal (aka. Elliptic Billiard)

with Incircle

Billiard's Excentrals

with Circumcircle

Homothetic

Dual

3.2.2 Non-Concentric, Axis-Parallel Pair

General Case

Poristic (aka. Bicentric)

Brocard's Porism

3.2.3 Concentric, Unaligned

3.2.4 Generic Pair

3.3 Blaschke's Parametrization

Here we consider 3-periodics inscribed in a circle and circumscribing a non-concentric ellipse. We will work in the complex plane and apply Blaschke Product techniques Daepp et al. (2019) and Helman, Laurain, et al. (2021) which simplify our parametrization. Namely, 3-periodic vertices become symmetric with respect to the information of the circle-ellipse pair.

As a first step, identify points in $\mathbb{R}^2$ with points in the complex plane $\mathbb{C}$. Let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. By translation and scaling, we may assume the outer circle of the pair to be the unit circle $\mathbb{T}$. Let $\{f, g\}$ be the two foci of the inner ellipse. As in Daepp et al. (2019), define:

verificar o q é redundante abaixo

Note that if one wants to study the concentric setting, just substitute $g = -f$.

Following Daepp et al. (2019, Chapter 4), for each $\lambda \in \mathbb{T}$, the three solutions of $B(z) = \lambda$ are the vertices of a 3-periodic orbit of the Poncelet family of triangles in the complex plane, and as λ varies in $\mathbb{T}$, the whole family of triangles is covered. Clearing the denominator in this equation and passing everything to the left-hand side, we get

$$z^3 - (f + g + \lambda\overline{f}\overline{g})z^2 + (fg + \lambda(\overline{f} + \overline{g}))z - \lambda = 0$$

Let $z_1, z_2, z_3 \in \mathbb{C}$ denote the vertices of Poncelet 3-periodics in the pair with circumcircle. Using Viète's formula, we obtain the following parametrization of the elementary symmetric polynomials on z_1, z_2, z_3:

Definition 1 (Blaschke's Parametrization)**.**

$$\begin{aligned}\sigma_1 &:= z_1 + z_2 + z_3 = f + g + \lambda\overline{f}\overline{g}\\ \sigma_2 &:= z_1 z_2 + z_2 z_3 + z_3 z_1 = fg + \lambda(\overline{f} + \overline{g})\\ \sigma_3 &:= z_1 z_2 z_3 = \lambda\end{aligned}$$

where f, g are the foci of the inner ellipse and $\lambda \in \mathbb{T}$ is the varying parameter.

Referring to Figure **??**:

Proposition 4. *If a triangle center $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ is a fixed linear combination of X_2 and X_3 for some $\alpha, \beta \in \mathbb{C}$, its locus over 3-periodics in the non-concentric pair with a circumcircle is a circle centered on $\mathcal{O}_\alpha$ and of radius $\mathcal{R}_\alpha$ given by:*

$$\mathcal{O}_\alpha = \frac{\alpha(f+g)}{3}, \quad \mathcal{R}_\alpha = \frac{|\alpha f g|}{3}$$

Observation 1. *Notice that the center and radius of the locus do not depend on β since the circumcenter X_3 is stationary at the origin of this system.*

Proof. Since, z_1, z_2, z_3 are the 3 vertices of the Poncelet triangle inscribed in the unit circle, its barycenter and circumcenter are given by $X_2 = (z_1 + z_2 + z_3)/3$ and $X_3 = 0$, respectively. We define $\mathcal{X}_{\alpha,\beta} := \alpha X_2 + \beta X_3 = \alpha(z_1 + z_2 + z_3)/3$. Using Definition 1, we get $\mathcal{X}_{\alpha,\beta} = \alpha(f + g + \lambda\overline{f}\overline{g})/3 = \alpha(f + g)/3 + \lambda(\alpha\overline{f}\overline{g})/3$, where the parameter λ varies on the unit circle $\mathbb{T}$. Thus, the locus of $\mathcal{X}_\gamma$ over the Poncelet family of triangles is a circle with center $\mathcal{O}_\alpha := \alpha(f + g)/3$ and radius $\mathcal{R}_\alpha := |\alpha\overline{f}\overline{g}|/3 = |\alpha f g|/3$. □

Using $\alpha = 1 - \gamma, \beta = \gamma$ for a fixed $\gamma \in \mathbb{R}$ in **??**, we get:

Corollary 1. *If a triangle center $\mathcal{X}_\gamma = (1-\gamma)X_2 + \gamma X_3$ is a real affine combination of X_2 and X_3 for some $\gamma \in \mathbb{R}$, its locus over 3-periodics in the non-concentric pair with a circumcircle is a circle. Moreover, as we vary γ, the centers of these loci are collinear with the fixed circumcenter.*

Many triangle centers in Kimberling (2019) are affine combinations of the barycenter X_2 and circumcenter X_3. See **??** for a compilation of them.

Observation 2. *For a generic triangle, only X_{98}, and X_{99} are simultaneously on the Euler line and on the circumcircle. However these are not linear combinations of X_2 and X_3. Still, if a triangle center is always on the circumcircle of a generic triangle (there are many of these, see Eric Weisstein (2019, Circumcircle)), its locus over 3-periodics in the non-concentric pair with circumcircle is trivially a circle.*

Corollary 2. *Over the family of 3-periodics inscribed in a circle and circumscribing a non-concentric inellipse centered at O_c, the locus of X_k, k in 2,4,5,20 are circles whose centers are collinear. The locus of X_5 is centered on O_c. The centers and radii of these circular loci are given by:*

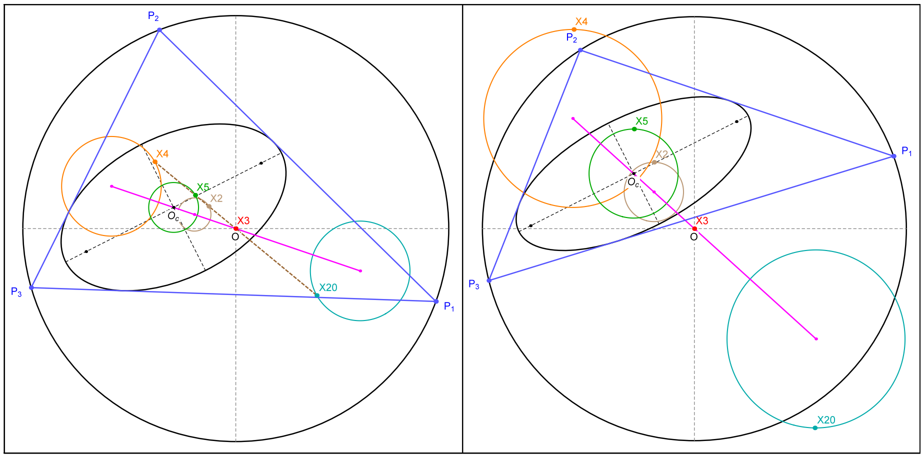

Figure 3.1: **Left:** 3-periodic family (blue) in the pair with circumcircle where the caustic contains X_3, i.e., all 3-periodics are acute. The loci of X_4 and X_{20} are interior to the circumcircle. **Right:** X_3 is exterior to the caustic, and 3-periodics can be either acute or obtuse. Equivalently, the locus of X_4 intersects the circumcircle. In both cases (left and right), the loci of X_k, k in 2,4,5,20 are circles with collinear centers (magenta line). The locus of X_5 is centered on O_c. The center of the X_2 locus is at $2/3$ along OO_c. Video

$$
\begin{aligned}
O_2 &= \frac{f+g}{3}, \quad O_4 = f+g, \quad O_5 = \frac{f+g}{2}, \quad O_{20} = -(f+g) \\
r_2 &= \frac{|fg|}{3}, \qquad r_4 = |fg|, \qquad r_5 = \frac{|fg|}{2}, \qquad r_{20} = |fg|
\end{aligned}
$$

Proof. As in Corollary **??**, we can use Proposition **??** with $\gamma = 0, -2, -1/2, 4$ to get the center and radius for X_2, X_4, X_5, X_{20}, respectively. All of these centers are real multiples of $f+g$, so they are all collinear. Moreover, the center O_5 of the circular loci of X_5 is $(f+g)/2$, that is, the midpoint of the foci of the inellipse, or in other words, the center O_c of the inellipse. □

Referring to Figure **??**:

Observation 3. *The family of 3-periodics in the pair with circumcircle includes obtuse triangles if and only if X_3 is exterior to the caustic.*

This is due to the fact that when X_3 is interior to the caustic, said triangle center can never be exterior to the 3-periodic. Conversely, if X_3 is exterior, it must also be external to some 3-periodic, rendering the latter obtuse.

Consider the parametrization of a triangular orbit $\{z_1, z_2, z_3\}$ given in Definition 1.

Let also the affine transformation $T(z) = pz + q\bar{z}$.

figura do blaschke

Theorem 5. *Over the family of 3-periodics interscribed in a generic nested pair of ellipses (non-concentric, non-axis-aligned), if $\mathcal{X}_{\alpha,\beta}$ is a fixed linear combination of X_2 and X_3, i.e., $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.*

Lemma 1. *If $u, v, w \in \mathbb{C}$ and λ is a parameter that varies over the unit circle $\mathbb{T} \subset \mathbb{C}$, then the curve parametrized by*

$$F(\lambda) = u\lambda + \frac{v}{\lambda} + w$$

is an ellipse centered at w, with semiaxis $|u| + |v|$ and $\big||u| - |v|\big|$, rotated with respect to the horizontal axis of $\mathbb{C}$ by an angle of $(\arg u + \arg v)/2$.

Proof. □

Consider the Moebius map $M_{z_0} = (z_0 - z)/(1 - \overline{z_0} z)$ and the Blaschke product of degree 3 given by $B = M_{z_0} M_{z_1} M_{z_2}$.

Theorem 6. *Let B be a Blaschke product of degree 3 with zeros $0, f, g$. For $\lambda \in \mathbb{T}$, let z_1, z_2, z_3 denote the three distinct solutions to $B(z) = \lambda$. Then the lines joining z_j and z_k, $(j \neq k)$ are tangent to the ellipse given by*

$$|w - f| + |w - g| = |1 - \overline{f} g|.$$

Theorem 7. *Given two points $f, g \in \mathbb{D}$. Then there exists a unique conic $\mathcal{E}$ with the foci f, g which is 3-Poncelet caustic with respect to $\mathbb{T}$. Moreover, $\mathcal{E}$ is an ellipse. That ellipse is the Blaschke ellipse with the major axis of length $|1 - \overline{f} g|$.*

Proof. □

3.4 Analyzing Loci of Triangle Centers

- Triangle centers
- Are Loci Algebraic (method of resultants, why so many)
- Are Loci Elliptic (Blaschke's parametrization Daepp et al. (2019))
- Monotonicity and Turning Number (ballet + paper 25)
- Table of results

3.4.1 Generic Nested Ellipses

In this Section we prove the locus of a given fixed linear combination of X_2 and X_3 is an ellipse. We will continue to use Blaschke product techniques since a generic non-concentric pair can always be seen as the affine image of a pair with circumcircle.

Consider the generic pair of nested ellipses $\mathcal{E} = (O, a, b)$ and $\mathcal{E}_c = (O_c, a_c, b_c.\theta)$ in Figure **??**. Let sθ, cθ denote the sine and cosine of θ, respectively. Define $c_c^2 = a_c^2 - b_c^2$. The Cayley condition for the pair to admit a 3-periodic family is given by:

$$
\begin{aligned}
& b^4x_c^4 + 2\,a^2b^2x_c^2y_c^2 + \left(2c_c^2\left(-b^2(a^2+b^2)\right)\mathrm{c}\theta^2 - 2\left(b^2 - \,b_c^2\right)b^2a^2 - 2\,b^4b_c^2\right)x_c^2 \\
& - 8\,a^2b^2x_c\,y_c\,c_c^2\mathrm{s}\theta\mathrm{c}\theta + a^4y_c^4 + \left(2c_c^2a^2\left(a^2+b^2\right)\mathrm{c}\theta^2 - 2\left(b_c^2+b^2\right)a^4 + 2\,a^2b^2b_c^2\right)y_c^2 \\
& + c_c^4c^4\left(\mathrm{c}\theta^4 - 2\,c_c^2c^2\left(a^2a_c^2 - b^2a^2 + b_c^2b^2\right)\mathrm{c}\theta^2\right. \\
& + \left(aa_c+ab-bb_c\right)\left(aa_c-ab-bb_c\right)\left(aa_c+ab+bb_c\right)\left(aa_c-ab+bb_c\right) = 0
\end{aligned}
\tag{3.1}
$$

Before moving on, we first prove a small parametrization lemma for complex coordinates:

Lemma 2. *If $u, v, w \in \mathbb{C}$ and λ is a parameter that varies over the unit circle $\mathbb{T} \subset \mathbb{C}$, then the curve parametrized by*

$$F(\lambda) = \lambda u + \frac{v}{\lambda} + w$$

is an ellipse centered at w, with semiaxis $|u| + |v|$ and $\big||u| - |v|\big|$, rotated with respect to the canonical axis of $\mathbb{C}$ by an angle of $(\arg u + \arg v)/2$.

Proof. If either $u = 0$ or $v = 0$, the curve $h(\mathbb{T})$ is clearly the translation of a multiple of the unit circle $\mathbb{T}$, and the result follows. Thus, we may assume $u \neq 0$ and $v \neq 0$.

Choose $k \in \mathbb{C}$ such that $k^2 = u/v$. Write k in polar form, as $k = r\mu$, where $r > 0$ ($r \in \mathbb{R}$) and $|\mu| = 1$. We define the following complex-valued functions:

$$R(z) := \mu z,\ S(z) := rz + (1/r)\overline{z},\ H(z) := kvz,\ T(z) := z + w$$

One can straight-forwardly check that $F = T \circ H \circ S \circ R$.

Since $|\mu| = 1$, R is a rotation of the plane, thus R sends the unit circle $\mathbb{T}$ to itself. Since $r \in \mathbb{R}$, $r > 0$, if we identify $\mathbb{C}$ with $\mathbb{R}^2$, S can be seen as a linear transformation that sends $(x, y) \mapsto ((r + 1/r)\,x, (r - 1/r)\,y)$. Thus, S sends $\mathbb{T}$ to an axis-aligned, origin-centered ellipse $\mathcal{E}_1$ with semiaxis $r + 1/r$ and $|r - 1/r|$. H is the composition of a rotation and a homothety. H sends the ellipse $\mathcal{E}_1$ to an origin-centered ellipse $\mathcal{E}_2$ rotated by an angle of $\arg(kv) = \arg(k) + \arg(v) = (\arg(u) - \arg(v))/2 + \arg(v) = (\arg(u) + \arg(v))/2$. The semiaxis of $\mathcal{E}_2$ have length

$$
\begin{aligned}
&|kv|(r+1/r) = r|v|(r+1/r) = |r^2v| + |v| = |k^2v| + |v| = |u| + |v|, \text{ and} \\
&|kv||r-1/r| = r|v||r-1/r| = \big||r^2v| - |v|\big| = \big||k^2v| - |v|\big| = \big||u| - |v|\big|
\end{aligned}
$$

Finally, T is a translation, thus T sends $\mathcal{E}_2$ to an ellipse $\mathcal{E}_3$ centered at w, rotated by an angle $(\arg(u) + \arg(v))/2$ from the axis, with semiaxis lengths $|u| + |v|$ and $\big||u| - |v|\big|$, as desired. □

Recall that over Poncelet N-periodics interscribed in a generic pair of conics, the locus of vertex and area centroids is an ellipse Schwartz and Sergei Tabachnikov (2016b) as is that of the circumcenter-of-mass Sergei Tabachnikov and Tsukerman (2014), a generalization of X_3 for $N > 3$. Referring to Figure 3.1:

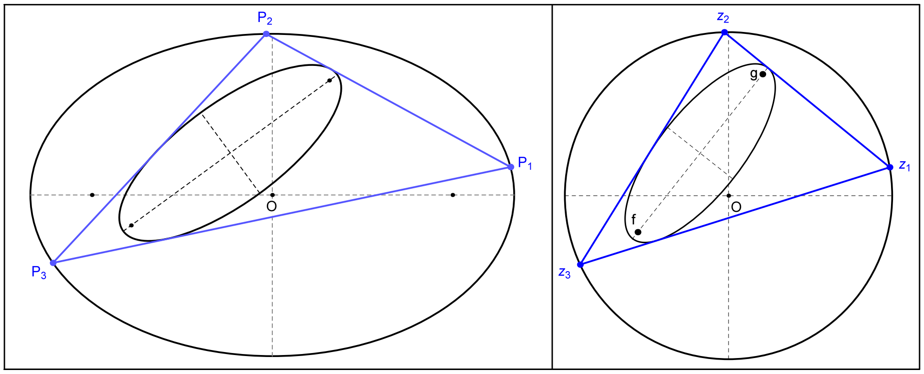

Figure 3.2: Affine transformation that sends a generic ellipse pair and its 3-periodic family (left) to a new pair with circumcircle (right). We parametrize the 3-periodic orbit with vertices z_i in the circumcircle pair using the foci of the latter's caustic f and g, and then apply the inverse affine transformation to get a parametrization of the vertices P_i of the original Poncelet pair. Video

Theorem 8. *Over the family of 3-periodics interscribed in an ellipse pair in general position (non-concentric, non-axis-aligned), if $\mathcal{X}_{\alpha,\beta}$ is a fixed linear combination of X_2 and X_3, i.e., $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.*

Proof. Consider a general $N = 3$ Poncelet pair of ellipses that forms a 1-parameter family of triangles. Without loss of generality, by translation and rotation, we may assume the outer ellipse is centered at the origin and axis-aligned with the plane $\mathbb{R}^2$, which we will also identify with the complex plane $\mathbb{C}$. Let a, b be the semi-axis of the outer ellipse, and a_c, b_c the semi-axis of the inner ellipse, as usual.

Referring to Figure **??**, consider the linear transformation that takes $(x, y) \mapsto (x/a, y/b)$. This transformation takes the outer ellipse to the unit circle $\mathbb{T}$ and the inner ellipse to another ellipse. Thus, it transforms the general Poncelet $N = 3$ system into a pair where the outer ellipse is the circumcircle, which we can parametrize using Blaschke products Daepp et al. (2019). In fact, to get back to the original system, we must apply the inverse transformation that takes $(x, y) \mapsto (ax, by)$. As a linear transformation from $\mathbb{C}$ to $\mathbb{C}$, we can write it as $L(z) := pz + q\overline{z}$, where $p := (a + b)/2, q := (a - b)/2$.

Let $z_1, z_2, z_3 \in \mathbb{T} \subset \mathbb{C}$ be the three vertices of the circumcircle family, parametrized as in Definition 1, and let $v_1 := L(z_1), v_2 := L(z_2), v_3 := L(z_3)$ be the three vertices of the original general family. The barycenter X_2 of the original family is given by $(v_1 + v_2 + v_3)/3$, and the circumcenter X_3 is given by Tak (n.d.):

$$X_3 = \begin{vmatrix} v_1 & |v_1|^2 & 1 \\ v_2 & |v_2|^2 & 1 \\ v_3 & |v_3|^2 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} v_1 & \overline{v_1} & 1 \\ v_2 & \overline{v_2} & 1 \\ v_3 & \overline{v_3} & 1 \end{vmatrix}$$

Since $\overline{z_1} = 1/z_1, \overline{z_2} = 1/z_2, \overline{z_3} = 1/z_3$, we can write v_1, v_2, v_3 as rational functions of z_1, z_2, z_3, respectively. Thus, both X_2 and X_3 are symmetric rational functions on z_1, z_2, z_3. Defining $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$, we have consequently that $\mathcal{X}_{\alpha,\beta}$ is also a symmetric rational function on z_1, z_2, z_3. Hence, we can reduce its numerator and denominator to functions on the elementary symmetric polynomials on z_1, z_2, z_3. This is exactly what we need in order to use the parametrization by Blaschke products.

In fact, we explicitly compute:

$$\mathcal{X}_{\alpha,\beta} = \frac{p^2 q\left(\sigma_2(\alpha+3\beta)+3\beta\sigma_3^2\right)+\alpha p^3\sigma_1\sigma_3 - pq^2(3\beta+\sigma_1\sigma_3(\alpha+3\beta)) - \alpha q^3\sigma_2}{3\sigma_3(p-q)(p+q)}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric polynomials on z_1, z_2, z_3.

Let $f, g \in \mathbb{C}$ be the foci of the inner ellipse in the circumcircle system. Using Definition 1, with the parameter λ varying on the unit circle $\mathbb{T}$, we get:

$$\mathcal{X}_{\alpha,\beta} = u\lambda + v\frac{1}{\lambda} + w \tag{3.2}$$

where:

$$\begin{aligned}
u :=& \frac{p\left(\overline{f}\overline{g}\left(\alpha p^2 - q^2(\alpha+3\beta)\right)+3\beta pq\right)}{3(p-q)(p+q)}\\
v :=& \frac{\beta pq(q-fgp)}{(q-p)(p+q)} + \frac{1}{3}\alpha fgq\\
w :=& \frac{q\left(\overline{f}+\overline{g}\right)\left(p^2(\alpha+3\beta)-\alpha q^2\right)+p(f+g)\left(\alpha p^2 - q^2(\alpha+3\beta)\right)}{3(p-q)(p+q)}
\end{aligned}$$

By Lemma 1, this is the parametrization of an ellipse centered at w, as desired. As in Lemma 1, it is also possible to explicitly calculate its axis and rotation angle, but these expressions become very long.

□

Corollary 3. *Over the family of 3-periodics interscribed in an ellipse pair in general position (non-concentric, non-axis-aligned), if $\mathcal{X}_\gamma$ is a real affine combination of X_2 and X_3, i.e., $\mathcal{X}_\gamma = (1-\gamma)X_2 + \gamma X_3$ for some fixed $\gamma \in \mathbb{R}$, then its locus is an ellipse. Moreover, as we vary γ, the centers of the loci of the $\mathcal{X}_\gamma$ are collinear.*

Proof. Apply Theorem 3 with $\alpha = 1-\gamma, \beta = \gamma$ to get the elliptical loci. As in the end of the proof of Theorem 3, the center of the locus of $\mathcal{X}_\gamma$ can be computed explicitly as

$$\begin{gathered}
w = w_0 + w_1\gamma, \text{ where}\\
w_0 = \frac{1}{3}\left(q\left(\overline{f}+\overline{g}\right)+p(f+g)\right)\\
w_1 = \frac{q\left(2p^2+q^2\right)\left(\overline{f}+\overline{g}\right)-p(f+g)\left(p^2+2q^2\right)}{3(p-q)(p+q)}
\end{gathered}$$

As $\gamma \in \mathbb{R}$ varies, it is clear the center w sweeps a line. □

We proved that all of the following triangle centers have elliptic loci in the general N=3 Poncelet system, including the barycenter, circumcenter, orthocenter, nine-point center, and de Longchamps point:

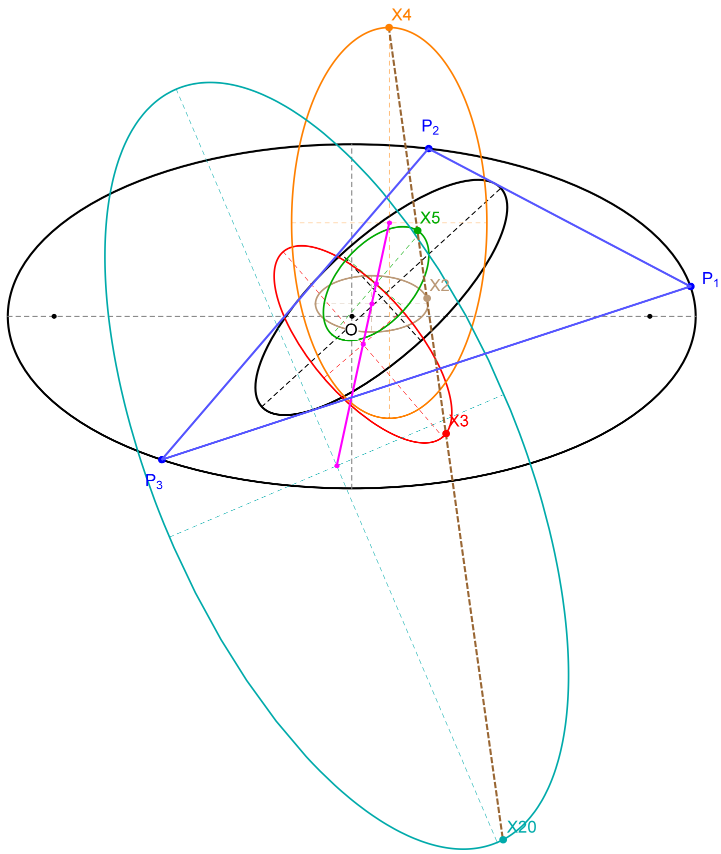

Figure 3.3: A 3-periodic is shown interscribed between two nonconcentric, non-aligned ellipses (black). The loci of X_k, $k = 2, 3, 4, 5, 20$ (and many others) remain ellipses. Those of X_2 and X_4 remain axis-aligned with the outer one. Furthermore the centers of all said elliptic loci are collinear (magenta line). Video

Observation 4. *Amongst the 40k+ centers listed on Kimberling (2019), about 4.9k triangle centers lie on the Euler line Kimberling (2020). Out of these, only 226 are fixed affine combinations of X_2 and X_3. For $k < 1000$, these amount to $X_k, k =$2, 3, 4, 5, 20, 140, 376, 381, 382, 546, 547, 548, 549, 550, 631, 632.*

Observation 5. *The elliptic loci of X_2 and X_4 are axis-aligned with the outer ellipse.*

We conclude this section with phenomenon specific to the case where $\mathcal{E}_c$ is a circle, **??**:

Observation 6. *Over the family of 3-periodics inscribed in an ellipse and circumscribing a non-concentric circle centered on $O_c = X_1$, the locus of X_3 and X_5 are ellipses whose major axes pass through X_1.*

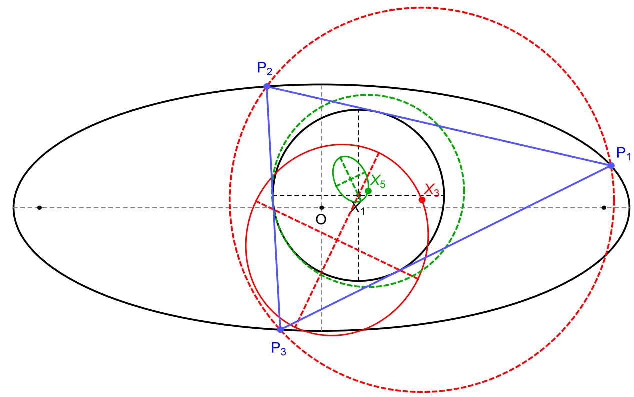

Figure 3.4: A 3-periodic (blue) is shown inscribed in an outer ellipse and an inner non-concentric circle centered on O_c. The loci of both circumcenter (solid red) and Euler center (solid green) are ellipses whose major axes pass through O_c. Video

3.5 Incenter and Excenters

Theorem 9. *Over 3-periodics in the elliptic billiard (confocal pair) the locus of the incenter X_1 is an ellipse given by $x^2/a_1^2 + y^2/b_1^2 = 1$, where*

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}, \quad \delta = \sqrt{a^4 - a^2b^2 + b^4}.$$

The locus of the Excenters (triangle formed by the intersection of external bisectors) is an ellipse with axes:

$$a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}$$

Notice it is similar to the X_1 locus, i.e., $a_1/b_1 = b_e/a_e$.

A list of the elliptic loci of centers in the X_1 to X_{200} range can be found here.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk bounded by $\mathbb{T}$.

Proposition 5. *Over Poncelet 3-periodics in the pair with an outer circle and an ellipse in generic position, the locus X_1 given by:*

$$\begin{aligned} X_1 : \; & z^4 - 2((\bar{f} + \bar{g})\lambda + fg)z^2 + 8\lambda z \\ & + (\bar{f} - \bar{g})^2\lambda^2 + 2(|f|^2 g + f|g|^2 - 2f - 2g)\lambda + f^2 g^2 = 0 \\ & : \; z^4 - 2\beta z^2 + 8\lambda z + (\beta^2 - 4\alpha\lambda) = 0 \end{aligned}$$

Proof. The incenter of a triangle with vertices $\{z_1, z_2, z_3\}$ is given by:

$$X_1 = \frac{\sqrt{a}\, z_1 + \sqrt{b}\, z_2 + \sqrt{c}\, z_3}{\sqrt{a} + \sqrt{b} + \sqrt{c}}$$
$$a = |z_2 - z_3|^2,\ b = |z_1 - z_3|^2,\ \ c = |z_2 - z_1|^2$$

Using that $z_i \in \mathbb{T}$ it follows that

$$a = 2 - (\frac{z_3}{z_2} + \frac{z_3}{z_2}),\ \ b = 2 - (\frac{z_1}{z_3} + \frac{z_3}{z_1}),\ \ c = 2 - (\frac{z_1}{z_2} + \frac{z_2}{z_1})$$

Eliminating the square roots in the equation $X_1 - z = 0$ and using the relations σ_i (i=1,2,3) given in Blaschke's parametrization the result follows. □

Proposition 6. *Over Poncelet 3-periodics in a generic nested ellipse pair, the locus of X_1 is given by the following sextic polynomial in z, λ:*

$$\begin{aligned}
X_1:\ &\lambda^2\left(p^2 - q^2\right) z^4 + 4\,\lambda\,\left(\alpha\,\lambda\,pq^2 - q\,\lambda^2 p^2 - \beta\,p^2\,q + p\,q^2\right) z^3\\
&+ (4\,\alpha\,\lambda^3 p^3 q - 4\,\alpha^2\lambda^2 p^2 q^2 + 2\,\alpha\,\beta\,\lambda\,p^3\,q - 2\,\alpha\,\beta\,\lambda\,pq^3 - 2\,\beta\,\lambda^2 p^4 + 6\,\beta\,\lambda^2 p^2 q^2\\
&- 6\,\alpha\,\lambda\,p^2\,q^2 + 2\,\alpha\,\lambda\,q^3 q + 4\,\beta^2 p^2\,q^2 + 6\,\lambda^2 p^3\,q - 6\,\lambda^2 pq^3 - 4\,\beta\,p\,q^3)z^2\\
&+ (4q(\alpha^2\beta p^2 q^2 + 2\alpha^2 p^2 pq - \alpha^2 pq^3 + \beta^2 p^4 - 2\beta^2 p^2 q^2 + 4\beta pq^3 - p^2 q^2 - 2q^4)\lambda\\
&- 4\alpha pq^2(\beta^2 p^2 - q^2) - 4p^3(\beta pq - 2p^2 - q^2)\lambda^3 - 16\alpha\lambda^2 p^4 q)z\\
&- \lambda^2(4\alpha\lambda - \beta^2)p^6 + 4p^5 q\lambda^4 + 2\lambda(4\alpha^2\lambda - \alpha\beta^2 - 3\beta\lambda)p^5 q - \lambda^2(8\alpha\lambda - 3\beta^2)p^4 q^2\\
&+ (\alpha^2\beta^2 - 4\alpha^3\lambda + 4\alpha\beta\lambda + 5\lambda^2)p^4 q^2 + 2\lambda(2\alpha^2\lambda - \alpha\beta^2 + \beta\lambda)p^3 q^3\\
&+ (2\alpha^2\beta - 2\alpha\lambda - 4\beta^2)p^3 q^3 - (\alpha^2\beta^2 + 4\alpha\beta\lambda - 4\beta^3 + 5\lambda^2)p^2 q^4 + (8\beta - 3\alpha^2)p^2 q^4\\
&+ (2\alpha^2\beta + 6\alpha\lambda - 8\beta^2)pq^5 - 4q^5 p + (4\beta - \alpha^2)q^6 = 0
\end{aligned}$$

Proof. Let $p, q \in \mathbb{R}$. Consider the affine transformation $T(z) = pz + q\bar{z}$ and set $w_i = T(z_i)$. The proof is similar to that given in Proposition 2. □

Proposition 7. *In the confocal pair the locus X_1 is defined by:*

$$2\,ab\lambda^2 z^2 + 2\,\lambda\,\left(a^3\lambda^2 - b^3\lambda^2 - a^3 - b^3\right) z + c^2\left(c^2\lambda^4 - 2\,ab\lambda^2 - c^2\right) = 0$$

Proof. We have that

$$f = \frac{1}{c}\sqrt{-a^2 - b^2 + 2\,\delta},\ \ g = -\frac{1}{c}\sqrt{-a^2 - b^2 + 2\,\delta}$$

□

Corollary 4. *The locus X_1 is the ellipse with semiaxes given by $a_1 = (a^2 - \delta)/b$ and $b_1 = (\delta - b^2)/a$.*

Proof. The quartic polynomial is factorizable as $p_1 p_2$, where

$$p_1 = z - \left(\frac{(a-b)\left(-a^2 - ab - b^2 + \delta\right)\lambda}{2\,ab} - \frac{(a+b)\left(-a^2 + ab - b^2 + \delta\right)}{2\,ab\lambda} \right)$$

$$\begin{aligned} p_2 = {} & 2ab\lambda^2 z^3 - ((a-b)(a^2+3ab+b^2+\delta)\lambda^3 - (a+b)(a^2-3ab+b^2+\delta)\lambda)z^2 \\ & + 6ab(a^2-b^2)\lambda^2 z + (a+b)^3(a^2-ab+b^2+\delta)\lambda^3 - (a-b)^3(a^2+ab+b^2+\delta)\lambda \end{aligned}$$

Follows directly from Lemma 1 and Proposition 4. □

Schwartz and Sergei Tabachnikov (2016a)

Conjecture 1. *Over 3-periodics interscribed between two ellipses in general position, the locus of a triangle center X_k is an ellipse if and only if X_k is a fixed linear combination of X_3 and X_4.*

Conjecture 2. *The locus of the incenter is an ellipse if and only if the Poncelet ellipse pair is confocal.*

3.6 Exercises

Chapter 4

Invariants in the Elliptic Billiard

Chapter 5

Invariants of the Bicentric Family

Chapter 6

Invariants of the Homothetic and Brocard Families

Chapter 7

Experimental Techniques

Chapter 8

Epilogue: Properties of Pairs of Conics

8.1 Focal Properties of Conics

The polar line associated to a point $P_0 = (x_0, y_0)$ with respect to an ellipse is given by

$$b^2 x_0 x + a^2 y_0 y - a^2 b^2 = 0.$$

The pair of tangent lines passing through the point $P_0 = (x_0, y_0)$ is given by

$$(xy_0 - yx_0)^2 - a^2 (x - x_0)^2 - b^2 (y - y_0)^2 = 0$$

Proposition 8. *Consider an ellipse $\mathcal{E}$ with foci F_1 and F_2. Let ℓ_P be the tangent line to $\mathcal{E}$ at $P \in \mathcal{E}$. Then ℓ_P is a bisector of the lines PF_1 and PF_2, see* **??**.

Proof. □

Proposition 9. *Consider an ellipse $\mathcal{E}$ with foci F_1 and F_2. Let P_0 be an exterior point and ℓ_{P_0} be the polar line associated to P_0 intersecting the ellipse $\mathcal{E}$ at P_1 and P_2. Suppose that $F_2 \in \ell_{P_0}$. Then P_0 is the center of a circle $\mathcal{C}_0$ tangent to the polar line line at F_2 and to the lines F_1P_1 and F_1P_2, see* **??**.

Proof. See A. V. Akopyan and Zaslavsky (2007a, page 10). By Proposition **??** the line P_0P_2 (resp. P_0P_1) is the bisector line of F_1P_2 and ℓ_{P_0} (resp. of F_1P_1 and ℓ_{P_0}). Therefore, there exists a circle $\mathcal{C}_0$ centered at P_0 and tangent to ℓ_{P_0}, P_1F_1 and F_1P_2. In fact, $\mathcal{C}_0$ is the excircle of the triangle $F_1P_1P_2$. Finally we observe that $\ell_{P_0} \cap \mathcal{C}_0$ is F_2. □

Proposition 10. *Consider an ellipse and two tangent lines passing through an exterior point P_0 as shown in Appendix A. Let also the line passing through the P_0 and focus F_2. Then it follows that that $\alpha_1 = \alpha_2$.*

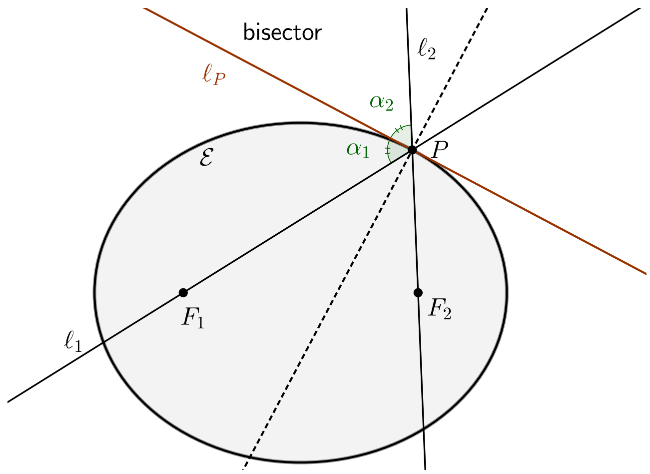

Figure 8.1: The tangent line ℓ_p is a bisector of the external angle, i.e., the angles α_1 and α_2 are equal.

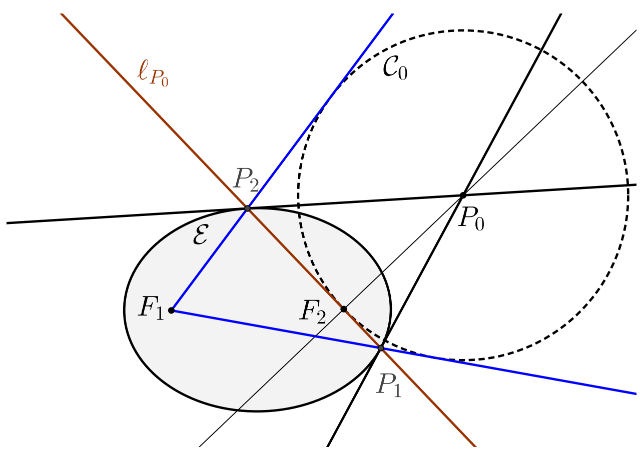

Figure 8.2: The polar line ℓ_{P_0} passing through the focus F_2 is tangent to the excircle $\mathcal{C}_0$ which is centered at P_0.

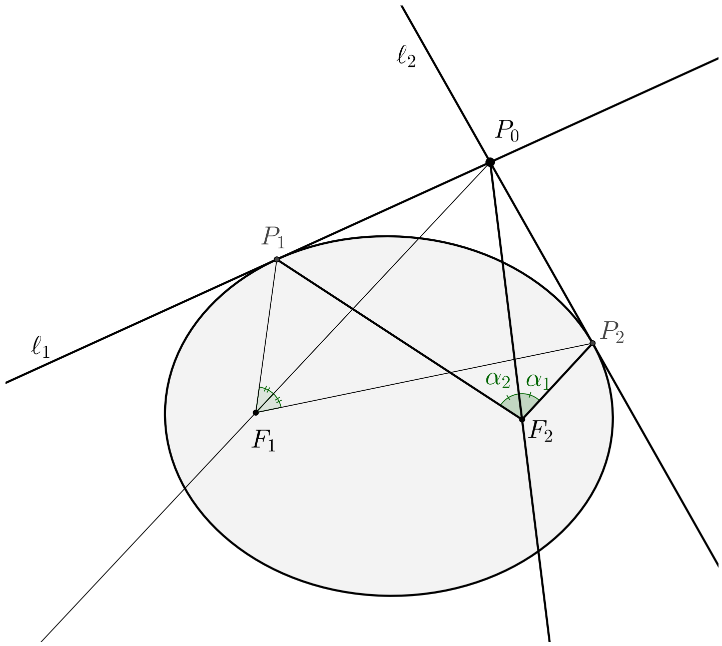

Figure 8.3: Angles α_1 and α_2 are equal.

Proof. Consider the quadrilateral $F_1P_2F_2P_1$. Observe that $|F_1P_2|+|P_2F_2| = |F_2P_1|+|P_1F_1|$. Therefore, there is a circle tangent to the sidelines F_1P_2, F_1P_1, F_2P_2 and F_2P_1, which is centered at P_0. So, the line F_2P_0 is a bisector of F_2P_1 and F_2P_2. □

Proposition 11. *Consider an ellipse and two tangent lines passing through an exterior point as shown in Appendix A. Let also the two lines passing through the foci and P_0. Then we have that $\theta_1 = \theta_2$.*

Proof. Let O be the point of intersection of the lines F_1P_1 and F_2P_2. Consider the triangle $P_0F_2P_2$ and the line F_2Q tangent to the circle centered at P_0. See **??**.

We have that

$$\begin{aligned}\theta_2 &= \angle F_2P_0P_2 = \angle P_0P_2Q - \angle P_0F_2P_2\\ &= \angle P_0P_2Q - \angle P_0F_2P_1\\ &= \frac{1}{2}\angle F_1P_2Q - \frac{1}{2}\angle P_1F_2Q\\ &= \frac{1}{2}\angle P_2OF_2\end{aligned}$$

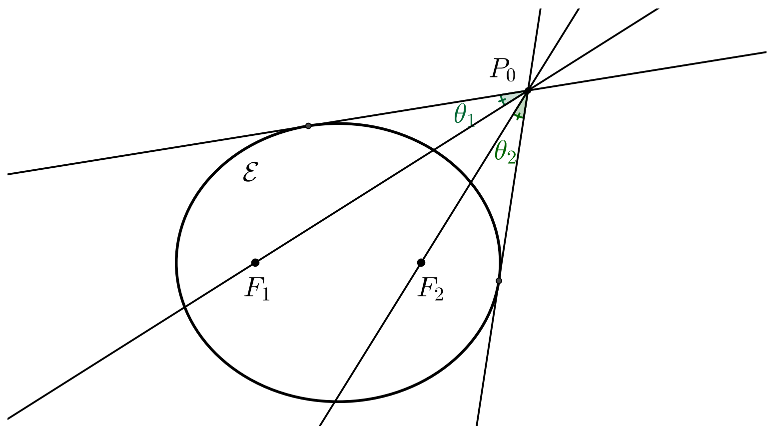

Figure 8.4: Angles θ_1 and θ_2 are equal.

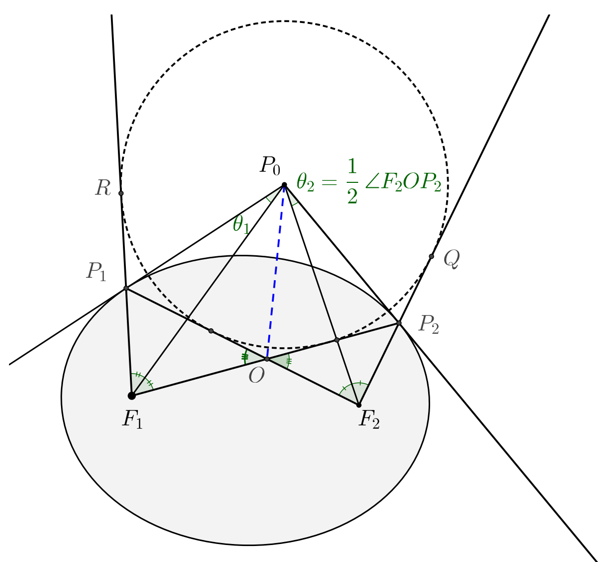

Figure 8.5: Angle θ_2 is equal to $\frac{1}{2}\angle F_2OP_2$.

The relations above follows from the fact that P_0F_2 is a bisector of the lines P_1F_2 and F_2P_2 and that $\angle F_1P_2Q$ is an external angle of triangle OF_2P_2. Analogously, $\theta_1 = \angle P_1OF_1$. So, $\theta_1 = \theta_2$. □

Proposition 12. *Consider a pair of confocal ellipses $\mathcal{E}$ and $\mathcal{E}'$ with semi-axes (a, b) and (a_c, b_c) respectively. Referring to **??** it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} + \frac{|P_0P_2|}{|P_2F_2|} = \frac{2a_c(a - a_c)}{b_c^2}$$

is constant (independent of P_0).

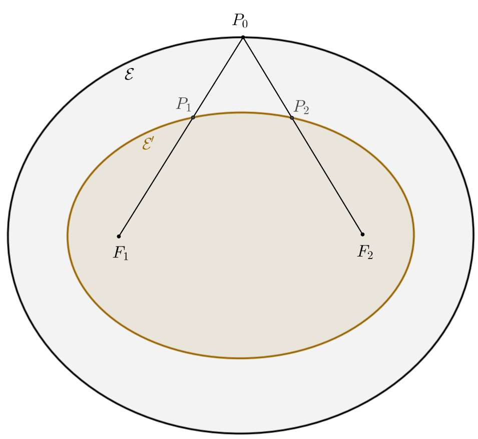

Figure 8.6: The sum of relation of focal distances is constant.

Proof. Follows from straightforward calculations, computing the quantities involved with help of CAS. □

Proposition 13. *Consider a confocal pair of an ellipse $\mathcal{E}$ and a hyperbola $\mathcal{H}$. Referring to Figure A.4 it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} + \frac{|P_0P_2|}{|P_2F_2|} = \frac{2a_c(a + a_c)}{b_c^2} \quad \textit{and} \quad \frac{|P_0F_1|}{|P_1'F_1|} + \frac{|P_0F_2|}{|P_2'F_2|} = \frac{2a_c(a - ac)}{b_c^2}$$

are constant (independent of P_0). The relations are valid for P_0 in the arc of ellipse where the points $P_i,\ P_i'$ (i=1,2) are well defined.

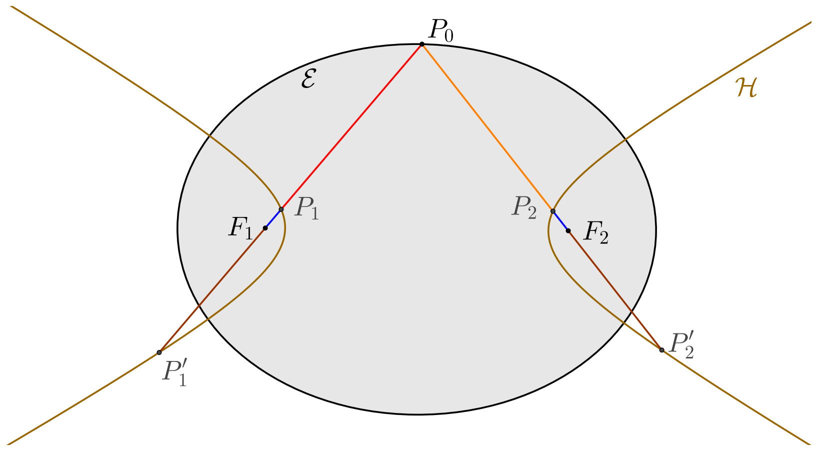

Figure 8.7: Sum of relations of focal distances in a confocal pair (ellipse and hyperbola) is constant.

Proof. Follows from straightforward calculations, computing the quantities involved with help of CAS. □

Proposition 14. *Consider a pair of confocal hyperbolas $\mathcal{H}$ and $\mathcal{H}_1$. Referring to Figure A.5 it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} / \frac{|P_0P_2|}{|P_2F_2|} =$$

is constant (independent of P_0).

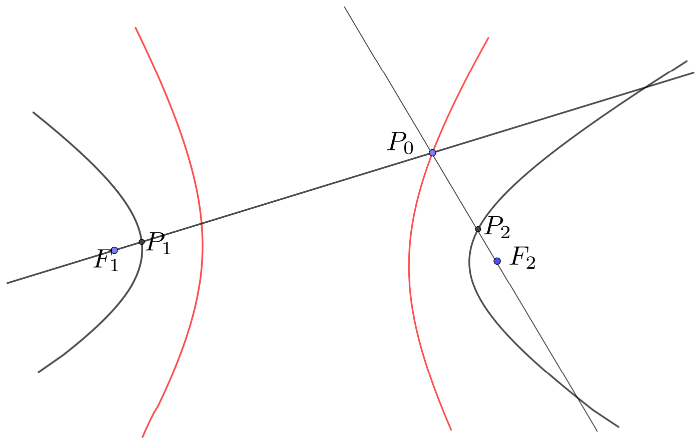

Figure 8.8: Relation of focal distances

Proof. □

Proposition 15. *Consider an ellipse $\mathcal{E}$ with foci $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P_0 = (x_0, y_0) \in \mathcal{E}$ and Q_0 the pedal of F_2 with respect to the tangent line passing through P_0. Let also Q_2 the reflection of F_2 with respect to the pedal point Q_0. Then*

$$|Q_2 - F_1| = 2a$$

Therefore the locus of points as constructed above is a circle $\mathcal{C}$ of radius $2a$ centered at F_1. Also the locus of pedal points is a circle centered at the origin and radius a.

The pair $\{\mathcal{E}, \mathcal{C}\}$ is a Poncelet pair having all periodic orbits of period 3.

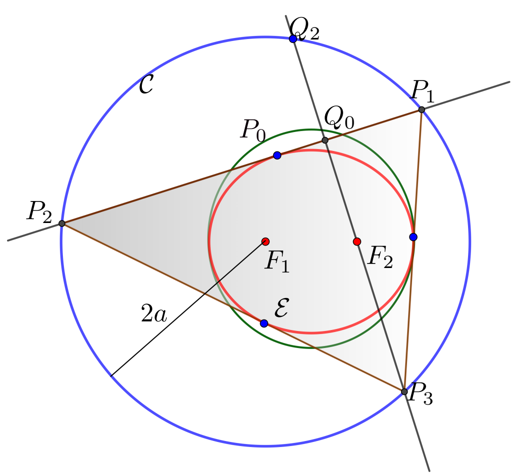

Figure 8.9:

Proof. □

Theorem 10. *Consider a pair of confocal conics with foci F_1 and F_2 in the plane (see Figure A.7). Consider the right branch of the hyperbola associated with F_2. Let P and Q be the points of intersection of the ellipse with the right branch of the hyperbola. Consider a ray starting at F_1 and intersecting the right branch of the hyperbola. Denote by X, A the intersection points of this ray with the ellipse and with the branch of the hyperbola. Suppose the focus F_2 lies on the line PQ. Then PQ is the bisector of the angle $\langle AF_2B$.*

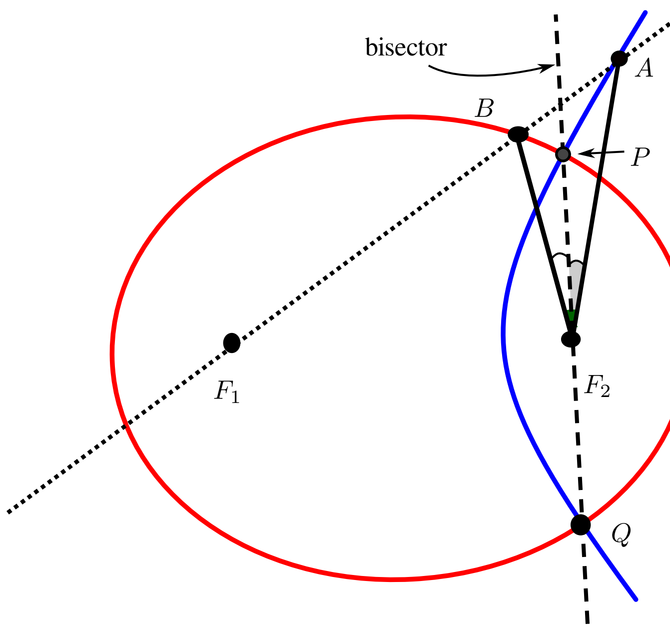

Figure 8.10: The line PQ passing through the focus F_2 and the intersection of the ellipse with the right branch of the hyperbola is a bisector of the angle AF_2B.

Theorem 11. *Considear a pair of confocal conics (ellipse and hyperbola) Consider an arbitrary point M (exterior to the ellipse) on the line passing through the intersection points P and Q of the ellipse and the right branch of the hyperbola. Draw two tangent lines to the ellipse and to the hyperbola. Then the lines P_1Q_1, P_2Q_2 (respec. P_1Q_2 and P_2Q_1) through the tangent points as shown in Figure A.8 passe through the focus F_1 (respec. F_2).*

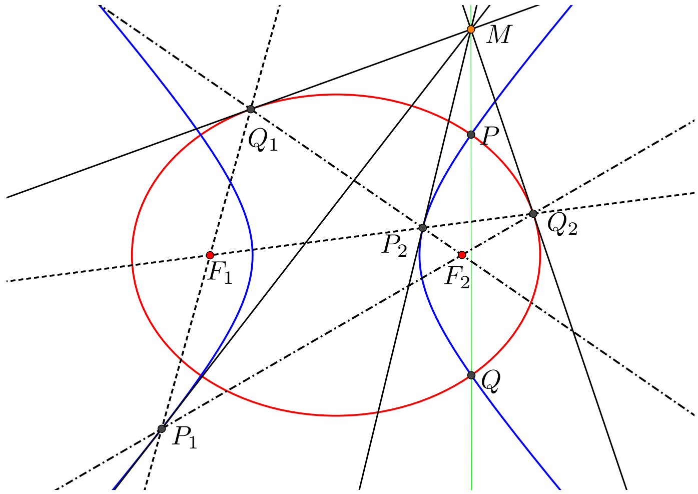

Figure 8.11: The four lines passing through the point M and tangent to the confocal conics determine four lines passing through the foci.

Proof. □

Theorem 12. *Let two confocal ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with foci F_1 and F_2 are given. Let a ray with the origin at F_1 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at A and B, respectively. Let a ray with the origin at F_2 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at C and D, respectively. Suppose the points B and C lie on a branch $\mathcal{H}_1$ of the hyperbola with the foci at F_1 and F_2. Then:*

a) the points A and D lie on a branch $\mathcal{H}_2$ of the hyperbola with the foci at F_1 and F_2. (see Appendix A)

b) Consider a ray starting at F_1 intersecting the branch $\mathcal{H}_1$ at P_1. Consider the ray F_2P_1 intersecting the ellipse $\mathcal{E}_2$ at P_2. Analogously, we define the points P_3, P_4, P_5. Then $P_5 = P_1$ (see Appendix A).

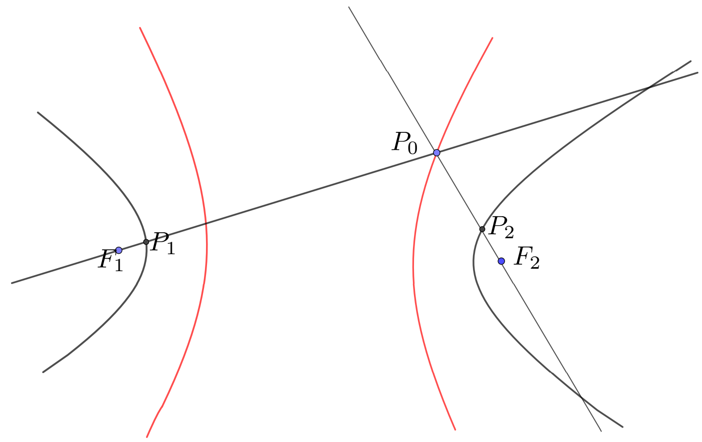

Figure 8.12:

Proof. □

8.2 Exercises

Exercise 5. *Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the circle $(x+c)^2 + y^2 = 4a^2$ defines a Poncelet pair such that all orbits have period 3.*

8.3 Confocal Properties of Conics

8.4 Ivory's Theorem

Theorem 13. *Consider the family of confocal conics defined by*

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} - 1 = 0$$

Then the two diagonals of a quadrangle made of arcs of ellipses and hyperbolas have equal length. In Figure B.1 we have that $|A-C| = |B-D|$.

Proof. Let

$$\alpha(u,v) = \left[\sqrt{\frac{(a^2-v)\,(a^2-u)}{a^2-b^2}}, \sqrt{-\frac{(b^2-v)\,(b^2-u)}{a^2-b^2}}\,\right]$$

with $u \in [b^2, a^2]$ and $v \in (-\infty, b^2) \cup (a^2, \infty)$. □

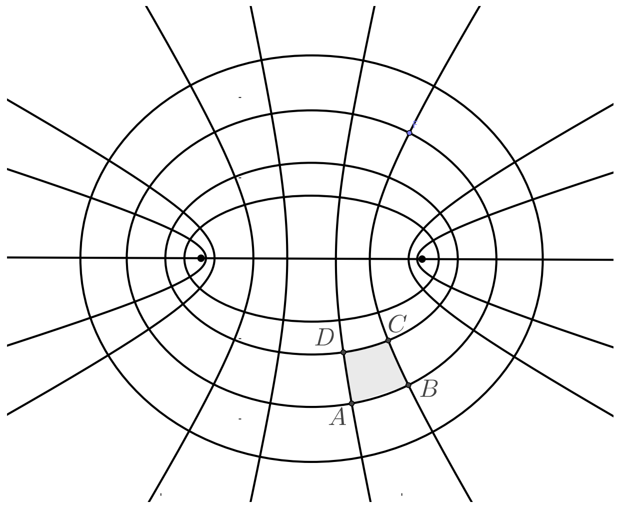

Figure 8.13: Confocal conics and quadrangles made of arcs of ellipses and hyperbolas.

8.5 Graves' Theorem and Periodicity

Proposition 16 (Darboux (1917, Chapitre III))**.** *Consider two confocal ellipses $\mathcal{E}$ and $\mathcal{E}'$ and a point $M \in \mathcal{E}$. Consider the two tangents ℓ_P and ℓ_Q, as shown in Figure B.2, intersecting $\mathcal{E}_1$ in P and Q. Then $|MP| + |MQ| - arc(P, Q) = \mathrm{cte}$, where $arc(P, Q)$ is the length of the elliptic arc with extremal points P and Q. In particular, in a billiard triangle conv$[P_1, P_2, P_3]$, $|P_1P_2| + |P_2P_3| + |P_3P_1| - L(\mathcal{E}_1) = c_1$, where $L(\mathcal{E}_1)$ is the length of $\mathcal{E}_1$, and all the billiard triangles have the same perimeter.*

Proof. See Chasles (1843), Darboux (1917, pp. 283-284) and Ragazzo, Dias Carneiro, and Addas Zanata (2005, pp. 115-116). □

The above result is valid for any billiard in a convex curve having caustics.

Proposition 17. *Let $\mathcal{E}$ be an ellipse of length L. For P_0 outside $\mathcal{E}$ let $L(P_0)$ be the length of a string through P_0 stretched tightly around $\mathcal{E}$. For each $r > L$, let $C_r = \{P_0 \in \mathbb{R}^2 : L(P_0) = r\}$. Then C_r is a confocal ellipse with $\mathcal{E}$. See* **??**.

Proof. See A. V. Akopyan and Zaslavsky (2007a, page 14). □

8.6 Skew-Hodograph map

Consider an ellipse $\mathcal{E} = \{p \in \mathbb{R}^2 : \langle Ap, p\rangle = 1\}$, where A is a positive self-adjoint matrix.

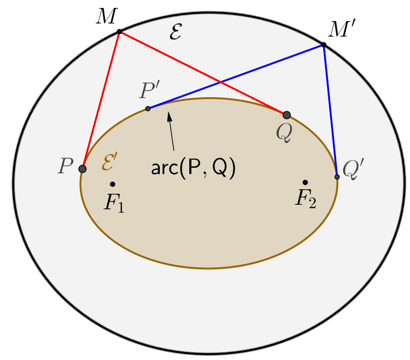

Figure 8.14: Tangents to a confocal ellipse $\mathcal{E}'$ and invariance of the length of chords.

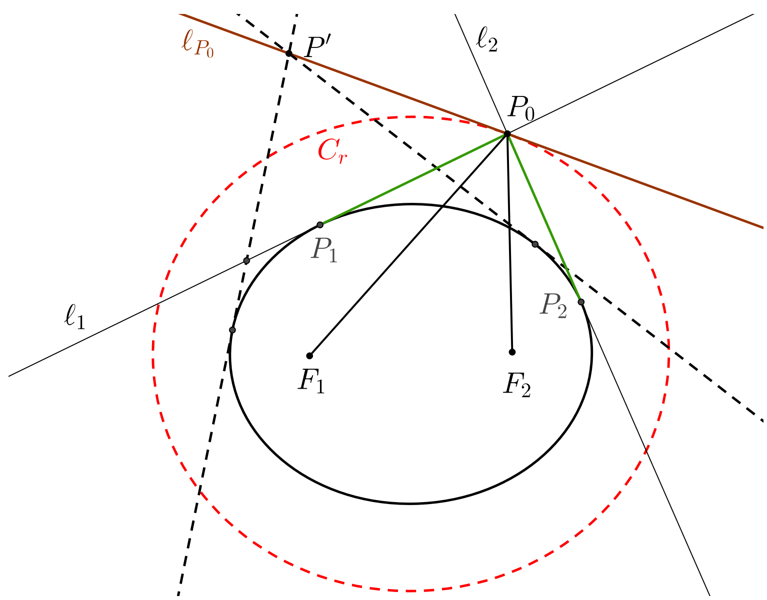

Figure 8.15: C_r is a confocal ellipse with $\mathcal{E}$.

In an elliptic billiard orbit (x_k, y_k) with $x_k \in \mathcal{E}$ and a unit vector $y_k \in \mathbb{R}^2$ we have that:

$$x_{k+1} = x_k \ + \mu_k y_{k+1}, \quad y_{k+1} = y_k + \ \nu_k A x_k$$
$$\nu_k = \ -\frac{2\langle A x_k, y_k\rangle}{\langle A x_k, A y_k\rangle}, \quad \mu_k = \ -\frac{2\langle A y_{k+1}, x_k\rangle}{\langle A y_{k+1}, y_{k+1}\rangle}, \quad y_{k+1} = \ \frac{x_{k+1} - x_k}{|x_{k+1} - x_k|}$$

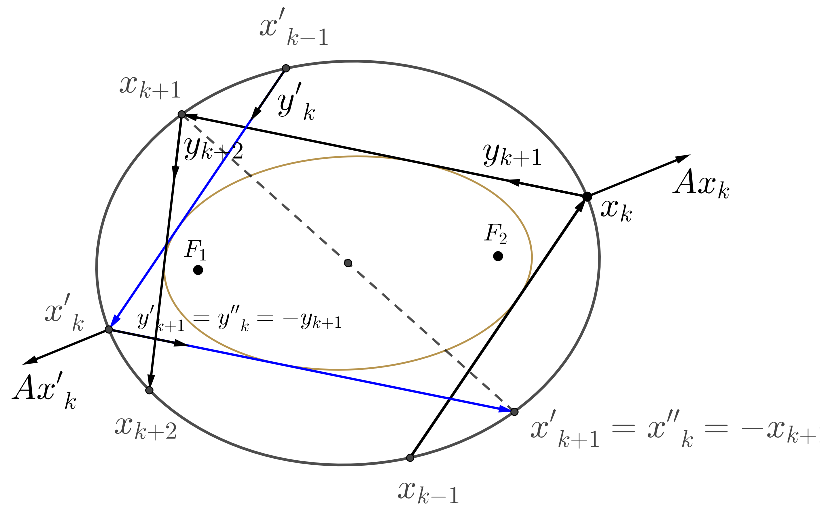

Figure 8.16: The skew-hodograph map $\phi(x_k, y_k) = (x'_k, y'_k)$ commutes with T and $\phi^2 = -T$.

Let the skew-hodograph mapping $\phi(x, y) = (x', y')$ defined by

$$\begin{aligned} x'_k &= C y_{k+1} = C(y_k + \nu_k A x_k) \\ y'_k &= -C^{-1} x_k, \ \ C = A^{-\frac{1}{2}} \end{aligned} \tag{8.1}$$

Proposition 18. *Let $T(x_k, y_k) = (x_{k+1}, y_{k+1})$ the billiard map. Then $\phi \circ T = T \circ \phi$ and $\phi^2 = \phi \circ \phi = -T$.*

Proof. Since A and C are selfadjoint matrices it follows that:

$$\begin{aligned}
\langle Ax'_k, x'_k\rangle &= \langle ACy_{k+1}, Cy_{k+1}\rangle = \langle AA^{-\frac{1}{2}}y_{k+1}, A^{-\frac{1}{2}}y_{k+1}\rangle = \langle A^{\frac{1}{2}}y_{k+1}, A^{-\frac{1}{2}}y_{k+1}\rangle \\
&= \langle A^{-\frac{1}{2}}A^{\frac{1}{2}}y_{k+1}, y_{k+1}\rangle = \langle y_{k+1}, y_{k+1}\rangle = 1 \\
\langle y'_k, y'_k\rangle &= \langle -C^{-1}x_k, -C^{-1}x_k\rangle = \langle A^{\frac{1}{2}}x_k, A^{\frac{1}{2}}x_k\rangle = \langle Ax_k, x_k\rangle = 1.
\end{aligned}$$

Straightforward calculations shows that

$$\nu'_k = -\mu_k, \quad \mu'_k = -\nu_{k+1}.$$

Therefore,

$$\begin{aligned}
x'_{k+1} - x'_k &= C(y_{k+2} - y_{k+1}) = C\nu_{k+1}Ax_{k+1} = \nu_{k+1}A^{\frac{1}{2}}x_{k+1} \\
&= -\nu_{k+1}C^{-1}x_{k+1} = -\nu_{k+1}y'_{k+1} = \mu'_k y'_{k+1} \\
y'_{k+1} - y'_k &= -C^{-1}(x_{k+1} - x_k) = -C^{-1}(\mu_k y_{k+1}) = -\mu_k C^{-1}(C^{-1}x'_k) \\
&= -\mu_k A^{\frac{1}{2}}A^{\frac{1}{2}}x'_k = -\mu_k Ax'_k = \nu'_k Ax'_k.
\end{aligned}$$

This means that the (x'_k, y'_k) is also a billiard orbit and so $\phi \circ T = T \circ \phi$. Finally,

$$\begin{aligned}
x''_k &= Cy'_{k+1} = C(-C^{-1}x_{k+1}) = -x_{k+1} \\
y''_k &= -C^{-1}x'_k = -C^{-1}(Cy_{k+1}) = -y_{k+1}
\end{aligned}$$

So, $\phi^2 = -T$. □

8.7 Properties of Convex Billiards

8.8 Properties of the chords and variation of length

In this section we obtain some properties of chords of convex curves and applications in billiard orbits.

Consider two regular convex curves γ and Γ parametrized by arc lengths s and t. Let $l(s,t) = |\gamma(s) - \Gamma(t)|$, $\theta(s,t)$ the angle between $\gamma'(s)$ and $V(s,t) = \Gamma(t) - \gamma(s)$ and $\eta(s,t)$ the angle between $\Gamma'(t)$ and $V(s,t)$. See Fig. C.1

Consider the Frenet frames $\{\gamma'(s), N_\gamma\}$ and $\{\Gamma'(t), N_\Gamma\}$ along γ and Γ. Denote the curvatures by k_γ and k_Γ.

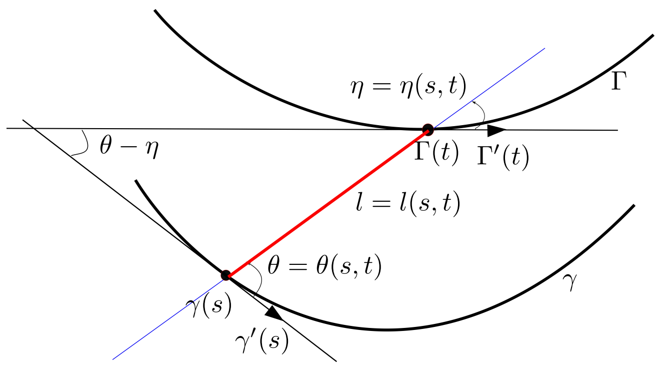

Figure 8.17: Pair of curves and variations of length and angles.

Proposition 19. *In the above conditions it follows that:*

$$\begin{aligned} dl &= -\cos\theta\, ds + \cos\eta\, dt \\ d\theta &= \left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{\sin\eta}{l} dt \\ d\eta &= \frac{\sin\theta}{l} ds - \left(\frac{\sin\eta}{l} + k_\Gamma(t)\right) dt \end{aligned} \tag{8.2}$$

Proof. We have that

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$$

From the equation

$$l^2 = \langle \Gamma(t) - \gamma(s), \Gamma(t) - \gamma(s) \rangle$$

it follows that

$$\begin{aligned} 2l\frac{\partial l}{\partial s} &= -2\langle \gamma'(s), \Gamma(t) - \gamma(s)\rangle = -2l\cos\theta \quad \Rightarrow \quad l_s = -\cos\theta \\ 2l\frac{\partial l}{\partial t} &= 2\langle \Gamma'(t), \Gamma(t) - \gamma(s)\rangle = 2l\cos\eta \quad \Rightarrow \quad l_t = \cos\eta \end{aligned}$$

From the equations

$$l(s,t)\cos\theta = \langle \gamma'(s), \Gamma(t) - \gamma(s)\rangle, \quad l(s,t)\cos\eta = \langle \Gamma'(t), \Gamma(t) - \gamma(s)\rangle$$

it follows that

$$
\begin{aligned}
l_s \cos\theta - l\theta_s \sin\theta &= \langle \gamma''(s), \Gamma(t) - \gamma(s)\rangle - \langle \gamma'(s), \gamma'(s)\rangle \\
&= \langle \gamma''(s), l\cos\theta\gamma' + l\sin\theta N_\gamma(s)\rangle - 1 \\
&= l\sin\theta k_\gamma(s) - 1 \\
l_t \cos\theta - l\theta_t \sin\theta &= \langle \gamma'(s), \Gamma'(t)\rangle = \cos(\theta - \eta) = \cos(\eta - \theta) \\
l_s \cos\eta - l\eta_s \sin\eta &= \langle \gamma'(s), \Gamma'(t)\rangle = \cos(\theta - \eta) \\
l_t \cos\eta - l\eta_t \sin\eta &= \langle \Gamma''(t), \Gamma(t) - \gamma(s)\rangle + \langle \Gamma'(t), \Gamma'(t)\rangle \\
&= \langle \Gamma''(t), l\cos\eta\Gamma' + l\sin\eta N_\Gamma(t)\rangle + 1 \\
&= \ k_\Gamma l \sin\eta + 1
\end{aligned}
$$

Performing the calculations leads to the result. ☐

Proposition 20. *In the same conditions above but with arc length parameters s and t it follows that*

$$
\begin{aligned}
l_{ss} &= \sin\theta\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) \\
l_{st} &= \frac{\sin\theta\sin\eta}{l} \\
l_{tt} &= \sin\eta\left(\frac{\sin\eta}{l} - k_\Gamma(s)\right)
\end{aligned}
\tag{8.3}
$$

Proof. Follows directly from differentiation of equation (C.1). ☐

Proposition 21. *In the same conditions above but with arbitrary parameters s and t it follows that*

$$
\begin{aligned}
dl &= -|\gamma'(s)|\cos\theta\ ds + |\Gamma'(t)|\cos\eta\ dt \\
d\theta &= |\gamma'(s)|\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{|\Gamma'(t)|\sin\eta}{l}\ dt \\
d\eta &= \frac{|\gamma'(s)|\sin\theta}{l}\ ds - |\Gamma'(t)|\left(\frac{\sin\eta}{l} + k_\Gamma(t)\right)\ dt
\end{aligned}
\tag{8.4}
$$

Proposition 22. *Consider a billiard in a region with boundary a convex curve Γ. Let γ be the caustic of a family of orbits as shown in Fig. C.2. Then for any $x \in \Gamma$*

$$|x - y| + |x - z| - arc(y, z) = cte.$$

Here $y, z \in \gamma$ are the points of tangency of the billiard orbit passing through x with the caustic and $arc(x, z)$ is the length of caustic between y and z.

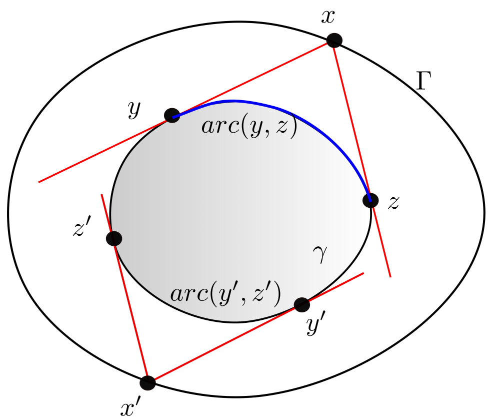

Figure 8.18: Tangents to a caustic and length of chords.

Sketch of Proof. Let $\Gamma(t)$ be a parametrization of the boundary. Consider also local parametrizations $\gamma_1(t)$ and $\gamma_2(t)$ of the caustic γ with $\gamma_1(0) = z$, $\gamma_2(0) = y$, and $\Gamma(0) = x$. Suppose that all curves are counterclockwise oriented. Let also the caustic parametrized by natural parameter s. Then

$$\gamma(s) = \gamma_1(t) = \Gamma(t) + \lambda(t)d_1(t), \quad \gamma(s) = \gamma_2(t) = \Gamma(t) + \lambda(t)d_21(t).$$

Here d_1 and d_2 are the directions of the tangent lines xy and xz to the caustic.

Let $l_1(t) = |\Gamma(t) - \gamma_1(t)|$ with $l_1(0) = |x - z|$. Also define $l_2(t) = |\Gamma(t) - \gamma_2(t)|$ with $l_2(0) = |x - y|$.

By Proposition 17 it follows that

$$\begin{aligned} dl_1 &= \cos\eta|\Gamma'(t)|dt - |\gamma_1'(s)|ds \\ dl_2 &= |\gamma_2'(s)|ds - \cos\eta|\Gamma'(t)|dt \end{aligned}$$

Here we used the condition of billiard orbit at the point x (angle of incidence is equal to angle of reflection) and that $\cos\theta_{1,2} = \pm 1$ (caustic is tangent to billiard orbits, taking into account the orientation). Therefore it follows that

$$d(l_1 + l_2) - |\gamma_2'(s)|ds + |\gamma_1'(s)|ds = 0.$$

Integrating it follows that

$$l_1(a) - l_1(0) + l_2(a) - l_2(0) = arc(\gamma_1(0), \gamma_1(a)) - arc(\gamma_2(0), \gamma_2(a))$$

Therefore,

$$l_1(a) + l_2(a) - arc(\gamma_1(a), \gamma_2(a)) = l_1(0) + l_2(0) - arc(\gamma_1(0), \gamma_2(0)).$$

$\square$

8.9 Joachimsthal's Integral

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Proposition 23. *Consider an ellipse $\mathcal{E}$ defined by $\langle Ap, p\rangle = 1$. Let u be an inward unit vector in the direction of the billiard orbit passing through the point $p_0 \in \mathcal{E}$. Let $T(p_1, u) = (p_2, v)$ the billiard map as shown in Figure C.3. Then*

$$\langle Ap_1, u\rangle = -\langle Ap_2, u\rangle = \langle Ap_2, v\rangle$$

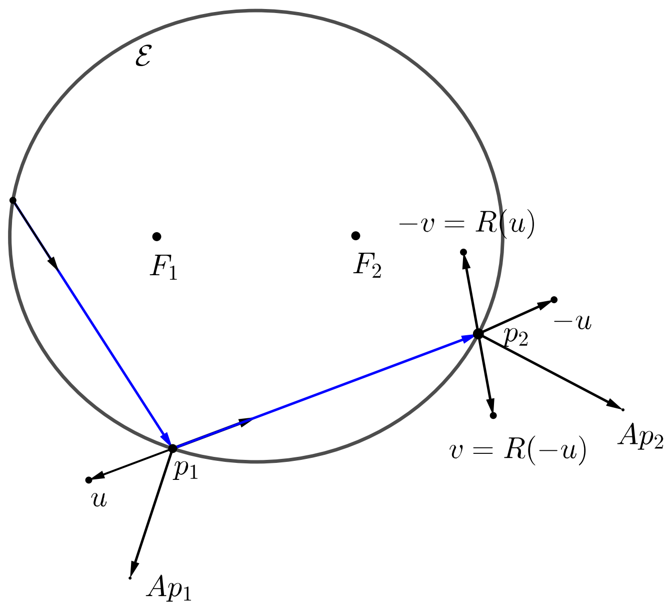

Figure 8.19: Joachimsthal's first integral $\langle Ap_1, u\rangle$ is T-invariant.

Proof. The tangent space $T_p\mathcal{E}$ is formed of the vectors u such that $\langle Ap, u\rangle = 0$. Therefore Ap is a normal vector to the ellipse at the point p. The vector u is proportional to $p_2 - p_1$.

Therefore,

$$\begin{aligned}\langle Ap_1 + Ap_2, p_2 - p_1\rangle &= \langle Ap_1, p_2\rangle + \langle Ap_2, p_1\rangle - \langle Ap_1, p_1\rangle + \langle Ap_2, p_2\rangle \\ &= \langle p_1, Ap_2\rangle - \langle Ap_2, p_1\rangle = 0.\end{aligned}$$

Then,

$$\langle Ap_1, u\rangle = \langle Ap_2, -u\rangle = \langle Ap_2, R(-u)\rangle = \langle Ap_2, v\rangle$$

□

8.10 Other Types of Billiards

8.11 Exercises

Chapter 9

Conclusion

55

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— (Apr. 2020a). *Angles $\phi = \pi - \theta_i$ (resp. $\phi' = \phi - \theta_i'$), so equivalent to invariant sum (resp. product) of cosines.* Private Communication.

— (Jan. 2020b). *Corollary of Theorem 6 in Akopyan et al., "Billiards in Ellipses Revisited" (2020).* Private Communication.

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