Invariants of Poncelet Families: an Experimental Promenade

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Chapter 1

Introduction

Chapter 2

Poncelet Preliminaries

2.1 Poncelet's Great Theorem

Consider a pair of nested ellipses $\mathcal{D} \subset \mathcal{C}$ as shown in Figure 2.1. Let $P_0 \in \mathcal{C}$ and draw a tangent line L_1 to $\mathcal{D}$ and passing through P_0. Call P_1 the second intersection of L_1 with $\mathcal{C}$. From P_1 draw the the second tangent line L_2 to $\mathcal{D}$. Clearly this process can be iterated to order n. The sequence of points $\{P_0, P_1, P_2, \ldots, P_n, \ldots\}$ will be called the *Poncelet orbit*.

When $P_n = P_0$ the Poncelet orbit is called periodic and the polygon $\mathcal{P}_n$ with vertices $\{P_0, \ldots, P_{n-1}, P_n\}$ will be called an $n-$gon. So, we obtain a polygon interscribed in the pair of ellipses $\{\mathcal{D}, \mathcal{C}\}$.

Theorem 1. *Consider a pair of nested ellipses $\{\mathcal{D}, \mathcal{C}\}$ as shown in Figure 2.1. If there is a $n-$gon interscribed between the pair $\mathcal{D}$ and $\mathcal{C}$, then for every $Q_0 \in \mathcal{C}$ there is an $n-$gon interscribed between $\mathcal{D}$ and $\mathcal{C}$ having Q_0 as one of its vertices.*

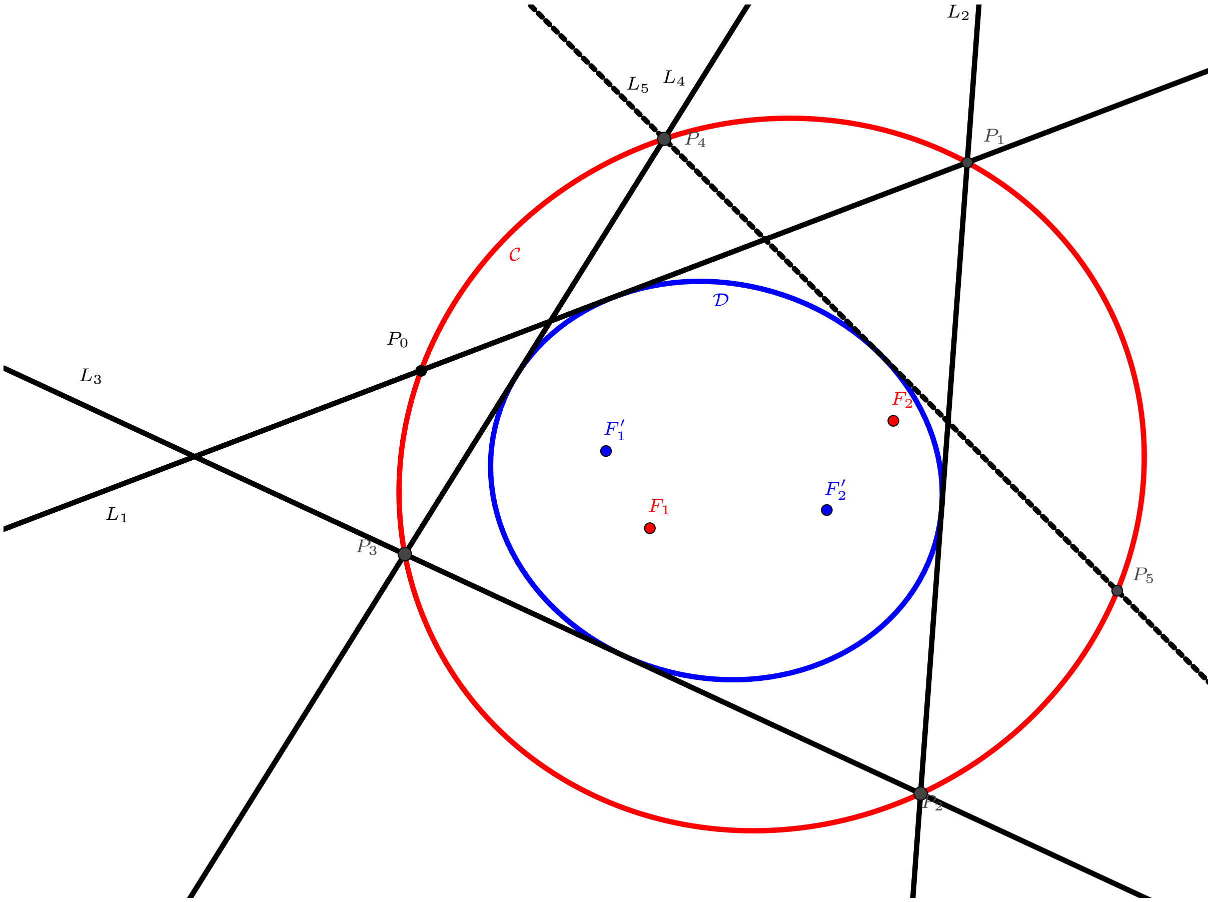

Figure 2.1: Poncelet map

2.2 Cayley's Conditions

Consider a pair of conics (ellipses) defined by two quadratic forms $q_1(x, y, z) = \frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - z^2 = 0$ and $q_2(x, y, z) = \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - z^2 = 0$ in projective coordinates (x, y, z).

Let $f(t) = \sqrt{\det(q_1 + tq_2)}$ where

$$q_i = \begin{pmatrix} \frac{1}{a_i^2} & 0 & 0 \\ 0 & \frac{1}{b_i^2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

2.3 Jacobi's Proof

Let $0 < k < 1$ and consider the elliptic integral

$$u = F(\varphi, k) = \int_0^{\varphi} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}}$$

The inverse of F will be denoted by $\varphi = \mathrm{am}(u, k)$ and is called the amplitude Jacobi function.

The functions

$$\begin{aligned}
\mathrm{cn}(u,k) &= \mathrm{JacobiCN}(u,k) = \cos(\mathrm{am}(u,k)) \\
\mathrm{sn}(u,k) &= \mathrm{JacobiSN}(u,k) = \sin(\mathrm{am}(u,k)) \\
\mathrm{dn}(u,k) &= \sqrt{1 - k^2 \mathrm{sn}^2(u,k)}
\end{aligned}$$

are called the Jacobi's elliptic functions. For k fixed they will denoted simply by $\mathrm{cn}(u)$ and $\mathrm{sn}(u)$. From definition basic properties are:

$$\mathrm{cn}(0) = 1,\ \mathrm{sn}(0) = 0,\ \mathrm{dn}(0) = 1;$$
$$\mathrm{cn}(K) = 0,\ \mathrm{sn}(K) = 1,\ \mathrm{dn}(K) = \sqrt{1 - k^2} = k$$
$$\mathrm{cn}(2K) = -1,\ \mathrm{sn}(2K) = 0,\ \mathrm{dn}(2K) = 1.$$

Also,

$$\begin{aligned}
\mathrm{sn}^2(u) + \mathrm{cn}^2(u) &= 1, \\
\mathrm{dn}^2(u) + k^2\mathrm{sn}^2(u) &= 1, \\
\mathrm{sn}'(u) &= \mathrm{cn}(u)\mathrm{dn}(u), \\
\mathrm{cn}'(u) &= -\mathrm{sn}(u)\mathrm{dn}(u), \\
\mathrm{dn}'(u) &= -k^2\mathrm{sn}(u)\mathrm{cn}(u).
\end{aligned}$$

$$\begin{aligned}
\mathrm{cn}(u+v) &= \frac{\mathrm{cn}(u)\mathrm{cn}(v) - \mathrm{sn}(u)\mathrm{sn}(v)\mathrm{dn}(u)\mathrm{dn}(v)}{\Delta(u,v)} \\
\mathrm{sn}(u+v) &= \frac{\mathrm{sn}(u)\mathrm{cn}(v)\mathrm{dn}(v) + \mathrm{sn}(v)\mathrm{cn}(u)\mathrm{dn}(u)\mathrm{dn}(u)}{\Delta(u,v)} \\
\mathrm{dn}(u+v) &= \frac{\mathrm{dn}(u)\mathrm{dn}(v) - k^2\mathrm{sn}(u)\mathrm{sn}(v)\mathrm{cn}(u)\mathrm{cn}(v)}{\Delta(u,v)} \\
\Delta(u,v) &= 1 - k^2\mathrm{sn}^2(u)\,\mathrm{sn}^2(v)
\end{aligned}$$

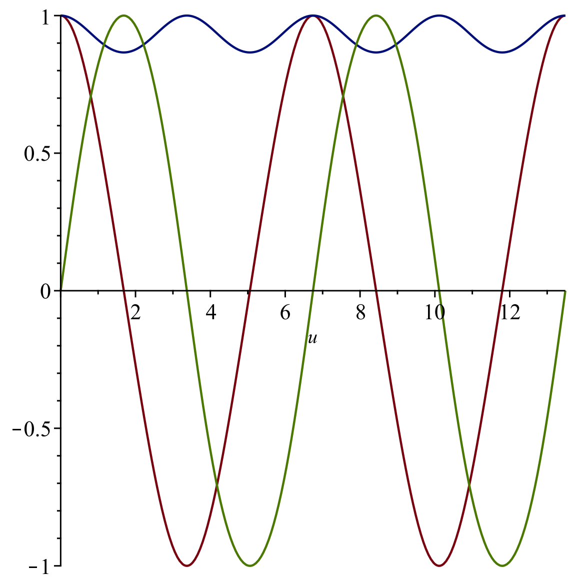

Figure 2.2: Jacobi elliptic functions

Below we recall some facts about three of Jacobi's elliptic functions extended to the complex plane $sn(z,k) = \sin(\mathrm{am}(z,k))$, $cn(z,k) = \cos(\mathrm{am}(z,k))$ and $dn(z,k) = \sqrt{1-k^2 sn^2(z,k)}$, where $z \in \mathbb{C}$, and $0 < k < 1$ is the elliptic modulus.

These functions have two independent periods and also have simple poles at the same points. In fact:

$$\begin{aligned}
\mathrm{sn}(u+4K) &= \mathrm{sn}(u+2iK') = \mathrm{sn}(u) \\
\mathrm{cn}(u+4K) &= \mathrm{cn}(u+2K+2iK') = \mathrm{cn}(u) \\
\mathrm{dn}(u+2K) &= \mathrm{dn}(u+4iK') = \mathrm{dn}(u) \\
K' &= K(k'), \quad k' = \sqrt{1-k^2}
\end{aligned}$$

The poles of these three functions, which are simple, occur at the points

$$2mK + i(2n+1)K', \quad m, n \in \mathbb{Z}$$

They also display a certain symmetry around the poles. Namely, if z_p is a pole of $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$, then, for every $w \in \mathbb{C}$, we have Armitage and Eberlein (2006, Chapter 2):

$$\begin{aligned}
\mathrm{sn}(z_p+w) &= -\,\mathrm{sn}(z_p-w) \\
\mathrm{cn}(z_p+w) &= -\,\mathrm{cn}(z_p-w) \\
\mathrm{dn}(z_p+w) &= -\,\mathrm{dn}(z_p-w)
\end{aligned} \tag{2.1}$$

Proposition 1. *A billiard orbit P_n $(n = 1, \dots, N)$ of period N is parametrized by*

$$P_n = \left[a \text{ JacobiSN}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right), b \text{ JacobiCN}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right)\right]$$
$$= \left[a\text{sn}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right), b\text{cn}\left(u + \frac{4n\tau K}{N}, \frac{c}{a}\right)\right]$$

where

$$K = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - (\frac{c}{a})^2 \sin^2 x}} dx$$

and $0 \leq u \leq 4K$.

2.4 Elliptic billiard

Theorem 2. *Consider an elliptic billiard defined in the ellipse $\mathcal{E}$ given by $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$. Let $F_1 = (-c, 0)$ and $F_2 = (c, 0)$ the foci of $\mathcal{E}$. Let $(P_n) = (P_n)_{n\in\mathbb{Z}}$ be a billiard orbit inscribed in $\mathcal{E}$. Then:*

i) If the segment of orbit P_0P_1 is outside the segment F_1F_2 then the caustic of the orbit (P_n) is a confocal ellipse $\mathcal{E}_1$ and the orbit is periodic or dense in the annulus defined by the pair $\{\mathcal{E}, \mathcal{E}_1\}$.

ii) If the segment of orbit P_0P_1 intersects the segment F_1F_2 then the caustic of the orbit is a confocal hyperbola $\mathcal{H}_1$ and the orbit is periodic or dense in the disk defined by the ellipse $\mathcal{E}$ and the caustic $\mathcal{H}_1$.

iii) If the segment of orbit P_0P_1 pass through a focus then the orbit pass through the other focus and is asymptotic to the 2-periodic orbit (diameter of the ellipse $\mathcal{E}$) in the past (backward) and the future(forward).

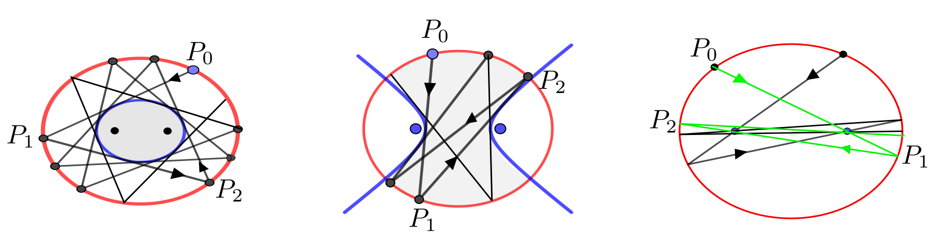

Figure 2.3: Three types of billiard orbits in the ellipse.

Proof. We follow **bry** to obtain the billiard map as a composition of two deck transformations. Consider the pair of nested ellipses parametrized by

$$\mathcal{E}: \; f(z,w) = \frac{z^2}{a^2} + \frac{w^2}{b^2} - 1 = 0$$
$$\mathcal{E}_1: \; g(x,y) = \frac{x^2}{a_c^2} + \frac{y^2}{b_c^2} - 1 = 0.$$

A tangent (oriented) line to $\mathcal{E}_1$ (caustic), passing through $q_0 = (x,y)$ is given by

$$h(x,y,z,w) = \frac{xz}{a_c^2} + \frac{yw}{b_c^2} - 1 = 0.$$

Now consider the set $\Sigma = \{(x,y,z,w) : f(z,w) = g(x,y) = h(x,y,z,w) = 0\}$. The set Σ is the union of two disjoint circles (curves diffeomorphic to circles) given by $\Sigma_+ = \{p \in \Sigma : xw - yz > 0\}$ and $\Sigma_- = \{p \in \Sigma : xw - yz < 0\}$. Given $q_0 \in \mathcal{E}_1$, let $p_0 = (z,w) \in \mathcal{E}$ such that $(q_0, p_0) \in \Sigma_+$. A line passing through p_0 and tangent to $\mathcal{E}_1$ passes through the point $q_1 = (u,v)$ and $(u,v,z,w) \in \Sigma_-$.

The projection $\pi_1 : \Sigma \to \mathcal{E}_1$ is a double cover. The same for the projection and $\pi : \Sigma \to \mathcal{E}$. Now we observe that there is a unique map $\tau : \Sigma_\pm \to \Sigma_\mp$ such that $\tau(x,y,z,w) = (x,y,\bar{z},\bar{w})$. Here $(\bar{z},\bar{w})$ is the other point of intersection of the tangent line passing (x,y) with the outer ellipse $\mathcal{E}$.

Also there is a unique map $\sigma : \Sigma_\pm \to \Sigma_\mp$ such that $\tau(x,y,z,w) = (\bar{x},\bar{y},z,w)$. The point $q_1 = (\bar{x},\bar{y}) \in \mathcal{E}_1$ is the in polar line of $p_0 = (z,w)$. Therefore the billiard orbit can be defined as follows. For each $q_i \in \mathcal{E}_1$, let $p_i \in \mathcal{E}$ the point of intersection of tangent line at q_i to $\mathcal{E}_1$ meets $\mathcal{E}$ with $(q_i,p_i) \in \Sigma_+$. Now let q_{i+1} the unique point on $\mathcal{E}_1$ such that $\{q_i, q_{i+1}\}$ are on the two tangent lines to $\mathcal{E}_1$ that pass through p_i. Therefore, the map $q_i \to q_{i+1}$ is given by $\sigma \circ \tau$ (resp. $p_i \to p_{i+2}$) is an orientation preserving diffeomorphism on $\mathcal{E}_1$ (resp. on $\mathcal{E}$).

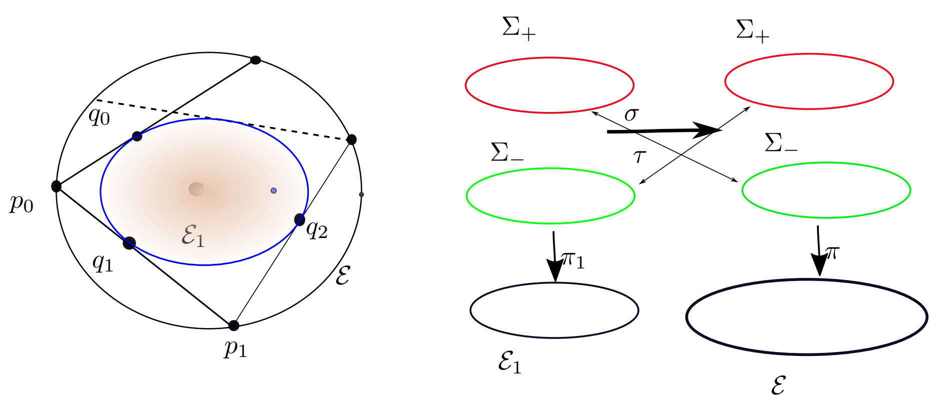

Figure 2.4: Three types of billiard orbits in the ellipse.

When the caustic is a hyperbola is necessary to consider the second iteration to obtain an orientation diffeomorphism. See **birkhoff** and **kolod**.

Finally, when the orbit pass through a focus the billiard map is conjugated to a diffeomorphism of the circle having two hyperbolic fixed points. □

2.5 Griffith and Harris' Proof

2.6 Exercises

Exercise 1. *Show that a pair ellipses* $x^2/A^2+y^2/B^2=1$ *and* $x^2/a^2+y^2/b^2=1$ *with semiaxes* (A,B) *and* (a,b) *(*$A>a,\ B>b$*) has a porism of pentagons (5-periodic orbits) then*

$$\frac{a^3}{A^3}+\frac{b^3}{B^3}+\left(\frac{a}{A}+\frac{b}{B}\right)^2=1+\left(\frac{a}{A}+\frac{b}{B}\right)\left(1+\frac{ab}{AB}\right)$$

Chapter 3

Loci of Triangle Centers over N=3 Poncelet Families

3.1 History of Result

Early videos 2011 com Jair Koiller Reznik (2011d) and Reznik (2011b), proof by complexification Romaskevich (2014), proof by Affine Curvature Garcia (2019), circumcenter Fierobe (2021). Schwartz Schwartz and Sergei Tabachnikov (2016a), Circumcenter of Mass Sergei Tabachnikov and Tsukerman (2014).

3.2 Triangle Centers

3.3 Some Concentric N=3 Families

3.4 Locus of Incenter and Excenters

3.5 Elliptic Loci in Generic Pairs

3.6 Exercises

Exercise 2. *blablabla*

3.7 Videos

Referring to Figure 3.1:

Theorem 3. *Over the family of 3-periodics interscribed in a generic nested pair of ellipses (non-concentric, non-axis-aligned), if $\mathcal{X}_{\alpha,\beta}$ is a fixed linear combina-*

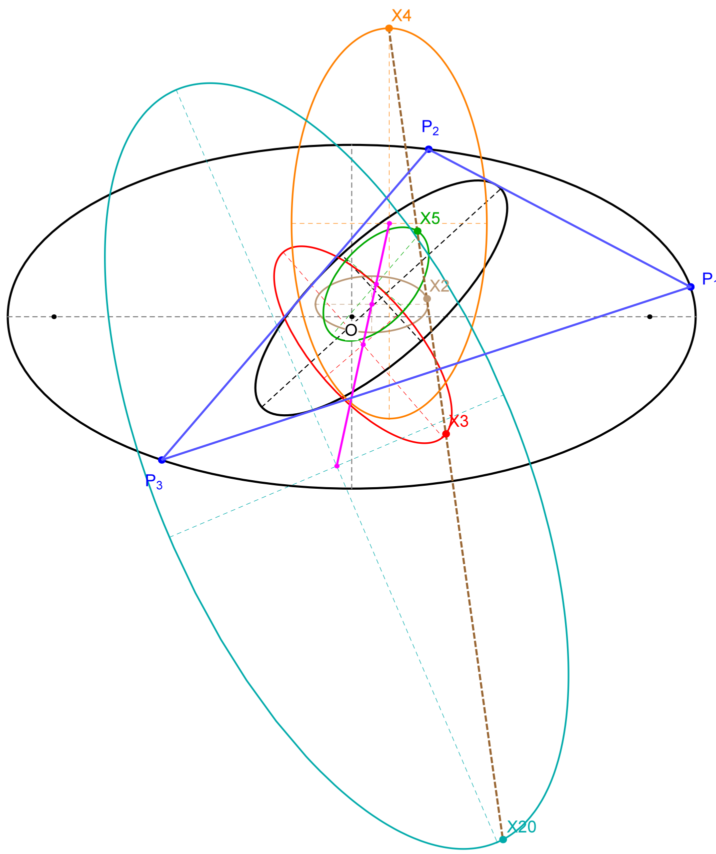

Figure 3.1: A 3-periodic is shown interscribed between two nonconcentric, non-aligned ellipses (black). The loci of X_k, $k = 2, 3, 4, 5, 20$ (and many others) are elliptic. Those of X_2 and X_4 are axis-aligned with the outer ellipse. Furthermore, the centers of all elliptic loci are collinear (magenta line).

tion of X_2 and X_3, i.e., $\mathcal{X}_{\alpha,\beta} = \alpha X_2 + \beta X_3$ for some fixed $\alpha, \beta \in \mathbb{C}$, then its locus is an ellipse.

Theorem 4. *Over 3-periodics in the elliptic billiard (confocal pair) the locus of the incenter X_1 is an ellipse given by $x^2/a_1^2 + y^2/b_1^2 = 1$, where*

$$a_1 = \frac{\delta - b^2}{a}, \quad b_1 = \frac{a^2 - \delta}{b}, \quad \delta = \sqrt{a^4 - a^2 b^2 + b^4}.$$

The locus of the Excenters (triangle formed by the intersection of external bisectors) is an ellipse with axes:

$$a_e = \frac{b^2 + \delta}{a}, \quad b_e = \frac{a^2 + \delta}{b}$$

Notice it is similar to the X_1 locus, i.e., $a_1/b_1 = b_e/a_e$.

3.8 Future Work

Conjecture 1. *The locus of the incenter is an ellipse if and only if the Poncelet ellipse pair is confocal.*

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk bounded by $\mathbb{T}$.

Lemma 1. *If $u, v, w \in \mathbb{C}$ and λ is a parameter that varies over the unit circle $\mathbb{T} \subset \mathbb{C}$, then the curve parametrized by*

$$F(\lambda) = u\lambda + \frac{v}{\lambda} + w$$

is an ellipse centered at w, with semiaxis $|u| + |v|$ and $\big||u| - |v|\big|$, rotated with respect to the horizontal axis of $\mathbb{C}$ by an angle of $(\arg u + \arg v)/2$.

Proof. □

Consider the Moebius map $M_{z_0} = (z_0 - z)/(1 - \overline{z_0} z)$ and the Blaschke product of degree 3 given by $B = M_{z_0} M_{z_1} M_{z_2}$.

Theorem 5. *Let B be a Blaschke product of degree 3 with zeros $0, f, g$. For $\lambda \in \mathbb{T}$, let z_1, z_2, z_3 denote the three distinct solutions to $B(z) = \lambda$. Then the lines joining z_j and z_k, $(j \neq k)$ are tangent to the ellipse given by*

$$|w - f| + |w - g| = |1 - \overline{f} g|.$$

Theorem 6. *Given two points $f, g \in \mathbb{D}$. Then there exists a unique conic $\mathcal{E}$ with the foci f, g which is 3-Poncelet caustic with respect to $\mathbb{T}$. Moreover, $\mathcal{E}$ is an ellipse. That ellipse is the Blaschke ellipse with the major axis of length $|1 - \overline{f} g|$.*

Consider the parametrization of a triangular orbit $\{z_1, z_2, z_3\}$ as given in Helman, Laurain, et al. (2021). Let also the affine transformation $T(z) = pz + q\bar{z}$.

Definition 1 (Blaschke's Parametrization)**.**

$$\begin{aligned} \sigma_1 &:= z_1 + z_2 + z_3 = f + g + \lambda \overline{f}\overline{g} = \alpha \\ \sigma_2 &:= z_1 z_2 + z_2 z_3 + z_3 z_1 = fg + \lambda(\overline{f} + \overline{g}) = \beta \\ \sigma_3 &:= z_1 z_2 z_3 = \lambda \end{aligned}$$

where f, g are the foci of the inner ellipse and $\lambda \in \mathbb{T}$ is the varying parameter.

Proposition 2. *Over Poncelet 3-periodics in the pair with an outer circle and an ellipse in generic position, the locus X_1 given by:*

$$\begin{aligned} X_1 : {} & z^4 - 2((\bar{f} + \bar{g})\lambda + fg)z^2 + 8\lambda z \\ & + (\bar{f} - \bar{g})^2 \lambda^2 + 2(|f|^2 g + f|g|^2 - 2f - 2g)\lambda + f^2 g^2 = 0 \\ & : \; z^4 - 2\beta z^2 + 8\lambda z + (\beta^2 - 4\alpha\lambda) = 0 \end{aligned}$$

Proof. The incenter of a triangle with vertices $\{z_1, z_2, z_3\}$ is given by:

$$X_1 = \frac{\sqrt{a}\, z_1 + \sqrt{b}\, z_2 + \sqrt{c}\, z_3}{\sqrt{a} + \sqrt{b} + \sqrt{c}}$$
$$a = |z_2 - z_3|^2, \; b = |z_1 - z_3|^2, \;\; c = |z_2 - z_1|^2$$

Using that $z_i \in \mathbb{T}$ it follows that

$$a = 2 - (\frac{z_3}{z_2} + \frac{z_3}{z_2}), \;\; b = 2 - (\frac{z_1}{z_3} + \frac{z_3}{z_1}), \;\; c = 2 - (\frac{z_1}{z_2} + \frac{z_2}{z_1})$$

Eliminating the square roots in the equation $X_1 - z = 0$ and using the relations σ_i (i=1,2,3) given in Blaschke's parametrization the result follows. □

Proposition 3. *Over Poncelet 3-periodics in a generic nested ellipse pair, the locus of X_1 is given by the following sextic polynomial in z, λ:*

$$\begin{aligned} X_1 : \; & \lambda^2 \left(p^2 - q^2\right) z^4 + 4\,\lambda\, \left(\alpha\,\lambda\, pq^2 - q\,\lambda^2 p^2 - \beta\, p^2\, q + p\, q^2\right) z^3 \\ & + (4\,\alpha\,\lambda^3 p^3 q - 4\,\alpha^2 \lambda^2 p^2 q^2 + 2\,\alpha\,\beta\,\lambda\, p^3\, q - 2\,\alpha\,\beta\,\lambda\, pq^3 - 2\,\beta\,\lambda^2 p^4 + 6\,\beta\,\lambda^2 p^2 q^2 \\ & - 6\,\alpha\,\lambda\, p^2\, q^2 + 2\,\alpha\,\lambda\, q^3 q + 4\,\beta^2 p^2\, q^2 + 6\,\lambda^2 p^3\, q - 6\,\lambda^2 pq^3 - 4\,\beta\, p\, q^3)z^2 \\ & + (4q(\alpha^2 \beta p^2 q^2 + 2\alpha^2 p^2 pq - \alpha^2 pq^3 + \beta^2 p^4 - 2\beta^2 p^2 q^2 + 4\beta pq^3 - p^2 q^2 - 2q^4)\lambda \\ & - 4\alpha pq^2(\beta^2 p^2 - q^2) - 4p^3(\beta pq - 2p^2 - q^2)\lambda^3 - 16\alpha\lambda^2 p^4 q)z \\ & - \lambda^2(4\alpha\lambda - \beta^2)p^6 + 4p^5 q\lambda^4 + 2\lambda(4\alpha^2\lambda - \alpha\beta^2 - 3\beta\lambda)p^5 q - \lambda^2(8\alpha\lambda - 3\beta^2)p^4 q^2 \\ & + (\alpha^2\beta^2 - 4\alpha^3\lambda + 4\alpha\beta\lambda + 5\lambda^2)p^4 q^2 + 2\lambda(2\alpha^2\lambda - \alpha\beta^2 + \beta\lambda)p^3 q^3 \\ & + (2\alpha^2\beta - 2\alpha\lambda - 4\beta^2)p^3 q^3 - (\alpha^2\beta^2 + 4\alpha\beta\lambda - 4\beta^3 + 5\lambda^2)p^2 q^4 + (8\beta - 3\alpha^2)p^2 q^4 \\ & + (2\alpha^2\beta + 6\alpha\lambda - 8\beta^2)pq^5 - 4q^5 p + (4\beta - \alpha^2)q^6 = 0 \end{aligned}$$

Proof. Let $p, q \in \mathbb{R}$. Consider the affine transformation $T(z) = pz + q\bar{z}$ and set $w_i = T(z_i)$. The proof is similar to that given in Proposition 2. □

Proposition 4. *In the confocal pair the locus X_1 is defined by:*

$$2\,ab\lambda^2 z^2 + 2\,\lambda\,\left(a^3\lambda^2 - b^3\lambda^2 - a^3 - b^3\right) z + c^2\left(c^2\lambda^4 - 2\,ab\lambda^2 - c^2\right) = 0$$

Proof. We have that

$$f = \frac{1}{c}\sqrt{-a^2 - b^2 + 2\,\delta},\quad g = -\frac{1}{c}\sqrt{-a^2 - b^2 + 2\,\delta}$$

□

Corollary 1. *The locus X_1 is the ellipse with semiaxes given by $a_1 = (a^2 - \delta)/b$ and $b_1 = (\delta - b^2)/a$.*

Proof. The quartic polynomial is factorizable as $p_1 p_2$, where

$$p_1 = z - \left(\frac{(a-b)\left(-a^2 - ab - b^2 + \delta\right)\lambda}{2\,ab} - \frac{(a+b)\left(-a^2 + ab - b^2 + \delta\right)}{2\,ab\lambda}\right)$$

$$\begin{aligned} p_2 = {} & 2ab\lambda^2 z^3 - ((a-b)(a^2 + 3ab + b^2 + \delta)\lambda^3 - (a+b)(a^2 - 3ab + b^2 + \delta)\lambda)z^2 \\ & + 6ab(a^2 - b^2)\lambda^2 z + (a+b)^3(a^2 - ab + b^2 + \delta)\lambda^3 - (a-b)^3(a^2 + ab + b^2 + \delta)\lambda \end{aligned}$$

Follows directly from Lemma 1 and Proposition 4. □

Schwartz and Sergei Tabachnikov (2016a)

Conjecture 2. *Over 3-periodics interscribed between two ellipses in general position, the locus of a triangle center X_k is an ellipse if and only if X_k is a fixed linear combination of X_3 and X_4.*

Chapter 4

Invariants in the Elliptic Billiard

4.1 Main Result

History of experiment. Phone call with Ronaldo, graph of r/R, Feuerbach's thm implies sum of cosines. Phone call with Jair about $N=4$, supplementary angles, sum of cosines vanishes. generalization for all N. Tried $N=5$ and it worked.

Dominique placed an expression as youtube comment for r/R which we recognized as $JL-3$ given ronaldo's explicit expressions for J and L. We tried $JL-N$ and it worked for all N.

Theorem 7. *The sum of cosines of elliptic billiard N-periodic cosines is invariant for all N and given by $JL-N$.*

4.2 Proof for $N=3$

Reznik, Garcia, and Koiller (2020a) and Garcia, Reznik, and Koiller (2020b).

Let $r(t)$ and $R(t)$ be the radius of the incircle and circumcircle aof 3-periodic billiard orbit $\mathcal{P}_3(t)=\{p_1(t),p_2(t),p_3(t)\}$, respectively.

Theorem 8. *r/R is invariant over the 3-periodic orbit family and given by*

$$\frac{r}{R}=\frac{2(\delta-b^2)(a^2-\delta)}{(a^2-b^2)^2}. \tag{4.1}$$

Proof. Let r and R be the radius of the incircle and circumcircle, respectively. For any triangle Coxeter and Greitzer (1967) we have

$$rR=\frac{s_1 s_2 s_3}{2L},$$

where $L = s_1 + s_2 + s_3$ is the perimeter, constant for 3-periodic orbits.

$$\frac{r}{R} = \frac{1}{2L}\frac{s_1 s_2 s_3}{R^2}. \tag{4.2}$$

Next, with $P_1 = (a, 0)$, obtain a *candidate* expression for r/R. This yields Equation (4.1) exactly. Using explicit expressions for orbit vertices (see) derive an expression for the square of the right-hand side of Equation (4.2) as a function of x_1 and subtract from it the square of Equation (4.1). It can be shown $\left(s_1 s_2 s_3/R^2\right)^2$ is rational on x_1 Garcia, Reznik, and Koiller (2020a). For simplification, use $R = s_1 s_2 s_3/(4A)$, where A is the triangle area. With a computer algebra system (CAS), show said difference is identically zero for all $x_1 \in (-a, a)$. ☐

Corollary 2.

$$\cos\theta_1(t) + \cos\theta_2(t) + \cos\theta_3(t) = 1 + \frac{r(t)}{R(t)}$$

is invariant over the 3-periodic orbit family.

Proof. The relation stated is valid for any triangle. Therefore the result follows directly from Theorem 8. ☐

4.3 Case of $N = 4$

Theorem 9. *Let $\mathcal{P}_4(t) = \{p_1(t), p_2(t), p_3(t), p_4(t)\}$ be the family of 4-periodic billiard orbits. Then,*

$$\sum_{i=1}^{4} \theta_i(t) = 0$$

Proof. If the orbit is simple it is a parallelogram and the result follows. When the orbit is self intersected it is inscribed in a circle and therefore the opposite angles are supplementary and so the result follows. ☐

4.4 Proof for all N via Linear Algebra

This section is based in A. Akopyan, Schwartz, and Serge Tabachnikov (2020). Consider the ellipse $\mathcal{E}$ defined by $\langle Ap, p\rangle = 1$, where A is the diagonal matrix

$$A = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}$$

Let also $p_i^* = Ap_i$, We have that p_i^* is a normal vector to the ellipse $\mathcal{E}$ at p_i and $\langle p_i, p_i^* \rangle = 1$.

Proposition 5. *The function*

$$J(p,u) = -\langle p^*, u\rangle$$

is invariant under the billiard map $T(p,u) = (q,v)$.

Proof. □

Theorem 10. *Let $\mathcal{P}_N = \{p_1, \dots, p_N\}$ be a $N-$periodic orbit of an elliptic billiard $\mathcal{E}$ with internal angle θ_i at the vertex p_i. Then,*

$$\sum_{i=1}^{N} \cos\theta_i = JL - N$$

Proof. Consider a $N-$periodic orbit $\mathcal{P}_n = \{p_1, \dots, p_N\}$ and let u_i be the unitary vector defined by

$$u_i = \frac{p_{i+1} - p_i}{|p_{i+1} - p_i|.}$$

Let $p_i^* = Ap_i$. We have that p_i^* is a normal vector to the ellipse $\mathcal{E}$ at p_i and $\langle p_i, p_i^*\rangle = 1$.

Then, the polygonal orbit $\mathcal{P}_n$ has perimeter L given by

$$L = \sum \langle p_{i+1} - p_i, u_i\rangle = \sum \langle p_i, u_{i-1}\rangle - \sum \langle p_i, u_i\rangle = \sum \langle p_i, u_{i-1} - u_i\rangle$$

By the reflection law we have that:

$$u_{i-1} - u_i = 2\sin\left(\frac{\pi - \theta_i}{2}\right)\frac{p_i^*}{|p_i^*|} = 2\cos\left(\frac{\theta_i}{2}\right)\frac{p_i^*}{|p_i^*|}$$

Also,

$$J = -\langle u_i, p_i^*\rangle = -\cos\left(\pi - \frac{\theta_i}{2}\right)|p_i^*| = \cos\left(\frac{\theta_i}{2}\right)|p_i^*|$$

Since $\langle p_i, p_i^*\rangle = 1$ and J is invariant it follows that

$$\begin{aligned} JL &= \sum \cos\left(\frac{\theta_i}{2}\right)|p_i^*|\langle p_i, u_{i-1} - u_i\rangle = \sum \cos\left(\frac{\theta_i}{2}\right)|p_i^*|\langle p_i, 2\cos\left(\frac{\theta_i}{2}\right)\frac{p_i^*}{|p_i^*|}\rangle \\ &= \sum 2\cos^2\left(\frac{\theta_i}{2}\right)\langle |p_i^*|p_i, \frac{p_i^*}{|p_i^*|}\rangle = \sum(1 + \cos\theta_i) = N + \sum \cos\theta_i \end{aligned}$$

□

4.5 Proof for all N via Liouville

This section is based in A. Akopyan, Schwartz, and Serge Tabachnikov (2020) and Bialy and Sergei Tabachnikov (2020).

Theorem 11. *Consider a Poncelet orbit $\{p_1, p_2, \dots, p_N\}$ inscribed in a circle $\mathcal{C}$ and circumscribed in an ellipse $\mathcal{E}$, both centered at 0. Then*

$$\sum_{i=1}^{N} \cos\theta_i = \sum_{i=1}^{N} \cos \angle p_{i-1} 0 p_i$$

is constant in the 1-parameter family of Poncelet N-gons.

Proof. The idea is to complexify the problem and show that the sum is bounded.

Consider the rational parametrization

$$p(t) = \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1} \right),$$

of the conic $x^2 + y^2 = 1$ with $t \in \mathbb{C}$.

We have that

$$S_N = \sum_{i=1}^{N} \cos \angle p_{i-1} 0 p_i = \langle p_{i-1}, p_i \rangle.$$

The sum S_N can be unbounded only when the complexified orbit contains points at infinity which are given by $p(I)$ and $p(-I)$. Here I is the complex number with $I^2 = -1$.

Claim: If $p(I)$ is a vertex of a Poncelet polygon, then the adjacent (neighboring) vertices to $p(I)$, say $a(I)$ and $b(I)$, are opposite in $\mathcal{C}$.

The complex lines $p(I)a(I)$ and $p(I)b(I)$ are tangent to the complex conic $\mathcal{E}$ and are parallel, therefore the tangency points of these lines with $\mathcal{E}$ are symmetric with respect to O, and hence their intersection points with $\mathcal{C}$ are also symmetric (the point $p(I)$ is invariant under the reflection in O, given by the Moebius map $M(t) = 1/t$).

Claim: For any finite point q on $\mathcal{C}$, it follows that

$$\langle q, a(I) \rangle + \langle q, b(I) \rangle = \langle q, a(I) + b(I) \rangle = \langle q, 0 \rangle = 0.$$

Now, consider the point $p(t+I)$ with t tending to zero and its adjacent (neighboring) vertices $a(I+t)$ and $b(I+t)$ of the Poncelet polygon. Notice that $p(t+I)$ tends to infinity as $0(1/t)$, while $a(t+I), b(t+I)$ tend to their limits $a(I), b(I)$ linearly. Furthermore, due to the symmetry, as t goes to zero, the linear in t terms are vectors with the same absolute value and opposite directions:

$$a(t+I) = a(I) + \mathbf{k}t + O_1(t^2), \quad b(t+I) = -a(I) - \mathbf{k}t + O_2(t^2).$$

Therefore, for small t we have:

$$\begin{aligned}\langle p(t+I), a(t+I)\rangle + \langle p(t+I), b(t+I)\rangle &= \langle p(t+I), a(t+I) + b(t+I)\rangle \\ &= \langle 0(1/t), 0(t^2)\rangle = 0(t).\end{aligned}$$

Then,

$$\langle p(I), a(I)\rangle + \langle p(I), b(I)\rangle = 0.$$

So the function S_N is bounded and by Liouville's theorem is constant. □

In the case $N = 3$ we have that

p(I)=(∞, ∞) $a(I) = [a\frac{1}{\sqrt{2\,a-1}}, i\,(a-1)\,\frac{1}{\sqrt{2\,a-1}}],\ b(I) = -a(I)$
$p(I+t) =$

Remark 1. *The internal angles α_i of the Poncelet polygon $\{p_1, p_2, \ldots, p_N\}$ satisfy $\alpha_i = \pi - \angle p_{i-1} 0 p_i$ and therefore the Theorem 11 is equivalent to Theorem 10.*

Proposition 6. *Consider the Poncelet pair $x^2 + y^2 = 1$ and $x^2/a^2 + y^2/(1-a)2 = 1$, with $0 < a < 1$. Then*

$$\sum_{i=1}^{3} \cos\theta_i = 2a^2 - 2a - 1.$$

4.6 Average Cosine Sum with Spatial Integrals

4.7 Future Work

In Reznik, Garcia, and Koiller (2021) we provide a complete list of invariants, many conjectured.

Chapter 5

Invariants of the Bicentric Family

5.1 Main Result

History of experiment. Pedro found out about our work via Youtube (insomnia). Asked us about an invariant involving billiard curvatures at the vertices, we found $\sum k^{2/3}$ but this is a direct corollary of the sum of cosines. This implied $\sum 1/(d_1 d_2)$ was invariant. Jair gave the idea to check if the stronger result $\sum 1/d_1$ was invariant, which it was. This corresponds to the sum of the inverse lengths of "focal spokes", or the sum of the lengths of the focal spokes to the inversive polygon, both which are invariant. We then simply tested the perimeter of the inversive polygon which was unexpectedly constant.

In Reznik and Garcia (2020) we showed:

Theorem 12. *Over 3-periodics in the elliptic billiard (confocal pair) the perimeter of the focus-inversive polygon is invariant and given by:*

...

Furthermore the inversive family is a 3-periodic of a rigidly rotating elliptic billiard whose axes are given by:

...

Theorem 13. *Over N-periodics in the elliptic billiard (confocal pair) the perimeter of the focus-inversive polygon is invariant.*

5.2 Proof by Jacobi Elliptic Functions

Lemma 2. *The polar curve of the ellipse $\mathcal{E}$ is the circle*

$$C(x,y) = (x + \frac{c(b^2+\rho^2)}{b^2})^2 + y^2 - \frac{a^2\rho^4}{b^4} = 0$$

ron:checar x0 abaixo

Lemma 3. *The polar curve of the hyperbola $\mathcal{H}$ is the circle*

$$C(x,y) = \left(x - \frac{c(\rho^2-b^2)}{b^2}\right)^2 + y^2 - \frac{a^2\rho^4}{b^4} = 0$$

Lemma 4. *The limit points of a pair of polar circles associated to a pair of confocal ellipes are*

$$[-c,0]\,,\quad \left[-c+\frac{\rho^2}{c},0\right]$$

5.3 Mapping a Confocal to a Bicentric Pair

Under the polar transformation an origin-centered ellipse $\mathcal{E}$ is sent to the circle:

$$\left(x + \frac{c\left(b^2+\rho^2\right)}{b^2}\right)^2 + y^2 - \frac{a^2\rho^4}{b^4} = 0$$

and the confocal ellipse $\mathcal{E}_c$ is sent to

$$\left(x + \frac{c\left(b_c^2+\rho^2\right)}{b_c^2}\right)^2 + y^2 - \frac{a_c^2\rho^4}{b_c^4} = 0$$

Therefore:

$$\begin{aligned}
r &= \frac{a\,\rho^2}{b^2},\;\; R = \frac{a_c\,\rho^2}{b_c^2} \\
d &= \frac{c\left(b_c^2+\rho^2\right)}{b_c^2} - \frac{c\left(b^2+\rho^2\right)}{b^2} = \frac{c\rho^2(b^2-b_c^2)}{b^2\,b_c^2} = \frac{\rho^2 c\,(a^2-a_c^2)}{b^2\,b_c^2} \\
k^2 &= \frac{4Rd}{(R+d)^2-r^2} = \frac{4\,c\,a_c\,(a_c-c)^2}{b_c^4} \\
\delta_\pm &= \pm\frac{\rho^6 a^4}{2\,c\,b^8} + \frac{\left(a^2b_c^4 - a_c^2b^4 - c^6\right)\rho^2}{2\,b^2b_c^2c^3}
\end{aligned}$$

The limiting points ℓ_1, ℓ_2 are given by: $[-c,0]$ and $[-c+\frac{\rho^2}{c},0]$.

Ron: calculos abaixos foram simplificados na secao seguinte

Reciprocally, given a pair of circles C_R : $(x+d)^2+y^2=R^2$ and C_r : $x^2+y^2=r^2$, by the polar transformation, it is associated a pair of confocal ellipses $\{\mathcal{E},\mathcal{E}_c\}$ with semiaxes given by:

$$
\begin{aligned}
a &= \sqrt{b^2+c^2} = \frac{\sqrt{2}\rho^2\sqrt{(R^2-d^2-r^2)\sqrt{\alpha}+(R^2-d^2)^2-r^2(2R^2-r^2)}}{2r\alpha}\\
b &= \frac{\sqrt{2}\rho^2}{2\,\alpha\,r}\sqrt{\alpha\,\left(\alpha+\sqrt{4\,\alpha\,r^2d^2+\alpha^2}\right)}\\
a_c &= \sqrt{b_c^2+c^2} = \frac{\sqrt{2}\rho^2\sqrt{(R^2+d^2-r^2)\sqrt{\alpha}+R^2(R^2-2r^2)+(d^2-r^2)^2}}{2R\alpha}\\
b_c &= \frac{\sqrt{2}\rho^2}{2\alpha\,R}\sqrt{\alpha\,\left(\alpha+\sqrt{4\,\alpha R^2\,d^2+\alpha^2}\right)}\\
c &= \frac{d\rho^2}{2\,\alpha\,(R^2-r^2)}\left(\sqrt{\alpha\,(4\,r^2d^2+\alpha)}+\sqrt{\alpha\,(4\,R^2d^2+\alpha)}\right)\\
\alpha &= (R-d+r)(R+d+r)(R-d-r)(R+d-r)
\end{aligned}
$$

Under the polar transformation an origin-centered ellipse $\mathcal{E}$ is sent to the circle:

$$\left(x+\frac{c\left(b^2+\rho^2\right)}{b^2}\right)^2+y^2-\frac{a^2\rho^4}{b^4}=0$$

and the confocal ellipse $\mathcal{E}_c$ is sent to

$$\left(x+\frac{c\left(b_c^2+\rho^2\right)}{b_c^2}\right)^2+y^2-\frac{a_c^2\rho^4}{b_c^4}=0$$

Therefore:

$$
\begin{aligned}
r &= \frac{a\,\rho^2}{b^2},\; R=\frac{a_c\,\rho^2}{b_c^2}\\
d &= \frac{c\left(b_c^2+\rho^2\right)}{b_c^2}-\frac{c\left(b^2+\rho^2\right)}{b^2}=\frac{c\rho^2(b^2-b_c^2)}{b^2\,b_c^2}=\frac{\rho^2c\,(a^2-a_c^2)}{b^2\,b_c^2}\\
k^2 &= \frac{4Rd}{(R+d)^2-r^2}=\frac{4\,c\,a_c\,(a_c-c)^2}{b_c^4}\\
\delta_{\pm} &= \pm\frac{\rho^6a^4}{2\,c\,b^8}+\frac{\left(a^2b_c^4-a_c^2b^4-c^6\right)\rho^2}{2\,b^2b_c^2c^3}
\end{aligned}
$$

The limiting points ℓ_1,ℓ_2 are given by: $[-c,0]$ and $[-c+\frac{\rho^2}{c},0]$.

5.4 Hyperbolas

Consider the pair of circles $x^2 + y^2 = r^2$ and $(x + d)^2 + y^2 = R^2$ and the limit points $\ell_1 = (R^2 - d^2 - r^2 - \Delta)/(2d)$ and $\ell_2 = (R^2 - d^2 - r^2 + \Delta)/(2d)$, where

$$\Delta = \sqrt{(d + R + r)\,(R - d + r)\,(R + d - r)\,(R - d - r)}$$

Lemma 5. *The polar image of the circle $x^2 + y^2 = r^2$ with respect to the limit point ℓ_2 is the hyperbola centered at*

$$\left[\frac{\Delta^2 - 2d^2k^2}{2d\Delta} + \frac{R^2 - d^2 - r^2}{2d}, 0\right]$$

and semiaxes given by

$$\begin{aligned} a^2 &= \frac{k^4(2d^2r^2 - \Delta(R^2 - d^2 - r^2 - \Delta))}{2r^2\Delta^2} \\ b^2 &= \frac{k^4(R^2 - d^2 - r^2 - \Delta)}{2\Delta r^2} \\ c^2 &= a^2 + b^2 = \frac{k^4d^2}{\Delta^2} \end{aligned}$$

Lemma 6. *The polar image of the circle $(x + d)^2 + y^2 = R^2$ with respect to the limit point ℓ_2 is the hyperbola centered at*

$$\left[\frac{\Delta^2 - 2d^2k^2}{2d\Delta} + \frac{R^2 - d^2 - r^2}{2d}, 0\right]$$

and semiaxes given by

$$\begin{aligned} a^2 &= \frac{k^4(2R^2d^2 - \Delta(R^2 + d^2 - r^2 - \Delta))}{2R^2\Delta^2} \\ b^2 &= \frac{(R^2 + d^2 - r^2 - \Delta)k^4}{2\Delta R^2} \\ c^2 &= a^2 + b^2 = \frac{k^4d^2}{\Delta^2} \end{aligned}$$

Lemma 7. *The polar image of the circle $x^2 + y^2 = r^2$ with respect to the limit point ℓ_1 is the ellipse centered at*

$$\left[\frac{2d^2k^2 - \Delta^2}{2\Delta d} + \frac{R^2 - d^2 - r^2}{2d}, 0\right]$$

and semiaxes given by

$$
\begin{aligned}
a^2 &= \frac{((\Delta + R^2 - d^2 - r^2)\Delta + 2d^2r^2)k^4}{2\Delta r^2} \\
b^2 &= \frac{(R^2 + \Delta - d^2 - r^2)k^4}{2\Delta r^2} \\
c^2 &= a^2 - b^2 = \frac{k^4 d^2}{\Delta^2}
\end{aligned}
$$

Lemma 8. *The polar image of the circle* $(x+d)^2 + y^2 = R^2$ *with respect to the limit point* ℓ_1 *is the ellipse centered at*

$$
\left[\frac{2d^2k^2 - \Delta^2}{2\Delta d} + \frac{R^2 - d^2 - r^2}{2d}, 0\right]
$$

and semiaxes given by

$$
\begin{aligned}
a^2 &= \frac{(2R^2d^2 + \Delta(\Delta + R^2 + d^2 - r^2))k^4}{2\Delta R^2} \\
b^2 &= \frac{(\Delta + R^2 + d^2 - r^2)k^4}{2\Delta R^2} \\
c^2 &= a^2 - b^2 = \frac{k^4 d^2}{\Delta^2}
\end{aligned}
$$

5.5 Poristic

Definition 2. *Let* $\mathcal{P} = \{p_1, \ldots p_n\}$ *be an* $n-$*gon with vertices* p_i *with* $p_{n+1} = p_1$. *From each vertex* p_i *draw a perpendicular to the segment* $p_{i-1}p_{i+1}$. *In general, these perpendiculars have no common point, but when they meet at a single point, this point will be called the orthocentre of the* $n-$*gon* $\mathcal{P}$.

Theorem 14. *Let* Γ *be the circle inscribed in an n-gon* $\mathcal{P}$, *let* I *be the inversion with respect to* Γ, a_i *the centres of the circles equal to the* $I-$*images of the straight lines containing sides of* $\mathcal{P}$, *and let* A *be the base* $n-$*gon with vertices* a_i. *The polygon* $\mathcal{P}$ *admits a circumscribed circle* $\mathcal{C}$ *if and only if the corresponding base* $n-$*gon* A *has an orthocentre. This orthocentre is the centre of the circle* $I(\mathcal{C})$, *the image of the circumscribed circle* $\mathcal{C}$ *under* I .

Pedro's paper.

5.6 Future Work

Conjecture 3. *Over N-periodics in the elliptic billiard, the sum of cosines of the inversive polygon is invariant except for simple* $N = 4$.

Conjecture 4. *The product of areas of the two focus-inversive polygons is invariant for all odd* N.

Conjecture 5. *The ratio of areas of polar to inversive polygons is invariant for all N.*

Chapter 6

Invariants of the Homothetic and Brocard Families

6.1 Main Result

History of experiment. Jair referenced a book Daepp et al. (2019) on whatsapp, not knowing of our work on loci. Mark quickly learned Blaschke products and noticed its usefulness for 3-periodic Poncelet.

Theorem 15. *Over 3-periodics in Poncelet pair (concentric or not) with a circumcircle, the locus of a triangle center which is a fixed linear combination of X_3 and X_4 is a circle given by:*

$$...$$

Furthermore all locus centers are collinear with the origin.

Theorem 16. *Over 3-periodics interscribed in a concentric, axis-aligned pair of ellipses, the power of the common center with respect to either the circurmcircle or Euler's circle is invariant and given by:*

$$...$$

Mark.

6.2 Future Work

Chapter 7

Experimental Techniques

7.1 Main Result

History. How to obtain vertices of N-periodic. Cayley determinants. Birkhoff counting of self-intersected.

Theorem 17. *For $N = 5$, given the semi-axes (a, b) of the elliptic billiard, those of the caustic are a root of the following sextic equation:*

$$...$$

7.2 Vertices via Numerical Optimization

7.3 Elliptic N-Periodics App

Chapter 8

Conclusion

Animations illustrating some of the above phenomena are listed on Table 8.1.

The following questions are posed to the reader:

id	Title	**youtu.be/<.>**
01	Cayley-Poncelet Phenomena I: Basics	`virCpDtEvJU`
02	Cayley-Poncelet Phenomena II: Intermediate	`4xsm_hQU-dE`

Table 8.1: Videos of some focus-inversive phenomena. The last column is clickable and provides the YouTube code.

Appendix A

Focal Properties of Conics

The polar line associated to a point $P_0 = (x_0, y_0)$ with respect to an ellipse is given by

$$b^2 x_0 x + a^2 y_0 y - a^2 b^2 = 0.$$

The pair of tangent lines passing through the point $P_0 = (x_0, y_0)$ is given by

$$(xy_0 - yx_0)^2 - a^2 (x - x_0)^2 - b^2 (y - y_0)^2 = 0$$

Proposition 7. *Consider an ellipse and two tangent lines passing through an exterior point as shown in Appendix A. Let also the two lines passing through the foci and P_0. Then we have that $\theta_1 = \theta_2$.*

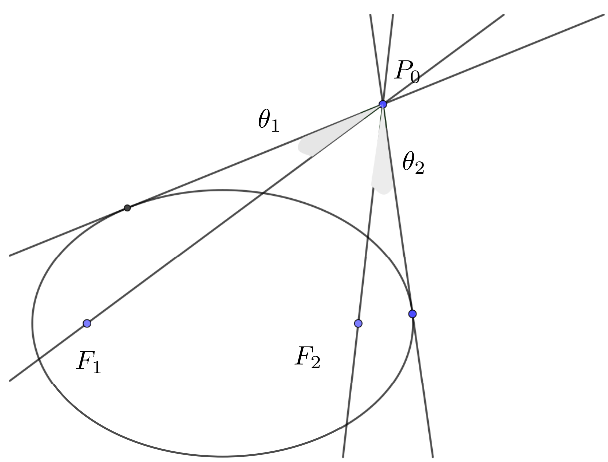

Figure A.1: Angles θ_1 and θ_2 are equal.

Proposition 8. *Consider an ellipse and two tangent lines passing through an exterior point P_0 as shown in Appendix A. Let also the line passing through the P_0 and focus F_2. Then it follows that that $\alpha_1 = \alpha_2$.*

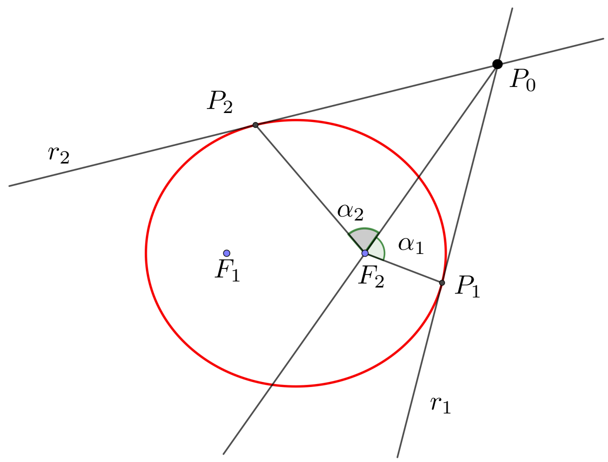

Figure A.2: Angles α_1 and α_2 are equal.

Proof. □

Proposition 9. *Consider a pair of confocal ellipses $\mathcal{E}$ and $\mathcal{E}_1$ with semi-axes (a, b) and (a_c, b_c) respectively. Referring to* **??** *it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} + \frac{|P_0P_2|}{|P_2F_2|} = \frac{2a_c(a - a_c)}{b_c^2}$$

is constant (independent of P_0).

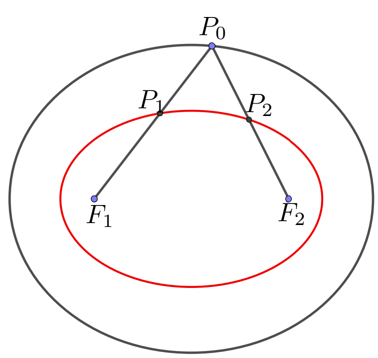

Figure A.3: Relation of focal distances

Proof. □

Proposition 10. *Consider a confocal pair of an ellipse $\mathcal{E}$ and a hyperbola $\mathcal{H}$. Referring to Figure A.4 it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} + \frac{|P_0P_2|}{|P_2F_2|} = \frac{2a_c(a+a_c)}{b_c^2} \quad \textit{and} \quad \frac{|P_0P_1'|}{|P_1'F_1|} + \frac{|P_0P_2'|}{|P_2'F_2|}$$

are constant (independent of P_0). parte 2 ao quadrado (ver maple)

$$4a_c^2(ab_c^2 + a_cb^2)^2/(b_c^4(2aa_c(a_c^2+b_c^2) + 2a_c^4 + b^2a_c^2 + 3a_c^2b_c^2 + b_c^4))$$

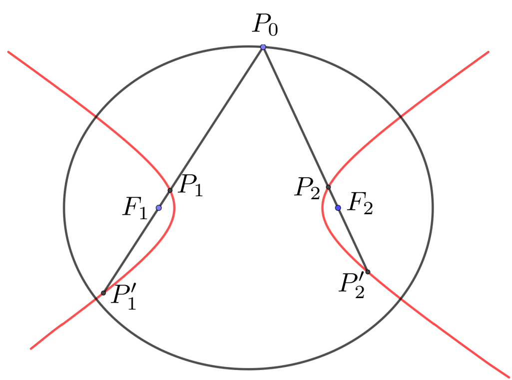

Figure A.4: Relation of focal distances

Proof. □

Proposition 11. *Consider a pair of confocal hyperbolas $\mathcal{H}$ and $\mathcal{H}_1$. Referring to Figure A.5 it follows that:*

$$\frac{|P_0P_1|}{|P_1F_1|} / \frac{|P_0P_2|}{|P_2F_2|} =$$

is constant (independent of P_0).

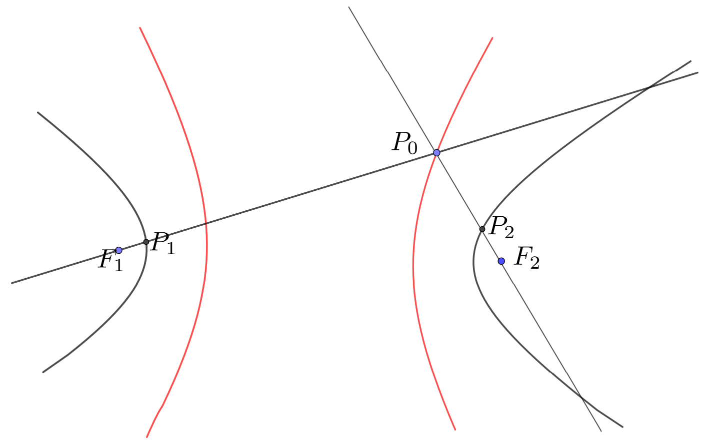

Figure A.5: Relation of focal distances

Proof. □

Proposition 12. *Consider an ellipse $\mathcal{E}$ with foci $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P_0 = (x_0, y_0) \in \mathcal{E}$ and Q_0 the pedal of F_2 with respect to the tangent line passing through P_0. Let also Q_2 the reflection of F_2 with respect to the pedal point Q_0. Then*

$$|Q_2 - F_1| = 2a$$

Therefore the locus of points as constructed above is a circle $\mathcal{C}$ of radius $2a$ centered at F_1. Also the locus of pedal points is a circle centered at the origin and radius a.

The pair $\{\mathcal{E}, \mathcal{C}\}$ is a Poncelet pair having all periodic orbits of period 3.

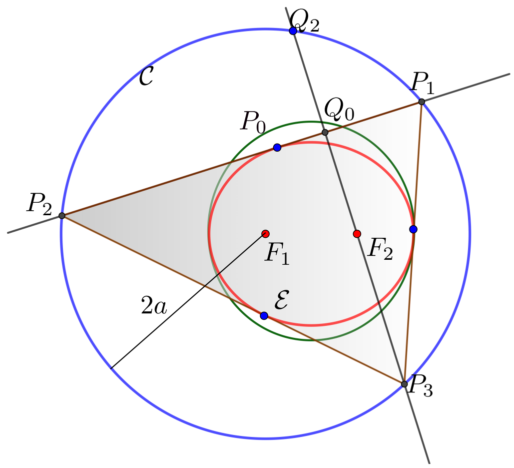

Figure A.6:

Proof. $\square$

Theorem 18. *Consider a pair of confocal conics with foci F_1 and F_2 in the plane (see Figure A.7). Consider the right branch of the hyperbola associated with F_2. Let P and Q be the points of intersection of the ellipse with the right branch of the hyperbola. Consider a ray starting at F_1 and intersecting the right branch of the hyperbola. Denote by X, A the intersection points of this ray with the ellipse and with the branch of the hyperbola. Suppose the focus F_2 lies on the line PQ. Then PQ is the bisector of the angle $\langle AF_2B$.*

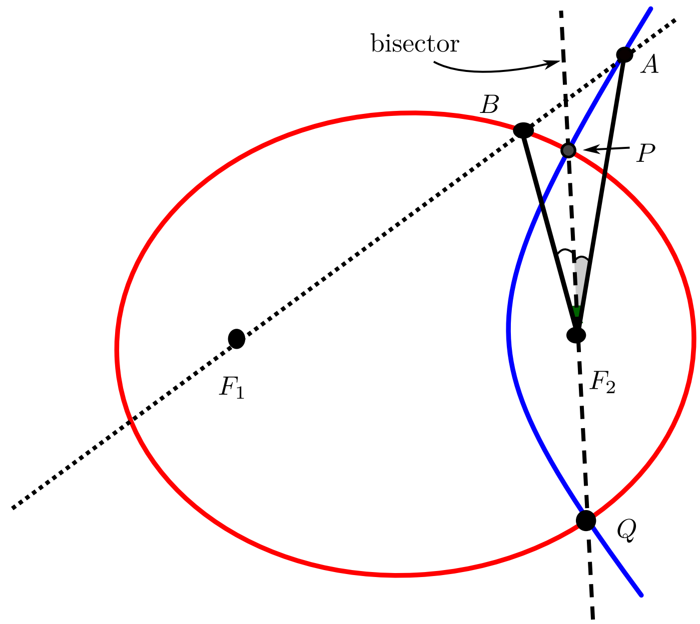

Figure A.7: The line PQ passing through the focus F_2 and the intersection of the ellipse with the right branch of the hyperbola is a bisector of the angle AF_2B.

Theorem 19. *Considear a pair of confocal conics (ellipse and hyperbola) Consider an arbitrary point M (exterior to the ellipse) on the line passing through the intersection points P and Q of the ellipse and the right branch of the hyperbola. Draw two tangent lines to the ellipse and to the hyperbola. Then the lines P_1Q_1, P_2Q_2 (respec. P_1Q_2 and P_2Q_1) through the tangent points as shown in Figure A.8 passe through the focus F_1 (respec. F_2).*

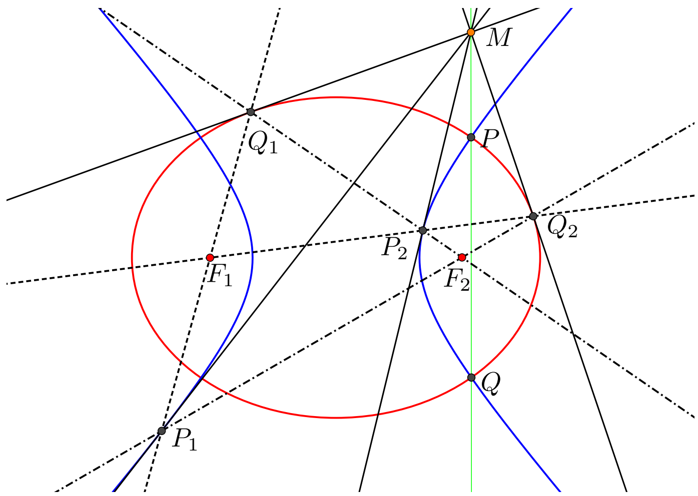

Figure A.8: The four lines passing through the point M and tangent to the confocal conics determine four lines passing through the foci.

Proof. □

Theorem 20. *Let two confocal ellipses $\mathcal{E}_1$ and $\mathcal{E}_2$ with foci F_1 and F_2 are given. Let a ray with the origin at F_1 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at A and B, respectively. Let a ray with the origin at F_2 intersects $\mathcal{E}_1$ and $\mathcal{E}_2$ at C and D, respectively. Suppose the points B and C lie on a branch $\mathcal{H}_1$ of the hyperbola with the foci at F_1 and F_2. Then:*

a) the points A and D lie on a branch $\mathcal{H}_2$ of the hyperbola with the foci at F_1 and F_2. (see Appendix A)

b) Consider a ray starting at F_1 intersecting the branch $\mathcal{H}_1$ at P_1. Consider the ray F_2P_1 intersecting the ellipse $\mathcal{E}_2$ at P_2. Analogously, we define the points P_3, P_4, P_5. Then $P_5 = P_1$ (see Appendix A).

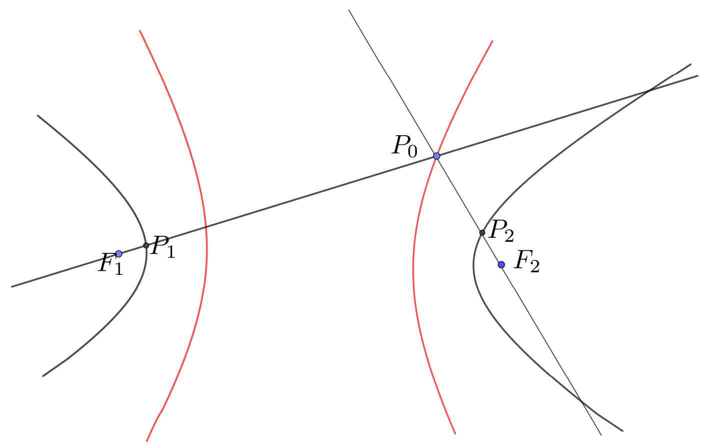

Figure A.9:

Proof. $\square$

A.1 Hodograph map

Consider an ellipse $\mathcal{E} = \{p \in \mathbb{R}^2 : \langle Ap, p\rangle = 1\}$, where A is a positive selfadjoint matrix.

In an elliptic billiard orbit (x_k, y_k) with $x_k \in \mathcal{E}$ and a unit vector $y_k \in \mathbb{R}^2$ we have that:

$$x_{k+1} = x_k + \mu_k y_{k+1}, \quad y_{k+1} = y_k + \nu_k A x_k$$

$$\nu_k = -\frac{2\langle Ax_k, y_k\rangle}{\langle Ax_k, Ay_k\rangle}, \quad \mu_k = -\frac{2\langle Ay_{k+1}, x_k\rangle}{\langle Ay_{k+1}, y_{k+1}\rangle}, \quad y_{k+1} = \frac{x_{k+1} - x_k}{|x_{k+1} - x_k|}$$

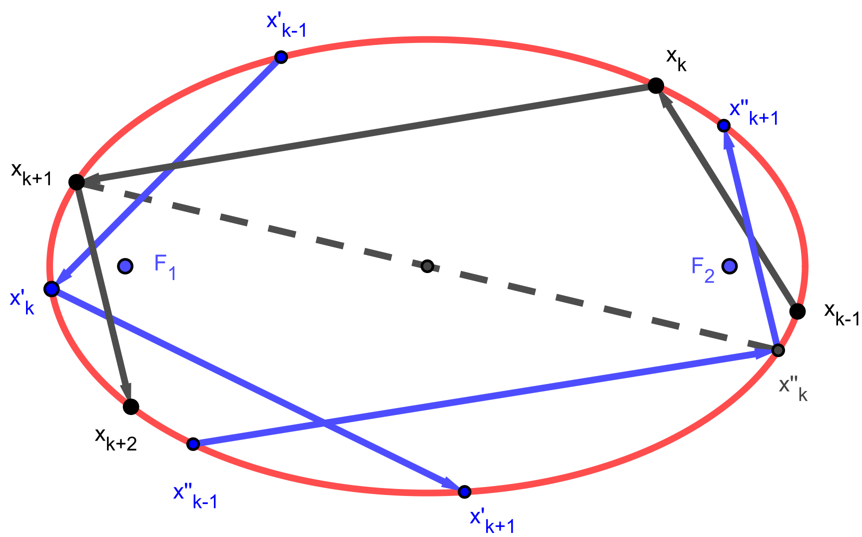

Figure A.10: The skew-hodograph map $\phi(x_k, y_k) = (x'_k, y'_k)$ commutes with T and $\phi^2 = -T$.

Let the skew-hodograph mapping $\phi(x, y) = (x', y')$ defined by

$$\begin{aligned} x'_k &= Cy_{k+1} = C(y_k + \nu_k Ax_k) \\ y'_k &= -C^{-1}x_k, \quad C = A^{-\frac{1}{2}} \end{aligned} \tag{A.1}$$

Proposition 13. *Let $T(x_k, y_k) = (x_{k+1}, y_{k+1})$ the billiard map. Then $\phi \circ T = T \circ \phi$ and $\phi^2 = \phi \circ \phi = -T$.*

Proof. Since A and C are selfadjoint matrices it follows that:

$$\begin{aligned}
\langle Ax'_k, x'_k\rangle &= \langle ACy_{k+1}, Cy_{k+1}\rangle = \langle AA^{-\frac{1}{2}}y_{k+1}, A^{-\frac{1}{2}}y_{k+1}\rangle = \langle A^{\frac{1}{2}}y_{k+1}, A^{-\frac{1}{2}}y_{k+1}\rangle \\
&= \langle A^{-\frac{1}{2}}A^{\frac{1}{2}}y_{k+1}, y_{k+1}\rangle = \langle y_{k+1}, y_{k+1}\rangle = 1 \\
\langle y'_k, y'_k\rangle &= \langle -C^{-1}x_k, -C^{-1}x_k\rangle = \langle A^{\frac{1}{2}}x_k, A^{\frac{1}{2}}x_k\rangle = \langle Ax_k, x_k\rangle = 1.
\end{aligned}$$

Straightforward calculations shows that

$$\nu'_k = -\mu_k, \quad \mu'_k = -\nu_{k+1}.$$

Therefore,

$$\begin{aligned}
x'_{k+1} - x'_k &= C(y_{k+2} - y_{k+1}) = C\nu_{k+1}Ax_{k+1} = \nu_{k+1}A^{\frac{1}{2}}x_{k+1} \\
&= -\nu_{k+1}C^{-1}x_{k+1} = -\nu_{k+1}y'_{k+1} = \mu'_k y'_{k+1} \\
y'_{k+1} - y'_k &= -C^{-1}(x_{k+1} - x_k) = -C^{-1}(\mu_k y_{k+1}) = -\mu_k C^{-1}(C^{-1}x'_k) \\
&= -\mu_k A^{\frac{1}{2}}A^{\frac{1}{2}}x'_k = -\mu_k Ax'_k = \nu'_k Ax'_k.
\end{aligned}$$

This means that the (x'_k, y'_k) is also a billiard orbit and so $\phi \circ T = T \circ \phi$. Finally,

$$\begin{aligned}
x''_k &= Cy'_{k+1} = C(-C^{-1}x_{k+1}) = -x_{k+1} \\
y''_k &= -C^{-1}x'_k = -C^{-1}(Cy_{k+1}) = -y_{k+1}
\end{aligned}$$

So, $\phi^2 = -T$. □

A.2 Exercises

Exercise 3. *Show that the ellipse $x^2/a^2 + y^2/b^2 = 1$ and the circle $(x+c)^2 + y^2 = 4a^2$ defines a Poncelet pair such that all orbits have period 3.*

Appendix B

Confocal Properties of Conics

B.1 Ivory's Theorem

Theorem 21. *Consider the family of confocal conics defined by*

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} - 1 = 0$$

Then the two diagonals of a quadrangle made of arcs of ellipses and hyperbolas have equal length. In Figure B.1 we have that $|A - C| = |B - D|$.

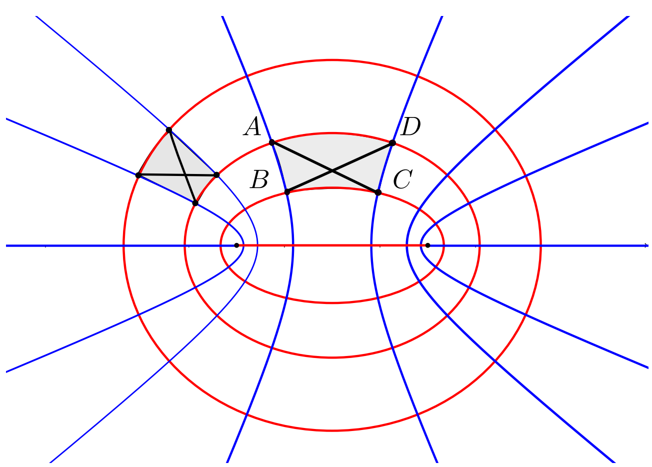

Figure B.1: Confocal conics and quadrangles made of arcs of ellipses and hyperbolas.

Proof. Let

$$\alpha(u,v) = \left[\sqrt{\frac{(a^2-v)\,(a^2-u)}{a^2-b^2}}, \sqrt{-\frac{(b^2-v)\,(b^2-u)}{a^2-b^2}}\right]$$

with $u \in [b^2, a^2]$ and $v \in (-\infty, b^2) \cup (a^2, \infty)$. □

B.2 Graves' Theorem and Periodicity

Proposition 14 (Darboux (1917, Chapitre III))**.** *Consider two confocal ellipses $\mathcal{E}$ and $\mathcal{E}_1$ and a point $M \in \mathcal{E}$. Consider the two tangents ℓ_P and ℓ_Q, as shown in Figure B.2, intersecting $\mathcal{E}_1$ in P and Q. Then $|MP| + |MQ| - arc(P,Q) = cte$, where $arc(P,Q)$ is the length of the elliptic arc with extremal points P and Q. In particular, in a billiard triangle $conv[P_1, P_2, P_3]$, $|P_1P_2| + |P_2P_3| + |P_3P_1| - L(\mathcal{E}_1) = c_1$, where $L(\mathcal{E}_1)$ is the length of $\mathcal{E}_1$, and all the billiard triangles have the same perimeter.*

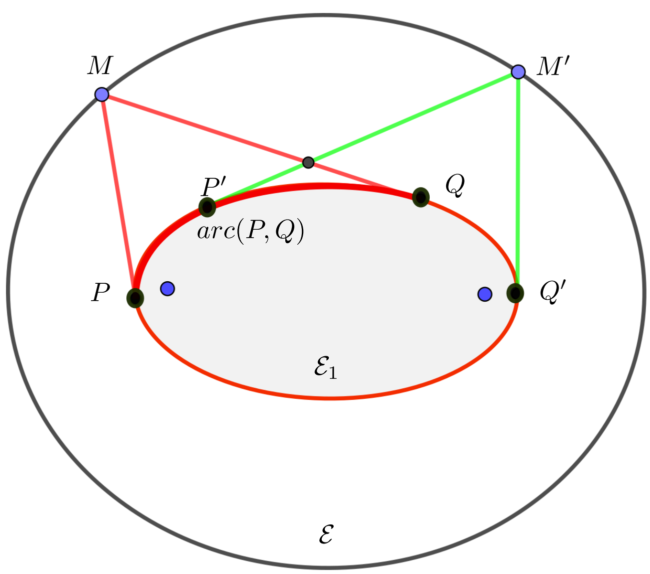

Figure B.2: Tangents to a confocal ellipse $\mathcal{E}_1$ and invariance of the length of chords.

Proof. See Chasles (1843), Darboux (1917, pp. 283-284) and Ragazzo, Dias Carneiro, and Addas Zanata (2005, pp. 115-116). It would be useful to obtain a proof using only the properties of the confocal pair of ellipses. □

The above result is valid for any billiard in a convex curve having caustics.

Appendix C

Properties of Convex Billiards

C.1 Properties of the chords and variation of length

In this section we obtain some properties of chords of convex curves and applications in billiard orbits.

Consider two regular convex curves γ and Γ parametrized by arc lengths s and t. Let $l(s,t) = |\gamma(s) - \Gamma(t)|$, $\theta(s,t)$ the angle between $\gamma'(s)$ and $V(s,t) = \Gamma(t) - \gamma(s)$ and $\eta(s,t)$ the angle between $\Gamma'(t)$ and $V(s,t)$. See Fig. C.1

Consider the Frenet frames $\{\gamma'(s), N_\gamma\}$ and $\{\Gamma'(t), N_\Gamma\}$ along γ and Γ. Denote the curvatures by k_γ and k_Γ.

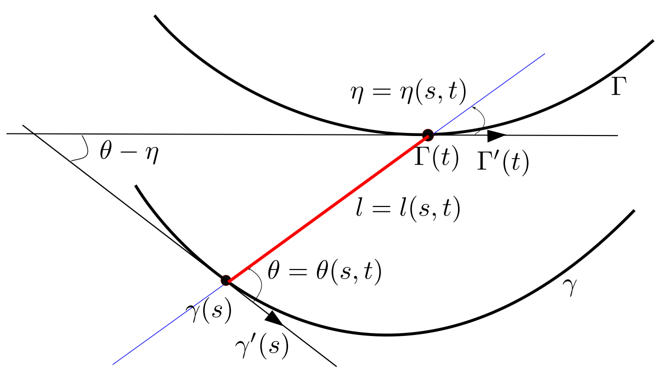

Figure C.1: Pair of curves and variations of length and angles.

Proposition 15. *In the above conditions it follows that:*

$$\begin{aligned} dl &= -\cos\theta\; ds + \cos\eta\; dt \\ d\theta &= \left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{\sin\eta}{l} dt \\ d\eta &= \frac{\sin\theta}{l} ds - \left(\frac{\sin\eta}{l} + k_\Gamma(t)\right) dt \end{aligned} \tag{C.1}$$

Proof. We have that

$$df = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial t} dt$$

From the equation

$$l^2 = \langle \Gamma(t) - \gamma(s), \Gamma(t) - \gamma(s) \rangle$$

it follows that

$$\begin{aligned} 2l\frac{\partial l}{\partial s} &= -2\langle \gamma'(s), \Gamma(t) - \gamma(s) \rangle = -2l\cos\theta \quad \Rightarrow \quad l_s = -\cos\theta \\ 2l\frac{\partial l}{\partial t} &= 2\langle \Gamma'(t), \Gamma(t) - \gamma(s) \rangle = 2l\cos\eta \quad \Rightarrow \quad l_t = \cos\eta \end{aligned}$$

From the equations

$$l(s,t)\cos\theta = \langle \gamma'(s), \Gamma(t) - \gamma(s) \rangle, \quad l(s,t)\cos\eta = \langle \Gamma'(t), \Gamma(t) - \gamma(s) \rangle$$

it follows that

$$\begin{aligned} l_s\cos\theta - l\theta_s\sin\theta &= \langle \gamma''(s), \Gamma(t) - \gamma(s) \rangle - \langle \gamma'(s), \gamma'(s) \rangle \\ &= \langle \gamma''(s), l\cos\theta\gamma' + l\sin\theta N_\gamma(s) \rangle - 1 \\ &= l\sin\theta k_\gamma(s) - 1 \\ l_t\cos\theta - l\theta_t\sin\theta &= \langle \gamma'(s), \Gamma'(t) \rangle = \cos(\theta - \eta) = \cos(\eta - \theta) \\ l_s\cos\eta - l\eta_s\sin\eta &= \langle \gamma'(s), \Gamma'(t) \rangle = \cos(\theta - \eta) \\ l_t\cos\eta - l\eta_t\sin\eta &= \langle \Gamma''(t), \Gamma(t) - \gamma(s) \rangle + \langle \Gamma'(t), \Gamma'(t) \rangle \\ &= \langle \Gamma''(t), l\cos\eta\Gamma' + l\sin\eta N_\Gamma(t) \rangle + 1 \\ &= \ k_\Gamma l\sin\eta + 1 \end{aligned}$$

Performing the calculations leads to the result. □

Proposition 16. *In the same conditions above but with arc length parameters s and t it follows that*

$$\begin{aligned} l_{ss} &= \sin\theta\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) \\ l_{st} &= \frac{\sin\theta\sin\eta}{l} \\ l_{tt} &= \sin\eta\left(\frac{\sin\eta}{l} - k_\Gamma(s)\right) \end{aligned} \tag{C.2}$$

Proof. Follows directly from differentiation of equation (C.1). □

Proposition 17. *In the same conditions above but with arbitrary parameters s and t it follows that*

$$\begin{aligned} dl &= -|\gamma'(s)|\cos\theta\, ds + |\Gamma'(t)|\cos\eta\, dt \\ d\theta &= |\gamma'(s)|\left(\frac{\sin\theta}{l} - k_\gamma(s)\right) ds - \frac{|\Gamma'(t)|\sin\eta}{l}\, dt \\ d\eta &= \frac{|\gamma'(s)|\sin\theta}{l}\, ds - |\Gamma'(t)|\left(\frac{\sin\eta}{l} + k_\Gamma(t)\right)\, dt \end{aligned} \tag{C.3}$$

Proposition 18. *Consider a billiard in a region with boundary a convex curve Γ. Let γ be the caustic of a family of orbits as shown in Fig. C.2. Then for any $x \in \Gamma$*

$$|x-y| + |x-z| - arc(y,z) = cte.$$

Here $y, z \in \gamma$ are the points of tangency of the billiard orbit passing through x with the caustic and $arc(x,z)$ is the length of caustic between y and z.

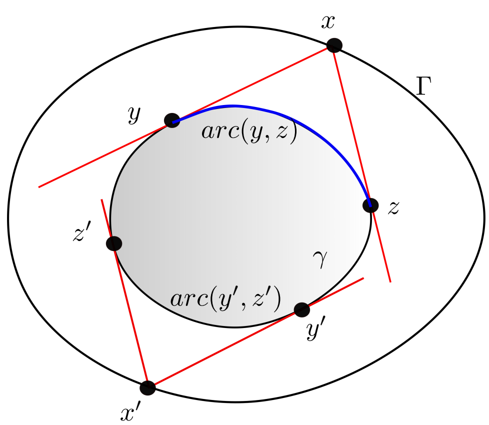

Figure C.2: Tangents to a caustic and length of chords.

Sketch of Proof. Let $\Gamma(t)$ be a parametrization of the boundary. Consider also local parametrizations $\gamma_1(t)$ and $\gamma_2(t)$ of the caustic γ with $\gamma_1(0) = z$, $\gamma_2(0) = y$, and $\Gamma(0) = x$. Suppose that all curves are counterclockwise oriented. Let also the caustic parametrized by natural parameter s. Then

$$\gamma(s) = \gamma_1(t) = \Gamma(t) + \lambda(t) d_1(t), \;\; \gamma(s) = \gamma_2(t) = \Gamma(t) + \lambda(t) d_2 1(t).$$

Here d_1 and d_2 are the directions of the tangent lines xy and xz to the caustic.

Let $l_1(t) = |\Gamma(t) - \gamma_1(t)|$ with $l_1(0) = |x-z|$. Also define $l_2(t) = |\Gamma(t) - \gamma_2(t)|$ with $l_2(0) = |x-y|$.

By Proposition 17 it follows that

$$\begin{aligned} dl_1 &= \cos\eta|\Gamma'(t)|dt - |\gamma_1'(s)|ds \\ dl_2 &= |\gamma_2'(s)|ds - \cos\eta|\Gamma'(t)|dt \end{aligned}$$

Here we used the condition of billiard orbit at the point x (angle of incidence is equal to angle of reflection) and that $\cos\theta_{1,2} = \pm 1$ (caustic is tangent to billiard orbits, taking into account the orientation). Therefore it follows that

$$d(l_1 + l_2) - |\gamma_2'(s)|ds + |\gamma_1'(s)|ds = 0.$$

Integrating it follows that

$$l_1(a) - l_1(0) + l_2(a) - l_2(0) = arc(\gamma_1(0), \gamma_1(a)) - arc(\gamma_2(0), \gamma_2(a))$$

Therefore,

$$l_1(a) + l_2(a) - arc(\gamma_1(a), \gamma_2(a)) = l_1(0) + l_2(0) - arc(\gamma_1(0), \gamma_2(0)).$$

□

C.2 Joachimsthal's Integral

ron: introduzir e uniformizar notacao

Proposition 19. *Consider an ellipse $\mathcal{E}$ defined by $\langle Ap, p\rangle = 1$. Let u be an inward unit vector in the direction of the billiard orbit passing through the point $p_0 \in \mathcal{E}$. Let $T(p_0, u) = (p_1, v)$ the billiard map as shown in Fig. . Then*

$$\langle Ap_0, u\rangle = -\langle Ap_1, u\rangle = \langle Ap_1, v\rangle$$

Proof. The tangent space $T_p\mathcal{E}$ is formed of the vectors u such that $\langle Ap, u\rangle = 0$. Therefore Ap is a normal vector to the ellipse at the point p. The vector u is proportional to $p_1 - p_0$.

Therefore,

$$\begin{aligned} \langle Ap_0 + Ap_1, p_1 - p_0\rangle &= \langle Ap_0, p_1\rangle + \langle Ap_1, p_0\rangle - \langle Ap_0, p_0\rangle + \langle Ap_1, p_1\rangle \\ &= \langle p_0, Ap_1\rangle - \langle Ap_1, p_0\rangle = 0. \end{aligned}$$

Then,

$$\langle Ap_0, u\rangle = \langle Ap_1, -u\rangle = \langle Ap_1, r(-u)\rangle = \langle Ap_1, v\rangle$$

□

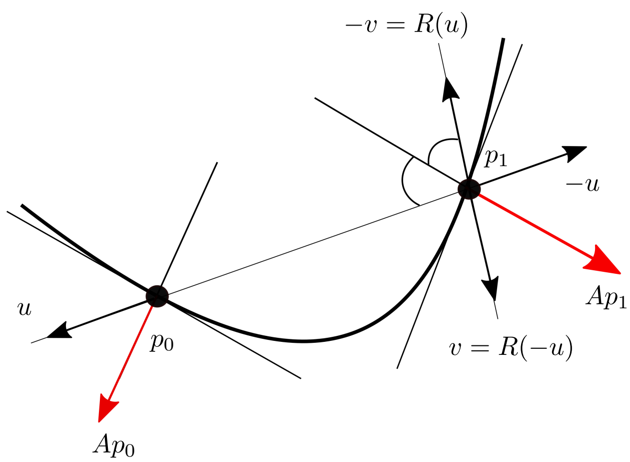

Figure C.3: Joachimsthal's first integral $\langle Ap_0, u\rangle$ is T-invariant

Appendix D

Other Types of Billiards

Appendix E

Table of Symbols

symbol	meaning
$\mathcal{E}, \mathcal{E}_c$	outer and inner ellipses
a, b	outer ellipse semi-axes' lengths
a_c, b_c	inner ellipse semi-axes' lengths
O, O_c	centers of $\mathcal{E}$,$\mathcal{E}_c$
(dx, dy)	translation $O_c - O$
θ	major semi-axis tilt $\mathcal{E}_c$ wrt $\mathcal{E}$
P_i, s_i	3-periodic vertices and sidelengths
r, R	3-periodic inradius and circumradius
a_i, b_i	semiaxes of the locus of X_i
r_i	radius of the locus of X_i (if $a_i = b_i$)
X_1	Incenter
X_2	Barycenter
X_3	Circumcenter
X_4	Orthocenter
X_5	Euler's circle center
X_{20}	de Longchamps point

Table E.1: Symbols of euclidean geometry used

Appendix F

Original Proposal

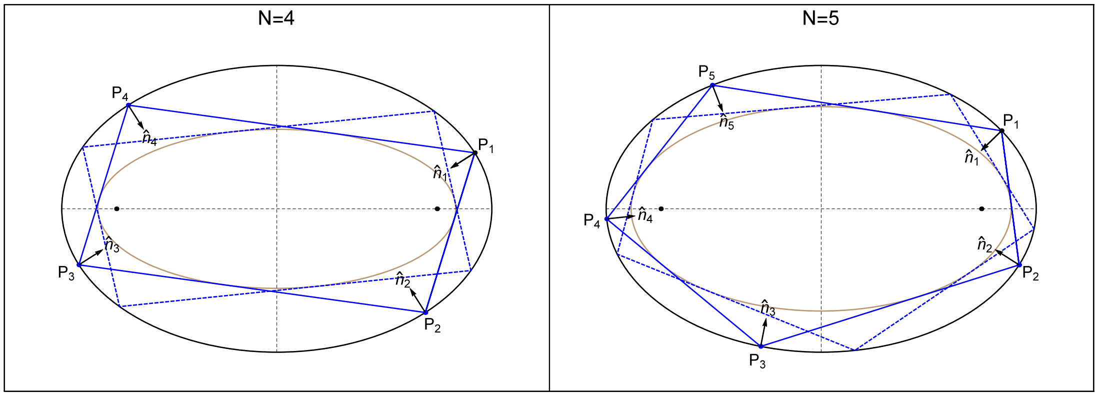

Figure F.1: Bilhar Elíptico e (4,5)-órbitas periódicas (azul) . Em todo vértice o vetor normal $\hat{n}_i$ bissecta os segmentos de órbitas bilhares $P_{i-1}P_i$ e P_iP_{i+1} que são tangentes à cáustica (marrom). Uma segunda órbita também é mostrada (azul pontilhado). Note que o perímetro é conservado sobre toda a família de N-periódicas. Video

1. **Nível**: Introdutório.
2. **Duração**: 5 aulas de 45 minutos.
3. **Descrição detalhada**:
   - **Objetivos**: Introdução à geometria dos bilhares elípticos Darboux (1917), Lebesgue (1942), Rozikov (2018), and Sergei Tabachnikov (2005), tema de grande influência na matemática dos últimos 200 anos. Divulgar novos invariantes ali manifestados, o método experimental de descoberta, e esboçar algumas provas.
   - **Público-Alvo**: aluno(a)s de graduação ou pós, professores, ou quaisquer interessado(a)s em conhecer novas e belas propriedades do bilhar elíptico encontradas experimentalmente.
   - **Conteúdo**: Geometria do bilhar elíptico, porisma de Poncelet e condições de Cayley Dragovi and Radnovi (2011), loci de centros triangulares Kimberling (2019) sobre a família de 3-periódicas, órbitas auto-intersectadas, polígonos inversivos, outras famílias Ponceletianas. Exemplificar alguns fenômenos por vídeos Reznik (2020a) e/ou ferramenta interativa Reznik and Darlan (2020).
   - **Monitoria**: co-autores e/ou alunos de iniciação científica farão revezamento.
4. **Distribuição de capítulos**:
   - Capítulo 1: Propriedades focais das cônicas A. V. Akopyan and Zaslavsky (2007a), Berger (1987), Darboux (1917), and Lebesgue

(1942). Família confocal de cônicas. Relações entre elipses e triângulos (inscritos e circunscritos). Introdução ao bilhar elíptico e ao porisma de Poncelet. Condições de Cayley. Exercícios e projetos.

- Capítulo 2: Centros triangulares (incentro, baricentro, circuncentro, ortocentro, mittenpunkt etc). Coordenadas trilineares e baricêntricas no plano. Propriedades básicas de alguns centros triangulares Coxeter and Greitzer (1967) and Kimberling (1998) e triângulos derivados (pedal, tangente, excentral etc). Loci e Invariantes Básicas: Loci elípticos de centros triangulares, soma de cossenos, razão de áreas, potência de um ponto em relação a um círculo. Método experimental e algebro-computacional Garcia (2019), Garcia, Reznik, and Koiller (2020a), and Garcia, Reznik, and Koiller (2020b). Exercícios e projetos.
- Capítulo 3: N-periodicas auto-intersectadas, condições de Birkhoff G. D. Birkhoff (1927). *Tour* de N-periódicas auto-intersectadas. Exercícios e projetos.
- Capítulo 4: Inversão com respeito a um círculo. O talentoso polígono foco-inversivo, seus invariantes e propriedades. Exceções em invariantes. Exercícios e projetos.
- Capítulo 5: Alguns invariantes em outras famílias Ponceletianas, e.g., homotética, porística, com incírculo, circumcírculo etc. Exercícios e projetos.
- Capítulo 6: Tópicos de bilhares em curvas convexas e polígonos. Uma conexão rápida com vários outros ramos da matemática, contextualizando os problemas em aberto. Resenha de problemas de investigação no contexto de bilhares.

5. Pré-requisitos: conhecimentos básicos de Construções Geométricas, Geometria Analítica, Álgebra Linear e Cálculo Diferencial.

6. Número de páginas: Uma estimativa preliminar é de 120 páginas com muitas figuras ilustrando as propriedades geométricas observadas em experimentos computacionais no bilhar elíptico e links para vídeos no youtube.

7. Outras informações: a boa recepção da nossa palestra de divulgação "Aventuras com Triângulos e Bilhares" ministrada no 32º CBM do IMPA (2019) Reznik, Garcia, and Koiller (2019a) and Reznik, Garcia, and Koiller (2019b) além de nossas publicações em 2020 Reznik, Garcia, and Koiller (2020a), Garcia, Reznik, and Koiller (2020a), Garcia, Reznik, and Koiller (2020b), Reznik, Garcia, and Koiller (2020b), Reznik and Garcia (2021a), Reznik, Garcia, and Koiller (2021), Garcia and Reznik (2021), Garcia, Reznik, Stachel, et al. (2020), Reznik and Garcia (2021b), Garcia and Reznik (2020b), and Reznik and Garcia (2020), nos motivou em propor esse curso com novos resultados e mais detalhes das técnicas utilizadas.

Anexo segue artigo aceito para publicação na revista Amer. Math. Monthly ilustrando alguns resultados obtidos pelos proponentes. Também anexamos o artigo publicado na revista Math. Intelligencer.

Bibliography

Ahlfors, Lars V. (1979). *Complex Analysis: an Introduction to Theory of Analytic Functions of One Complex Variable*. McGraw Hill.

Akopyan, A. V. and A. A. Zaslavsky (2007a). *Geometry of Conics*. Providence, RI: Amer. Math. Soc. (cit. on p. 60).

— (2007b). *Geometry of Conics*. Russian 2nd edition. Providence, RI. URL: https://bit.ly/32n7t3t.

Akopyan, Arseniy (2012). "Conjugation of lines with respect to a triangle". In: *Journal of Classical Geometry* 1, pp. 23–31. URL: users.mccme.ru/akopyan/papers/conjugation_en.pdf.

— (Apr. 2020a). *Angles $\phi = \pi - \theta_i$ (resp. $\phi' = \phi - \theta_i'$), so equivalent to invariant sum (resp. product) of cosines.* Private Communication.

— (Jan. 2020b). *Corollary of Theorem 6 in Akopyan et al., "Billiards in Ellipses Revisited" (2020).* Private Communication.

— (Jan. 2020c). *Follows from previous results: the construction is affine and holds for any two concentric conics.* Private Communication.

— (Apr. 2020d). *Perpendicular feet to N-periodic or its tangential polygon are cyclic.* Private Communication.

Akopyan, Arseniy, Richard Schwartz, and Serge Tabachnikov (Sept. 2020). "Billiards in Ellipses Revisited". In: *Eur. J. Math.* DOI: 10.1007/s40879-020-00426-9 (cit. on pp. 20, 22).

Armitage, J. V. and W. F. Eberlein (2006). *Elliptic Functions*. London: Cambridge University Press. DOI: doi.org/10.1017/CBO9780511617867 (cit. on p. 8).

Berger, Marcelo (1987). *Geometry I, II*. New York: Springer Verlag (cit. on p. 60).

Bialy, Misha and Sergei Tabachnikov (Sept. 2020). "Dan Reznik's Identities and More". In: *Eur. J. Math.* URL: doi:10.1007/s40879-020-00428-7 (cit. on p. 22).

Birkhoff, G. (1927). "On the periodic motions of dynamical systems". In: *Acta Mathematica* 50.1, pp. 359–379. DOI: 10.1007/BF02421325. URL: doi.org/10.1007/BF02421325.

Birkhoff, G. D. (1927). *Dynamical Systems*. 1966th ed. Vol. 9. Colloqium Publications. Providence, RI: American Mathematical Society. URL: www.freeinfosociety.com/media/pdf/2219.pdf (cit. on p. 61).

Bradley, Christopher (2011). *The Geometry of the Brocard Axis and Associated Conics*. CJB/2011/170. URL: people.bath.ac.uk/masgcs/Article116.pdf.

Bradley, Christopher and Geoff Smith (2007). "On a Construction of Hagge". In: *Forum Geometricorum* 7, pp. 231–247. URL: forumgeom.fau.edu/FG2007volume7/FG200730.pdf.

Carneiro, M. J. and R. Garcia (2019). "Teorema dos Quatro Vértices e a sua Recíproca". In: 32^o Colóquio Brasileiro de Matemática.

Centina, Andrea del (2016). "Poncelet's Porism: a long story of renewed discoveries I". In: *Arch. Hist. Exact Sci.* 70.2, pp. 1–122. DOI: 10.1007/s00407-015-0163-y. URL: doi.org/10.1007/s00407-015-0163-y.

Chasles, M (1843). "Propriétés genérales des arcs d'une section conique dont la différence est rectifiable." In: *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* 17, pp. 838–844 (cit. on p. 48).

Chavez-Caliz, A.C. (Aug. 2020). "More About Areas and Centers of Poncelet Polygons". In: *Arnold Math J.* URL: doi:10.1007/s40598-020-00154-8.

Connes, Alain and Don Zagier (2007). "A Property of Parallelograms Inscribed in Ellipses". In: *The American Mathematical Monthly* 114.10, pp. 909–914. URL: people.mpim-bonn.mpg.de/zagier/files/amm/114/fulltext.pdf.

Coolidge, J. L. (1971). *A Treatise on the Geometry of the Circle and Sphere*. New York,NY: Chelsea.

Coxeter, Harold S. M. and Samuel L. Greitzer (1967). *Geometry Revisited*. Vol. 19. New Mathematical Library. New York: Random House, Inc., pp. xiv+193 (cit. on pp. 19, 61).

Daepp, Ulrich et al. (2019). *Finding Ellipses: What Blaschke Products, Poncelets Theorem, and the Numerical Range Know about Each Other*. MAA Press/AMS (cit. on p. 31).

Darboux, Gaston (1870). "Sur les polygones inscrits et circonscrits à l'ellipsoïde". In: *Bulletin de la Société Philomatique de Paris*. 6e serie 7, pp. 92–95. URL: www.biodiversitylibrary.org/item/98693#page/98/mode/1up.

— (1917). *Principes de Géométrie Analytique*. Paris: Gauthier-Villars (cit. on pp. 48, 60).

Darlan, Iverton and Dan Reznik (2020). *Loci of Ellipse-Mounted Triangle Centers*. URL: dan-reznik.github.io/ellipse-mounted-triangles/.

Dolgirev, P. E. (2014). "On some properties of confocal conics". In: *J. of Classical Geometry* 3, pp. 4–11.

Dragovi, Vladimir and Milena Radnovi (2011). *Poncelet Porisms and Beyond: Integrable Billiards, Hyperelliptic Jacobians and Pencils of Quadrics*. Frontiers in Mathematics. Basel: Springer. URL: books.google.com.br/books?id=QcOmDAEACAAJ (cit. on p. 60).

— (2019). "Caustics of Poncelet polygons and classical extremal polynomials". In: *Regul. Chaotic Dyn.* 24.1, pp. 1–35. DOI: 10.1134/S1560354719010015. URL: doi.org/10.1134/S1560354719010015.

Escudero, Carlos A. and Agust Reventós (2007). "An interesting property of the evolute". In: *Amer. Math. Monthly* 114.7, pp. 623–628. DOI: 10.1080/

00029890.2007.11920452. URL: doi.org/10.1080/00029890.2007.11920452.

Ferréol, Robert (2017). *Encyclopédie des Formes Mathématiques Remarquables (Deltoid Curve)*. URL: mathcurve.com/courbes2d.gb/deltoid/deltoid.shtml.

— (July 2020a). *Équation de la larme de Brocard 1.* Private Communication.

— (2020b). *Mathcurve Portal*. URL: mathcurve.com/courbes2d.gb/strophoiddroite/strophoiddroite.shtml.

Fierobe, Corentin (2021). "On the circumcenters of triangular orbits in elliptic billiard". In: *Journal of Dynamical and Control Systems*. Accepted. URL: arxiv.org/pdf/1807.11903.pdf (cit. on p. 13).

Fischer, G. (2001). *Plane Algebraic Curves*. Providence, RI: American Mathematical Society.

Fitzgibbon, Andrew, Maurizio Pilu, and Robert Fisher (May 1999). "Direct Least Square Fitting of Ellipses". In: *Pattern Analysis and Machine Intelligence* 21.5.

Gallatly, William (1914). *The modern geometry of the triangle*. Francis Hodgson.

Garcia, Ronaldo (2016). "Centers of Inscribed Circles in Triangular Orbits of an Elliptic Billiard". arxiv. URL: arxiv.org/pdf/1607.00179v1.pdf.

— (2019). "Elliptic Billiards and Ellipses Associated to the 3-Periodic Orbits". In: *American Mathematical Monthly* 126.06, pp. 491–504. eprint: doi:10.1080/00029890.2019.1593087 (cit. on pp. 13, 61).

Garcia, Ronaldo and Dan Reznik (Dec. 2020a). *Family Ties: Relating Poncelet 3-Periodics by their Properties*. arXiv:2012.11270.

— (Oct. 2020b). *Invariants of Self-Intersected and Inversive N-Periodics in the Elliptic Billiard* (cit. on p. 61).

— (June 2021). *Related By Similarity I: Poristic Triangles and 3-Periodics in the Elliptic Billiard* (cit. on p. 61).

Garcia, Ronaldo, Dan Reznik, and Jair Koiller (2020a). *Loci of 3-periodics in an Elliptic Billiard: why so many ellipses?* arXiv:2001.08041. submitted (cit. on pp. 20, 61).

— (2020b). "New Properties of Triangular Orbits in Elliptic Billiards". In: *Amer. Math. Monthly* to appear. eprint: arXiv:2001.08054 (cit. on pp. 19, 61).

Garcia, Ronaldo, Dan Reznik, Hellmuth Stachel, et al. (2020). "Steiner's Hat: a Constant-Area Deltoid Associated with the Ellipse". In: *J. Croatian Soc. for Geom. Gr. (KoG)* 24. DOI: 10.31896/k.24.2 (cit. on p. 61).

Georgiev, Vladimir and Veneta Nedyalkova (2012). "Poncelets porism and periodic triangles in ellipse". In: *Dynamat*. URL: www.dynamat.oriw.eu/upload_pdf/20121022_153833__0.pdf.

Gheorghe, Gabriela and Dan Reznik (2021). "A special conic associated with the Reuleaux Negative Pedal Curve". In: *Intl. J. of Geom.* 10.2, pp. 33–49.

Gibert, Bernard (2020). *Brocard Triangles and Related Cubics*. URL: https://bernard-gibert.pagesperso-orange.fr/gloss/brocardtriangles.html.

Giblin, P. J. and J. P. Warder (2014). "Evolving evolutoids". In: *Amer. Math. Monthly* 121.10, pp. 871–889. DOI: 10.4169/amer.math.monthly.121.10.871. URL: doi.org/10.4169/amer.math.monthly.121.10.871.

Glaeser, Georg, Hellmuth Stachel, and Boris Odehnal (2016). *The Universe of Conics: From the ancient Greeks to 21st century developments*. Springer.

Glutsyuk, Alexey (2014). "On odd-periodic orbits in complex planar billiards". In: *J. Dyn. Control Syst.* 20.3, pp. 293–306. DOI: 10.1007/s10883-014-9236-5. URL: doi.org/10.1007/s10883-014-9236-5.

Griffiths, Philipp and Joseph Harris (1978). "On Cayley's explicit solution to Poncelet's porism". In: *Enseign. Math. (2)* 24, pp. 31–40.

Grozdev, Sara and Deko Dekov (June 2014). "The computer program Discoverer as a tool of mathematical investigation". In: *International Journal of Computer Discovered Mathematics (IJCDM)*. URL: www.ddekov.eu/j/2014/JCGM201405.pdf.

— (Nov. 2015). "A Survey of Mathematics Discovered by Computers". In: *International Journal of Computer Discovered Mathematics (IJCDM)*. URL: www.journal-1.eu/2015/01/Grozdev-Dekov-A-Survey-pp.3-20.pdf.

Grozdev, Sava, Hiroshi Okumura, and Deko Dekov (2017). "Computer Discovered Mathematics: Triangles homothetic with the Orthic triangle". In: *Int. J. of Comp. Disc. Mathematics (IJCDM)* 2, pp. 46–54. URL: www.journal-1.eu/2017/Grozdev-Okumura-Dekov-Orthic-triangles-pp.46-54.pdf.

Gruber, P.M. (1993). "History of Convexity". In: *Handbook of Convex Geometry, Part A*. Ed. by P.M. Gruber and J.M. Wills. Vol. A. Elsevier, pp. 1–15.

Hartmann, Frederick and Robert Jantzen (Aug. 2010). "Apollonius's Ellipse and Evolute Revisited – The Alternative Approach to the Evolute". In: *Convergence*. URL: www.maa.org/book/export/html/116803.

Helman, Mark (Aug. 2019). *The Loci of* $X_i, i = 7, 40, 57, 63, 142, 144$ *are elliptic*. Private Communication. Houston, TX.

— (Jan. 2020). *Proofs related to locus of* X_{88}. Private Communication. Houston, TX.

Helman, Mark, Ronaldo Garcia, and Dan Reznik (Oct. 2020). *Intriguing Invariants of Centers of Ellipse-Inscribed Triangles*. arXiv:2010.09408.

Helman, Mark, Dominique Laurain, et al. (Feb. 2021). *Center Power and Loci of Poncelet Triangles*. arXiv:2102.09438 (cit. on p. 16).

Himmelstrand, Markus, Victor Wilén, and Maria Saprykina (2012). *A Survey of Dynamical Billiards*. URL: bit.ly/2kfYHkC.

Honsberger, Ross (1995). *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*. Mathematical Association of America.

Izmestiev, Ivan and Serge Tabachnikov (Jan. 2017). "Ivorys theorem revisited". In: *Journal of Integrable Systems* 2 (1). URL: doi.org/10.1093/integr/xyx006.

Izmestiev, Ivan and Sergei Tabachnikov (Jan. 2017). "Ivory's theorem revisited". In: *Journal of Integrable Systems* 2.1. URL: doi.org/10.1093/integr/xyx006.

J. Moser, A. P. Veselov (1991). “Discrete versions of some classical integrable systems and factorization of matrix polynomials”. In: *Commun. Math. Phys.* 139, pp. 217–243.

Jacobi, C. G. J. (1828). “Über die Anwendung der elliptischen Transcendentten auf ein bekanntes Problem der Elementargeometrie”. In: *J. reine angew. Math.* 3, pp. 376–389. DOI: 10.1515/crll.1828.3.376.

— (1837). “Zur Theorie der Variations-Rechnung und der Differential-Gleichungen”. In: *J. reine angew. Math.* 17, pp. 68–82. DOI: 10.1515/crll.1837.17.68. URL: doi.org/10.1515/crll.1837.17.68.

— (1839). “Note von der geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwürdigen analytischen Substitution [The geodesic on an ellipsoid and various applications of a remarkable analytical substitution].” In: *J. reine angew. Math.* 19, pp. 309–313. DOI: doi:10.1515/crll.1839.19.309.

— (1884). *Vorlesungen über Dynamik*. Ed. by Alfred Clebsch. Gehalten an der Universität zu Königsberg im Wintersemester 1842–1843. Berlin: Druck & Verlag von George Reimer. URL: quod.lib.umich.edu/u/umhistmath/AAS8078.0001.001/5?rgn=full+text;view=pdf.

Johnson, Roger A. (1929). *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*. Boston, MA: Houghton Mifflin.

— (1960). *Advanced Euclidean Geometry*. 2nd. editor John W. Young. New York, NY: Dover. URL: bit.ly/33cbrvd.

Jovanovi, Boidar (2011). “What are completely integrable Hamilton Systems”. In: *The Teaching of Mathematics* 13.1, pp. 1–14.

Kaiser, M.J., T.L. Morin, and T.B. Trafalis (1990). “Centers and invariant points of convex bodies”. In: *Applied Geometry and Discrete Mathematics: The Victor Klee Festschrift*. Ed. by Peter Gritzmann and Bernd Sturmfels. Vol. 4. Providence, RI: American Mathematical Society, pp. 367–386.

Kaloshin, Vadim and Alfonso Sorrentino (2018). “On the Integrability of Birkhoff Billiards”. In: *Phil. Trans. R. Soc.* A.376. DOI: doi.org/10.1098/rsta.2017.0419.

Kimberling, Clark (1993a). “Functional equations associated with triangle geometry”. In: *Aequationes Math.* 45, pp. 127–152. DOI: 10.1007/BF01855873. URL: doi.org/10.1007/BF01855873.

— (1993b). “Triangle centers as functions”. In: *Rocky Mountain J. Math.* 23.4, pp. 1269–1286. DOI: 10.1216/rmjm/1181072493. URL: doi.org/10.1216/rmjm/1181072493.

— (1998). *Triangle Centers and Central Triangles*. Vol. 129. Congr. Numer. Utilitas Mathematica Publishing, Inc. (cit. on p. 61).

— (2019). *Encyclopedia of Triangle Centers*. URL: faculty.evansville.edu/ck6/encyclopedia/ETC.html (cit. on p. 60).

— (2020). *Central Lines of Triangle Centers*. URL: bit.ly/34vVoJ8.

Kozlov, V. V. and D. V. Treshchëv (1991). *Billiards: A Genetic Introduction to the Dynamics of Systems with Impacts*. Translations of mathematical monographs. Providence, RI: American Mathematical Society. URL: books.google.com.br/books?id=b18buAEACAAJ.

Lang, Serge (2002). *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, pp. xvi+914. DOI: 10.1007/978-1-4613-0041-0. URL: doi.org/10.1007/978-1-4613-0041-0.

Laurain, Dominique (Aug. 2019). *Formula for the Radius of the Orbits' Excentral Cosine Circle*. Private Communication. France.

Lebesgue, H. (1942). *Les coniques*. Paris: Gauthier-Villars (cit. on p. 60).

Levi, Mark and Sergei Tabachnikov (2007). "The Poncelet Grid and Billiards in Ellipses". In: *The American Mathematical Monthly* 114.10, pp. 895–908. DOI: 10.1080/00029890.2007.11920482. URL: doi.org/10.1080/00029890.2007.11920482.

Lockhart, Paul (2009). *A Mathematicians Lament*. Bellevue Library Press.

Lockwood, E.H. (1961). *A Book of Curves*. Cambridge University Press.

Mandan, Sahib Ram (1979). "Orthologic Desargues' figure". In: *J. Austral. Math. Soc. Ser. A* 28.3, pp. 295–302.

Minevich, Igor (July 2019). *The Intouchpoints of the Orbit's Anticomplementary Triangle Sweep the Billiard*. Private Communication. Boston, MA.

Minevich, Igor and Patrick Morton (2017). "Synthetic foundations of cevian geometry, III: The generalized orthocenter". In: *Journal of Geometry* 108, pp. 437–455. DOI: 10.1007/s00022-016-0350-2. URL: arxiv.org/pdf/1506.06253.pdf.

Mitrea, Dorina and Marius Mitrea (1994). "A generalization of a theorem of Euler". In: *Amer. Math. Monthly* 101.1, pp. 55–58. DOI: 10.2307/2325125. URL: doi.org/10.2307/2325125.

Moore, Ramon, R. Baker Kearfott, and Michael J. Cloud (2009). *Introduction to Interval Analysis*. SIAM.

Moses, Peter (Sept. 2020). *Family of triangles with fixed Brocard points*. Private Communication.

Neumann, C. (1859). "De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur". In: *J. reine angew. Math.* 56, pp. 46–63. DOI: 10.1515/crll.1859.56.46.

Odehnal, Boris (2011). "Poristic loci of triangle centers". In: *J. Geom. Graph.* 15.1, pp. 45–67.

Ostermann, A. and G. Wanner (2012). *Geometry by Its History*. Springer Verlag.

Ovsienko, Valentin, Richard Schwartz, and Sergei Tabachnikov (2010). "The Pentagram Map: A Discrete Integrable System". In: *S. Commun. Math. Phys.* 299.409. URL: doi.org/10.1007/s00220-010-1075-y.

Pamfilos, Paris (2004). "On Some Actions of D_3 on a Triangle". In: *Forum Geometricorum* 4, pp. 157–176. URL: forumgeom.fau.edu/FG2004volume4/FG200420.pdf.

— (2020). "Triangles Sharing their Euler Circle and Circumcircle". In: *International Journal of Geometry* 9.1, pp. 5–24. URL: ijgeometry.com/wp-content/uploads/2020/03/1.-5-24.pdf.

Ptracu, Ion and Florentin Smarandache (2020). *The Geometry of the Orthological Triangles*. Pons Editions. URL: fs.unm.edu/GeometryOrthologicalTriangles.pdf.

Pedoe, D. (1970). *A course of geometry for colleges and universities*. Cambridge University Press, London-New York, pp. xvi+449.

Preparata, Franco and Michael Shamos (1988). *Computational Geometry - An Introduction*. 2nd. Springer-Verlag.

Pugh, Charles C. (2015). *Real mathematical analysis*. Second. Undergraduate Texts in Mathematics. Springer, Cham, pp. xi+478. DOI: 10.1007/978-3-319-17771-7. URL: doi.org/10.1007/978-3-319-17771-7.

Ragazzo, Clodoaldo G., Mário J. Dias Carneiro, and Salvador Addas Zanata (2005). *Introdução à dinâmica de aplicações do tipo twist*. Publicações Matemáticas do IMPA. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, pp. ii+168 (cit. on p. 48).

Reznik, Dan (2011a). *"Dynamic Billiards in Ellipse"*. URL: demonstrations.wolfram.com/DynamicBilliardsInEllipse/.

— (2011b). *Locus of the incircle touchpoints is a higher-order curve*. URL: youtu.be/9xU6T7hQMzs (cit. on p. 13).

— (2011c). *Path of Incenter for Family of Triangular Orbits in Elliptical Billiard*. YouTube. URL: https://youtu.be/BBsyM7RnswA.

— (2011d). *Triangular Orbits in Elliptic Billiards: Locus of the Incenter is an Ellipse*. URL: youtu.be/9xU6T7hQMzs (cit. on p. 13).

— (2019a). *Applet Showing the Locus of Several Triangular Centers*. URL: editor.p5js.org/dreznik/full/i1Lin7lt7.

— (2019b). *New Properties of Polygonal Orbits in Elliptic Billiards: Main Videos*. URL: bit.ly/2k5GXbB.

— (2019c). *Triangular Orbits in Elliptic Billards: Loci of Points X(1) X(100)*. URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/loci_6tri.html.

— (2019d). *Triangular Orbits in Elliptic Billards: Loci of Points X(1) X(100)*. URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/loci_6tri.html.

— (2020a). *Animations of Dynamic Geometry Phenomena*. YouTube. URL: https://www.youtube.com/user/dreznik/videos (cit. on p. 60).

— (2020b). *Playlist for "Invariants of N-Periodic Trajectories in Elliptic Billiards"*. URL: bit.ly/3aNqgqU.

— (2020c). *Playlist for "Invariants of N-Periodics in the Elliptic Billiard"*. URL: bit.ly/2xeVGYw.

— (2020d). *Playlist for "Loci of Triangular Orbits in an Elliptic Billiard"*. URL: bit.ly/3072LV3.

— (2020e). *Playlist for "New Properties of Triangular Orbits in an Elliptic Billiard"*. URL: bit.ly/379mk1I.

— (2020f). *YouTube Playlist for Constant-Area Deltoid*. URL: bit.ly/3gqfIRm.

Reznik, Dan and Iverton Darlan (2020). *Loci of Ellipse-Mounted Triangle Centers*. URL: https://dan-reznik.github.io/ellipse-mounted-triangles/ (cit. on p. 60).

Reznik, Dan and Ronaldo Garcia (Dec. 2020). *The Talented Mr. Inversive Triangle in the Elliptic Billiard*. arXiv:2012.03020 (cit. on pp. 25, 61).

Reznik, Dan and Ronaldo Garcia (2021a). "Circuminvariants of 3-periodics in the elliptic billiard". In: *Intl. J. Geometry* 10.1, pp. 31–57 (cit. on p. 61).

— (Mar. 2021b). "Related By Similarity II: Poncelet 3-Periodics in the Homothetic Pair and the Brocard Porism". In: *Intl. J. of Geom.* 10.4, pp. 18–31 (cit. on p. 61).

Reznik, Dan, Ronaldo Garcia, and Jair Koiller (Aug. 2019a). *Adventures with Triangles and Billiards.* YouTube. URL: youtu.be/t872mftaI2g (cit. on p. 61).

— (July 2019b). *Adventures with Triangles and Billiards.* Rio de Janeiro. URL: impa.br/en_US/eventos-do-impa/eventos-2019/%2032o-coloquio-brasileiro-de-matematica (cit. on p. 61).

— (July 2019c). *Adventures with Triangles and Billiards.* Rio de Janeiro. URL: impa.br/en_US/eventos-do-impa/eventos-2019/%2032o-coloquio-brasileiro-de-matematica.

— (Aug. 2019d). *Adventures with Triangles and Billiards.* URL: youtu.be/t872mftaI2g.

— (2019e). *Media for Elliptic Billards and Family of Orbits.* GitHub. URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/videos.html.

— (2019f). *Media for Elliptic Billards and Family of Orbits.* URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/videos.html.

— (2019g). *New Properties of Triangular Orbits in Elliptic Billiards.* GitHub. URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/.

— (2019h). *New Properties of Triangular Orbits in Elliptic Billiards.* URL: dan-reznik.github.io/Elliptical-Billiards-Triangular-Orbits/.

— (2020a). "Can the Elliptic Billiard still surprise us?" In: *Math Intelligencer* 42, pp. 6–17. DOI: 10.1007/s00283-019-09951-2. URL: rdcu.be/b2cg1 (cit. on pp. 19, 61).

— (2020b). "The Ballet of Triangle Centers on the Elliptic Billiard". In: *Journal for Geometry and Graphics* 24.1, pp. 079–101 (cit. on p. 61).

— (Feb. 2021). "Fifty New Invariants of N-Periodics in the Elliptic Billiard". In: *Arnold Math. J.* URL: https://rdcu.be/cftyF (cit. on pp. 23, 61).

Reznik, Dan, Ronaldo Garcia, and Hellmuth Stachel (2020). *Area-Invariant Pedal-Like Curves Derived from the Ellipse.* submitted.

Richardson, Bill (2010). *I was just looking for an example.* URL: www.math.wichita.edu/~richardson/Orthic-story/orthic-triangle-complete.html.

Romaskevich, Olga (2014). "On the incenters of triangular orbits on elliptic billiards". In: *Enseign. Math.* 60, pp. 247–255. DOI: 10.4171/LEM/60-3/4-2. URL: arxiv.org/pdf/1304.7588.pdf (cit. on p. 13).

— (May 2019). *Proof the Mittenpunkt is Stationary.* Private Communication. Rennes, France.

Ronaldo Garcia Dan Reznik, Jair Koiller (2021). *Loci of 3-periodics in the Elliptic Billiard: Why so many ellipses?* GitHub. URL: https://dan-reznik.github.io/why-so-many-ellipses.

Rozikov, Utkir A (2018). *An Introduction To Mathematical Billiards*. World Scientific Publishing Company (cit. on p. 60).

Schwartz, Richard (2007). "The Poncelet grid". In: *Adv. Geometry* 7, pp. 157–175.

Schwartz, Richard and Sergei Tabachnikov (2016a). "Centers of mass of Poncelet polygons, 200 years after". In: *Math. Intelligencer* 38.2, pp. 29–34. DOI: 10.1007/s00283-016-9622-9. URL: www.math.psu.edu/tabachni/prints/Poncelet5.pdf (cit. on pp. 13, 17).

— (2016b). "Centers of mass of Poncelet polygons, 200 years after". In: *Math. Intelligencer* 38.2, pp. 29–34. DOI: 10.1007/s00283-016-9622-9. URL: www.math.psu.edu/tabachni/prints/Poncelet5.pdf.

Shail, Ron (1996). "Some properties of Brocard points". In: *The Mathematical Gazette* 80.489, pp. 485–491. DOI: 10.2307/3618511.

Snyder, John M. (1992). "Interval Analysis for Computer Graphics". In: *Computer Graphics* 26.2, pp. 121–129.

Stachel, Hellmuth (May 2020). *Proofs for k_{113} and k_{116}*. Private Communication. Technical University of Vienna.

Steiner, Jakob (1838). "Über den Krümmungs-Schwerpunkt ebener Curven". In: *Abhandlungen der Königlichen Akademie der Wissenshaften zu Berlin*, pp. 19–91.

Tabachnikov, Sergei (2002). "Ellipsoids, Complete Integrability and Hyperbolic Geometry". In: *Moscow Mathematical Journal* 2.1, pp. 185–198. URL: ftp.math.psu.edu/tabachni/prints/integr.pdf.

— (2005). *Geometry and Billiards*. Vol. 30. Student Mathematical Library. Mathematics Advanced Study Semesters, University Park, PA. Providence, RI: American Mathematical Society, pp. xii+176. DOI: 10.1090/stml/030. URL: www.personal.psu.edu/sot2/books/billiardsgeometry.pdf (cit. on p. 60).

— (2019a). "Projective configuration theorems: old wine into new wineskins". In: *Geometry in History*. Ed. by S. Dani and A. Papadopoulos. Springer Verlag, pp. 401–434. URL: arxiv.org/pdf/1607.04758.pdf.

— (Oct. 2019b). *Proof of Expression for Constant Cosine Sum for any N*. Private Communication.

— (July 2019c). *Proofs of Stationary Circle Radius for $N > 3$*. Private Communication.

— (July 2020). *Invariant sum of squared sidelengths for N-periodics in the Homothetic Poncelet Pair*. Private Communication.

Tabachnikov, Sergei and Richard Schwartz (July 2019). *Proof of Constant Cosine Sum $N > 3$*. Private Communication.

Tabachnikov, Sergei and Emmanuel Tsukerman (2014). "Circumcenter of Mass and Generalized Euler Line". In: *Discrete Comput. Geom.* 51, pp. 815–836. DOI: 10.1007/s00454-014-9597-2 (cit. on p. 13).

Tak, Ng Chung (n.d.). *General Formula for Equidistant Locus of Three Points*. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/1628779 (version: 2016-01-

27). eprint: https://math.stackexchange.com/q/1628779. URL: https://math.stackexchange.com/q/1628779.

Thébault, Victor (1952). "Perspective and orthologic triangles and tetrahedrons". In: *Amer. Math. Monthly* 59, pp. 24–28. DOI: 10.2307/2307184. URL: doi.org/10.2307/2307184.

Traité sur les propriétés projectives des figures (1822). Paris: Bachelier. URL: gallica.bnf.fr/ark:/12148/bpt6k9608143v.texteImage.

Veselov, A. P. (1988). "Integrable discrete-time systems and difference operators". In: *Functional Analysis and Applications* 22, pp. 1–13.

Weisstein, E. (2019). *Orthic Triangle*. URL: mathworld.wolfram.com/OrthicTriangle.html.

Weisstein, Eric (2019). "Mathworld". In: *MathWorld–A Wolfram Web Resource*. URL: mathworld.wolfram.com.

Wells, David (1991). *The Penguin Dictionary of Curious and Interesting Geometry*. Penguin Books.

Wikipedia (2019). *Geodesics on an ellipsoid*. URL: en.wikipedia.org/wiki/Geodesics_on_an_ellipsoid#Geodesics_on_a_triaxial_ellipsoid.

Winston, Roland, Juan Minano, and Pablo G. Benitez (2005). *Non-Imaging Optics*. Elsevier.

Wolfram, Stephen (2019). *Mathematica, Version 10.0*. Champaign, IL.

Wu, Pei Yuan (2000). "Polygons and numerical ranges". In: *Amer. Math. Monthly* 107.6, pp. 528–540. DOI: 10.2307/2589348. URL: https://doi.org/10.2307/2589348.

Yiu, Paul (2001). "Introduction to the Geometry of the Triangle". URL: math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry121226.pdf.

— (2003). "A Tour of Triangle Geometry". URL: math.fau.edu/Yiu/TourOfTriangleGeometry/MAAFlorida37040428.pdf.

Zwikker, C. (2005). *The Advanced Geometry of Plane Curves and Their Applications*. Dover Publications.