$X(i)$ LOCI OF BILLIARDS

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Resumo. $X(6)$ is given by a quartic equation

1. Locus X(1)

The ellipse

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1,$$

where

$$\begin{aligned} a_1 =& \frac{-b^2+\sqrt{a^4-a^2b^2+b^4}}{a} \\ b_1 =& -\frac{-a^2+\sqrt{a^4-a^2b^2+b^4}}{b} \end{aligned}$$

is the geometric locus of $X(1)$.

The confocal ellipse is given

$$\begin{aligned} a_c =& -\frac{a\left(b^2-\sqrt{a^4-a^2b^2+b^4}\right)}{a^2-b^2} \\ b_c =& \frac{b\left(a^2-\sqrt{a^4-a^2b^2+b^4}\right)}{a^2-b^2} \end{aligned}$$

The excentric ellipse

$$\begin{aligned} a_e =& \frac{b^2+\sqrt{a^4-a^2b^2+b^4}}{a} \\ b_e =& \frac{a^2+\sqrt{a^4-a^2b^2+b^4}}{b}, \quad \delta=\sqrt{a^4-a^2b^2+b^4} \end{aligned}$$

Date: 21 de agosto de 2019.

2. Locus X(2)-Circuncenter

The ellipse

$$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=1,$$

where

$$\begin{aligned} a_2 =& \frac{1}{2}\,\frac{a^2-\sqrt{a^4-a^2b^2+b^4}}{a} \\ b_2 =& -\frac{1}{2}\,\frac{b^2-\sqrt{a^4-a^2b^2+b^4}}{b} \end{aligned}$$

is the geometric locus of $X(2)$.

3. Locus X(3)-Barycenter

The ellipse

$$\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=1,$$

where

$$\begin{aligned} a_3^2 =& \frac{a^2\left(5\,a^4+5\,b^4-\left(4\,a^2+4\,b^2\right)\sqrt{a^4-a^2b^2+b^4}-2\,a^2b^2\right)}{9\left(a^2-b^2\right)^2} \\ a_3 =& \frac{a\left(-a^2-b^2+2\sqrt{a^4-a^2b^2+b^4}\right)}{3\left(a^2-b^2\right)} \\ b_3^2 =& \frac{b^2\left(5\,a^4+5\,b^4-\left(4\,a^2+4\,b^2\right)\sqrt{a^4-a^2b^2+b^4}-2\,a^2b^2\right)}{9\left(a^2-b^2\right)^2} \\ b_3 =& \frac{\left(-a^2-b^2+2\sqrt{a^4-a^2b^2+b^4}\right)b}{3\left(a^2-b^2\right)} \end{aligned}$$

is the geometric locus of $X(3)$.

4. Locus X(4)-Ortocenter

The ellipse

$$\frac{x^2}{a_4^2}+\frac{y^2}{b_4^2}=1,$$

where

$$
\begin{aligned}
a_4^2 =& \frac{-4\,a^2b^2\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}+\left(a^4+b^4\right)^2+a^2b^2\left(a^2+b^2\right)^2}{a^2\left(a^2-b^2\right)^2}\\
b_4^2 =& \frac{-4\,a^2b^2\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}+\left(a^4+b^4\right)^2+a^2b^2\left(a^2+b^2\right)^2}{b^2\left(a^2-b^2\right)^2}\\
a_4 =& \frac{\left(a^2+b^2\right)\sqrt{a^4-b^2a^2+b^4}-2\,b^2a^2}{\left(a^2-b^2\right)a}\\
b_4 =& \frac{\left(a^2+b^2\right)\sqrt{a^4-b^2a^2+b^4}-2\,b^2a^2}{\left(a^2-b^2\right)b}
\end{aligned}
$$

is the geometric locus of $X(4)$.

$b_4 = b$ if and only if $b = \frac{1}{7}\sqrt{7+14\sqrt{2}}a = (0.7395391544...)a$

$b_4 = a$ if and only if $b = (0.6622640780...)a$ where $a^6 + a^4b^2 - 4\,a^3b^3 - a^2b^4 - b^6 = 0$.

5. Locus X(6)- Semymedian

The locus $X(6)$ is the quartic defined by:

$$
\begin{aligned}
Q_4(x,y) =& b^4\left(5\,a^4-6\,a^2b^2+5\,b^4-4\,a^2\delta+4\,b^2\delta\right)x^4-a^2b^4\left(5\,a^4-5\,a^2b^2+2\,b^4-4\,a^2\delta+2\,b^2\delta\right)x^2\\
&+a^4\left(5\,a^4-6\,a^2b^2+5\,b^4+4\,a^2\delta-4\,b^2\delta\right)y^4-a^4b^2\left(2\,a^4-5\,a^2b^2+5\,b^4+2\,a^2\delta-4\,b^2\delta\right)y^2\\
&+2\,b^2a^2\left(3\,a^4-2\,a^2b^2+3\,b^4\right)x^2y^2\\
\delta =& \sqrt{a^4-a^2b^2+b^4}
\end{aligned}
$$

The ellipse

$$\frac{x^2}{a_6^2}+\frac{y^2}{b_6^2}=1,$$

where

$$
\begin{aligned}
a_6 =& \frac{\left(-a^2b^2-b^4+3\,\sqrt{a^4-a^2b^2+b^4}a^2-\sqrt{a^4-a^2b^2+b^4}b^2\right)a}{3\,a^4-2\,a^2b^2+3\,b^4}\\
b_6 =& \frac{\left(a^4+a^2b^2+\sqrt{a^4-a^2b^2+b^4}a^2-3\,\sqrt{a^4-a^2b^2+b^4}b^2\right)b}{3\,a^4-2\,a^2b^2+3\,b^4}
\end{aligned}
$$

is tangent to the quartic $Q_4(x,y) = 0$.

6. Locus X(5)

The ellipse

$$\frac{x^2}{a_5^2}+\frac{y^2}{b_5^2}=1,$$

where

$$a_5 = \frac{a^2\left(a^2+3\,b^2\right)-\sqrt{a^4-a^2b^2+b^4}\left(3\,a^2+b^2\right)}{4a\left(a^2-b^2\right)}$$

$$b_5 = \frac{-b^2\left(3\,a^2+b^2\right)+\sqrt{a^4-a^2b^2+b^4}\left(a^2+3\,b^2\right)}{4b\left(a^2-b^2\right)}$$

is the geometric locus of $X(5)$.

The conversion formula from the trilinear coordinates x, y, z to the vector of Cartesian coordinates P of the point is given by

$$P = \frac{ax}{ax+by+cz}A+\frac{by}{ax+by+cz}B+\frac{cz}{ax+by+cz}C$$

where the side lengths are $|C-B|=a$, $|A-C|=b$ and $|B-A|=c$.

7. Locus X(7) – Gergonne Point

The ellipse

$$\frac{x^2}{a_7^2}+\frac{y^2}{b_7^2}=1,$$

where

$$a_7 = -\frac{\left(a^2+b^2-2\sqrt{a^4-a^2b^2+b^4}\right)a}{a^2-b^2}$$

$$b_7 = -\frac{\left(a^2+b^2-2\sqrt{a^4-a^2b^2+b^4}\right)b}{a^2-b^2}$$

is the geometric locus of $X(7)$. Is is similar to the billiard (ellipse).

8. Locus X(8) – Nagel Point

The ellipse

$$\frac{x^2}{a_8^2}+\frac{y^2}{b_8^2}=1,$$

where

$$a_8 = -\frac{-a^4 + a^2b^2 - 2\,b^4 + 2\,b^2\sqrt{a^4 - a^2b^2 + b^4}}{a\,(a^2 - b^2)}$$

$$b_8 = \frac{-2\,a^4 + a^2b^2 - b^4 + 2\,a^2\sqrt{a^4 - a^2b^2 + b^4}}{b\,(a^2 - b^2)}$$

is the geometric locus of $X(8)$.

9. Locus X(10)

The ellipse

$$\frac{x^2}{a_{10}^2} + \frac{y^2}{b_{10}^2} = 1,$$

where

$$a_{10} = \frac{\left(a^2 + b^2\right)\sqrt{a^4 - a^2b^2 + b^4} - a^4 - b^4}{2\,(a^2 - b^2)\,a}$$

$$b_{10} = \frac{\left(a^2 + b^2\right)\sqrt{a^4 - a^2b^2 + b^4} - a^4 - b^4}{2\,(a^2 - b^2)\,b}$$

is the geometric locus of $X(10)$.

10. X(11) Feuerbach Point

It is the caustic

11. Locus X(12) X(1),X(5)-HARMONIC CONJUGATE OF X(11)

The ellipse

$$\frac{x^2}{a_{12}^2} + \frac{y^2}{b_{12}^2} = 1,$$

where

$$a_{12} = -\frac{-b^2\left(15\,a^6 + 12\,b^2a^4 + 3\,a^2b^4 + 2\,b^6\right) + \left(7\,a^6 + 12\,b^2a^4 + 11\,a^2b^4 + 2\,b^6\right)\sqrt{a^4 - a^2b^2 + b^4}}{a\,(7\,a^6 + 11\,b^2a^4 - 11\,a^2b^4 - 7\,b^6)}$$

$$b_{12} = \frac{-2\,a^8 - 3\,b^2a^6 - 12\,a^4b^4 - 15\,b^6a^2 + \left(2\,a^6 + 11\,b^2a^4 + 12\,a^2b^4 + 7\,b^6\right)\sqrt{a^4 - a^2b^2 + b^4}}{b\,(7\,a^6 + 11\,b^2a^4 - 11\,a^2b^4 - 7\,b^6)}$$

is the geometric locus of $X(12)$.

12. Locus X(20)- DE LONGCHAMPS POINT

The ellipse

$$\frac{x^2}{a_{20}^2}+\frac{y^2}{b_{20}^2}=1,$$

where

$$\begin{aligned} a_{20} =&\frac{-a^2\left(a^2-3\,b^2\right)-2b^2\,\sqrt{a^4-a^2b^2+b^4}}{a(a^2-b^2)}\\ b_{20} =&-\frac{b^2\left(3\,a^2-b^2\right)-2a^2\,\sqrt{a^4-a^2b^2+b^4}}{b\,(a^2-b^2)} \end{aligned}$$

is the geometric locus of $X(20)$.

13. Locus X(21)

The ellipse

$$\frac{x^2}{a_{21}^2}+\frac{y^2}{b_{21}^2}=1,$$

where

$$\begin{aligned} a_{21} =&\frac{-a^4-a^2b^2-2\,b^4+2\,\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}}{a\,(3\,a^2+5\,b^2)}\\ b_{21} =&\frac{2\,a^4+a^2b^2+b^4-2\,\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}}{b\,(5\,a^2+3\,b^2)} \end{aligned}$$

is the geometric locus of $X(21)$.

14. Locus X(35)

The ellipse

$$\frac{x^2}{a_{35}^2}+\frac{y^2}{b_{35}^2}=1,$$

where

$$\begin{aligned} a_{35} =&\frac{\left(7\,a^2+b^2\right)\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}-b^2\left(11\,a^4+4\,a^2b^2+b^4\right)}{a\,(7\,a^4+18\,a^2b^2+7\,b^4)}\\ b_{35} =&\frac{\left(a^2+7\,b^2\right)\left(a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}-a^2\left(a^4+4\,a^2b^2+11\,b^4\right)}{b\,(7\,a^4+18\,a^2b^2+7\,b^4)} \end{aligned}$$

is the geometric locus of $X(35)$.

15. Locus X(36) – Inverse-in-Circumcircle of Incenter

The ellipse

$$\frac{x^2}{a_{36}^2}+\frac{y^2}{b_{36}^2}=1,$$

where

$$a_{36}=\frac{\left(3\,a^2-b^2\right)\sqrt{a^4-a^2b^2+b^4}+b^2\left(a^2+b^2\right)}{3a\left(a^2-b^2\right)}$$

$$b_{36}=\frac{\left(a^2-3\,b^2\right)\sqrt{a^4-a^2b^2+b^4}+a^2\left(a^2+b^2\right)}{3b(\,a^2-\,b^2)}$$

is the geometric locus of $X(36)$.

16. Locus X(40)

The ellipse

$$\frac{x^2}{a_{40}^2}+\frac{y^2}{b_{40}^2}=1,$$

where

$$a_{40}=\frac{a^2-b^2}{a}$$

$$b_{40}=\frac{a^2-b^2}{b}$$

is the geometric locus of $X(40)$.

17. Locus X(46)

The ellipse

$$\frac{x^2}{a_{46}^2}+\frac{y^2}{b_{46}^2}=1,$$

where

$$a_{46}=\frac{\left(a^2-b^2\right)\left(\left(5\,a^2+b^2\right)\sqrt{a^4-a^2b^2+b^4}+b^2\left(3\,a^2-b^2\right)\right)}{a\left(5\,a^4-6\,a^2b^2+5\,b^4\right)}$$

$$b_{46}=-\frac{\left(a^2\left(a^2-3\,b^2\right)-\sqrt{a^4-a^2b^2+b^4}\left(a^2+5\,b^2\right)\right)\left(a^2-b^2\right)}{b\left(5\,a^4-6\,a^2b^2+5\,b^4\right)}$$

is the geometric locus of $X(40)$.

18. Locus X(55) –

The ellipse

$$\frac{x^2}{a_{55}^2}+\frac{y^2}{b_{55}^2}=1,$$

where

$$a_{55}=\frac{\left(-b^2+\sqrt{a^4-b^2a^2+b^4}\right)a}{a^2+b^2}$$

$$b_{55}=\frac{\left(a^2-\sqrt{a^4-b^2a^2+b^4}\right)b}{a^2+b^2}$$

is the geometric locus of $X(55)$. It is similar to the confocal ellipse.

19. Locus X(56) –

The ellipse

$$\frac{x^2}{a_{56}^2}+\frac{y^2}{b_{56}^2}=1,$$

where

$$a_{56}=\frac{-b^2\left(a^4-a^2b^2+2\,b^4\right)+\left(5\,a^4-5\,a^2b^2+2\,b^4\right)\sqrt{a^4-a^2b^2+b^4}}{a\,(5\,a^4-6\,a^2b^2+5\,b^4)}$$

$$b_{56}=-\frac{-a^2\left(2\,a^4-a^2b^2+b^4\right)+\left(2\,a^4-5\,a^2b^2+5\,b^4\right)\sqrt{a^4-a^2b^2+b^4}}{b\,(5\,a^4-6\,a^2b^2+5\,b^4)}$$

is the geometric locus of $X(56)$.

20. Locus X(57) – X(57) ISOGONAL CONJUGATE OF X(9)

$$a_{57}=\frac{a(a^2-b^2)}{\sqrt{a^4-a^2b^2+b^4}},\quad b_{57}=\frac{b(a^2-b^2)}{\sqrt{a^4-a^2b^2+b^4}}$$

is the geometric locus of $X(57)$.

21. Locus X(63)

$$a_{63}=\frac{a\left(a^2-b^2\right)}{a^2+b^2}$$

$$b_{63}=\frac{b\left(a^2-b^2\right)}{a^2+b^2}$$

is the geometric locus of $X(63)$.

22. Locus X(65)

$$a_{65} = -\frac{-a^4b^2 - a^2b^4 - 2\,b^6 - \left(a^4 - 3\,a^2b^2 - 2\,b^4\right)\sqrt{a^4 - a^2b^2 + b^4}}{a\left(a^2 - b^2\right)^2}$$
$$b_{65} = \frac{-2\,a^6 - a^4b^2 - a^2b^4 + \left(2\,a^4 + 3\,b^2a^2 - b^4\right)\sqrt{a^4 - b^2a^2 + b^4}}{b\left(a^2 - b^2\right)^2}$$

is the geometric locus of $X(65)$.

23. Locus X(72) –

The ellipse

$$\frac{x^2}{a_{72}^2} + \frac{y^2}{b_{72}^2} = 1,$$

where

$$a_{72} = \frac{a^6 + 2\,a^2b^4 + b^6 - \sqrt{a^4 - a^2b^2 + b^4}b^2\left(3\,a^2 + b^2\right)}{a\left(a^2 - b^2\right)^2}$$
$$b_{72} = \frac{-a^6 - 2\,a^4b^2 - b^6 + a^2\left(a^2 + 3\,b^2\right)\sqrt{a^4 - a^2b^2 + b^4}}{b\left(a^2 - b^2\right)^2}$$

is the geometric locus of $X(72)$.

24. Locus X(78) – ISOGONAL CONJUGATE OF X(34)

The ellipse

$$\frac{x^2}{a_{78}^2} + \frac{y^2}{b_{78}^2} = 1,$$

where

$$a_{78} = \frac{5\,a^6 - 4\,a^4b^2 + a^2b^4 + 2\,b^6 - 2\sqrt{a^4 - b^2a^2 + b^4}b^2\left(a^2 + b^2\right)}{a\left(5\,a^4 - 6\,b^2a^2 + 5\,b^4\right)}$$
$$b_{78} = \frac{-2\,a^6 - a^4b^2 + 4\,a^2b^4 - 5\,b^6 + 2\sqrt{a^4 - b^2a^2 + b^4}a^2\left(a^2 + b^2\right)}{b\left(5\,a^4 - 6\,b^2a^2 + 5\,b^4\right)}$$

is the geometric locus of $X(78)$.

25. Locus X(79) – ISOGONAL CONJUGATE OF X(35)

The ellipse

$$\frac{x^2}{a_{79}^2}+\frac{y^2}{b_{79}^2}=1,$$

where

$$a_{79}=\frac{-b^2\left(11\,a^4+4\,b^2a^2+b^4\right)+\left(3\,a^4+12\,b^2a^2+b^4\right)\sqrt{a^4-b^2a^2+b^4}}{a\,(a^2-b^2)\,(3\,a^2+5\,b^2)}$$

$$b_{79}=\frac{-a^2\left(a^4+4\,b^2a^2+11\,b^4\right)+\left(a^4+12\,b^2a^2+3\,b^4\right)\sqrt{a^4-b^2a^2+b^4}}{b\,(a^2-b^2)\,(5\,a^2+3\,b^2)}$$

is the geometric locus of $X(79)$.

26. Locus X(80) – REFLECTION OF INCENTER IN FEUERBACH POINT

The ellipse

$$\frac{x^2}{a_{80}^2}+\frac{y^2}{b_{80}^2}=1,$$

where

$$a_{80}=\frac{\left(-b^2+\sqrt{a^4-b^2a^2+b^4}\right)(a^2+b^2)}{(a^2-b^2)\,a}$$

$$b_{80}=\frac{\left(a^2-\sqrt{a^4-b^2a^2+b^4}\right)(a^2+b^2)}{b\,(a^2-b^2)}$$

is the geometric locus of $X(80)$.

27. Locus X(84)

The ellipse

$$\frac{x^2}{a_{84}^2}+\frac{y^2}{b_{84}^2}=1,$$

where

$$a_{84}=\frac{\left(b^2+\sqrt{a^4-b^2a^2+b^4}\right)(a^2-b^2)}{a^3}$$

$$b_{84}=\frac{\left(a^2+\sqrt{a^4-b^2a^2+b^4}\right)(a^2-b^2)}{b^3}$$

is the geometric locus of $X(84)$.

28. X(88)- Isogonal Conjugate of X_{44}

X(88) is the billiard.

29. Locus X(90)

The ellipse

$$\frac{x^2}{a_{90}^2} + \frac{y^2}{b_{90}^2} = 1,$$

where

$$a_{90} = \frac{\left(b^2\left(3\,a^2 - b^2\right) + \left(a^2 + b^2\right)\sqrt{a^4 - b^2a^2 + b^4}\right)\left(a^2 - b^2\right)}{a\left(a^4 + 2\,b^2a^2 - 7\,b^4\right)}$$

$$b_{90} = -\frac{\left(-a^2\left(a^2 - 3\,b^2\right) + \left(a^2 + b^2\right)\sqrt{a^4 - b^2a^2 + b^4}\right)\left(a^2 - b^2\right)}{b\left(7\,a^4 - 2\,b^2a^2 - b^4\right)}$$

is the geometric locus of $X(90)$.

30. X(100) – Anticomplement of Feuerbach Point

It is the billiard