Poncelet Invariants:
an Experimental Promenade

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Chapter 1

Introduction

Chapter 2

Poncelet Preliminaries

Chapter 3

Loci of Triangle Centers over N=3 Poncelet Families

3.1 Introduction

Reader used to projective geometry may find it redundant to list so many Poncelet family cases. However, locus phenomena are euclidean.

History of the Results

Early videos 2011 com Jair Koiller Reznik (2011d) and Reznik (2011b), proof by complexification Romaskevich (2014), proof by Affine Curvature Garcia (2019), circumcenter Fierobe (2021). Centers of Mass of Poncelet Polygon Schwartz and Sergei Tabachnikov (2016a), Circumcenter of Mass Sergei Tabachnikov and Tsukerman (2014).

3.2 Some N=3 Poncelet Families

3.3 3-periodic orbits

Consider a pair of ellipses $\mathcal{E}$ and $\mathcal{E}_c$ given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\;\; \frac{x^2}{a_c^2}+\frac{y^2}{b_c^2}-1=0$$

Let $P_1=(x_1,y_1)\in\mathcal{E}$ and consider the two tangent lines to $\mathcal{E}_c$ passing through P_1. The intersection of these two lines with $\mathcal{E}$ are given by:

$$
\begin{aligned}
P_2 &= [\frac{p_1 x_1 + p_2 y_1}{b p_3}, \frac{-q_1 x_1 + q_2 y_1}{a p_3}] \\
P_3 &= [\frac{p_1 x_1 - p_2 y_1}{b p_3}, \frac{q_1 x_1 + q_2 y_1}{a p_3}] \\
p_1 &= b(a^4 b_c^4 - (a^2 - a_c^2)^2 b^4) \\
p_2 &= 2a((a^2 + a_c^2) b^2 - b_c^2 a^2) \sqrt{b^2 b_c^2 (a^2 - a_c^2) x_1^2 + a_c^2 a^2 (b^2 - b_c^2) y_1^2} \\
p_3 &= \frac{(a^2 b^2 + a^2 b_c^2 - a_c^2 b^2)^2 x_1^2}{a^2} + \frac{(a^2 b^2 - a^2 b_c^2 + a_c^2 b^2)^2 y_1^2}{b^2} \\
q_1 &= 2b((b^2 + b_c^2) a^2 - a_c^2 b^2) \sqrt{b^2 b_c^2 (a^2 - a_c^2) x_1^2 + a_c^2 a^2 (b^2 - b_c^2) y_1^2} \\
q_2 &= -a(a^2 b^2 - a^2 b_c^2 - a_c^2 b^2)(a^2 b^2 - a^2 b_c^2 + a_c^2 b^2).
\end{aligned}
\tag{3.1}
$$

3.3.1 Concentric, Axis-Parallel Pair

General Case

- Cayley's condition $a_c/a + b_c/b = 1$
- Vertex Parametrization

  $$\{P_1, P_2, P_3\}$$

  given by Equation (3.1), with $a_c/a + b_c/b = 1$.
- Figure

Below we introduce a few special cases.

Confocal (aka. Elliptic Billiard)

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with

$$a_c = \frac{a(\delta - b^2)}{c^2}, \quad b_c = \frac{b(a^2 - \delta)}{c^2}, \quad \delta^2 = a^4 - a^2 b^2 + b^4, \; c^2 = a^2 - b^2$$

with Incircle

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with $a_c = b_c = r = ab/(a+b)$.

Billiard's Excentrals

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with the ellipse pair $\{\mathcal{E}_e, \mathcal{E}\}$, where E_e is given by $x^2/a_e^2 + y^2/b_e^2 = 1$, with

$$a_e = \frac{\delta + b^2}{a}, \quad b_e = \frac{a^2 + \delta)}{b}.$$

with Circumcircle

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with $a = b = R = a_c + b_c$.

Homothetic

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with $a = 2a_c,\ b = 2b_c$.

Dual

Quem e o dual: é aquele cuja caustica é anti-homotetica a externa, ou seja $a'/b' = b/a$

$$\{P_1, P_2, P_3\}$$

given by Equation (3.1), with $a = 2b_c,\ b = 2a_c$.

3.3.2 Non-Concentric, Axis-Parallel Pair

General Case

Poristic (aka. Bicentric)

Brocard's Porism

3.3.3 Concentric, Unaligned

3.3.4 Generic Pair

3.4 Blaschke's Parametrization

Here we consider 3-periodics inscribed in a unit circle and circumscribing a non-concentric ellipse with foci f and g; see Figure 3.1. We will work in the complex plane and apply Blaschke Products Daepp et al. (2019) whereby Poncelet 3-periodic vertices become symmetric with respect to the information of the circle-ellipse pair. These were first used to analyze loci of Poncelet triangle centers in Helman, Laurain, et al. (2021).

Let $z_1, z_2, z_3 \in \mathbb{C}$ denote the vertices of Poncelet 3-periodics in a generic $N = 3$ family with fixed (unit) circumcircle denoted $\mathbb{T}$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Let f, g be the foci of the caustic. Using Viète's formula, we obtain the following parametrization of the elementary symmetric polynomials on z_1, z_2, z_3 Daepp et al. (2019):

Definition 1 (Blaschke's Parametrization)**.**

$$\begin{aligned}\sigma_1 := z_1 + z_2 + z_3 =& f + g + \lambda\overline{f}\overline{g}\\ \sigma_2 := z_1 z_2 + z_2 z_3 + z_3 z_1 =& fg + \lambda(\overline{f} + \overline{g})\\ \sigma_3 := z_1 z_2 z_3 =& \lambda\end{aligned}$$

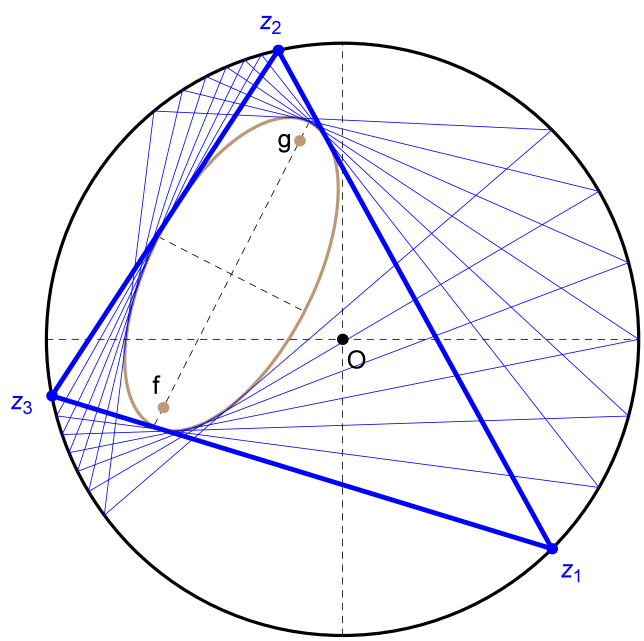

Figure 3.1: Blaschke parametrization after Daepp et al. (2019). The vertices of the 3-periodic (blue) are z_1, z_2, z_3. The foci of the caustic are f, g.

where $\lambda \in \mathbb{T}$ is the varying parameter.

Note that the concentric case occurs when $g = -f$.

For each $\lambda \in \mathbb{T}$, the three solutions of $B(z) = \lambda$ are the vertices of a 3-periodic orbit of the Poncelet family of triangles in the complex plane Daepp et al. (2019, Chapter 4). Furthermore, as λ varies in $\mathbb{T}$, the whole family of triangles is covered. Clearing the denominator in this equation and passing everything to the left-hand side, we get

$$z^3 - (f + g + \lambda\overline{f}\overline{g})z^2 + (fg + \lambda(\overline{f} + \overline{g}))z - \lambda = 0$$

Lemma 1. *If $u, v, w \in \mathbb{C}$ and λ is a parameter that varies over the unit circle $\mathbb{T} \subset \mathbb{C}$, then the curve parametrized by*

$$F(\lambda) = u\lambda + \frac{v}{\lambda} + w$$

is an ellipse centered at w, with semiaxis $|u| + |v|$ and $\big||u| - |v|\big|$, rotated with respect to the horizontal axis of $\mathbb{C}$ by an angle of $(\arg u + \arg v)/2$.

Proof. If either $u = 0$ or $v = 0$, the curve $h(\mathbb{T})$ is clearly the translation of a multiple of the unit circle $\mathbb{T}$, and the result follows. Thus, we may assume $u \neq 0$ and $v \neq 0$.

Choose $k \in \mathbb{C}$ such that $k^2 = u/v$. Write k in polar form, as $k = r\mu$, where $r > 0$ ($r \in \mathbb{R}$) and $|\mu| = 1$. We define the following complex-valued functions:

$$R(z) := \mu z,\ S(z) := rz + (1/r)\overline{z},\ H(z) := kvz,\ T(z) := z + w$$

One can straight-forwardly check that $F = T \circ H \circ S \circ R$.

Since $|\mu| = 1$, R is a rotation of the plane, thus R sends the unit circle $\mathbb{T}$ to itself. Since $r \in \mathbb{R}$, $r > 0$, if we identify $\mathbb{C}$ with $\mathbb{R}^2$, S can be seen as a linear transformation that sends $(x, y) \mapsto ((r + 1/r)\,x, (r - 1/r)\,y)$. Thus, S sends $\mathbb{T}$ to an axis-aligned, origin-centered ellipse $\mathcal{E}_1$ with semiaxis $r + 1/r$ and $|r - 1/r|$. H is the composition of a rotation and a homothety. H sends the ellipse $\mathcal{E}_1$ to an origin-centered ellipse $\mathcal{E}_2$ rotated by an angle of $\arg(kv) = \arg(k) + \arg(v) = (\arg(u) - \arg(v))/2 + \arg(v) = (\arg(u) + \arg(v))/2$. The semiaxis of $\mathcal{E}_2$ have length

$$|kv|(r + 1/r) = r|v|(r + 1/r) = |r^2 v| + |v| = |k^2 v| + |v| = |u| + |v|, \text{ and}$$
$$|kv||r - 1/r| = r|v||r - 1/r| = \big||r^2 v| - |v|\big| = \big||k^2 v| - |v|\big| = \big||u| - |v|\big|$$

Finally, T is a translation, thus T sends $\mathcal{E}_2$ to an ellipse $\mathcal{E}_3$ centered at w, rotated by an angle $(\arg(u) + \arg(v))/2$ from the axis, with semiaxis lengths $|u| + |v|$ and $\big||u| - |v|\big|$, as desired. ☐

Consider the Moebius map $M_{z_0} = (z_0 - z)/(1 - \overline{z_0}z)$ and the Blaschke product of degree 3 given by $B = M_{z_0}M_{z_1}M_{z_2}$.

Theorem 1. *Let B be a Blaschke product of degree 3 with zeros $0, f, g$. For $\lambda \in \mathbb{T}$, let z_1, z_2, z_3 denote the three distinct solutions to $B(z) = \lambda$. Then the lines joining z_j and z_k, $(j \neq k)$ are tangent to the ellipse given by*

$$|w - f| + |w - g| = |1 - \overline{f}g|.$$

Theorem 2. *Given two points $f, g \in \mathbb{D}$. Then there exists a unique conic $\mathcal{E}$ with the foci f, g which is 3-Poncelet caustic with respect to $\mathbb{T}$. Moreover, $\mathcal{E}$ is an ellipse. That ellipse is the Blaschke ellipse with the major axis of length $|1 - \overline{f}g|$.*

Proof. □

3.5 Analyzing Loci of Triangle Centers

- Triangle centers
- Are Loci Algebraic (method of resultants, why so many)
- Are Loci Elliptic (Blaschke's parametrization Daepp et al. (2019))
- Monotonicity and Turning Number (ballet + paper 25)
- Table of results

3.5.1 Generic Nested Ellipses

In this Section we prove the locus of a given fixed linear combination of X_2 and X_3 is an ellipse. We will continue to use Blaschke product techniques since a generic non-concentric pair can always be seen as the affine image of a pair with circumcircle.

Consider the generic pair of nested ellipses $\mathcal{E} = (O, a, b)$ and $\mathcal{E}_c = (O_c, a_c, b_c.\theta)$ in Figure **??**. Let sθ, cθ denote the sine and cosine of θ, respectively. Define $c_c^2 = a_c^2 - b_c^2$. The Cayley condition for the pair to admit a 3-periodic family is given by:

$$\begin{aligned} & b^4 x_c^4 + 2\,a^2 b^2 x_c^2 y_c^2 + \left(2c_c^2\left(-b^2(a^2+b^2)\right)\mathrm{c}\theta^2 - 2\left(b^2 - b_c^2\right) b^2 a^2 - 2\,b^4 b_c^2\right) x_c^2 \\ & - 8\,a^2 b^2 x_c\, y_c\, c_c^2 \mathrm{s}\theta\mathrm{c}\theta + a^4 y_c^4 + \left(2c_c^2 a^2\left(a^2+b^2\right)\mathrm{c}\theta^2 - 2\left(b_c^2 + b^2\right) a^4 + 2\,a^2 b^2 b_c^2\right) y_c^2 \\ & + c_c^4 c^4 \left(\mathrm{c}\theta^4 - 2\,c_c^2 c^2 \left(a^2 a_c^2 - b^2 a^2 + b_c^2 b^2\right)\mathrm{c}\theta^2\right. \\ & + \left(a a_c + ab - b b_c\right)\left(a a_c - ab - b b_c\right)\left(a a_c + ab + b b_c\right)\left(a a_c - ab + b b_c\right) = 0 \end{aligned} \tag{3.2}$$

Recall that over Poncelet N-periodics interscribed in a generic pair of conics, the locus of vertex and area centroids is an ellipse Schwartz and Sergei Tabachnikov (2016b) as is that of the circumcenter-of-mass Sergei Tabachnikov and Tsukerman (2014), a generalization of X_3 for $N > 3$. Referring to Figure 3.3:

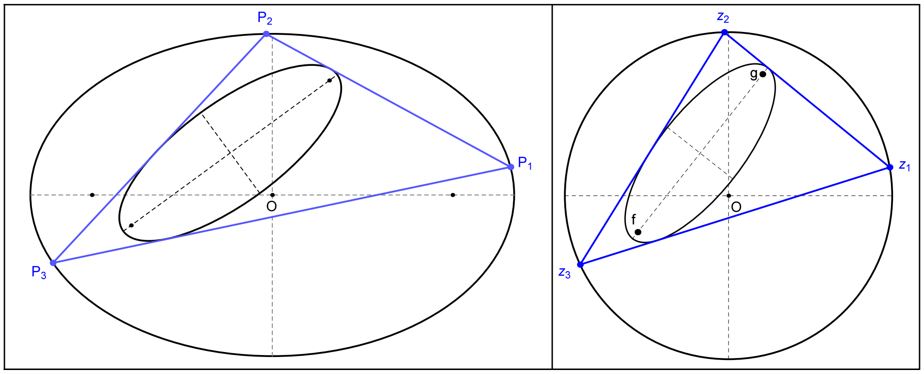

Figure 3.2: Affine transformation that sends a generic ellipse pair and its 3-periodic family (left) to a new pair with circumcircle (right). We parametrize the 3-periodic orbit with vertices z_i in the circumcircle pair using the foci of the latter's caustic f and g, and then apply the inverse affine transformation to get a parametrization of the vertices P_i of the original Poncelet pair. Video

Theorem 3. *Over the family of 3-periodics interscribed in an ellipse pair in general position (non-concentric, non-axis-aligned), if $\mathcal{X}_{\alpha,\beta}$ is a fixed linear combination of X_2 and X_3, i.e., $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.*

Proof. Consider a general $N = 3$ Poncelet pair of ellipses that forms a 1-parameter family of triangles. Without loss of generality, by translation and rotation, we may assume the outer ellipse is centered at the origin and axis-aligned with the plane $\mathbb{R}^2$, which we will also identify with the complex plane $\mathbb{C}$. Let a, b be the semi-axis of the outer ellipse, and a_c, b_c the semi-axis of the inner ellipse, as usual.

Referring to Figure 3.2, consider the linear transformation that takes $(x, y) \mapsto (x/a, y/b)$. This transformation takes the outer ellipse to the unit circle $\mathbb{T}$ and the inner ellipse to another ellipse. Thus, it transforms the general Poncelet $N = 3$ system into a pair where the outer ellipse is the circumcircle, which we can parametrize using Blaschke products Daepp et al. (2019). In fact, to get back to the original system, we must apply the inverse transformation that takes $(x, y) \mapsto (ax, by)$. As a linear transformation from $\mathbb{C}$ to $\mathbb{C}$, we can write it as $L(z) := pz + q\bar{z}$, where $p := (a + b)/2, q := (a - b)/2$.

Let $z_1, z_2, z_3 \in \mathbb{T} \subset \mathbb{C}$ be the three vertices of the circumcircle family, parametrized as in Definition 1, and let $v_1 := L(z_1), v_2 := L(z_2), v_3 := L(z_3)$ be the three vertices of the original general family. The barycenter X_2 of the original family is given by $(v_1 + v_2 + v_3)/3$, and the circumcenter X_3 is given by Tak (n.d.):

$$X_3 = \begin{vmatrix} v_1 & |v_1|^2 & 1 \\ v_2 & |v_2|^2 & 1 \\ v_3 & |v_3|^2 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} v_1 & \overline{v_1} & 1 \\ v_2 & \overline{v_2} & 1 \\ v_3 & \overline{v_3} & 1 \end{vmatrix}$$

Since $\overline{z_1} = 1/z_1, \overline{z_2} = 1/z_2, \overline{z_3} = 1/z_3$, we can write v_1, v_2, v_3 as rational functions of z_1, z_2, z_3, respectively. Thus, both X_2 and X_3 are symmetric rational functions on z_1, z_2, z_3. Defining $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$, we have consequently that $\mathcal{X}_{\alpha,\beta}$ is also a symmetric rational function on z_1, z_2, z_3. Hence, we can reduce its numerator and denominator to functions on the elementary symmetric polynomials on z_1, z_2, z_3. This is exactly what we need in order to use the parametrization by Blaschke products.

In fact, we explicitly compute:

$$\mathcal{X}_{\alpha,\beta} = \frac{p^2 q\left(\sigma_2(\alpha+3\beta) + 3\beta\sigma_3^2\right) + \alpha p^3\sigma_1\sigma_3 - pq^2(3\beta + \sigma_1\sigma_3(\alpha+3\beta)) - \alpha q^3\sigma_2}{3\sigma_3(p-q)(p+q)}$$

where $\sigma_1, \sigma_2, \sigma_3$ are the elementary symmetric polynomials on z_1, z_2, z_3.

Let $f, g \in \mathbb{C}$ be the foci of the inner ellipse in the circumcircle system. Using Definition 1, with the parameter λ varying on the unit circle $\mathbb{T}$, we get:

$$\mathcal{X}_{\alpha,\beta} = u\lambda + v\frac{1}{\lambda} + w \tag{3.3}$$

where:

$$\begin{aligned} u :=& \frac{p\left(\overline{f}\overline{g}\left(\alpha p^2 - q^2(\alpha+3\beta)\right) + 3\beta pq\right)}{3(p-q)(p+q)} \\ v :=& \frac{\beta pq(q - fgp)}{(q-p)(p+q)} + \frac{1}{3}\alpha fgq \\ w :=& \frac{q\left(\overline{f}+\overline{g}\right)\left(p^2(\alpha+3\beta) - \alpha q^2\right) + p(f+g)\left(\alpha p^2 - q^2(\alpha+3\beta)\right)}{3(p-q)(p+q)} \end{aligned}$$

By Lemma 1, this is the parametrization of an ellipse centered at w, as desired. As in Lemma 1, it is also possible to explicitly calculate its axis and rotation angle, but these expressions become very long.

□

Corollary 1. *Over the family of 3-periodics interscribed in an ellipse pair in general position (non-concentric, non-axis-aligned), if $\mathcal{X}_\gamma$ is a real affine combination of X_2 and X_3, i.e., $\mathcal{X}_\gamma = (1-\gamma)X_2 + \gamma X_3$ for some fixed $\gamma \in \mathbb{R}$, then its locus is an ellipse. Moreover, as we vary γ, the centers of the loci of the $\mathcal{X}_\gamma$ are collinear.*

Proof. Apply Theorem 4 with $\alpha = 1-\gamma, \beta = \gamma$ to get the elliptical loci. As in the end of the proof of Theorem 4, the center of the locus of $\mathcal{X}_\gamma$ can be computed explicitly as

$$w = w_0 + w_1\gamma, \text{ where}$$
$$w_0 = \frac{1}{3}\left(q\left(\overline{f}+\overline{g}\right)+p(f+g)\right)$$
$$w_1 = \frac{q\left(2p^2+q^2\right)\left(\overline{f}+\overline{g}\right)-p(f+g)\left(p^2+2q^2\right)}{3(p-q)(p+q)}$$

As $\gamma \in \mathbb{R}$ varies, it is clear the center w sweeps a line. □

We proved that all of the following triangle centers have elliptic loci in the general N=3 Poncelet system, including the barycenter, circumcenter, orthocenter, nine-point center, and de Longchamps point:

Observation 1. *Amongst the 40k+ centers listed on Kimberling (2019), about 4.9k triangle centers lie on the Euler line Kimberling (2020). Out of these, only 226 are fixed affine combinations of X_2 and X_3. For $k < 1000$, these amount to $X_k, k =$2, 3, 4, 5, 20, 140, 376, 381, 382, 546, 547, 548, 549, 550, 631, 632.*

Observation 2. *The elliptic loci of X_2 and X_4 are axis-aligned with the outer ellipse.*

We conclude this section with phenomenon specific to the case where $\mathcal{E}_c$ is a circle, Figure 3.4:

Observation 3. *Over the family of 3-periodics inscribed in an ellipse and circumscribing a non-concentric circle centered on $O_c = X_1$, the locus of X_3 and X_5 are ellipses whose major axes pass through X_1.*

Referring to Figure 3.5:

Proposition 1. *If a triangle center $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ is a fixed linear combination of X_2 and X_3 for some $\alpha, \beta \in \mathbb{C}$, its locus over 3-periodics in the non-concentric pair with a circumcircle is a circle centered on $\mathcal{O}_\alpha$ and of radius $\mathcal{R}_\alpha$ given by:*

$$\mathcal{O}_\alpha = \frac{\alpha(f+g)}{3}, \quad \mathcal{R}_\alpha = \frac{|\alpha f g|}{3}$$

Observation 4. *Notice that the center and radius of the locus do not depend on β since the circumcenter X_3 is stationary at the origin of this system.*

Proof. Since, z_1, z_2, z_3 are the 3 vertices of the Poncelet triangle inscribed in the unit circle, its barycenter and circumcenter are given by $X_2 = (z_1+z_2+z_3)/3$ and $X_3 = 0$, respectively. We define $\mathcal{X}_{\alpha,\beta} := \alpha X_2 + \beta X_3 = \alpha(z_1+z_2+z_3)/3$. Using Definition 1, we get $\mathcal{X}_{\alpha,\beta} = \alpha(f+g+\lambda\overline{f}\overline{g})/3 = \alpha(f+g)/3 + \lambda(\alpha\overline{f}\overline{g})/3$, where the parameter λ varies on the unit circle $\mathbb{T}$. Thus, the locus of $\mathcal{X}_\gamma$ over the Poncelet family of triangles is a circle with center $\mathcal{O}_\alpha := \alpha(f+g)/3$ and radius $\mathcal{R}_\alpha := |\alpha\overline{f}\overline{g}|/3 = |\alpha f g|/3$. □

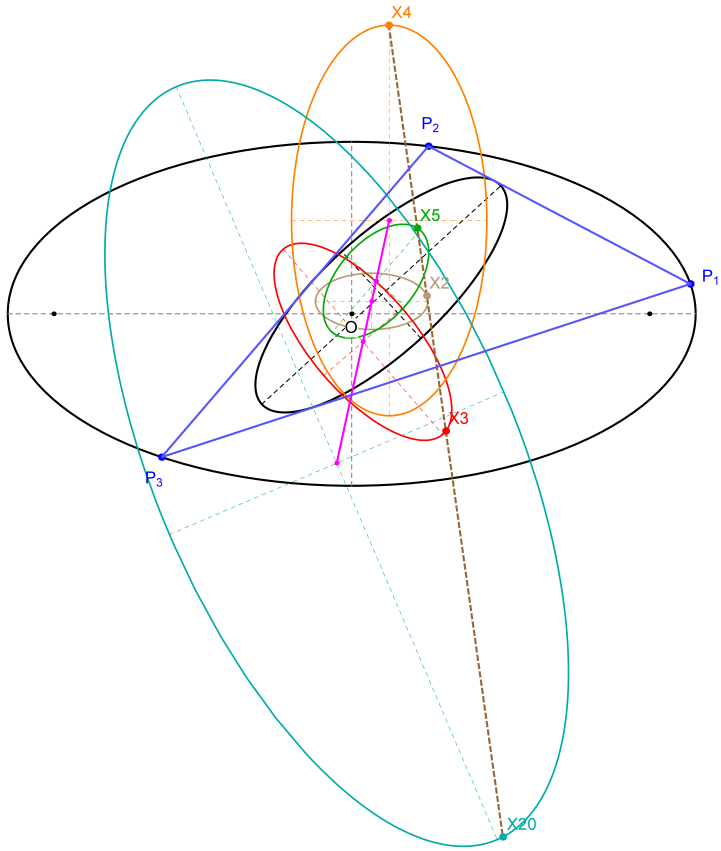

Figure 3.3: A 3-periodic is shown interscribed between two nonconcentric, non-aligned ellipses (black). The loci of X_k, $k = 2, 3, 4, 5, 20$ (and many others) remain ellipses. Those of X_2 and X_4 remain axis-aligned with the outer one. Furthermore the centers of all said elliptic loci are collinear (magenta line). Video

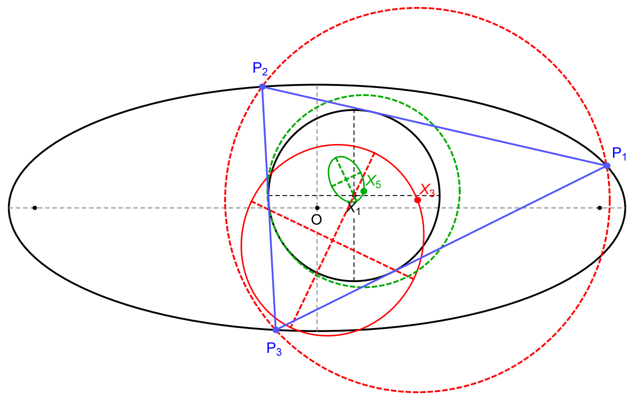

Figure 3.4: A 3-periodic (blue) is shown inscribed in an outer ellipse and an inner non-concentric circle centered on O_c. The loci of both circumcenter (solid red) and Euler center (solid green) are ellipses whose major axes pass through O_c. Video

Using $\alpha = 1 - \gamma, \beta = \gamma$ for a fixed $\gamma \in \mathbb{R}$ in Proposition 1, we get:

Corollary 2. *If a triangle center $\mathcal{X}_\gamma = (1-\gamma)X_2 + \gamma X_3$ is a real affine combination of X_2 and X_3 for some $\gamma \in \mathbb{R}$, its locus over 3-periodics in the non-concentric pair with a circumcircle is a circle. Moreover, as we vary γ, the centers of these loci are collinear with the fixed circumcenter.*

Many triangle centers in Kimberling (2019) are affine combinations of the barycenter X_2 and circumcenter X_3. See Observation 1 for a compilation of them.

Observation 5. *For a generic triangle, only X_{98}, and X_{99} are simultaneously on the Euler line and on the circumcircle. However these are not linear combinations of X_2 and X_3. Still, if a triangle center is always on the circumcircle of a generic triangle (there are many of these, see Eric Weisstein (2019, Circumcircle)), its locus over 3-periodics in the non-concentric pair with circumcircle is trivially a circle.*

Corollary 3. *Over the family of 3-periodics inscribed in a circle and circumscribing a non-concentric inellipse centered at O_c, the locus of X_k, k in 2,4,5,20 are circles whose centers are collinear. The locus of X_5 is centered on O_c. The centers and radii of these circular loci are given by:*

$$
\begin{aligned}
O_2 &= \frac{f+g}{3}, \quad O_4 = f+g, \quad O_5 = \frac{f+g}{2}, \quad O_{20} = -(f+g)\\
r_2 &= \frac{|fg|}{3}, \qquad r_4 = |fg|, \qquad r_5 = \frac{|fg|}{2}, \qquad r_{20} = |fg|
\end{aligned}
$$

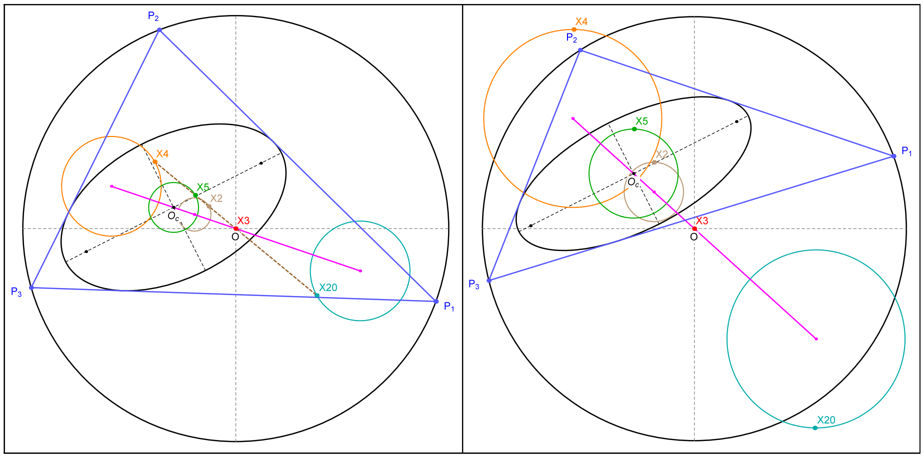

Figure 3.5: **Left:** 3-periodic family (blue) in the pair with circumcircle where the caustic contains X_3, i.e., all 3-periodics are acute. The loci of X_4 and X_{20} are interior to the circumcircle. **Right:** X_3 is exterior to the caustic, and 3-periodics can be either acute or obtuse. Equivalently, the locus of X_4 intersects the circumcircle. In both cases (left and right), the loci of X_k, k in 2,4,5,20 are circles with collinear centers (magenta line). The locus of X_5 is centered on O_c. The center of the X_2 locus is at 2/3 along OO_c. Video

Proof. As in Corollary 2, we can use Proposition 1 with $\gamma = 0, -2, -1/2, 4$ to get the center and radius for X_2, X_4, X_5, X_{20}, respectively. All of these centers are real multiples of $f + g$, so they are all collinear. Moreover, the center O_5 of the circular loci of X_5 is $(f + g)/2$, that is, the midpoint of the foci of the inellipse, or in other words, the center O_c of the inellipse. □

Referring to Figure 3.5:

Observation 6. *The family of 3-periodics in the pair with circumcircle includes obtuse triangles if and only if X_3 is exterior to the caustic.*

This is due to the fact that when X_3 is interior to the caustic, said triangle center can never be exterior to the 3-periodic. Conversely, if X_3 is exterior, it must also be external to some 3-periodic, rendering the latter obtuse.

Consider the parametrization of a triangular orbit $\{z_1, z_2, z_3\}$ given in Definition 1.

Let also the affine transformation $T(z) = pz + q\bar{z}$.

figura do blaschke

Theorem 4. *Over the family of 3-periodics interscribed in a generic nested pair of ellipses (non-concentric, non-axis-aligned), if $\mathcal{X}_{\alpha,\beta}$ is a fixed linear combination of X_2 and X_3, i.e., $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.*

3.6 Incenter and Excenters

Theorem 5. *Over 3-periodics in the elliptic billiard (confocal pair) the locus of the incenter X_1 is an ellipse given by $x^2/a_1^2 + y^2/b_1^2 = 1$, where*

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}, \quad \delta = \sqrt{a^4 - a^2b^2 + b^4}.$$

The locus of the Excenters (triangle formed by the intersection of external bisectors) is an ellipse with axes:

$$a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}$$

Notice it is similar to the X_1 locus, i.e., $a_1/b_1 = b_e/a_e$.

A list of the elliptic loci of centers in the X_1 to X_{200} range can be found here.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk bounded by $\mathbb{T}$.

Proposition 2. *Over Poncelet 3-periodics in the pair with an outer circle and an ellipse in generic position, the locus X_1 given by:*

$$\begin{aligned} X_1 : &\, z^4 - 2((\bar{f} + \bar{g})\lambda + fg)z^2 + 8\lambda z \\ &+ (\bar{f} - \bar{g})^2\lambda^2 + 2(|f|^2 g + f|g|^2 - 2f - 2g)\lambda + f^2g^2 = 0 \\ : &\; z^4 - 2\beta z^2 + 8\lambda z + (\beta^2 - 4\alpha\lambda) = 0 \end{aligned}$$

Proof. The incenter of a triangle with vertices $\{z_1, z_2, z_3\}$ is given by:

$$\begin{aligned} X_1 &= \frac{\sqrt{a}\, z_1 + \sqrt{b}\, z_2 + \sqrt{c}\, z_3}{\sqrt{a} + \sqrt{b} + \sqrt{c}} \\ a &= |z_2 - z_3|^2, \; b = |z_1 - z_3|^2, \;\; c = |z_2 - z_1|^2 \end{aligned}$$

Using that $z_i \in \mathbb{T}$ it follows that

$$a = 2 - (\frac{z_3}{z_2} + \frac{z_3}{z_2}), \;\; b = 2 - (\frac{z_1}{z_3} + \frac{z_3}{z_1}), \;\; c = 2 - (\frac{z_1}{z_2} + \frac{z_2}{z_1})$$

Eliminating the square roots in the equation $X_1 - z = 0$ and using the relations σ_i (i=1,2,3) given in Blaschke's parametrization the result follows. □

Proposition 3. *Over Poncelet 3-periodics in a generic nested ellipse pair, the* *locus of X_1 is given by the following sextic polynomial in z, λ:*

$$
\begin{aligned}
X_1:\ & \lambda^2\left(p^2-q^2\right)z^4+4\,\lambda\left(\alpha\,\lambda\,pq^2-q\,\lambda^2p^2-\beta\,p^2\,q+p\,q^2\right)z^3\\
&+(4\,\alpha\,\lambda^3p^3q-4\,\alpha^2\lambda^2p^2q^2+2\,\alpha\,\beta\,\lambda\,p^3\,q-2\,\alpha\,\beta\,\lambda\,pq^3-2\,\beta\,\lambda^2p^4+6\,\beta\,\lambda^2p^2q^2\\
&-6\,\alpha\,\lambda\,p^2\,q^2+2\,\alpha\,\lambda\,q^3q+4\,\beta^2p^2\,q^2+6\,\lambda^2p^3\,q-6\,\lambda^2pq^3-4\,\beta\,p\,q^3)z^2\\
&+(4q(\alpha^2\beta p^2q^2+2\alpha^2p^2pq-\alpha^2pq^3+\beta^2p^4-2\beta^2p^2q^2+4\beta pq^3-p^2q^2-2q^4)\lambda\\
&-4\alpha pq^2(\beta^2p^2-q^2)-4p^3(\beta pq-2p^2-q^2)\lambda^3-16\alpha\lambda^2p^4q)z\\
&-\lambda^2(4\alpha\lambda-\beta^2)p^6+4p^5q\lambda^4+2\lambda(4\alpha^2\lambda-\alpha\beta^2-3\beta\lambda)p^5q-\lambda^2(8\alpha\lambda-3\beta^2)p^4q^2\\
&+(\alpha^2\beta^2-4\alpha^3\lambda+4\alpha\beta\lambda+5\lambda^2)p^4q^2+2\lambda(2\alpha^2\lambda-\alpha\beta^2+\beta\lambda)p^3q^3\\
&+(2\alpha^2\beta-2\alpha\lambda-4\beta^2)p^3q^3-(\alpha^2\beta^2+4\alpha\beta\lambda-4\beta^3+5\lambda^2)p^2q^4+(8\beta-3\alpha^2)p^2q^4\\
&+(2\alpha^2\beta+6\alpha\lambda-8\beta^2)pq^5-4q^5p+(4\beta-\alpha^2)q^6=0
\end{aligned}
$$

Proof. Let $p, q \in \mathbb{R}$. Consider the affine transformation $T(z) = pz + q\overline{z}$ and set $w_i = T(z_i)$. The proof is similar to that given in Proposition 2. □

Proposition 4. *In the confocal pair the locus X_1 is defined by:*

$$2\,ab\lambda^2z^2+2\,\lambda\left(a^3\lambda^2-b^3\lambda^2-a^3-b^3\right)z+c^2\left(c^2\lambda^4-2\,ab\lambda^2-c^2\right)=0$$

Proof. We have that

$$f=\frac{1}{c}\sqrt{-a^2-b^2+2\,\delta},\quad g=-\frac{1}{c}\sqrt{-a^2-b^2+2\,\delta}$$

□

Corollary 4. *The locus X_1 is the ellipse with semiaxes given by $a_1 = (a^2-\delta)/b$ and $b_1 = (\delta - b^2)/a$.*

Proof. The quartic polynomial is factorizable as p_1p_2, where

$$p_1=z-\left(\frac{(a-b)\left(-a^2-ab-b^2+\delta\right)\lambda}{2\,ab}-\frac{(a+b)\left(-a^2+ab-b^2+\delta\right)}{2\,ab\lambda}\right)$$

$$
\begin{aligned}
p_2=&\,2ab\lambda^2z^3-((a-b)(a^2+3ab+b^2+\delta)\lambda^3-(a+b)(a^2-3ab+b^2+\delta)\lambda)z^2\\
&+6ab(a^2-b^2)\lambda^2z+(a+b)^3(a^2-ab+b^2+\delta)\lambda^3-(a-b)^3(a^2+ab+b^2+\delta)\lambda
\end{aligned}
$$

Follows directly from Lemma 1 and Proposition 4. □

Schwartz and Sergei Tabachnikov (2016a)

Conjecture 1. *Over 3-periodics interscribed between two ellipses in general position, the locus of a triangle center X_k is an ellipse if and only if X_k is a fixed linear combination of X_3 and X_4.*

Conjecture 2. *The locus of the incenter is an ellipse if and only if the Poncelet ellipse pair is confocal.*

3.7 Loci of Triangle Centers when Trilinears are Rational

Consider a Triangle Center X whose Trilinears $p : q : r$ are rational on the sidelengths s_1, s_2, s_3, i.e., the Triangle Center Function h is rational, equation (**??**).

Theorem 6. *In the family of 3-periodic orbits in a billiard the locus of a rational triangle center is an algebraic curve.*

Our proof is based on the following 3-steps which yield an algebraic curve $\mathcal{L}(x, y) = 0$ which contains the locus. We refer to Lemmas 2 and 3 appearing below.

Proof.

Step 1. *Introduce the symbolic variables u, u_1, u_2:*

$$u^2 + u_1^2 = 1, \quad \rho_1\, u^2 + u_2^2 = 1.$$

The vertices will be given by rational functions of u, u_1, u_2

$$P_1 = (a\, u, b\, u_1), \quad P_2 = (p_{2x}, p_{2y})/q_2, \quad P_3 = (p_{3x}, p_{3y})/q_3$$

Expressions for P_1, P_2, P_3 appear in Appendix **??** as do equations $g_i = 0$, $i = 1, 2, 3$, polynomial in s_i, u, u_1, u_2. Verificar equacao a referenciar

Step 2. *Express the locus X as a rational function on $u, u_1, u_2, s_1, s_2, s_3$.*

Convert $p : q : r$ to Cartesians $X = (x, y)$ via Equation (**??**). From Lemma 2, it follows that (x, y) is rational on $u, u_1, u_2, s_1, s_2, s_3$.

$$x = \mathcal{Q}/\mathcal{R}, \quad y = \mathcal{S}/\mathcal{T}$$

To obtain the polynomials $\mathcal{Q}, \mathcal{R}, \mathcal{S}, \mathcal{T}$ on said variables $u, u_1, u_2, s_1, s_2, s_3$, one substitutes the p, q, r by the corresponding rational functions of s_1, s_2, s_3 that define a specific Triangle Center X. Other than that, the method proceeds identically.

Step 3. *Computing resultants. Our problem is now cast in terms of the polynomial equations:*

$$E_0 = \mathcal{Q} - x\, \mathcal{R} = 0, \quad F_0 = \mathcal{S} - y\, \mathcal{T} = 0$$

Firstly, compute the resultants, in chain fashion:

$$\begin{aligned} E_1 =& \mathrm{Res}(g_1, E_0, s_1) = 0, \quad F_1 = \mathrm{Res}(g_1, F_0, s_1) = 0 \\ E_2 =& \mathrm{Res}(g_2, E_1, s_2) = 0, \quad F_2 = \mathrm{Res}(g_2, F_1, s_2) = 0 \\ E_3 =& \mathrm{Res}(g_3, E_2, s_3) = 0, \quad F_3 = \mathrm{Res}(g_3, F_2, s_3) = 0 \end{aligned}$$

It follows that $E_3(x, u, u_1, u_2) = 0$ and $F_3(y, u, u_1, u_2) = 0$ are polynomial equations. In other words, s_1, s_2, s_3 have been eliminated.

Now eliminate the variables u_1 and u_2 by taking the following resultants:

$$\begin{aligned} E_4(x, u, u_2) =& \mathrm{Res}(E_3, u_1^2 + u^2 - 1, u_1) = 0 \\ F_4(y, u, u_2) =& \mathrm{Res}(F_3, u_1^2 + u^2 - 1, u_1) = 0 \\ E_5(x, u) =& \mathrm{Res}(E_4, u_2^2 + \rho_1 u^2 - 1, u_2) = 0 \\ F_5(y, u) =& \mathrm{Res}(F_4, u_2^2 + \rho_1 u^2 - 1, u_2) = 0 \end{aligned}$$

This yields two polynomial equations $E_5(x, u) = 0$ and $F_5(y, u) = 0$.

Finally compute the resultant

$$\mathcal{L} = \mathrm{Res}(E_5, F_5, u) = 0$$

that eliminates u and gives the implicit algebraic equation for the locus X. □

Remark 1. *In practice, after obtaining a resultant, a human assists the CAS by factoring out spurious branches (when recognized), in order to get the final answer in more reduced form.*

When not rational in the sidelengths, except a few cases[1], Triangle Centers in Kimberling's list have explicit Trilinears involving fractional powers and/or terms containing the triangle area. Those can be made implicit, i.e, given by zero sets of polynomials involving p, q, r, s_1, s_2, s_3. The chain of resultants to be computed will be increased by three, in order to eliminate the variables p, q, r before (or after) s_1, s_2, s_3.

Supporting Lemmas

Lemma 2. *Let* $P_1 = (au, b\sqrt{1-u^2})$. *The coordinates of* P_2 *and* P_3 *of the 3-periodic billiard orbit are rational functions in the variables* u, u_1, u_2, *where* $u_1 = \sqrt{1-u^2}$, $u_2 = \sqrt{1-\rho_1 u^2}$ *and* $\rho_1 = c^4(b^2+\delta)^2/a^6$.

Proof. Follows directly from the parametrization of the billiard orbit, Appendix **??**. ver equacao a referenciar In fact, $P_2 = (x_2(u), y_2(u)) = (p_{2x}/q_2, p_{2y}/q_2)$ and $P_3 = (x_3(u), y_3(u)) = (p_{3x}/q_3, p_{3y}/q_3)$, where p_{2x}, p_{2y}, p_{3x} and p_{3y} have degree 4 in (u, u_1, u_2) and q_2, q_3 are algebraic of degree 4 in u. Expressions for u_1, u_2 appear in Appendix **??**. ver equacao a referenciar □

Lemma 3. *Let* $P_1 = (au, b\sqrt{1-u^2})$. *Let* s_1, s_2 *and* s_3 *the sides of the triangular orbit* $P_1P_2P_3$. *Then* $g_1(u, s_1) = 0$, $g_2(s_2, u_2, u) = 0$ *and* $g_3(s_3, u_2, u) = 0$ *for polynomial functions* g_i.

[1] For instance Hofstadter points $X(359), X(360)$.

Proof. Using the parametrization of the 3-periodic billiard orbit it follows that $s_1^2 - |P_2 - P_3|^2 = 0$ is a rational equation in the variables u, s_1. Simplifying, leads to $g_1(s_1, u) = 0$.

Analogously for s_2 and s_3. In this case, the equations $s_2^2 - |P_1 - P_3|^2 = 0$ and $s_3^2 - |P_1 - P_2|^2 = 0$ have square roots $u_2 = \sqrt{1 - \rho_1 u^2}$ and $u_1 = \sqrt{1 - u^2}$ and are rational in the variables s_2, u_2, u_1, u and s_3, u_2, u_1, u respectively. It follows that the degrees of g_1, g_2, and g_3 are 10. Simplifying, leads to $g_2(s_2, u_2, u_1, u) = 0$ and $g_3(s_3, u_2, u_1, u) = 0$. □

Theorem 7. *In the family of 3-periodic orbits in a Poncelet pair of conics the locus of a rational triangle center is an algebraic curve.*

Proof. Follows the same steps as in the case of an elliptic billiard. See proof of 6. □

3.8 Exercises

Exercise 1. *Show that any triangle is an orbit of a billiard.*

Exercise 2. *The power of a point Q with respect to a circle centered at C_0 of radius μ is given by $|Q - C_0|^2 - \mu^2$ Eric Weisstein (2019, Circle Power). Let $\mathcal{C}(t)$ be the (moving) circumcircle to the 3-periodic billiard orbits $\{P_1(t), P_2(t), P_3(t)\}$ centered at $X_3(t)$. The power of the billiard center O with respect to $\mathcal{C}(t)$ is invariant and equal to $-\delta$.*

Exercise 3. *The* cosine circle *(also known as the second Lemoine circle) Eric Weisstein (2019, Cosine Circle) of a triangle passes through 6 points: the 3 pairs of intersections of sides with lines drawn through the symmedian X_6 parallel to sides of the orthic triangle. Recall that the orthic vertices are the feet of altitudes. Its center is X_6 Eric Weisstein (2019, Cosine Circle). If one takes the excentral triangle of a billiard orbit as the reference triangle, its orthic is the orbit itself.*

Show that the cosine circle of the excentral triangle is invariant over the family of 3-periodic orbits. Its radius $r^* = (a^2 - b^2)/\sqrt{2\delta - a^2 - b^2}$ is constant and it is concentric and external to the elliptic billiard.

Chapter 4

Invariants in the Elliptic Billiard

Chapter 5

Invariants of the Bicentric Family

Chapter 6

Invariants of the Homothetic and Brocard Families

Chapter 7

Experimental Techniques

Chapter 8

Epilogue: Properties of Pairs of Conics

Chapter 9

Conclusion

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