Poncelet Invariants:
an Experimental Promenade

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Properties of N=3 Poncelet Families

3.1 Introduction

This research started upon conversations with Jair Koiller about the geometry of elliptic billiard trajectories, following one author's recent work in control of heliostat fields for solar energy plants[1]; see **gross2020-solar** for a recent publication. An early, naïve experimental artifact was an animation of 3-periodics in the elliptic billiard along with the locus of their incenter; see Reznik (2011d). A natural choice given that each vertex is bisected by the ellipse normal. At the time we did not know this locus could be an ellipse, and indeed, how rare a find this is (we have conjecture that amongst the 5d space of possible ellipse pairs, only in the confocal pair – a 1d subspace – can the locus of the incenter be a conic). A twin animation was also produced depicting the self-intersected locus of the intouchpoints; see Reznik (2011b).

Shortly thereafter Romaskevich (2014) produced a proof using methods of complex algebraic geometry. This was followed by an alternative proof using techniques of real analytic geometry given explicitly the equation of the locus; see Garcia (2019). The loci centroids of Poncelet polygons was studied in Schwartz and Sergei Tabachnikov (2016a). In Garcia (2019) the centroid locus was given explicitly. Fierobe (2021) and **garcia2018** proved that the locus of the circumcenter over billiard 3-periodics is also an ellipse.

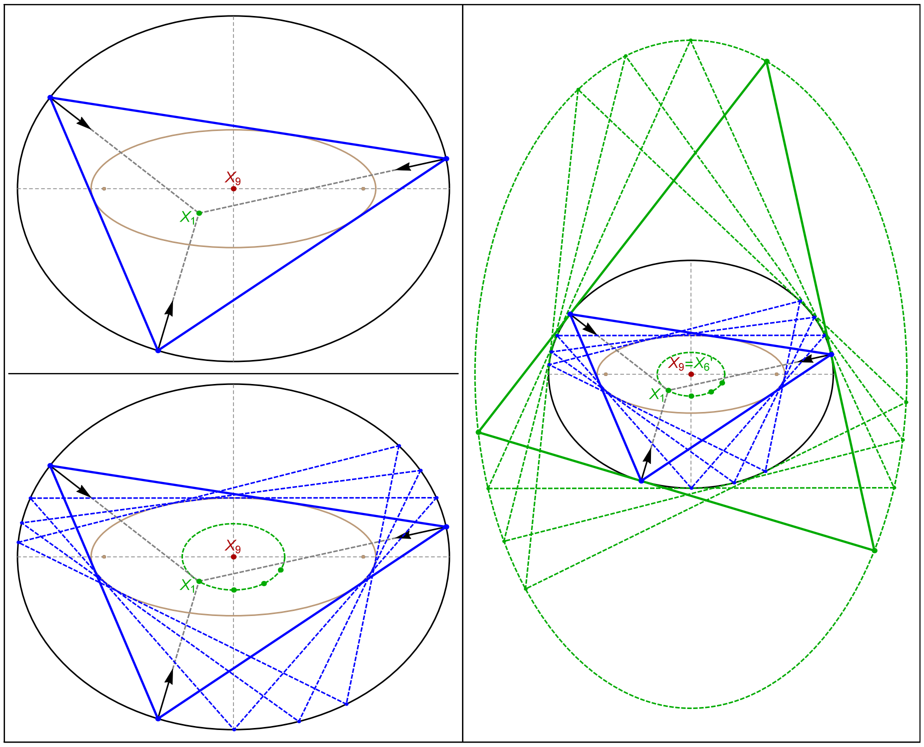

Figure 3.1: **Top Left:** An elliptic billiard 3-periodic (solid blue) is shown inscribed in an outer ellipse (black) and a confocal caustic (brown). Graves' theorem implies its internal angles will be bisected by ellipse normals (black arrows). Also shown is the incenter X_1 defined as the intersection of said bisectors. **Bottom Left:** Poncelet's porism implies a 1d family of such triangles exists. Some samples are shown (dashed blue). A classic invariant is perimeter. The Mittenpunkt X_9 remains stationary at the center. The incenter X_1 sweeps an ellipse (dashed green). **Right:** The excentral triangle (solid green) has sides perpendicular to the bisectors. Over billiard 3-periodics, the excentral is of variable perimeter. Its vertices (known as the "excenters") also sweep an ellipse (dashed green) whose aspect ratio is the reciprocal of that of the incenter locus. The Symmedian point X_6' of the excentral triangle coincides with X_9 of the reference and is therefore stationary.

3.2 The Confocal Family (Elliptic Billiard)

Henceforth let *billiard 3-periodics* refer to the 1d family of Poncelet triangles interscribed between pair of confocal ellipses $\mathcal{E}$ and $\mathcal{E}_c$ given by:

$$\mathcal{E}:\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad \mathcal{E}_c:\frac{x^2}{a_c^2}+\frac{y^2}{b_c^2}-1=0$$

where $c^2=a^2-b^2=a_c^2-b_c^2$.

The Cayley condition for a concentric, axis parallel pair of ellipses to admit a 3-periodic family was derived in ronaldo, and is reproduced below:

$$\frac{a_c}{a}+\frac{b_c}{b}=1 \tag{3.1}$$

In turn, this constrains the semi-axes of the confocal caustic.

Proposition 1. *The semi-axes a_c, b_c of the confocal caustic are given by:*

$$a_c=\frac{a\left(\delta-b^2\right)}{c^2},\quad b_c=\frac{b\left(a^2-\delta\right)}{c^2}.$$

where $\delta=\sqrt{a^4-a^2b^2+b^4}$, an oft-occurring quantity, will be henceforth called the Darboux constant.

Billiard N-periodics classically conserve perimeter L and Joachimsthal's constant J (see section xxx). When $N=3$, we can derive these explicitly.

Proposition 2. *For billiard 3-periodics, the perimeter and Joachimsthal's constant are given by:*

$$J=\frac{\sqrt{2\delta-a^2-b^2}}{c^2},\quad L=2(\delta+a^2+b^2)J$$

Proof. We compute the values considering an isosceles 3-periodic orbit with $P_1=[a,0]$, and

$$P_2=\left[\frac{a\left(b^2-\delta\right)}{a^2-b^2},\frac{b^2\sqrt{2\delta-a^2-b^2}}{a^2-b^2}\right],\quad P_3=\left[\frac{a\left(b^2-\delta\right)}{a^2-b^2},-\frac{b^2\sqrt{2\delta-a^2-b^2}}{a^2-b^2}\right] \tag{3.2}$$

We have that

$$L=|P_2-P_3|+2|P_1-P_2|,\quad J=\langle\frac{P_1-P_3}{|P_1-P_3|},[\frac{1}{a},0]\rangle$$

Straigthforward calculations leads to the result stated.

ronaldo: CAS+parametrization □

[1]Such fields are in fact a discretized/flattened focusing paraboloid, see **sundrop2016**; **esolar2017**.

Note: the use of J in this chapter refers to its value for the $N = 3$ case. Referring to Figure 3.1:

Theorem 1. *Over billiard 3-periodics, the locus of the incenter X_1 and excenter are ellipses $\mathcal{E}_1$ and $\mathcal{E}'$ concentric and axis-parallel with the confocal pair whose axes (a_1, b_1) and (a', b') are given by:*

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}$$
$$a' = \frac{b^2 + \delta}{a}, \quad b' = \frac{a^2 + \delta}{b}$$

Furthermore, $\mathcal{E}_1$ and $\mathcal{E}'$ have reciprocal aspect ratios, i.e., $a_1/b_1 = b_e/a_e$.

Proof. ronaldo □

It turns out δ has a curious geometric interpretation. Recall the power of a point Q with respect to a circle $\mathcal{C} = (C_0, R_0)$ is given by $|Q - C_0|^2 - R_0^2$, see Eric Weisstein (2019, Circle Power). Let $\mathcal{C}$ denote the (moving) circumcircle of billiard 3-periodic, and $O = X_9$ the billiard center.

Proposition 3. *The power of O with respect to $\mathcal{C}$ is constant and equal to $-\delta$.*

Proof. Consider an isosceles 3-periodic orbit given by Equation (3.2).

Its circumcircle will be centered at $C_0 = [\frac{b^2-\delta}{2b}, 0]$ with circumradius $R_0 = \frac{b^2+\delta}{2b}$. Therefore, the power of the center of the ellipse with respect to the circumcircle is given by

$$|OC_0|^2 - R_0^2 = \left(\frac{b^2 - \delta}{2b}\right)^2 - \left(\frac{b^2 + \delta}{2b}\right)^2 = -\delta.$$

For a generic 3-periodic orbit, the stated invariance is confirmed via a CAS, using the explicit vertex expressions given by localizar depois □

The Mittenpunkt X_9 is a triangle center where lines from each excenter thru the side midpoint meet. Referring to Figure 3.2:

Theorem 2. *Over the family of 3-periodics in the elliptic billiard, X_9 is stationary at the common center.*

An elegant syntethic proof was kindly contributed by Romaskevich (2019):

Proof. Let $\mathcal{E}$ be the outer ellipse in the confocal pair, O. By definition, the Mittenpunkt X_9 is where lines from the excenters E_i through the side midpoints M_i concur. Notice each side is an ellipse chord between tangents to $\mathcal{E}$ seen from the E_i (this is because in the confocal pair the excentral triangle is tangent to $\mathcal{E}$). Consider the image of lines E_iM_i under an affine transform which sends $\mathcal{E}$ to a circle $\mathcal{C}'$, let O' be its center. The transformed lines will pass through the midpoints of chords of $\mathcal{C}'$ between tangents seen from E_i' (the affine image of E_i). By circular symmetry, such lines must also pass through O', and therefore remain stationary. But O' is the affine image of O, so the result follows. □

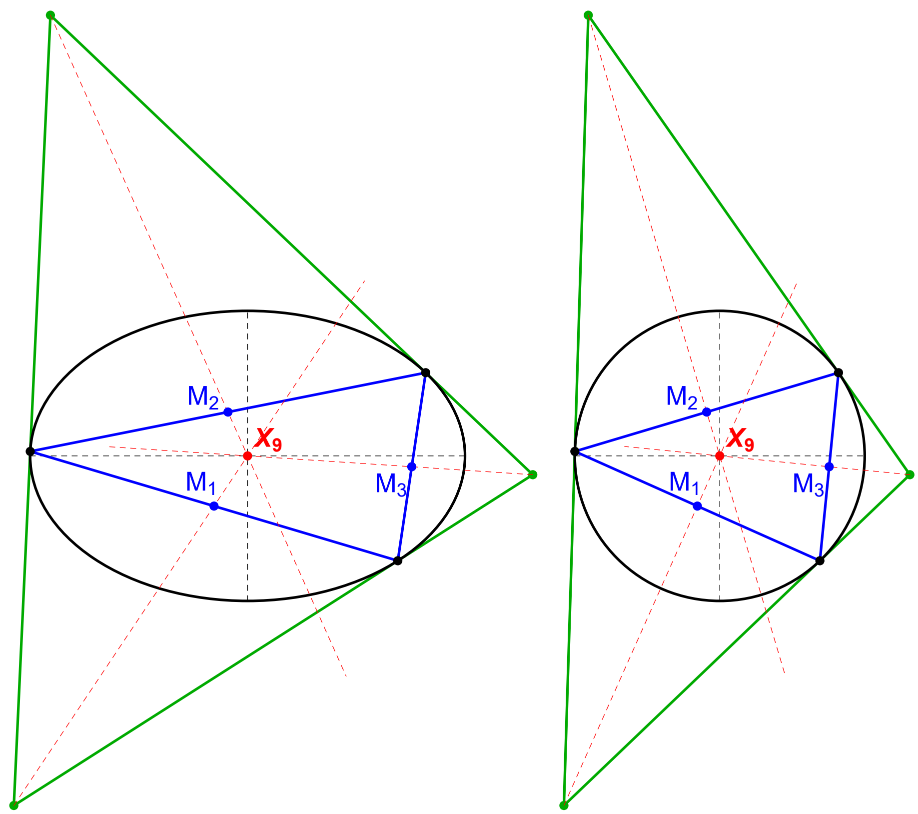

Figure 3.2: **Left**: 3-periodic billiard triangle (blue), its excentral triangle (green). The Mittenpunkt X_9 is the point of concurrence of lines drawn from the excenters through sides' midpoints M_i. **Right**: the affine image which sends the billiard to a circle. Lines from imaged excenters through sides' midpoints must pass through the origin. Since the latter is stationary, so must be its pre-image X_9, which is stationary at the billiard center. Video

Given a triangle, let r and R denote the radius of its incircle and circumcircle, known as the *inradius* and *circumradius*, respectively. Over billiard 3-periodics, note these two radii are variable. Referring to Figure 3.3:

Theorem 3. *r/R is invariant over the 3-periodic orbit family and given by:*

$$\frac{r}{R} = \frac{2(\delta - b^2)(a^2 - \delta)}{c^4}.$$

Proof. The following relation, found in Johnson (1960), holds for any triangle:

$$rR = \frac{s_1 s_2 s_3}{2L},$$

where $L = s_1 + s_2 + s_3$ is the perimeter, constant for 3-periodic orbits. Therefore:

$$\frac{r}{R} = \frac{1}{2L}\frac{s_1 s_2 s_3}{R^2}. \tag{3.3}$$

Next, let $P_1 = (a, 0)$ be a vertex of an isosceles 3-periodic. Obtain a candidate expression for r/R. This yields (3) exactly. Using explicit expressions for orbit vertices (see ronaldo), derive an expression for the square of the right-hand side of (3.3) as a function of x_1 and subtract from it the square of (3). In Garcia, Reznik, and Koiller (2020b) it is shown $\left(s_1 s_2 s_3/R^2\right)^2$ is rational on x_1. For simplification, use $R = s_1 s_2 s_3/(4A)$, where A is the triangle area. With a CAS, show said difference is identically zero for all $x_1 \in (-a, a)$. □

Let θ_i, r, R, and A denote the ith internal angle, inradius, circumradius, and area of a reference triangle. Primed quantities refer to the excentral triangle. The relations below, appearing in Johnson (1960), hold for any triangle:

$$\sum_{i=1}^{3} \cos\theta_i = 1 + \frac{r}{R} \tag{3.4}$$

$$\prod_{i=1}^{3} \cos\theta_i' = \frac{r}{4R} \tag{3.5}$$

$$\frac{A}{A'} = \frac{r}{2R} \tag{3.6}$$

Corollary 1. *Over billiard 3-periodics, also invariant are the sum of the orbit cosines, the product of excentral cosines, and the ratio of excentral-to-orbit areas.*

Direct calculations yields an expression for the invariant sum of cosines in terms of elliptic billiard constants J and L.

Corollary 2. $\sum_{i=1}^{3} \cos\theta_i = JL - 3$

At it will be seen later, the above generalizes to $JL - N$ for all N.

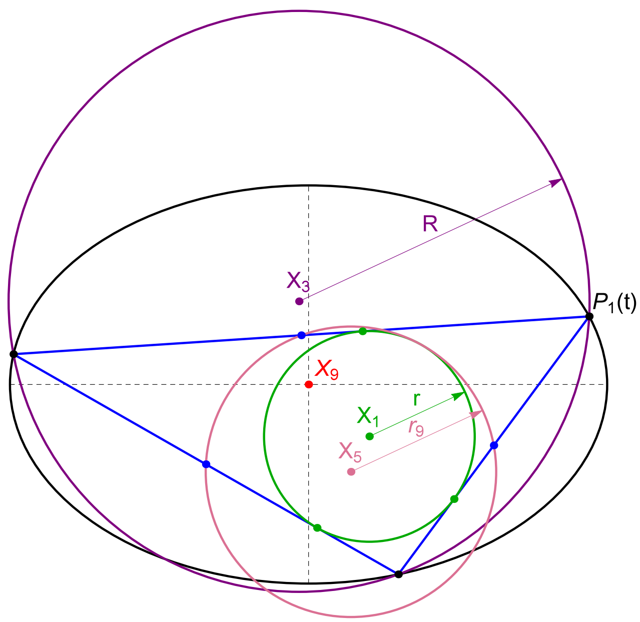

Figure 3.3: The incircle (green), circumcircle (purple), and 9-point (Euler's) circle (pink) of a billiard triangle (blue). These are centered on X_1, X_3, and X_5, respectively. Their radii are the inradius r, circumradius R, and 9-point circle radius $r_9 = 2R$. Over the family, the ratio r/R is invariant. In turn this implies an invariant sum of cosines.

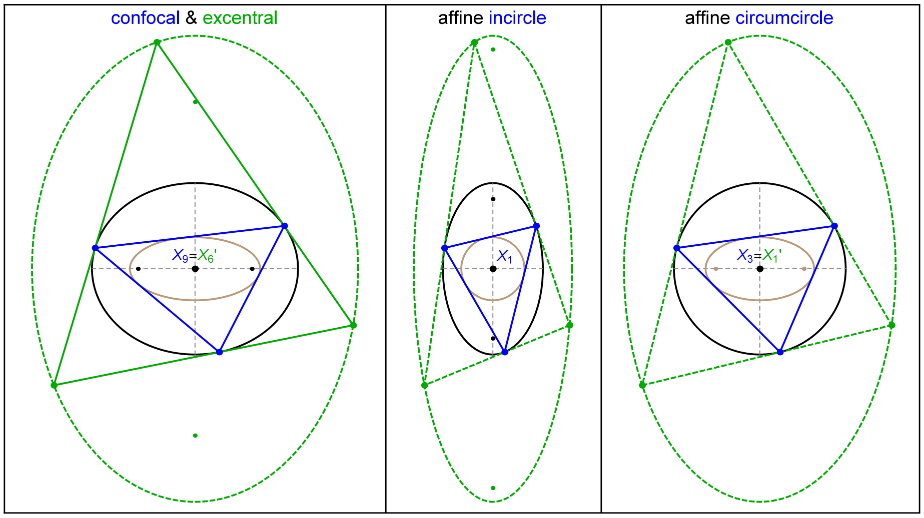

Figure 3.4: **Left:** billiard 3-periodic (blue) and its excentral triangle (green). The former conserves the sum of its cosines. The latter is inscribed in an ellipse (dashed green) and conserves the product of its cosines. **Middle:** Affine image of confocal family which sends caustic (brown) to a circle. This family also conserves the sum of cosines, equal to that conserved by its confocal pre-image. **Right:** Affine image of confocal family which sends billiard ellipse (black) to a circle. This family also conserves the product of cosines, equal to that conserved by the excentral family of its pre-image. Video

3.3 Other Notable Concentric, Axis-Parallel, Poncelet Families

Below we introduce five additional notable 3-periodic Poncelet families interscribed between a pair of concentric, axis-parallel ellipses. As before, the Cayley condition Equation (3.1) will be used to constrain the ellipse pair.

3.3.1 Excentral Family

This is the Ponceletian family of excentral triangles to billiard 3-periodics. If the axes of its caustic are a, b, this family is inscribed in an ellipse with a', b' given in Theorem 1; see Figure 3.4(left).

The symmedian point X_6 of the excentral triangle coincides with the mittenpunkt of its reference, see Kimberling (2019, X(6)). Therefore:

Corollary 3. *Over excentral 3-periodics, the symmedian point X_6 of the excentral family is stationary.*

Recall that in Corollary 1 two other notable invariants are mentioned: product of internal angle cosines, and ratio of its area by billiard 3-periodics.

Corollary 4. *The invariant product of cosines of excentral 3-periodics is a quarter of the quantity in Corollary 1. Furthermore the area ratio of billiard 3-periodics by excentrals is half the quantity in Corollary 1.*

Let s_i' denote the variable sidelengths of the excentral family, $i = 1, 2, 3$. Here is an additional curious invariant:

Proposition 4. *Over the excentral family, the sum squared sidelines divided by the product of sidelines is constant. Furthermore it is equal to Joachimsthal's constant J of its parent 3-periodic billiard family. Explicitly:*

$$\frac{\sum (s_i')^2}{\prod s_i'} = \frac{\sqrt{2\delta - a^2 - b^2}}{c^2} = J$$

Proof. Derive explicit expressions for excentral sidelengths and arrive at claim via CAS simplification. □

Referring to Figure 3.5, the *cosine circle* of a triangle is defined in Eric Weisstein (2019, Cosine Circle) as being centered on the symmedian point X_6 and containing the 6 intersections of lines through X_6 parallel to the sides of the orthic triangle. Its radius r^* is given by the product of sidelengths divided by the sum of their squares.

Corollary 5. *The cosine circle of the excentral family is stationary with radius $r^* = 1/J$.*

3.3.2 Incircle Family

The incircle family, shown in Figure 3.4(middle), is the Poncelet family in a concentric, axis-parallel pair for which the caustic is a circle (let r denote its radius). It follows immediately that the family's incenter X_1 is stationary. Let a_e, b_e be the axes of the ellipse the family is inscribed in. Cayley yields $r = a_c = b_c = (a_e b_e)/(a_e + b_e)$.

Proposition 5. *The incircle family has invariant circumradius given by $R = (a_e + b_e)/2$. Furthermore, the locus of its circumcenter X_3 is a circle of radius $d = R - b_e = a_e - R$ centered on the common center $O = X_1$.*

refer to figure with circular locus of X_3

Proof. Let $P_1 = (x_1, y_1)$ be a first vertex of the incircle family. Using an explicit parametrization for P_2 and P_3, obtain via CAS the following coordinates for the moving circumcenter X_3:

$$X_3 = \frac{a_e - b_e}{2}\left[-\frac{x_1\left(-x_1^2(a_e+b_e)^2 + {a_e}^2 b_e(2a_e+b_e)\right)}{a_e\left(({a_e}^2 - {b_e}^2)x_1^2 + {a_e}^2{b_e}^2\right)}, \frac{y_1\left(x_1^2(a_e+b_e)^2 - {a_e}^2{b_e}^2\right)}{b_e\left({a_e}^2x_1^2 + {b_e}^2\left({a_e}^2 - x_1^2\right)\right)}\right]$$

And circumradius $R = |P_1 - X_3| = (a_e + b_e)/2$. Also obtain that the locus of X_3 is a circle concentric with the incircle and of radius $(a_e - b_e)/2$. □

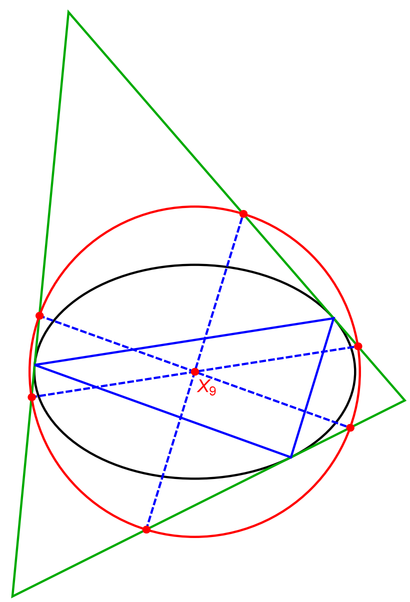

Figure 3.5: The cosine circle (red) of the excentral family (green) is stationary. It contains the 6 intersections of lines (dashed blue) through the common center (family's X_6 and billiard periodics' X_9) which are parallel to their orthic, i.e., sidelines of billiard 3 periodics. Video

Referring to Section 3.2: problema com cref

Corollary 6. *The incircle family conserves its sum of cosines given by:*

$$\sum \cos\theta_i = 1 + \frac{r}{R} = \frac{{a_e}^2 + 4a_e b_e + {b_e}^2}{(a_e + b_e)^2}$$

Affine image: As Figure 3.4(middle) depicts, the incircle family can also be obtained from an affine image of billiard 3-periodics which sends the confocal caustic to a circle. As before, let a, b be the semi-axes of its billiard ellipse pre-image.

Lemma 1. *The confocal family is sent to the incircle family by scaling it along the major axis by an amount s given by:*

$$s = \frac{b(\delta - c^2)}{a^3}$$

Proof. The scaled family will be inscribed in an ellipse with semi-axes $a_e = sa$, and $b_e = b$. Its caustic will be the circle $r = b_c$, where $b_c = b\left(a^2 - \delta\right)/c^2$ is the confocal caustic minor axis given in Proposition 1. The Cayley condition for the incircle family imposes that $r = b_c = (a_e b_e)/(a_e + b_e)$, i.e., the result follows from solving $b_c = (sab)/(sa + b)$ for s.

□

Surprisingly:

Proposition 6. *The sum of cosines conserved by the incircle family is identical to that conserved by billiard 3-periodics which are its affine pre-image.*

Proof. Let s be the scaling along the major axis in Lemma 1. Plug $a_e = sa$ and $b_e = b$ into Corollary 6, subtract one (to obtain r/R for the incircle family) and verify it yields the expression in Theorem 3. □

3.3.3 Circumcircle Family

The circumcircle family, shown in Figure 3.4(right), is the Poncelet family in a concentric, axis-parallel pair for which the outer ellipse is a circle (let R denote its radius). It follows immediately that the family's circumcenter X_3 is stationary. Let a', b' be the axes of its inellipse and s_i the sidelengths. Cayley imposes $a' + b' = R$.

Lemma 2. *Poncelet triangles in the circumcircle family are always acute.*

Proof. Since the stationary circumcenter X_3 is interior to the caustic caustic, it will be interior to circumcircle family triangles, and the result follows. □

Proposition 7. *The circumcircle family conserves the sum of squared side-lengths. This is given by:*

$$\sum_{i=1}^{3} s_i^2 = 4(a' + 2b')(2a' + b')$$

Proof. ronaldo CAS-assisted simplification from vertex parametrization. □

Proposition 8. *The circumcircle family conserves the product of its internal angle cosines. This is given by:*

$$\prod_{i=1}^{3} \cos\theta_i = \frac{a'b'}{2(a'+b')^2} = \frac{a'b'}{2R^2}$$

Proof. CAS-assisted simplification from vertex parametrization. □

Recall the orthic triangle has vertices at the feet a triangle's altitudes. Let R_h denote its circumradius. In , see Eric Weisstein (2019, Orthic Triangle, Eqn 7), one finds the relation $R_h = R/2$. Therefore R_h is invariant over the circumcircle family. Let r_h denote the orthic's inradius. Referring to Figure 3.6:

Proposition 9. *Over the circumcircle family r_h is invariant and given by $r_h = a'b'/(a'+b')$.*

Proof. In Eric Weisstein (2019, Orthic Triangle, Eqn. 5) one finds the relation $r_h = 2R\prod_{i=1}^{3}\cos\theta_i$. Recalling $R = a'+b'$, substitution into Proposition 8 yields the claim. □

Below we analyze a Poncelet family with fixed incircle and non-concentric fixed circumcircle, known as the "poristic" family. The previous result implies that the orthics derived from the circumcircle family can be regarded as a rigidly moving poristic family.

Affine image: As Figure 3.4(right) depicts, the circumcircle family can also be obtained from an affine image of billiard 3-periodics which sends the billiard ellipse with semi-axes a, b to a circle with radius $R = b$. Therefore billiard 3-periodics are sent to the circumcircle family by scaling it along the major axis by an amount $s' = b/a$. Therefore Proposition 1 implies:

Lemma 3. *The caustic semi-axes a', b' of the circumcircle family which is the s'-affine image of the confocal family are given by:*

$$a' = \frac{b}{a}a_c = \frac{b(\delta - b^2)}{c^2}, \quad b' = b_c = \frac{b(a^2-\delta)}{c^2}$$

Note that the s'-affine image of billiard excentrals becomes a Poncelet family with fixed incircle; see Figure 3.4(right, dashed green triangles). We have seen above such a family conserves its sum of cosines. Suprisingly, the following invariant "role reversal" takes place:

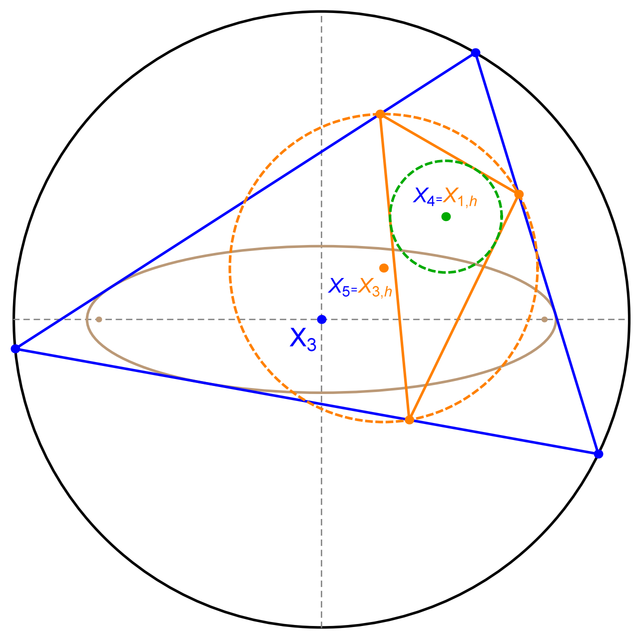

Figure 3.6: A circumcircle family 3-periodic (blue) and its orthic triangle (orange). Over the family the orthic's circumcircle (dashed orange) and incircle (dashed green) have invariant radii. Also shown are their centers $X_{3,h}$ and $X_{1,h}$ which, for any reference triangle, correspond the nine-point center X_5 and orthocenter X_1. Video

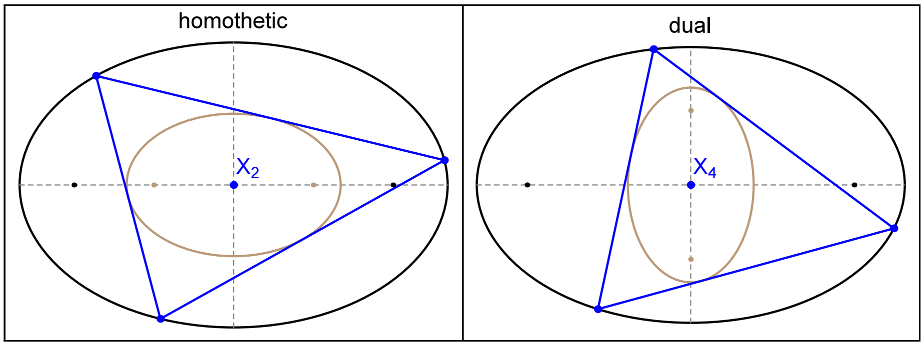

Figure 3.7: **Left:** a Poncelet 3-periodic (blue) interscribed in a concentric, axis-parallel pair of homothetic ellipses. These are affine images of a concentric system and therefore conserve area and the barycenter X_2 is stationary at the common center. **Right:** A pair of concentric, axis-parallel ellipses with reciprocal aspect ratios which admit a Poncelet 3-periodic family (blue). The ellipses are dual of each other and the orthocenter is stationary at the common center.

Proposition 10. *The sum of cosines conserved by billiard 3-periodics is the same as the one conserved by the s'-affine image of billiard excentrals. Furthermore product of cosines conserved by billiard excentrals is the same as the one conserved by the s'-affine image of billiard 3-periodics (circumcircle family). Furthermore,*

Proof. For the first statement it suffices to show that the s'-affine image of billiard excentrals has sides parallel to those of the s-image of billiard 3-periodics, i.e., the incircle family and use Proposition 6. ronaldo consegue seguir □

3.3.4 Homothetic

$$\{P_1, P_2, P_3\}$$

given by Equation (3.7), with $a = 2a_c,\ b = 2b_c$.

3.3.5 Dual

$$\{P_1, P_2, P_3\}$$

given by Equation (3.7), with $a_c = \lambda b,\ b_c = \lambda a$ and $\lambda = ab/(a^2 + b^2)$.

3.3.6 General Vertex Parametrization

Here we consider a general pair concentric, axes-parallel ellipses denoted $\mathcal{E}$ and $\mathcal{E}_c$. We will derive a generic parametrization for the vertices of 3-periodics in such a pair. A first calculation will be helpful. Referring to Figure 3.8(left),

Family	Fixed	Conserves	Notes
Confocal	X_9	L, J, r/R, $\sum \cos\theta_i$	i.e., billiard 3-periodics
Incircle	X_1	R, $\sum \cos\theta_i$	sum of cosines same as confocal affine pre-image
Circumcircle	X_3	$\sum s_i^2$, $\prod \cos\theta_i$, r_h,R_h	product of cosines same as excentrals' in confocal affine pre-image
Confocal Excentrals	X_6	A'/A, $\prod \cos\theta_i'$, $\sum (s_i')^2/\prod s_i'$	primed quantities refer to those of the excentral family
Homothetic	X_2	A, $\sum s_i^2$, ω, $\sum \cot\theta_i$	affine image of concentric circles
Dual	X_4	n/a	

Table 3.1: Summary of fixed points and (known) conserved quantites for the concentric families mentioned in this section.

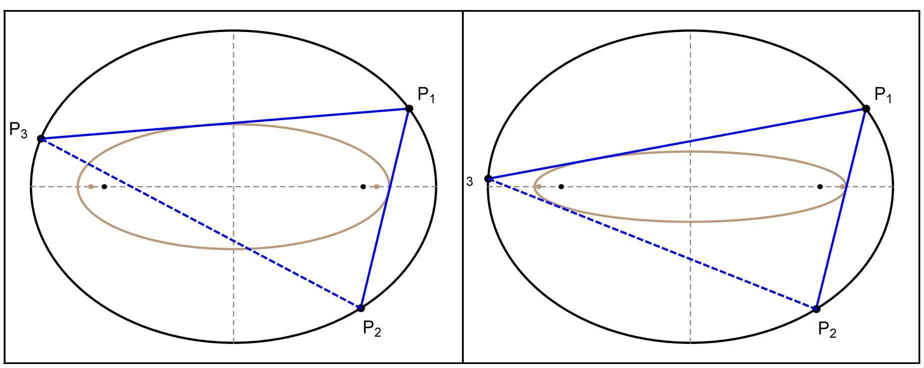

Figure 3.8: **Left:** Two concentric, axis-parallel ellipses (black and brown), and a point P_1 on the outer one. The lines thru P_1 tangent to the inner ellipse intersect the outer one at P_2 and P_3. Notice that P_2P_3 cut thru the inner ellipse, i.e., the pair of ellipses does not satisfy Cayley's conditions. **Right:** the minor axis of the inner ellipse has been scaled such that $P_1P_2P_3$ is now a Poncelet triangle.

the following are coordinates for the intersections P_2 and P_3 on $\mathcal{E}$ of the two tangents to $\mathcal{E}_c$ seen from a point $P_1 = [x_1, y_1]$ also on $\mathcal{E}$:

ronaldo reduzi a notação, pode checar? que tal minúsculas para escalares, maiúsculas para pontos; exceção historica: L e J

$$
\begin{aligned}
P_2 = [x_2, y_2] &= \frac{1}{k_2}\left[\frac{u_1 x_1 + u_2 y_1}{b}, \frac{w_1 x_1 + w_2 y_1}{a}\right] \\
P_3 = [x_3, y_3] &= \frac{1}{k_2}\left[\frac{w_1 x_1 - w_2 y_1}{b}, \frac{w_1 x_1 - w_2 y_1}{a}\right]
\end{aligned}
\tag{3.7}
$$

where:

$$
\begin{aligned}
u_1 &= b\left(a^4 b_c^4 - (a^2 - a_c^2)^2 b^4\right) \\
u_2 &= 2ak_1\left((a^2 + a_c^2)b^2 - b_c^2 a^2\right) \\
k_1 &= \sqrt{b^2 b_c^2 (a^2 - a_c^2) x_1^2 + a_c^2 a^2 (b^2 - b_c^2) y_1^2} \\
k_2 &= \left(\frac{a^2(b^2 + b_c^2) - a_c^2 b^2}{a} x_1\right)^2 + \left(\frac{a^2(b^2 - b_c^2) + a_c^2 b^2}{b} y_1\right)^2 \\
w_1 &= 2bk_1\left((b^2 + b_c^2)a^2 - a_c^2 b^2\right) \\
w_2 &= a\left(a_c^4 b^4 - a^4 (b^2 - b_c^2)^2\right)
\end{aligned}
$$

3.4 Some Non-Concentric Families

3.4.1 Poristics (Bicentric Triangles)

3.4.2 Poristic Excentrals

3.4.3 Brocard's Porism

3.4.4 General Case

3.5 Exercises

Exercise 1. *Show that every triangle has a circumbilliard, i.e., an ellipse to which it is inscribed and to which it is a billiard 3-periodic. Compute the axes of said circumbilliard with respect to triangle vertices.*

Exercise 2. *Prove that the power of the circumcircle with with respect to the common center in each of the following 3-periodic families is constant and given by the listed expressions. (i) incircle: $-a_e b_e$; (ii) homothetic: $-(a_e^2 + b_e^2)/2$, and (iii) excentral: $-a^2 - b^2 - 2\delta$.*

Exercise 3. *Prove the radius r^* of the stationary cosine circle of the excentral family is larger than the major axis a of its caustic.*

Family	Fixed	Conserves	Notes
Poristic (bicentric)	X_1, X_3, ...	$\sum \cos\theta_i$	polar image of Confocal family wrt to a focus
Poristic Excentrals	X_2, X_3, X_4, X_5	$\sum s_i^2$, $\prod \cos\theta_i$	Inscribed in circle; caustic is MacBeath inconic
Brocard	X_3, X_6, X_{15}, X_{16}, X_{39}, $X_{182}, \ldots$, Ω_1, Ω_2	$\sum s_i^{-2}$, ω, $\sum \cot\theta_i$	polar image of Homothetic family wrt caustic focus; inscribed in circle; caustic is Brocard inellipse
focus-Inversive	X_7	$L, \sum \cos\theta_i$	inversive image of Confocals wrt a focus; non-Ponceletian; inscribed in Pascal's limaçon; caustic non-ellipse

Table 3.2: Some Non-concentric Families, their stationary triangle centers, and known conservations.

Exercise 4. *Are there any conservations and/or fixed triangle centers for the family which is an s-affine image of billiard excentrals? Those are the dashed green triangles in Figure 3.4(middle).*

Chapter 4

Analyzing Loci of N=3 Poncelet Families

Chapter 5

Invariants in the Elliptic Billiard

Chapter 6

Invariants of the Bicentric Family

Chapter 7

Invariants of the Homothetic and Brocard Families

Chapter 8

Experimental Techniques

Chapter 9

Epilogue: Properties of Pairs of Conics

Chapter 10

Conclusion

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