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Caustics of Poncelet Polygons and Classical Extremal Polynomials

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Abstract—A comprehensive analysis of periodic trajectories of billiards within ellipses in the Euclidean plane is presented. The novelty of the approach is based on a relationship recently established by the authors between periodic billiard trajectories and extremal polynomials on the systems of d intervals on the real line and ellipsoidal billiards in d-dimensional space. Even in the planar case systematically studied in the present paper, it leads to new results in characterizing n periodic trajectories vs. so-called n elliptic periodic trajectories, which are n-periodic in elliptical coordinates. The characterizations are done both in terms of the underlying elliptic curve and divisors on it and in terms of polynomial functional equations, like Pell's equation. This new approach also sheds light on some classical results. In particular, we connect the search for caustics which generate periodic trajectories with three classical classes of extremal polynomials on two intervals, introduced by Zolotarev and Akhiezer. The main classifying tool are winding numbers, for which we provide several interpretations, including one in terms of numbers of points of alternance of extremal polynomials. The latter implies important inequality between the winding numbers, which, as a consequence, provides another proof of monotonicity of rotation numbers. A complete catalog of billiard trajectories with small periods is provided for $n = 3, 4, 5, 6$ along with an effective search for caustics. As a byproduct, an intriguing connection between Cayley-type conditions and discriminantly separable polynomials has been observed for all those small periods.

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1. INTRODUCTION

In our recent paper [20] we have developed a strong link between the theory of billiards within quadrics in d-dimensional space and the theory of extremal polynomials on the systems of d intervals on the real line. Using this link, we proved some fundamental properties of the billiard dynamics and paved a road to a comprehensive study of periodic trajectories of the billiards within ellipsoids in the d-dimensional Euclidean space.

The goal of the present paper is to provide a case study of the basic, planar case. It is well known that a geometric manifestation of integrability of elliptical billiards is the existence of a caustic, a conic confocal with the boundary of a billiard table, which is tangent to every segment of a given billiard trajectory. It is also well known that the elliptical billiard dynamics is equivalent to the projective-geometry situation considered by Poncelet in 1813/14, when the boundary conic and the

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caustic conic are in an arbitrary position, not necessarily confocal. It should also be mentioned that it is possible to redefine the billiard reflection in projective-geometric terms of harmonic conjugation of four lines in a pencil, to associate a generalized billiard system to any pair of conics, a boundary and a caustic. This is the reason why we call periodic trajectories also the Poncelet polygons, or if the period is n, the Poncelet n-gons. The Poncelet theorem ([34], see also [6, 19, 25, 32]) states that if such a polygon exists for the fixed boundary and the caustic, then there are infinitely many such polygons, sharing the boundary and the caustic. Cayley around 1853 derived a criterion which answers the question for two given conics, a boundary and a caustic, whether the dynamics they generate is periodic with a given period or not. Cayley answered the question by translating it to an equivalent question whether a given point of an elliptic curve is of a given order in the group structure defined by the elliptic curve. Some 35 years later, Halphen established a relationship between the Poncelet polygons and continued fractions and approximation theory, see [26, Part 2, p. 600].

Using the fact that the classical Cayley condition can be reformulated as a polynomial functional equation of Pell type, we are going to show that the search for caustics which generate periodic trajectories of a given period is intimately related to the classical extremal polynomials, namely, to the Zolotarev polynomials and Akhiezer polynomials. Zolotarev was a talented student of Chebyshev, the founding father of a famous Saint-Petersburg school in the second half of the 19th century. Chebyshev introduced celebrated Chebyshev polynomials in his study of extremal problems on an interval. It is interesting to point out that Chebyshev came to the extremal questions from the engineering problems from the theory of mechanisms. These problems were intimately related to the main technological questions brought about by the industrial revolution. Similar questions were also studied by Poncelet. Akhiezer, one of the prominent figures of the 20th century mathematics, significantly developed further the ideas of Chebyshev and his school. Based upon his deep results on orthogonal polynomials and their continuous analogues, the notion of Baker-Akhiezer functions emerged in the work of I. M. Krichever forty year ago. Soon it became one of the main tools of modern algebro-geometric theory of integrable systems, established by Novikov, Dubrovin, and others in 1970s, see [21] and references therein.

Apart from the connections between the Cayley condition and the Zolotarev and Akhiezer extremal polynomials, we also establish in this paper an unexpected relationship between the Cayley condition and another class of polynomials, the so-called discriminantly separable polynomials. This class has been introduced quite recently, less than a decade ago, in [12], related to the celebrated Kowalevski top and the Kowalevski integration procedure for equations of motion of the top [29]. Later, such polynomials were related to other continuous and discrete integrable systems, see [14–17].

Organization of this paper

The next Section 2 introduces the basic notions related to elliptical billiards, derives an algebro-geometric criterion for periodicity with a given period n for billiard trajectories within a given ellipse and a confocal caustic in terms of an elliptic curve which is isomorphic, but different from the so-called Cayley cubic, originally studied by Cayley. Then a detailed study and two characterizations of n elliptic periodic trajectories are provided. Let us recall that an n elliptic periodic trajectory has period n in elliptic coordinates, but is not necessarily periodic in Cartesian coordinates. One of the characterizations is given in terms of the underlying elliptic curve. The second one is done in Section 3 in terms of extremal polynomials on two intervals and solutions of polynomial functional equations, like the Pell equation. This section concludes with a detailed analysis of winding numbers and their different appearances. The derived properties of winding numbers serve as a main classifying tool for the caustics which generate periodic trajectories of a given period. They also lead to another proof of the monotonicity of rotation numbers. Section 4 employs the derived criteria for periodicity and winding numbers to study in detail the cases of small periods $n = 2, 3, 4, 5, 6$. Section 5 relates the results from the previous section to three classes of classical extremal polynomials on two intervals introduced by Zolotarev and Akhiezer. The last section, 6, collects come intriguing, yet not fully understood, experimental observations that Cayley-type conditions produce the so-called discriminantly separable polynomials.

2. ELLIPTICAL BILLIARDS AND PERIODIC TRAJECTORIES

2.1. Elliptical Billiard and Confocal Families

Mathematical billiard within a plane domain is a dynamical system where a particle moves without constraints within the domain, and obeys the billiard reflection law on the boundary [30], see Fig. 1. Billiard trajectories are polygonal lines with vertices on the boundary.

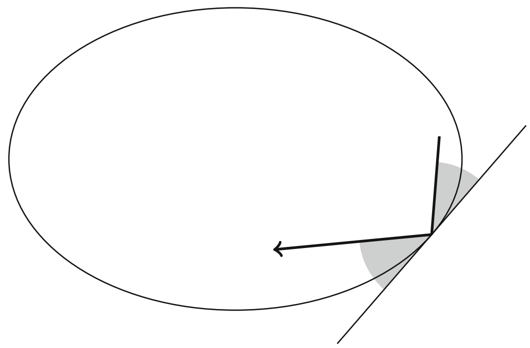

Fig. 1. The billiard reflection law: the impact and reflection angles are congruent.

Mathematical billiard is an idealized model where the billiard ball is replaced by a material point, and friction and spin are neglected. Such a billiard system in the Euclidean space is a good model for the motion of light rays, with mirror boundary. The dynamics has two different regimes: inside the billiard domain, and the impacts. We assume that the impacts are *absolutely elastic*, which means that the geometric billiard law is satisfied, i. e., the impact and reflection angles are congruent to each other, and the speed remains unchanged. Here, we assume that the material point is traveling under inertia between the impacts, although motion in a force field can also be considered, for example, in a gravitational field [28], with Hooke's potential [23, 36], or with integrable potentials [11].

In this work, we will study *elliptical billiards* — the mathematical billiards within an ellipse in the Euclidean plane:

$$\mathcal{E}\colon \frac{\mathsf{x}^2}{a} + \frac{\mathsf{y}^2}{b} = 1, \quad a > b > 0.$$

Each trajectory of the elliptic billiard has *a caustic*: a curve such that each segment of the trajectory lies on its tangent line. Moreover, the caustics of billiard trajectories within $\mathcal{E}$ belong to the family of conics confocal with the boundary:

$$\mathcal{C}_\lambda\colon \frac{\mathsf{x}^2}{a-\lambda} + \frac{\mathsf{y}^2}{b-\lambda} = 1. \tag{2.1}$$

We notice that the family (2.1) contains two types of smooth conics: ellipses, corresponding to $\lambda < b$, and hyperbolas, corresponding to $\lambda \in (b, a)$, see Fig. 2.

In addition, there are two degenerate conics and the family: the x-axis, corresponding to $\lambda = b$, and the y-axis, corresponding to $\lambda = a$.

Each point in the plane which is not a focus of the confocal family lies on exactly two conics $\mathcal{C}_{\lambda_1}$ and $\mathcal{C}_{\lambda_2}$ from (2.1) — one ellipse and one hyperbola, which are orthogonal to each other at the intersection point. In other words, we can join to such a point a unique pair of parameters (λ_1, λ_2), $\lambda_1 < \lambda_2$, corresponding to the two confocal conics which contain the point. The pair (λ_1, λ_2) is called *elliptic coordinates* of the point. We note that points symmetric with respect to the x- and y-axes have the same elliptic coordinates.

All segments of the billiard trajectories within $\mathcal{E}$ with a fixed caustic $\mathcal{C}_{\lambda_0}$ lie in a domain Ω_{λ_0}, that is:

- if $\mathcal{C}_{\lambda_0}$ is an ellipse, that is, $\lambda_0 \in (0, b)$, Ω_{λ_0} is the annulus between the billiard boundary $\mathcal{E}$ and the caustic;

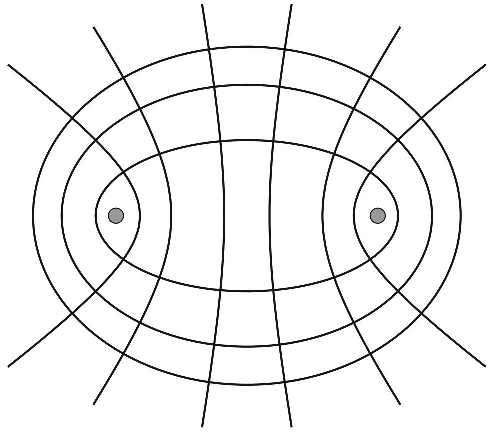

Fig. 2. A family of confocal conics in the plane.

- if $\mathscr{C}_{\lambda_0}$ is a hyperbola, that is, $\lambda_0 \in (b, a)$, Ω_{λ_0} is the part within the billiard boundary $\mathscr{E}$ which is between the branches of the caustic.

In the elliptic coordinates, Ω_{λ_0} is given by

$$(\lambda_1, \lambda_2) \in [0, \alpha_1] \times [\alpha_2, a], \quad \alpha_1 = \min\{b, \lambda_0\}, \ \alpha_2 = \max\{b, \lambda_0\}.$$

On any billiard trajectory, the value $\lambda_1 = 0$ is achieved at the reflection points on the boundary, the value $\lambda_2 = a$ at the intersection points with the y-axis, while the corresponding elliptic coordinate has the value λ_0 at the touching points with the caustic, and the value b at the intersection points with the x-axis. Between these points, the elliptic coordinates change monotonously.

For a periodic billiard trajectory, we introduce *the winding numbers* (m_0, m_1): m_0 is the number of its reflection points, and m_1 the number of its intersection with the y-axis. The number m_1 is always even, since the y-axis must be crossed an even number of times along the period.

2.2. Periodic Trajectories and Cayley's Cubic

It is of particular interest to condsider periodic billiard trajectories — the trajectories that become closed after a certain number of reflections. In the next lemma, we note that the type of the caustic may be determined by the period of a trajectory.

Lemma 1. *The period of a closed billiard trajectory with a hyperbola as a caustic is always even.*

Proof. Denote by F_1, F_2 the focal points of the boundary ellipse. Then every segment of the billiard trajectory intersects the segment F_1F_2. Thus, for a periodic trajectory there should be an even number of intersections of the trajectory with F_1F_2 and the period is even. □

The Poncelet theorem [19, 32, 34] implies that all trajectories sharing the same caustic with a periodic elliptical billiard trajectory are also periodic, and moreover, all these trajectories have the same period. It is natural to ask about an analytic condition that would determine if elliptical billiard trajectories with a given caustic will become closed after a certain number of bounces. Such a condition was derived by Cayley in the mid-19th century.

Here, in Theorems 1 and 2, we present a derivation of the analytic conditions following the ideas of Jacobi and Darboux [10, 27], see also [18]. In Theorem 3, we will present the classical Cayley conditions, since it has a slightly different form, and show that they are equivalent to the ones presented in Theorem 2.

Theorem 1. *The billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are n-periodic if and only if $nQ_0 \sim nQ_{\lambda_0}$ on the elliptic curve:*

$$\mathcal{C} \ : \ y^2 = (a - x)(b - x)(\lambda_0 - x), \tag{2.2}$$

with Q_0 being a point of $\mathcal{C}$ corresponding to $x = 0$, and Q_{λ_0} the point corresponding to $x = \lambda_0$.

Proof. Consider the integral

$$\frac{d\lambda_1}{\sqrt{(a-\lambda_1)(b-\lambda_1)(\lambda_0-\lambda_1)}}+\frac{d\lambda_2}{\sqrt{(a-\lambda_2)(b-\lambda_2)(\lambda_0-\lambda_2)}} \tag{2.3}$$

along the polygonal line, which represents n consecutive segments of a billiard trajectory within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$. That integral equals zero along each line touching $\mathscr{C}_{\lambda_0}$. Thus, considering the behavior of the elliptic coordinates along the trajectory with winding numbers (m_0, m_1), $m_0 = n$, we get that the first vertex of the polygonal line will coincide with the last one if and only if

$$n(Q_0 - Q_{\alpha_1}) + m_1(Q_{\alpha_2} - Q_a) \sim 0$$

on $\mathcal{C}$. Here, we have denoted by Q_{α_1}, Q_{α_2}, Q_a the points of $\mathcal{C}$ corresponding to x equal to α_1, α_2, a, respectively. Since $2Q_{\alpha_2} \sim 2Q_a$ and m_1 is even, we have found that the periodicity condition reduces to

$$n(Q_0 - Q_{\alpha_1}) \sim 0.$$

Now, if n is even, $2\mathscr{Q}_{\alpha_1} \sim 2Q_{\lambda_0}$ implies $n(Q_0 - Q_{\lambda_0}) \sim 0$. If n is odd, then, according to Lemma 1, the caustic of the trajectory must be an ellipse, so $\alpha_1 = \lambda_0$. □

Theorem 2. *The billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are n-periodic if and only if:*

$$C_2 = 0, \quad \begin{vmatrix} C_2 & C_3 \\ C_3 & C_4 \end{vmatrix} = 0, \quad \begin{vmatrix} C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \\ C_4 & C_5 & C_6 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 3, 5, 7, \dots$$

$$B_3 = 0, \quad \begin{vmatrix} B_3 & B_4 \\ B_4 & B_5 \end{vmatrix} = 0, \quad \begin{vmatrix} B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \\ B_5 & B_6 & B_7 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 4, 6, 8, \dots.$$

Here, we have denoted:

$$\sqrt{(a-x)(b-x)(\lambda_0-x)} = B_0 + B_1x + B_2x^2 + \dots,$$
$$\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{\lambda_0 - x} = C_0 + C_1x + C_2x^2 + \dots,$$

the Taylor expansions around $x = 0$.

Proof. Denote by Q_∞ the point of $\mathcal{C}$ (2.2) corresponding to $x = \infty$ and notice that

$$2Q_{\lambda_0} \sim 2Q_\infty. \tag{2.4}$$

Consider first n even. Because of (2.4), the condition $nQ_0 \sim nQ_{\lambda_0}$ is equivalent to $nQ_0 \sim nQ_\infty$, which is equivalent to the existence of a meromorphic function of $\mathcal{C}$ with the unique pole at Q_∞ and unique zero at Q_0, such that the pole and the zero are both of multiplicity n. The basis of $\mathcal{L}(nQ_\infty)$ is

$$1, x, x^2, \dots, x^{n/2}, y, xy, x^{n/2-2}y, \tag{2.5}$$

thus a nontrivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} B_{n/2+1} & B_{n/2} & \dots & B_3 \\ B_{n/2+2} & B_{n/2+1} & \dots & B_4 \\ \dots & & & \\ B_{n-1} & B_n & \dots & B_{n/2+1} \end{vmatrix} = 0.$$

Now suppose n is odd. Because of (2.4), the condition $nQ_0 \sim nQ_{\lambda_0}$ is equivalent to $nQ_0 \sim (n-1)Q_\infty + Q_{\lambda_0}$, which is equivalent to the existence of a meromorphic function of $\mathcal{C}$ with only two poles: of order $n-1$ at Q_∞ and a simple pole at Q_{λ_0}, and unique zero at Q_0. The basis $\mathcal{L}\big((n-1)Q_\infty + Q_{\lambda_0}\big)$ is

$$1, x, x^2, \dots, x^{(n-1)/2}, \frac{y}{\lambda_0 - x}, \frac{xy}{\lambda_0 - x}, \dots, \frac{x^{(n-1)/2-1}y}{\lambda_0 - x}, \tag{2.6}$$

thus a nontrivial linear combination of those functions with a zero of order n at $x = 0$ exists if and only if:

$$\begin{vmatrix} C_{(n-1)/2+1} & C_{(n-1)/2} & \dots & C_2 \\ C_{(n-1)/2+2} & C_{(n-1)/2+1} & \dots & C_3 \\ \dots & & & \\ C_{n-1} & C_n & \dots & C_{(n-1)/2+1} \end{vmatrix} = 0.$$

□

We can rewrite Eq. (2.1) for the family of confocal conics in matrix form:

$$(x\, y\, 1)M_\lambda(x\, y\, 1)^T = 0,$$

where

$$M_\lambda = \begin{pmatrix} \dfrac{1}{a-\lambda} & 0 & 0 \\ 0 & \dfrac{1}{b-\lambda} & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Following Cayley, we can use the matrices M_λ to define another elliptic curve, the so-called *Cayley cubic*

$$\mathcal{C}^* \ : \ y^2 = \det(M_0 + xM_{\lambda_0}). \tag{2.7}$$

The curve (2.7) can be explicitly written as

$$y^2 = -(x+1)\left(\frac{1}{a-\lambda_0} + \frac{x}{a}\right)\left(\frac{1}{b-\lambda_0} + \frac{x}{b}\right).$$

This is an elliptic curve with branching points corresponding to

$$x \in \left\{-1, \frac{a}{\lambda_0 - a}, \frac{b}{\lambda_0 - b}, \infty\right\}.$$

Using the bilinear transformation

$$(x, y) \mapsto (x_1, y_1),$$
$$x_1 = \frac{\lambda_0 x}{1+x}, \quad y_1 = \text{const} \cdot \frac{y}{1+x},$$

the branching points are transformed to $\{\infty, a, b, \lambda_0\}$, respectively, and the Cayley cubic into $y_1^2 = (a - x_1)(b - x_1)(\lambda_0 - x_1)$.

Thus, $nP_0 \sim nP_\infty$ on $\mathcal{C}^*$ is equivalent to $nQ_0 \sim nQ_{\lambda_0}$ on $\mathcal{C}$. We conclude that the previous theorem is equivalent to the following formulation of the classical Cayley condition.

Theorem 3 (Cayley's condition, [9, 24, 25, 32]). *There is a closed polygonal line with n vertices, inscribed in $\mathscr{E}$ and circumscribed about $\mathscr{C}_{\lambda_0}$ if and only if $nP_0 \sim nP_\infty$ on the Cayley cubic $\mathcal{C}^*$, given by (2.7), with P_0 being one of the points of the curve $\mathcal{C}^*$ corresponding to $x = 0$, and P_∞ the point corresponding to $x = \infty$.*

Further, $nP_0 \sim nP_\infty$ *is equivalent to*

$$A_2 = 0, \quad \begin{vmatrix} A_2 & A_3 \\ A_3 & A_4 \end{vmatrix} = 0, \quad \begin{vmatrix} A_2 & A_3 & A_4 \\ A_3 & A_4 & A_5 \\ A_4 & A_5 & A_6 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 3, 5, 7, \dots$$

$$A_3 = 0, \quad \begin{vmatrix} A_3 & A_4 \\ A_4 & A_5 \end{vmatrix} = 0, \quad \begin{vmatrix} A_3 & A_4 & A_5 \\ A_4 & A_5 & A_6 \\ A_5 & A_6 & A_7 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 4, 6, 8, \dots,$$

with

$$\sqrt{\det(M_0 + xM_{\lambda_0})} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots,$$

being the Taylor expansion about $x = 0$.

2.3. Elliptic Periodic Trajectories

Points of the plane which are symmetric with respect to the coordinate axes share the same elliptic coordinates, thus there is no bijection between the elliptic and the Cartesian coordinates. Thus, we introduce a separate notion of periodicity in elliptic coordinates.

Definition 1. A billiard trajectory is *n-elliptic periodic* if it is n-periodic in elliptic coordinates joined to the confocal family (2.1).

Theorem 4. *A billiard trajectory within $\mathscr{E}$ is $2n$-periodic if and only if it is n-elliptic periodic.*

Proof. The symmetry implies that each n-elliptic periodic trajectory is symmetric with respect to one of the axes or the origin. Consequently, such a trajectory is always $2n$-periodic.

Along any $2n$-periodic trajectory, the elliptic coordinate λ_1 will trace the segment $[0, \alpha_1]$ $2n$ times in each direction. Similarly, λ_2 will trace the segment $[\alpha_2, a]$ an even number of times in each direction, since each periodic trajectory intersects the y-axis an even number of times. Each of these intersections corresponds to $\lambda_2 = a$. This all together implies that the given trajectory will be elliptic periodic with period n. □

Thus, we see that in the Euclidean plane there are no $2n$-periodic trajectories which are not n-elliptic periodic. The situation is different in higher-dimensional spaces, as is shown in [20].

Now we will derive algebro-geometric conditions for elliptic periodic trajectories.

Theorem 5. *A billiard trajectory within $\mathscr{E}$ with the caustic $\mathscr{Q}_{\lambda_0}$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied on $\mathcal{C}$:*

(*a*) $\mathscr{Q}_{\lambda_0}$ *is an ellipse and* $n(Q_0 - Q_{\lambda_0}) + Q_b - Q_a \sim 0$;

(*b*) $\mathscr{Q}_{\lambda_0}$ *is a hyperbola and* $n(Q_0 - Q_b) + Q_{\lambda_0} - Q_a \sim 0$;

(*c*) $\mathscr{Q}_{\lambda_0}$ *is a hyperbola, n is odd, and* $n(Q_0 - Q_b) \sim 0$.

Moreover, such trajectories are always symmetric with respect to the origin in case (*a*) *and, when n is odd, in case* (*b*). *They are symmetric with respect to the larger axis in case* (*b*) *for even n, and with respect to the smaller axis in case* (*c*).

Proof. Let M_0 be the initial point of a given n-elliptic periodic trajectory, and M_1 the next point on the trajectory with the same elliptic coordinates. Then, integrating (2.3) from M_0 to M_1 along the trajectory, we get

$$n(Q_0 - Q_{\alpha_1}) + m(Q_{\alpha_2} - Q_a) \sim 0,$$

where m is the number of times that the particle crossed the y-axis. Along that part of the trajectory, the particle crossed the x-axis m' times, where $m' = n$ if $b = \alpha_1$, and $m' = m$ if $b = \alpha_2$.

To conclude the proof, we notice that at least one of m, m' must be odd if the trajectory is not n-periodic. $\square$

The explicit Cayley-type conditions for elliptic periodic trajectories are:

Theorem 6. *A billiard trajectory within $\mathscr{E}$ with the caustic $\mathscr{Q}_{\lambda_0}$ is n-elliptic periodic without being n-periodic if and only if one of the following conditions is satisfied:*

(*a*) *$\mathscr{Q}_{\lambda_0}$ is an ellipse and*

$$C_1 = 0, \quad \begin{vmatrix} C_1 & C_2 \\ C_2 & C_3 \end{vmatrix} = 0, \quad \begin{vmatrix} C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \\ C_3 & C_4 & C_5 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 2, 4, 6, \dots$$

$$B_2 = 0, \quad \begin{vmatrix} B_2 & B_3 \\ B_3 & B_4 \end{vmatrix} = 0, \quad \begin{vmatrix} B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 3, 5, 7, \dots;$$

(*b*) *$\mathscr{Q}_{\lambda_0}$ is a hyperbola and*

$$D_1 = 0, \quad \begin{vmatrix} D_1 & D_2 \\ D_2 & D_3 \end{vmatrix} = 0, \quad \begin{vmatrix} D_1 & D_2 & D_3 \\ D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 2, 4, 6, \dots$$

$$B_2 = 0, \quad \begin{vmatrix} B_2 & B_3 \\ B_3 & B_4 \end{vmatrix} = 0, \quad \begin{vmatrix} B_2 & B_3 & B_4 \\ B_3 & B_4 & B_5 \\ B_4 & B_5 & B_6 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 3, 5, 7, \dots;$$

(*c*) *$\mathscr{Q}_{\lambda_0}$ is a hyperbola, n is odd, and*

$$D_2 = 0, \quad \begin{vmatrix} D_2 & D_3 \\ D_3 & D_4 \end{vmatrix} = 0, \quad \begin{vmatrix} D_2 & D_3 & D_4 \\ D_3 & D_4 & D_5 \\ D_4 & D_5 & D_6 \end{vmatrix} = 0, \dots \quad \textit{for} \quad n = 3, 5, 7, \dots.$$

Here, we have denoted:

$$\frac{\sqrt{(a-x)(b-x)(\lambda_0 - x)}}{b-x} = D_0 + D_1 x + D_2 x^2 + \dots,$$

the Taylor expansion around $x = 0$, while Bs and Cs are as in Theorem 2.

Proof. (a) In this case, the caustic $\mathcal{Q}_{\lambda_0}$ is an ellipse.

Take first n even. Using Theorem 5, we have

$$nQ_0 \sim nQ_{\lambda_0} - Q_b + Q_a \sim nQ_\infty - Q_b + Q_a \sim nQ_\infty - Q_\infty + Q_{\lambda_0} \sim (n-1)Q_\infty + Q_{\lambda_0}.$$

The basis of $\mathcal{L}\big((n-1)Q_\infty + Q_{\lambda_0}\big)$ is

$$1, x, x^2, \dots, x^{n/2-1}, \frac{y}{x-\lambda_0}, \frac{xy}{x-\lambda_0}, \frac{x^{n/2-1}y}{x-\lambda_0},$$

thus a nontrivial linear combination of these functions with a zero of order n at $x=0$ exists if and only if

$$\begin{vmatrix} C_{n/2} & C_{n/2-1} & \dots & C_1 \\ C_{n/2+1} & C_{n/2} & \dots & C_2 \\ \dots & & & \\ C_{n-1} & C_{n-2} & \dots & C_{n/2} \end{vmatrix} = 0.$$

For odd n, we have

$$nQ_0 \sim nQ_{\lambda_0} - Q_b + Q_a \sim (n-1)Q_\infty + Q_{\lambda_0} - Q_b + Q_a \sim (n-1)Q_\infty + Q_\infty \sim nQ_\infty.$$

The basis of $\mathcal{L}(nQ_\infty)$ is

$$1, x, x^2, \dots, x^{(n-1)/2}, y, xy, x^{(n-1)/2-1}y,$$

thus a nontrivial linear combination of these functions with a zero of order n at $x=0$ exists if and only if

$$\begin{vmatrix} B_{(n-1)/2+1} & B_{(n-1)/2} & \dots & B_2 \\ B_{(n-1)/2+2} & B_{(n-1)/2+1} & \dots & B_3 \\ \dots & & & \\ B_{n-1} & B_{n-2} & \dots & B_{(n-1)/2+1} \end{vmatrix} = 0.$$

(b) In this case, the caustic $\mathcal{Q}_{\lambda_0}$ is a hyperbola.

Take first n even. Using Theorem 5, we have

$$nQ_0 \sim nQ_b - Q_{\lambda_0} + Q_a \sim nQ_\infty - Q_{\lambda_0} + Q_a \sim nQ_\infty - Q_\infty + Q_b \sim (n-1)Q_\infty + Q_b.$$

The basis of $\mathcal{L}((n-1)Q_\infty + Q_b)$ is

$$1, x, x^2, \dots, x^{n/2-1}, \frac{y}{x-b}, \frac{xy}{x-b}, \frac{x^{n/2-1}y}{x-b},$$

thus a nontrivial linear combination of these functions with a zero of order n at $x=0$ exists if and only if

$$\begin{vmatrix} D_{n/2} & D_{n/2-1} & \dots & D_1 \\ D_{n/2+1} & D_{n/2} & \dots & D_2 \\ \dots & & & \\ D_{n-1} & D_{n-2} & \dots & D_{n/2} \end{vmatrix} = 0.$$

For odd n, we have

$$nQ_0 \sim nQ_b + Q_{\lambda_0} - Q_a \sim (n-1)Q_\infty + Q_b + Q_{\lambda_0} - Q_a \sim (n-1)Q_\infty + Q_\infty \sim nQ_\infty.$$

The determinant conditions are then obtained as in part (a).

(c) We have $nQ_0 \sim nQ_b \sim (n-1)Q_\infty + Q_b$. The conditions are derived as in the proof of Theorem 2, just by replacing λ_0 with b. □

Corollary 1. *We notice that whenever a trajectory is n-elliptic periodic, the divisor $n(Q_0 - Q_{\lambda_0})$ is of order 2 on the elliptic curve (2.2), which means that it is equivalent to one of the following: $Q_{\lambda_0} - Q_\infty$, $Q_a - Q_\infty$, or $Q_b - Q_\infty$.*

More precisely, a billiard trajectory within $\mathscr{E}$ with the caustic $\mathscr{Q}_{\lambda_0}$ is n-elliptic periodic without being n-periodic if and only if

- *$\mathscr{Q}_{\lambda_0}$ is an ellipse and $n(Q_0 - Q_{\lambda_0}) \sim Q_{\lambda_0} - Q_\infty$;*
- *$\mathscr{Q}_{\lambda_0}$ is a hyperbola, n even, and $n(Q_0 - Q_{\lambda_0}) \sim Q_b - Q_\infty$;*
- *$\mathscr{Q}_{\lambda_0}$ is a hyperbola, n odd, and $n(Q_0 - Q_{\lambda_0})$ is equivalent to $Q_a - Q_\infty$ or $Q_{\lambda_0} - Q_\infty$.*

Remark 1. Theorem 2 implies that a billiard trajectory within $\mathscr{E}$ with the caustic $\mathscr{Q}_{\lambda_0}$ is $2n$-periodic if and only if $n(Q_0 - Q_{\lambda_0})$ is of order 2 or equivalent to the zero divisor. Thus, such a trajectory would be $2n$-periodic without being n-elliptic periodic if and only if

- $\mathscr{Q}_{\lambda_0}$ is an ellipse and $n(Q_0 - Q_{\lambda_0})$ is equivalent to $Q_a - Q_\infty$ or $Q_b - Q_\infty$;
- $\mathscr{Q}_{\lambda_0}$ is a hyperbola, n even, and $n(Q_0 - Q_{\lambda_0})$ is equivalent to $Q_a - Q_\infty$ or $Q_{\lambda_0} - Q_\infty$;
- $\mathscr{Q}_{\lambda_0}$ is a hyperbola, n odd, and $n(Q_0 - Q_{\lambda_0}) \sim Q_b - Q_\infty$.

Theorem 4 shows that none of these scenarios can be realized.

3. POLYNOMIAL FUNCTIONAL EQUATIONS AND PERIODICITY CONDITIONS

3.1. Pell's Equations, Extremal Polynomials, and Periodicity

The matrix conditions for periodic trajectories, presented in Sections 2.2 and 2.3, can be equivalently written in the form of polynomial functional equations. We will derive those equations in this section and relate them to the so-called *Pell equations*. For discussion including higher-dimensional cases, see [20] and references therein.

Theorem 7. *The billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2, respectively, and satisfying the following:*

(*a*) *if $n = 2m$ is even, then $d_1 = m$, $d_2 = m - 2$, and*

$$p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right)q_{m-2}^2(s) = 1;$$

(*b*) *if $n = 2m + 1$ is odd, then $d_1 = m$, $d_2 = m - 1$, and*

$$\left(s - \frac{1}{\lambda_0}\right)p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)q_{m-1}^2(s) = -1.$$

Proof. We note first that the proof of Theorem 2 implies that there is a nontrivial linear combination of the bases (2.5) for n even, or (2.6) for n odd, with the zero of order n at $x = 0$.

(*a*) For $n = 2m$, we get that there are polynomials $p_m^*(x)$ and $q_{m-2}^*(x)$ of degrees m and $m - 2$, respectively, such that the expression

$$p_m^*(x) - q_{m-2}^*(x)\sqrt{(a - x)(b - x)(\lambda_0 - x)}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$p_m^*(x) + q_{m-2}^*(x)\sqrt{(a - x)(b - x)(\lambda_0 - x)},$$

we get that the polynomial $\big(p_m^*(x)\big)^2 - (a-x)(b-x)(\lambda_0-x)\big(q_{m-2}^*(x)\big)^2$ has a zero of order $2m$ at $x=0$. Since the degree of that polynomial is $2m$, it follows that

$$\big(p_m^*(x)\big)^2 - (a-x)(b-x)(\lambda_0-x)\big(q_{m-2}^*(x)\big)^2 = cx^{2m}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of p_m^*. Dividing the last relation by cx^{2m} and introducing $s=1/x$, we get the required relation.

(b) On the other hand, for $n=2m+1$, we get that there are polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, respectively, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{\lambda_0-x}$$

has a zero of order $2m+1$ at $x=0$. Multiplying that expression by

$$(\lambda_0-x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{\lambda_0-x}\right),$$

we get that the polynomial $(\lambda_0-x)\big(p_m^*(x)\big)^2 - (a-x)(b-x)\big(q_{m-1}^*(x)\big)^2$ has a zero of order $2m+1$ at $x=0$. Since the degree of that polynomial is $2m+1$, it follows that

$$(\lambda_0-x)\big(p_m^*(x)\big)^2 - (a-x)(b-x)\big(q_{m-1}^*(x)\big)^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the opposite of the square of the leading coefficient of p_m^*. Dividing the last relation by $-cx^{2m+1}$ and introducing $s=1/x$, we get the required relation. □

Corollary 2. *If the billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_{\lambda_0}$ are n-periodic, then there exist real polynomials $\hat{p}_n$ and $\hat{q}_{n-2}$ of degrees n and $n-2$, respectively, which satisfy the Pell equation:*

$$\hat{p}_n^2(s) - s\left(s-\frac{1}{a}\right)\left(s-\frac{1}{b}\right)\left(s-\frac{1}{\lambda_0}\right)\hat{q}_{n-2}^2(s) = 1. \tag{3.1}$$

Proof. For $n=2m$, take $\hat{p}_n = 2p_m^2-1$ and $\hat{q}_{n-2} = 2p_mq_{m-2}$. For $n=2m+1$, we set $\hat{p}_n = 2\left(s-\dfrac{1}{\lambda_0}\right)p_m^2+1$ and $\hat{q}_{n-2} = 2p_mq_{m-1}$. □

Theorem 8. *The billiard trajectories within $\mathcal{E}$ with caustic $\mathcal{C}_{\lambda_0}$ are elliptic n-periodic without being n-periodic if and only if there exists a pair of real polynomials p_{d_1}, q_{d_2} of degrees d_1, d_2, respectively, and satisfying the following:*

(a) *$\mathcal{C}_{\lambda_0}$ is an ellipse and*

– *$n=2m$ is even, $d_1=d_2=m-1$,*

$$s\left(s-\frac{1}{\lambda_0}\right)p_{m-1}^2(s) - \left(s-\frac{1}{a}\right)\left(s-\frac{1}{b}\right)q_{m-1}^2(s) = -1;$$

– *$n=2m+1$ is odd, $d_1=m$, $d_2=m-1$,*

$$sp_m^2(s) - \left(s-\frac{1}{a}\right)\left(s-\frac{1}{b}\right)\left(s-\frac{1}{\lambda_0}\right)q_{m-1}^2(s) = 1;$$

(b) *$\mathcal{C}_{\lambda_0}$ is a hyperbola and*

– *$n=2m$ is even, $d_1=d_2=m-1$,*

$$s\left(s-\frac{1}{b}\right)p_{m-1}^2(s) - \left(s-\frac{1}{a}\right)\left(s-\frac{1}{\lambda_0}\right)q_{m-1}^2(s) = -1;$$

– *$n = 2m+1$ is odd, $d_1 = m$, $d_2 = m-1$,*

$$sp_m^2(s) - \left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right) q_{m-1}^2(s) = 1;$$

(*c*) *$\mathscr{C}_{\lambda_0}$ is a hyperbola, $n = 2m+1$ is odd, $d_1 = m$, $d_2 = m-1$,*

$$\left(s - \frac{1}{b}\right) p_m^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\lambda_0}\right) q_{m-1}^2(s) = -1.$$

Proof. (*a*) For $n = 2m$, the proof of Theorem 6 implies that there are polynomials $p_{m-1}^*(x)$ and $q_{m-1}^*(x)$ of degrees $m-1$, such that the expression

$$p_{m-1}^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{\lambda_0 - x}$$

has a zero of order $2m$ at $x = 0$. Multiplying that expression by

$$(\lambda_0 - x)\left(p_{m-1}^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{\lambda_0 - x}\right),$$

we get that the polynomial $(\lambda_0 - x)\big(p_{m-1}^*(x)\big)^2 - (a-x)(b-x)\big(q_{m-1}^*(x)\big)^2$ has a zero of order $2m$ at $x = 0$. Since the degree of that polynomial is $2m$, it follows that

$$(\lambda_0 - x)\big(p_{m-1}^*(x)\big)^2 - (a-x)(b-x)\big(q_{m-1}^*(x)\big)^2 = cx^{2m}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of q_{m-1}^*. Dividing the last relation by $-cx^{2m}$ and introducing $s = 1/x$, we get the required relation.

For $n = 2m+1$, the proof of Theorem 6 implies that there are polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\sqrt{(a-x)(b-x)(\lambda_0-x)}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$p_m^*(x) + q_{m-1}^*(x)\sqrt{(a-x)(b-x)(\lambda_0-x)},$$

we get that the polynomial $\big(p_m^*(x)\big)^2 - (a-x)(b-x)(\lambda_0-x)\big(q_{m-1}^*(x)\big)^2$ has a zero of order $2m+1$ at $x = 0$. Since the degree of that polynomial is $2m+1$, it follows that

$$\big(p_m^*(x)\big)^2 - (a-x)(b-x)(\lambda_0-x)\big(q_{m-1}^*(x)\big)^2 = cx^{2m+1}$$

for some constant c. Notice that c is positive, since it equals the square of the leading coefficient of q_{m-1}^*. Dividing the last relation by cx^{2m+1} and introducing $s = 1/x$, we get the required relation.

The result (*b*) is obtained in a similar way.

For (*c*), the proof of Theorem 6 implies that there are polynomials $p_m^*(x)$ and $q_{m-1}^*(x)$ of degrees m and $m-1$, such that the expression

$$p_m^*(x) - q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{b - x}$$

has a zero of order $2m+1$ at $x = 0$. Multiplying that expression by

$$(b - x)\left(p_m^*(x) + q_{m-1}^*(x)\frac{\sqrt{(a-x)(b-x)(\lambda_0-x)}}{b - x}\right),$$

we get that the polynomial $(b-x)\big(p_m^*(x)\big)^2 - (a-x)(\lambda_0 - x)\big(q_{m-1}^*(x)\big)^2$ has a zero of order $2m+1$ at $x=0$. Since the degree of that polynomial is $2m+1$, it follows that

$$(b-x)\big(p_m^*(x)\big)^2 - (a-x)(\lambda_0 - x)\big(q_{m-1}^*(x)\big)^2 = cx^{2m+1}$$

for some constant c. Notice that c is negative, since it is opposite to the square of the leading coefficient of p_m^*. Dividing the last relation by $-cx^{2m+1}$ and introducing $s = 1/x$, we get the required relation. □

From Corollary 2 and Theorem 8 we get:

Corollary 3. *The billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are n-elliptic periodic if and only if there exist a pair of real polynomials $\hat{p}_n$, $\hat{q}_{n-2}$ of degrees n and $n-2$, respectively, such that the Pell equation* (3.1) *holds.*

Proof. If the trajectories are n-periodic, the statement is true according to Corollary 2.

If the trajectories are n-elliptic periodic without being n-periodic, then one of the cases of Theorem 8 is satisfied.

In case (a), $n = 2m$, we have $\hat{p}_n = 2s\left(s - \dfrac{1}{\lambda_0}\right)p_{m-1}^2(s) + 1$, $\hat{q}_{n-2}(s) = 2p_{m-1}(s)q_{m-1}(s)$. For $n = 2m+1$, we have $\hat{p}_n = 2sp_m^2(s) - 1$, $\hat{q}_{n-2}(s) = 2p_m(s)q_{m-1}(s)$.

In case (b), $n = 2m$, we have $\hat{p}_n = 2s\left(s - \dfrac{1}{b}\right)p_{m-1}^2(s) + 1$, $\hat{q}_{n-2}(s) = 2p_{m-1}(s)q_{m-1}(s)$. For $n = 2m+1$, it is the same as in (a).

In case (c), we have $\hat{p}_n = 2\left(s - \dfrac{1}{b}\right)p_m^2(s) + 1$, $\hat{q}_{n-2}(s) = 2p_m(s)q_{m-1}(s)$.

Now suppose that the Pell equation is satisfied. Then, according to part (a) of Theorem 7, the trajectories are $2n$-periodic, and by Theorem 4 they are n-elliptic periodic. □

The next two theorems address the question of how to distinguish n-elliptic periodic trajectories which are not n-periodic from those that are.

Theorem 9. *Suppose that real polynomials $\hat{p}_{2n+1}$, $\hat{q}_{2n-1}$ of degrees $2n+1$, $2n-1$, respectively, satisfy the Pell equation:*

$$\hat{p}_{2n+1}^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right)\hat{q}_{2n-1}^2(s) = 1. \tag{3.2}$$

Then there exist unique real polynomials p_n, q_{n-1}, S_1, S_3 of degrees n, $n-1$, 1, 3, respectively, such that:

- *S_1, S_3 are monic and $S_1S_3 = s\left(s - \dfrac{1}{a}\right)\left(s - \dfrac{1}{b}\right)\left(s - \dfrac{1}{\lambda_0}\right)$;*
- *$S_1p_n^2 - S_3q_{n-1}^2$ equals 1 or -1.*

Moreover, the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are all $(2n+1)$-elliptic periodic. They are $(2n+1)$-periodic if and only if $S_1(s) = s - \dfrac{1}{\lambda_0}$.

Proof. Equation (3.2) is equivalent to

$$(\hat{p}_{2n+1}(s) - 1)(\hat{p}_{2n+1}(s) + 1) = s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right)\hat{q}_{2n-1}^2(s).$$

Notice that the two factors, $\hat{p}_{2n+1}(s) - 1$ and $\hat{p}_{2n+1}(s) + 1$, of the left-hand side are mutually prime, which implies that polynomial $\hat{p}_{2n+1}$ takes one of the values 1 and -1 at three of the points from the set $\{0, 1/a, 1/b, 1/\lambda_0\}$, and the opposite value at the remaining point of that set. Replacing $\hat{p}_{2n+1}$ by $-\hat{p}_{2n+1}$ if needed, we can assume that $\hat{p}_{2n+1}(s_1) = 1$ at only one point $s_1 \in \{0, 1/a, 1/b, 1/\lambda_0\}$.

Now we set S_1 and S_3 as

$$S_1(s) = s - s_1, \quad S_1(s)S_3(s) = s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right).$$

Polynomials p_n, q_{n-1} are such that

$$\hat{p}_{2n+1} - 1 = \sigma \cdot 2S_1 p_n^2, \quad \hat{p}_{2n+1} + 1 = \sigma \cdot 2S_3 q_{n-1}^2, \quad \hat{q}_{2n-1} = 2p_n q_{n-1}, \quad \sigma \in \{-1, 1\}.$$

From there we get

$$S_1 p_n^2 - S_3 q_n^2 = -\sigma. \tag{3.3}$$

Denote $\{c_1, c_2, c_3, c_4\} = \{0, 1/a, 1/b, 1/\lambda_0\}$, so that $c_4 < c_3 < c_2 < c_1$. Since $0 < b < a$ and $0 < \lambda_0 < a$, we have $c_4 = 0$, and $c_1 \in \{1/b, 1/\lambda_0\}$. If c_2 or c_3 is the root of S_1, then S_1 is negative on $[c_4, c_3]$ and positive on $[c_2, c_1]$, while S_3 has the opposite signs on these two segments. Thus $S_1 p_n^2 - S_3 q_{n-1}^2$ is negative on $[c_4, c_3]$, and positive on $[c_2, c_1]$, which is not possible. We conclude that the root of S_1 can be only c_1 or c_4.

If $S_1(s) = s - c_4 = s$, then $-\sigma$ equals the free coefficient of $-S_3 q_n^2$, which is positive, thus $-\sigma = 1$, and (3.3) becomes:

$$s p_n^2 - \left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right) q_n^2 = 1.$$

Now applying cases (a) and (b) of Theorem 8, we get that the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are $(2n+1)$-elliptic periodic without being $(2n+1)$-periodic.

If $S_1(s) = s - c_1$, then $-\sigma$ equals the free coefficient of $S_1 p_n^2$, which is negative, thus $-\sigma = -1$. Thus, when $c_1 = 1/b$, i.e., when $\mathscr{C}_{\lambda_0}$ is a hyperbola, (3.3) becomes:

$$\left(s - \frac{1}{b}\right) p_n^2 - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{\lambda_0}\right) q_n^2 = 1,$$

so the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are elliptic n-periodic without being n-periodic, according to case (c) of Theorem 8.

If $c_1 = 1/\lambda_0$, which means that $\mathscr{C}_{\lambda_0}$ is an ellipse, (3.3) becomes:

$$\left(s - \frac{1}{\lambda_0}\right) p_n^2 - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right) q_n^2 = 1,$$

so the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are $(2n+1)$-periodic, according to case (b) of Theorem 7. □

Remark 2. It is interesting here to notice that the proof of Theorem 9 represents an alternative, algebraic, proof of Lemma 1.

Theorem 10. *Suppose that real polynomials $\hat{p}_{2n}$, $\hat{q}_{2n-2}$ of degrees $2n$, $2n-2$, respectively, satisfy the Pell equation:*

$$\hat{p}_{2n}^2(s) - s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right)\hat{q}_{2n-2}^2(s) = 1. \tag{3.4}$$

Then there exist unique real polynomials p_{d_1}, q_{d_2}, S', S'' of degrees d_1, d_2, d', d'', respectively, such that either $d_1 = d_2 = n-1$, $d' = d'' = 2$ or $d_1 = n$, $d_2 = n-2$, $d' = 0$, $d'' = 4$, and

- *S', S'' are monic and $S'S'' = s\left(s - \dfrac{1}{a}\right)\left(s - \dfrac{1}{b}\right)\left(s - \dfrac{1}{\lambda_0}\right)$;*

- *$S' p_{d_1}^2 - S'' q_{d_2}^2$ equals 1 or -1.*

Moreover, the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are all $2n$-elliptic periodic. They are $2n$-periodic if and only if $S' = 1$.

Proof. Equation (3.4) is equivalent to

$$(\hat{p}_{2n}(s) - 1)(\hat{p}_{2n}(s) + 1) = s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right)\hat{q}^2_{2n-2}(s).$$

Notice that the two factors, $\hat{p}_{2n}(s) - 1$ and $\hat{p}_{2n}(s) + 1$, of the left-hand side are mutually prime, which implies that polynomial $\hat{p}_{2n}$ takes the value 1 at an even number of points from the set $\{0, 1/a, 1/b, 1/\lambda_0\}$, and the opposite value -1 at the remaining points of that set.

First, suppose that $\hat{p}_{2n}(s)$ takes the same value at each point of those four points. Replacing $\hat{p}_{2n}$ by $-\hat{p}_{2n}$ if needed, we can assume that the value is -1. Set S' and S'' as

$$S'(s) = 1, \quad S''(s) = s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right).$$

Polynomials p_n, q_{n-2} are such that

$$\hat{p}_{2n} - 1 = \sigma \cdot 2p_n^2, \quad \hat{p}_{2n} + 1 = \sigma \cdot 2S'' q_{n-2}^2, \quad \hat{q}_{2n-2} = 2p_n q_{n-2}, \quad \sigma \in \{-1, 1\},$$

from where we get $p_n^2 - S'' q_{n-2}^2 = -\sigma$. We can conclude that $-\sigma = 1$, since it equals the square of the free coefficient of p_n. Finally, applying case (a) of Theorem 7, we can conclude that the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are $2n$-periodic.

Second, we will suppose that $\hat{p}_{2n}(s)$ takes each of the values -1, 1 at two points from the set $\{0, 1/a, 1/b, 1/\lambda_0\}$. Replacing $\hat{p}_{2n}$ by $-\hat{p}_{2n}$ if needed, we can assume that $\hat{p}_{2n}(0) = \hat{p}_{2n}(s_1) = 1$, for $s_1 \in \{1/a, 1/b, 1/\lambda_0\}$. Set S' and S'' as

$$S'(s) = s(s - s_1), \quad S'(s)S''(s) = s\left(s - \frac{1}{a}\right)\left(s - \frac{1}{b}\right)\left(s - \frac{1}{\lambda_0}\right).$$

Polynomials p_{n-1}, q_{n-1} are such that

$$\hat{p}_{2n} - 1 = \sigma \cdot 2S' p_{n-1}^2, \quad \hat{p}_{2n} + 1 = \sigma \cdot 2S'' q_{n-1}^2, \quad \hat{q}_{2n-2} = 2p_{n-1}q_{n-1}, \quad \sigma \in \{-1, 1\},$$

so $S' p_{n-1}^2 - S'' q_{n-1}^2 = -\sigma$. From there, $-\sigma$ equals the free coefficient of $-S'' q_{n-1}^2$, which is negative, thus $-\sigma = -1$:

$$S' p_n^2 - S'' q_{n-2}^2 = -1. \tag{3.5}$$

As in the proof of Theorem 9, we denote by $c_4 < c_3 < c_2 < c_1$ the elements of $\{0, 1/a, 1/b, 1/\lambda_0\}$. If c_2 or c_3 is the root of S', then S' is negative on $[c_4, c_3]$ and positive on $[c_2, c_1]$, while S'' has the opposite signs on these two segments. Thus $S' p_{n-1}^2 - S'' q_{n-1}^2$ is negative on $[c_4, c_3]$ and positive on $[c_2, c_1]$, which is not possible. We conclude that $s_1 = c_1$, which can be either $1/\lambda_0$ or $1/b$.

If $c_1 = 1/\lambda_0$, that means $\mathscr{C}_{\lambda_0}$ is an ellipse, while it is a hyperbola for $c_1 = 1/b$. Thus, (3.5) implies that cases (a) and (b), respectively, of Theorem 7 are satisfied, i. e., the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_{\lambda_0}$ are $2n$-elliptic periodic without being $2n$-periodic. □

3.2. Generalized Chebyshev Polynomials and Rotation Function

From the previous consideration we see that a caustic $\mathscr{C}_{\lambda_0}$ generates n-elliptic periodic trajectories within $\mathscr{E}$ if and only if there exist a pair of real polynomials $\hat{p}_n$, $\hat{q}_{n-2}$ of degrees n and $n-2$, respectively, such that the Pell equation (3.1) holds:

$$\hat{p}_n^2(s) - \hat{\mathcal{P}}_4(s)\hat{q}_{n-2}^2(s) = 1.$$

Here $\hat{\mathcal{P}}_4(s) = \prod_{i=1}^4 (s - c_i)$, assuming that $\{c_1, c_2, c_3, c_4\} = \{1/\lambda_0, 1/a, 1/b, 0\}$ are, as in the proofs of Theorems 9 and 10, ordered by the condition $c_4 = 0 < c_3 < c_2 < c_1$.

The polynomials $\hat{p}_n$ are so-called *generalized Chebyshev polynomials* on two intervals $[c_4, c_3] \cup [c_2, c_1]$, with an appropriate normalization. Namely, one can consider the question of finding the monic polynomial of a certain degree n which minimizes the maximum norm on the union of two intervals. Denote such a polynomial as $\hat{P}_n$ and its norm L_n. The fact that polynomial $\hat{p}_n$ is a solution of the Pell equation on the union of intervals $[c_4, c_3] \cup [c_2, c_1]$ is equivalent to the following conditions:

(*i*) $\hat{p}_n = \hat{P}_n / \pm L_n$

(*ii*) the set $[c_4, c_3] \cup [c_2, c_1]$ is the maximal subset of $\mathbf{R}$ for which $\hat{P}_n$ is the minimal polynomial in the sense above.

Chebyshev was the first who considered a similar problem on one interval, and this was how the celebrated Chebyshev polynomials emerged in the 19th century. We are going to say a bit more about the original Chebyshev polynomials in Section 5. Now, following the principles formulated by Chebyshev and his school and also Borel (see [4]), we are going to study the structure of extremal points of $\hat{p}_n$, in particular, the set of points of alternance.

Notice that the roots of $\hat{\mathcal{P}}_4(s)$ are simple solutions of the equation $\hat{p}_n^2(s) = 1$, while the roots of $\hat{q}_{n-2}(s)$ are double solutions of the equation $\hat{p}_n^2(s) = 1$. Because of the degrees of the polynomials, these are all points where $\hat{p}_n^2(s)$ equals unity.

Let us recall that a set of *points of alternance* is, by definition, a subset of the solutions of the equation $\hat{p}_n^2(s) = 1$, with the maximal number of elements, such that the signs of $\hat{p}_n$ alter on it. Such a set is not uniquely determined, however, the number of its elements is fixed and equal to $n + 1$.

If we denote the number of points of alternance of the polynomial $\hat{p}_n$ on the segment $[c_4, c_3]$ as $1 + m_1$ and on the segment $[c_4, c_1]$ as $1 + m_0$, we see that the difference $m_0 - m_1$ is thus equal to the number of points of alternance on the interval $[c_3, c_1]$. According to the structure of the sets of the alternance, that number equals the sum of the numbers of the double points of alternance from the interval (c_2, c_1) and one simple point of alternance at one of the endpoints of the interval. Thus we get

$$m_0 = m_1 + \tau_1 + 1,$$

and

$$m_0 > m_1.$$

Here τ_1 is the number of zeros of the polynomial $\hat{q}_{n-2}$ on the interval (c_2, c_1). The pair (τ_1, τ_2), with $\tau_2 = m_1 - 1$ is called *the signature*, see [37].

From [31] it follows that the numbers $(n = m_0, m_1)$ satisfy the following condition:

$$n \int_{c_1}^{\infty} \frac{1}{\sqrt{\hat{\mathcal{P}}_4(s)}} ds = m_1 \int_{c_3}^{c_2} \frac{1}{\sqrt{\hat{\mathcal{P}}_4(s)}} ds. \tag{3.6}$$

Thus (m_0, m_1) will represent exactly the winding numbers, which we introduced at the end of Section 2.1, of the corresponding billiard trajectories.

For the reader's sake, we are going to review briefly the results about the winding numbers from [20] about winding numbers, specialized for the planar case.

Lemma 2 (Theorem 2.12 from [33]). *Let p_n, p_n^* be two polynomials of degree n which solve the Pell equations. Denote by*

$$\mathscr{I} = [c_4, c_3] \cup [c_2, c_1] \quad \textit{and} \quad \mathscr{I}^* = [c_4^*, c_3^*] \cup [c_2^*, c_1^*],$$

respectively, the sets $\{x \mid |p_n(x)| \leqslant 1\}$ and $\{x \mid |p_n^*(x)| \leqslant 1\}$. Suppose that:

(*i*) *at least one of the segments from $\mathscr{I}$ coincides with a segment from $\mathscr{I}^*$;*

(*ii*) *the other pair of segments from* $\mathscr{I}$ *and* $\mathscr{I}^*$ *have a joint either left or right endpoint;*

(*iii*) *in each pair of the corresponding segments* $[c_4, c_3]$, $[c_4^*, c_3^*]$ *and* $[c_2, c_1]$, $[c_2^*, c_1^*]$, *the polynomials* p_n, p_n^* *have the same number of extreme points.*

Then the polynomials p_n, p_n^* *coincide up to a constant multiplier and sets* $\mathscr{I}$ *and* $\mathscr{I}^*$ *coincide.*

Theorem 11. *Let* m_0 *and* m_1 *be given integers. Then there is at most one ellipse* $\mathscr{E}'$ *and at most one hyperbola* $\mathscr{H}$ *from the confocal family* (2.1), *such that the billiard trajectories within* $\mathscr{E}$ *and with caustics* $\mathscr{E}'$ *and* $\mathscr{H}$ *are periodic with winding numbers* (m_0, m_1).

Proof. All assumptions of Lemma 2 are satisfied. □

Corollary 4. *The rotation number*

$$\rho(\lambda) = \rho(\lambda, a, b) = \frac{\int_0^{\min\{b,\lambda\}} \frac{dt}{\sqrt{(\lambda-t)(b-t)(a-t)}}}{\int_{\max\{b,\lambda\}}^0 \frac{dt}{\sqrt{(\lambda-t)(b-t)(a-t)}}}$$

is a strictly monotone function on each of the intervals $(-\infty, b)$ *and* (b, a).

Proof. First, observe that the rotation number is rational if and only if the billiard trajectories within $\mathscr{E}$ with caustic $\mathscr{C}_\lambda$ are periodic, when $\rho(\lambda) = m_1/m_0$, see Eq. (3.6). According to Theorem 11, ρ can take any rational value at most once on each of the intervals $(-\infty, b)$ and (b, a). In addition, the rotation map for the billiard within the ellipse is a diffeomorphism at an open dense subset of each of the intervals $(-\infty, b)$ and (b, a), which can be proved by using the same arguments as in [35]. From there, ρ will be one-to-one on the whole intervals, thus, since it is continuous, also strictly monotone. □

Remark 3. In [22], there is another proof of Corollary 4, which uses the theory of algebraic surfaces. A different proof of injectivity of the rotation map, which works for higher-dimensional situations as well was presented recently in [20].

4. TRAJECTORIES WITH SMALL PERIODS: $n = 3, 4, 5, 6$

4.1. 3-periodic Trajectories

There is a 3-periodic trajectory of the billiard within $\mathscr{E}$, with a nondegenerate caustic $\mathscr{C}_{\lambda_0}$ if and only if the following conditions are satisfied:

- the caustics is an ellipse, i. e., $\lambda_0 \in (0, b)$; and
- $C_2 = 0$.

We can calculate:

$$C_2 = \frac{(a-b)^2\lambda_0^2 + 2ab(a+b)\lambda_0 - 3a^2b^2}{8(ab)^{3/2}\lambda_0^{5/2}}, \tag{4.1}$$

so $C_2 = 0$ is equivalent to

$$\lambda_0 = -\frac{ab(a+b) \pm 2ab\sqrt{a^2 - ab + b^2}}{(a-b)^2}.$$

Both solutions are always real, one of them being negative, and the other positive and smaller than b. Thus, there is a unique caustic giving 3-periodic trajectories that corresponds to real motion, the ellipse defined by

$$\lambda_0 = -\frac{ab(a+b) + 2ab\sqrt{a^2 - ab + b^2}}{(a-b)^2}. \tag{4.2}$$

The winding numbers of such trajectories satisfy $m_0 > m_1$, with $m_0 = 3$ and m_1 being even. Thus, $(m_0, m_1) = (3, 2)$, $(\tau_1, \tau_2) = (0, 1)$. The graph of the corresponding polynomial $\hat{p}_3(s)$ is shown in Fig. 3.

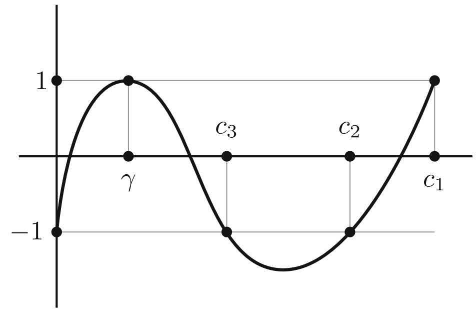

Fig. 3. The graph of $\hat{p}_3(s)$. The parameters are $c_1 = 1/\lambda_0$, $c_2 = 1/b$, $c_3 = 1/a$.

4.2. 4-periodic Trajectories

There is a 4-periodic trajectory of the billiard within $\mathscr{E}$, with a nondegenerate caustic $\mathscr{C}_{\lambda_0}$ if and only if $B_3 = 0$.

We can calculate:

$$B_3 = \frac{(a\lambda_0 - ab + \lambda_0 b)(a\lambda_0 + ab - \lambda_0 b)(-a\lambda_0 + ab + \lambda_0 b)}{16(ab\lambda_0)^{5/2}}, \tag{4.3}$$

so the coefficient equals zero if and only if

$$\lambda_0 \in \left\{ \frac{ab}{a+b}, \frac{ab}{a-b}, \frac{ab}{b-a} \right\}. \tag{4.4}$$

The first solution $\lambda_0 = ab/(a+b)$ is positive and smaller than b, so it corresponds to a confocal ellipse as a caustic. The second solution $\lambda_0 = ab/(a-b)$ is always bigger than b. It is also smaller than a if and only if $b < a/2$, so this is when a confocal hyperbola as a caustic for a 4-periodic trajectory exists. The third solution is negative, so it does not correspond to any real trajectories.

The winding numbers of 4-periodic trajectories satisfy $m_0 > m_1$, with $m_0 = 4$ and m_1 being even. Thus, $(m_0, m_1) = (4, 2)$, $(\tau_1, \tau_2) = (1, 1)$. The graph of the corresponding polynomial $\hat{p}_4(s)$ is shown in Fig. 4.

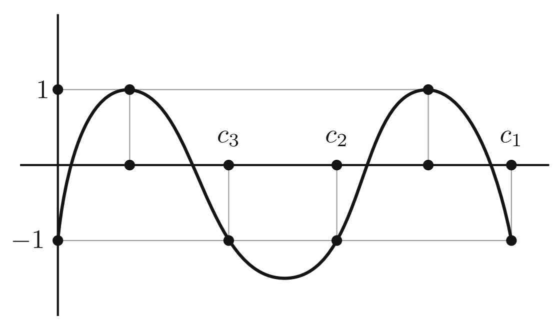

Fig. 4. The graph of $\hat{p}_4(s)$. The parameters are $c_1, c_2 \in \{1/\lambda_0, 1/b\}$, $c_3 = 1/a$.

4.3. 5-periodic Trajectories

There is a 5-periodic trajectory of the billiard within $\mathscr{E}$, with nondegenerate caustic $\mathscr{C}_{\lambda_0}$ if and only if the following conditions are satisfied:

- the caustics is an ellipse, i. e., $\lambda_0 \in (0, b)$; and
- $C_2 C_4 = C_3^2$.

We can calculate:

$$
\begin{aligned}
C_3^2 - C_2C_4 = & \frac{1}{1024a^5b^5\lambda_0^7}\times \\
& \times \left(-(a-b)^6\lambda_0^6 - 2ab(a-b)^2(a+b)(3a+b)(a+3b)\lambda_0^5 \right.\\
& \quad + a^2b^2(a-b)^2\left(29a^2+54ab+29b^2\right)\lambda_0^4 - 36a^3b^3(a-b)^2(a+b)\lambda_0^3 \\
& \quad \left. + a^4b^4\left(9a^2-34ab+9b^2\right)\lambda_0^2 + 10a^5b^5(a+b)\lambda_0 - 5a^6b^6\right).
\end{aligned}
$$

The winding numbers of 5-periodic trajectories satisfy $m_0 > m_1$, with $m_0 = 5$ and m_1 being even. Thus, $(m_0, m_1) \in \{(5,2),(5,4)\}$, with $(\tau_1,\tau_2) \in \{(2,1),(0,3)\}$, respectively. The graph of the corresponding polynomial $\hat{p}_5(s)$ is shown in Figs. 5 and 6.

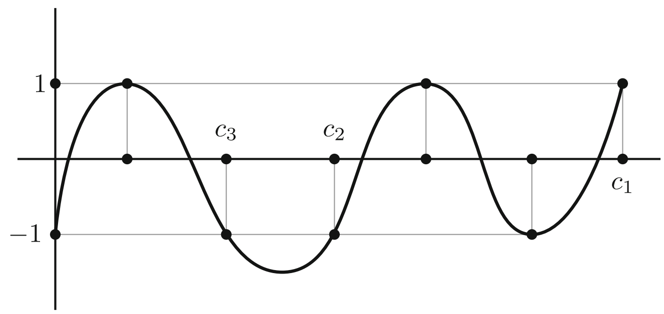

Fig. 5. The graph of $\hat{p}_5(s)$. The parameters are $c_1 = 1/\lambda_0$, $c_2 = 1/b$, $c_3 = 1/a$; $(m_0, m_1) = (5,2)$, $(\tau_1,\tau_2) = (2,1)$.

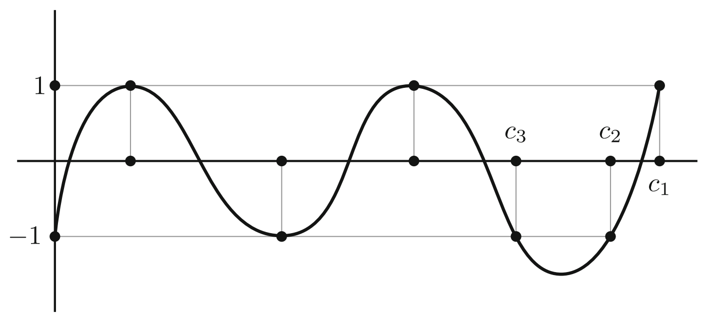

Fig. 6. The graph of $\hat{p}_5(s)$. The parameters are $c_1 = 1/\lambda_0$, $c_2 = 1/b$, $c_3 = 1/a$; $(m_0, m_1) = (5,4)$, $(\tau_1,\tau_2) = (0,3)$.

4.4. 6-periodic Trajectories

There is a 6-periodic trajectory of the billiard within $\mathscr{E}$, with nondegenerate caustic $\mathscr{C}_{\lambda_0}$ if and only if $B_4^2 - B_3B_5 = 0$.

We calculate:

$$
\begin{aligned}
B_4^2 - B_3B_5 = -\frac{B_2C_2}{256(ab)^4\lambda_0^3} & \times \left((a-b)(a+3b)\lambda_0^2 - 2ab(a-b)\lambda_0 + a^2b^2\right) \\
& \times \left((a-b)(3a+b)\lambda_0^2 - 2ab(a-b)\lambda_0 - a^2b^2\right).
\end{aligned}
$$

First, let us consider the condition $B_2 = 0$. We have

$$
B_2 = -\frac{a^2b^2 - 2ab\lambda_0(a+b) + \lambda_0^2(a-b)^2}{8(ab\lambda_0)^{3/2}}, \tag{4.5}
$$

so $B_2 = 0$ is equivalent to $\lambda_0 = ab/(\sqrt{a} \pm \sqrt{b})^2$. From the condition $\lambda_0 < b$, we have a unique solution which gives an ellipse as the caustic:

$$\lambda_0 = \frac{ab}{(\sqrt{a} + \sqrt{b})^2}.$$

For $a > 4b$, the option $\lambda_0 = ab/(\sqrt{a} - \sqrt{b})^2$ will provide a hyperbola as the caustic, see Fig. 7.

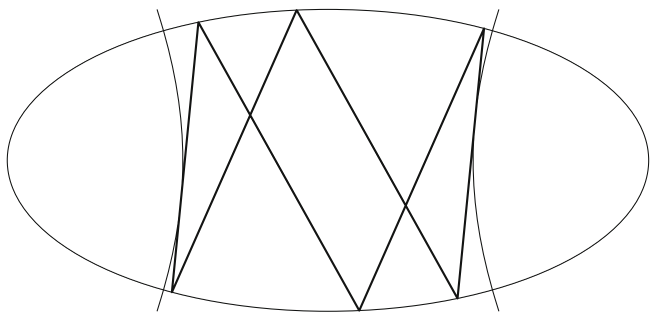

Fig. 7. A 6-periodic trajectory with a hyperbola as a caustic. The parameters are $a = 4.5$, $b = 1$, $\lambda_0 = ab/(\sqrt{a} - \sqrt{b})^2$; $(m_0, m_1) = (6, 2)$. The trajectory is symmetric with respect to the origin.

Remark 4. Notice that the winding numbers of 6-periodic trajectories obtained from the condition $B_2 = 0$ are $(m_0, m_1) = (6, 2)$. This is because $B_2 = 0$ is equivalent to $3P_0 \sim P_\infty$ on the elliptic curve. On the other hand, the trajectories obtained are 3-periodic in elliptic coordinates, and they transverse the y-axis only once along the period, so the condition of their periodicity is equivalent to

$$3(P_0 - P_b) + (P_{\lambda_0} - P_a) \sim 0, \qquad \text{for} \quad \lambda_0 \in (b, a),$$
$$3(P_0 - P_{\lambda_0}) + (P_b - P_a) \sim 0, \qquad \text{for} \quad \lambda_0 \in (0, b).$$

Both of the relations obtained are equivalent to $3P_0 \sim 3P_\infty$, since $2P_\infty \sim 2P_a \sim 2P_b \sim 2P_{\lambda_0}$ and $3P_\infty \sim P_a + P_b + P_{\lambda_0}$.

Next, $C_2 = 0$ gives 3-periodic trajectories.

The discriminant of $(a-b)(a+3b)\lambda_0^2 - 2ab(a-b)\lambda_0 + a^2b^2$ is $-16a^2b^3(a-b)$, which is negative, so the expression has no real roots in λ_0.

On the other hand, $(a-b)(3a+b)\lambda_0^2 - 2ab(a-b)\lambda_0 - a^2b^2$ has two real roots:

$$\lambda_0 = \frac{ab(a-b) \pm 2ab\sqrt{a(a-b)}}{(a-b)(3a+b)}.$$

The smaller one is always negative, so it does not correspond to a real billiard trajectory.

Let us analyze the bigger one:

$$\lambda_0 = \frac{ab(a-b) + 2ab\sqrt{a(a-b)}}{(a-b)(3a+b)}.$$

We have that always $\lambda_0 > b$, since

$$\frac{ab(a-b) + 2ab\sqrt{a(a-b)}}{(a-b)(3a+b)} > b \qquad \Leftrightarrow \qquad b^2(3a+b) > 0.$$

We need to check the condition $\lambda_0 < a$, which is equivalent to

$$\frac{ab(a-b) + 2ab\sqrt{a(a-b)}}{(a-b)(3a+b)} < a \qquad \Leftrightarrow \qquad a > \frac{4}{3}b.$$

Thus, for $a > 4b/3$, there is another class of 6-periodic trajectories which have a hyperbola as a caustic, see Fig. 8.

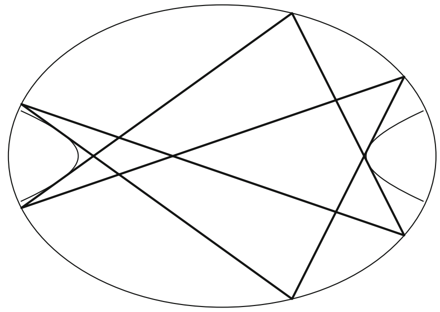

Fig. 8. A 6-periodic trajectory with a hyperbola as a caustic. The parameters are $a = 2$, $b = 1$, $\lambda_0 = \big(ab(a-b) + 2ab\sqrt{a(a-b)}\big)/\big((a-b)(3a+b)\big)$; $(m_0, m_1) = (6, 4)$. The trajectory is symmetric with respect to the longer axis of the ellipse.

Remark 5. The winding numbers of those 6-periodic trajectories are $(m_0, m_1) = (6, 4)$. The trajectories obtained are 3-periodic in elliptic coordinates, and they transverse the y-axis twice along the period, so the condition of their periodicity is equivalent to

$$3(P_0 - P_b) + 2(P_{\lambda_0} - P_a) \sim 0,$$

which is equivalent to $3P_0 \sim 3P_b$.

5. PERIODIC BILLIARD TRAJECTORIES AND CLASSICAL EXTREMAL POLYNOMIALS

We want to employ the classical theory of extremal polynomials on two intervals to get explicit formulae for the polynomials $\hat{p}_n$. As is well known, the extremal polynomials don't exist on any union of two disjoint real intervals. Thus, we want to exploit the relationship between the endpoints of such configurations of two intervals and to relate it to the formulae for the caustics which generate periodic billiard trajectories. To illustrate the main idea, we are going to start with families of polynomials introduced by Zolotarev in the 1870s [4, 5]. Later on, we will use two families of polynomials derived by Akhiezer — one in 1928 and the other, more general, in the 1930s to get the general formulae for our polynomials $\hat{p}_n$, [1–3].

Let us recall that the celebrated Chebyshev polynomials $T_n(x), n = 0, 1, 2, \dots$ defined by the recursion

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x), \tag{5.1}$$

for $n = 1, 2 \dots$ can be parameterized as

$$T_n(x) = \cos n\phi,\ x = \cos\phi, \tag{5.2}$$

or, alternatively,

$$T_n(x) = \frac{1}{2}\left(v^n + \frac{1}{v^n}\right), \quad x = \frac{1}{2}\left(v + \frac{1}{v}\right). \tag{5.3}$$

Denote $L_0 = 1$ and $L_n = 2^{1-n}, n = 1, 2, \dots$. Then the Chebyshev theorem states that the polynomials $L_nT_n(x)$ are characterized as the solutions of the following minmax problem:

find a polynomial of degree n with the leading coefficient equal to 1 which minimizes the uniform norm on the interval $[-1, 1]$.

5.1. Zolotarev Polynomials

Following the ideas of Chebyshev, his student Zolotarev posed and solved a handful of problems, including the following:

For a given real parameter σ and all polynomials of degree n of the form

$$p(x) = x^n - n\sigma x^{n-1} + p_2x^{n-2} + \dots p_n, \tag{5.4}$$

find the one with the minimal uniform norm on the interval $[-1, 1]$.

Denote this minimal uniform norm as $L_n = L(\sigma, n)$.

For $\sigma > \tan^2(\Pi/2n)$, the solution z_n has the following property ([4], p. 298), see Fig. 9:

Π1 — The equation $z_n^2(x) = L_n^2$ has $n-2$ double solutions in the open interval $(-1,1)$ and simple solutions at -1, 1, α, β, where $1 < \alpha < \beta$, while in the union of the intervals $[-1,1]\cup[\alpha,\beta]$ the inequality $z_n^2 \leqslant L_n^2$ is satisfied and $z_n^2 > L_n^2$ in the complement.

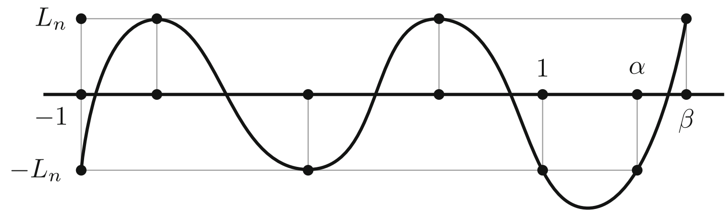

Fig. 9. The graph of $z_n(x)$.

The polynomials z_n are given by the following explicit formulae:

$$z_n = \ell_n\left(v(u)^n + \frac{1}{v(u)^n}\right), \quad x = \frac{\operatorname{sn}^2 u + \operatorname{sn}^2 \frac{K}{n}}{\operatorname{sn}^2 u - \operatorname{sn}^2 \frac{K}{n}}, \tag{5.5}$$

where

$$\ell_n = \frac{1}{2^n}\left(\frac{\sqrt{\kappa}\theta_1^2(0)}{H_1\left(\frac{K}{n}\right)\theta_1\left(\frac{K}{n}\right)}\right)^{2n}, \quad v(u) = \frac{H\left(\frac{K}{n}-u\right)}{H\left(\frac{K}{n}+u\right)}$$

and

$$\sigma = \frac{2\operatorname{sn}\frac{K}{n}}{\operatorname{cn}\frac{K}{n}\operatorname{dn}\frac{K}{n}}\left(\frac{1}{\operatorname{sn}\frac{2K}{n}} - \frac{\theta'\left(\frac{K}{n}\right)}{\theta\left(\frac{K}{n}\right)}\right) - 1.$$

Formulae for the endpoints of the second interval are

$$\alpha = \frac{1+\kappa^2\operatorname{sn}^2\frac{K}{n}}{\operatorname{dn}^2\frac{K}{n}}, \quad \beta = \frac{1+\operatorname{sn}^2\frac{K}{n}}{\operatorname{cn}^2\frac{K}{n}}, \tag{5.6}$$

with

$$\kappa^2 = \frac{(\alpha-1)(\beta+1)}{(\alpha+1)(\beta-1)}.$$

In order to derive the formulae for $\hat{p}_3$ in terms of z_3, let us construct an affine transformation

$$h : [-1,1]\cup[\alpha,\beta] \to [0,a^{-1}]\cup[b^{-1},\lambda_0^{-1}],\ h(x) = \hat{a}x + \hat{b}.$$

According to Cayley's condition (4.2)

$$\lambda_0 = -\frac{ab(a+b) + 2ab\sqrt{a^2-ab+b^2}}{(a-b)^2}.$$

We immediately get

$$\hat{a} = \hat{b},\ \hat{a} = \frac{1}{2a}$$

and

$$\alpha = 2t-1, \tag{5.7}$$

$$\beta = -\frac{2t^2-3t+3+2\sqrt{t^2-t+1}}{t+1+2\sqrt{t^2-t+1}}, \tag{5.8}$$

$$\beta = \frac{4}{3}\sqrt{t^2-t+1} + \frac{2}{3}t + \frac{5}{3}, \tag{5.9}$$

where $t = a/b$.

There is a relation between α and β as defined by the formulae above:

$$9\beta^2 - 3\alpha^2 - 6\alpha\beta + 12\alpha - 36\beta + 56 = 0. \tag{5.10}$$

Now we get the following

Proposition 1. *The polynomial $\hat{p}_3$ can be expressed in terms of the Zolotarev polynomial z_3 up to an inessential constant factor:*

$$\hat{p}_3(s) \sim z_3(2as - 1).$$

To verify the proposition, we should certify that α and β defined in (5.6) for $n = 3$ satisfy relation (5.7).

In order to do that we will use well-known identities for the Jacobi elliptic functions:

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u + 1, \tag{5.11}$$

$$\kappa^2 \operatorname{sn}^2 u + \operatorname{dn}^2 u = 1, \tag{5.12}$$

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - \kappa^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{5.13}$$

$$\operatorname{sn}(K - u) = \frac{\operatorname{cn} u}{\operatorname{dn} u}. \tag{5.14}$$

In particular, we get

$$\operatorname{sn}\left(\frac{2K}{3}\right) = \frac{2 \operatorname{sn} \frac{K}{3} \operatorname{cn} \frac{K}{3} \operatorname{dn} \frac{K}{3}}{1 - \kappa^2 \operatorname{sn}^4 \frac{K}{3}}, \tag{5.15}$$

$$\operatorname{sn}\left(\frac{2K}{3}\right) = \operatorname{sn}\left(K - \frac{K}{3}\right) = \frac{\operatorname{cn} \frac{K}{3}}{\operatorname{dn} \frac{K}{3}}. \tag{5.16}$$

Let us denote

$$Y = \operatorname{sn}\left(\frac{K}{3}\right),$$

then from the previous two relations we get

$$1 - 2Y + 2\kappa^2 Y^3 - \kappa^2 Y^4 = 0.$$

We can express κ in terms of Y and get

$$\kappa^2 = \frac{2Y - 1}{Y^3(2 - Y)}.$$

By plugging the last relation into (5.6) for $n = 3$ we get

$$\alpha = \frac{Y^2 - 4Y + 1}{Y^2 - 1}.$$

Since, at the same time from the Cayley condition we have $\alpha = 2t - 1$, with $t = a/b$, we can express Y in terms of t:

$$(t - 1)Y^2 + 2Y - t = 0,$$

and

$$Y = \frac{-1 \pm \sqrt{1 - t + t^2}}{t - 1}. \tag{5.17}$$

We plug the last relation into the formula for β from (5.6) for $n = 3$ and we get a formula for β in terms of t:

$$\beta = \frac{(t-1)^2 + (-1 \pm \sqrt{t^2 - t + 1})^2}{(t-1)^2 - (-1 \pm \sqrt{t^2 - t + 1})^2}, \tag{5.18}$$

$$\beta = -\frac{2t^2 - 3t + 3 \pm 2\sqrt{t^2 - t + 1}}{t + 1 \pm 2\sqrt{t^2 - t + 1}}. \tag{5.19}$$

We see that the last formula with the choice of the + sign corresponds to a formula for β from (5.7). This finalizes the verification. One can observe that the $-$ sign option from the formula above would correspond to the $-$ sign in the formula for λ_0 above the formula (4.2).

Among the polynomials $\hat{p}_n$ the property of type Π1 can be attributed only to those with $n = 2k+1$ and winding numbers $(2k+1, 2k)$, in other words, to those with the signature $(0, 2k-1)$. For example, this is satisfied for the polynomial $\hat{p}_5$ presented in Fig. 6, while it is not true for the polynomial $\hat{p}_5$ presented in Fig. 5.

5.2. Akhiezer Polynomials on Symmetric Intervals $[-1, -\alpha] \cup [\alpha, 1]$

The problem of finding polynomials of degree n with the leading coefficient 1 and minimizing the uniform norm on the union of two symmetric intervals $[-1, -\alpha] \cup [\alpha, 1]$, for given $0 < \alpha < 1$ appeared to be of significant interest in radio engineering applications. Following the ideas of Chebyshev and Zolotarev, Akhiezer derived in 1928 explicit formulae for such polynomials $A_n(x; \alpha)$ with the deviation $L_n(\alpha)$, [4, 5].

These formulae are especially simple in the case of even degrees $n = 2m$, when Akhiezer polynomials A_{2m} are obtained by a quadratic substitution from the Chebyshev polynomial T_m:

$$A_{2m}(x;\alpha) = \frac{(1-\alpha^2)^m}{2^{2m-1}} T_m\left(\frac{2x^2 - 1 - \alpha^2}{1-\alpha^2}\right), \tag{5.20}$$

with

$$L_{2m}(\alpha) = \frac{(1-\alpha^2)^m}{2^{2m-1}}.$$

We are going to construct $\hat{p}_4(s)$ up to an inessential constant factor as a composition of $A_4(x; \alpha)$ for a certain α and an affine transformation. We are going to study the possibility of having an affine transformation

$$g : [-1, -\alpha] \cup [\alpha, 1] \to [0, c_3] \cup [c_2, c_1], \quad g(x) = \hat{a}x + \hat{b}$$

in two versions, depending on whether the caustic corresponds to the reciprocical value of c_1 or of c_2. The former case corresponds to the case of the caustic being an ellipse, and the latter, to the case of it being a hyperbola. Thus, we will denote these two cases by (E) and (H), respectively.

Case (E).

For

$$g : [-1, -\alpha] \cup [\alpha, 1] \to [0, a^{-1}] \cup [b^{-1}, \lambda^{-1}], \quad g(x) = \hat{a}x + \hat{b}$$

we get

$$\hat{a} = \hat{b}, \quad \alpha = \frac{a-b}{a+b}, \quad \hat{a} = \frac{a+b}{2ab}.$$

Thus,

$$g(1) = 2\hat{a} = \frac{a+b}{ab} = \frac{1}{\lambda}$$

implies

$$\lambda = \frac{ab}{a+b},$$

which coincides with the formula for the elliptical caustic for 4 periodic billiard trajectories derived from the Cayley condition.

Proposition 2. *In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to*

$$\hat{p}_4(s) \sim \frac{2a^2b^2}{(a+b)^4} T_2\big(2abs^2 - 2(a+b)s + 1\big), \tag{5.21}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial.

Case (H).

For

$$g : [-1, -\alpha] \cup [\alpha, 1] \to [0, a^{-1}] \cup [\lambda^{-1}, b^{-1}], \quad g(x) = \hat{a}x + \hat{b}$$

we get

$$\hat{a} = \hat{b}, \quad \alpha = \frac{a - 2b}{a}, \quad \hat{a} = \frac{1}{2b}.$$

Thus,

$$g(\alpha) = \hat{a}(1 + \alpha) = \frac{a - b}{ab} = \frac{1}{\lambda}$$

implies

$$\lambda = \frac{ab}{a - b},$$

which coincides with the formula for the hyperbolic caustic for 4 periodic billiard trajectories derived from the Cayley condition.

Proposition 3. *In this case the polynomial $\hat{p}_4(s)$ is equal up to a constant multiplier to*

$$\hat{p}_4(s) \sim \frac{2b^2(a - b)^2}{a^4} T_2 \left(\frac{a^2s^2 - 4a^2bs + 8b^3(a - b)}{8b^3(a - b)} \right), \tag{5.22}$$

where $T_2(x) = 2x^2 - 1$ is the second Chebyshev polynomial.

The polynomials $\hat{p}_4$ from (5.21) and (5.22) can be rewritten in canonical form, respectively, as

$$\hat{p}_4(s) \sim 8a^2b^2s^4 - 16ab(a + b)s^3 + 8(a^2 + 3ab + b^2)s^2 - 8(a + b)s + 1$$

and

$$\hat{p}_4(s) \sim a^4s^4 - 8a^4bs^3 + 16a^2b^2(a^2 + ab - b^2)s^2 + 64a^2b^4(b - a)s + 32b^9(b - 2a).$$

By analyzing the structure of the extremal points of the polynomials A_n, we generalize the last two propositions as follows:

Proposition 4. *For trajectories with period $4k$ and winding numbers $(4k, 2k)$, the corresponding polynomials $\hat{p}_{4k}$ are equal up to an inessential constant factor to*

$$\hat{p}_{4k}(s) \sim T_{2k}\big(2abs^2 - 2(a + b)s + 1\big),$$

for the elliptical caustic, and to

$$\hat{p}_{4k}(s) \sim T_{2k} \left(\frac{a^2s^2 - 4a^2bs + 8b^3(a - b)}{8b^3(a - b)} \right),$$

in the case of the caustic being a hyperbola, where T_{2k} is the $2k$th Chebyshev polynomial.

Since the polynomials A_n have symmetrically distributed extremal points in the intervals $[-1, -\alpha]$ and $[\alpha, 1]$, they can't serve as models for polynomials $\hat{p}_k$ except in the cases listed in the above proposition.

5.3. General Akhiezer Polynomials on Unions of Two Intervals

So far, we have managed to express polynomials $\hat{p}_3, \hat{p}_4$, $\hat{p}_5$ only in the case of the signature $(0, 3)$, $\hat{p}_{2n+1}$ with the signature $(0, 2n-1)$ and $\hat{p}_{4k}$ only in the case of the signature $(2k-1, 2k-1)$, by use of polynomials of Zolotarev and Akhiezer. However, we were unable to get, for example, the polynomial $\hat{p}_5$ with the signature $(2, 1)$, as well as polynomials $\hat{p}_{4k+2}$ and $\hat{p}_{4k}, \hat{p}_{2k+1}$ with signatures different from those listed above. Thus, in order to get the general formulae for the general polynomials $\hat{p}_n$, we need to employ a more general theory of extremal polynomials on two intervals, developed by Akhiezer [1–3].

Following Akhiezer, let us consider the union of two intervals $[-1, \alpha] \cup [\beta, 1]$, where

$$\alpha = 1 - 2\,\mathrm{sn}^2 \frac{m}{n} K, \quad \beta = 2\,\mathrm{sn}^2 \frac{n-m}{n} K - 1.$$

Define

$$TA_n(x, m, k) = L\left(v(u)^n + \frac{1}{v(u)^n}\right), \tag{5.23}$$

where

$$v(u) = \frac{H\left(u - \frac{mK}{n}\right)}{H\left(\frac{mK}{n} + u\right)},$$

$$x = \frac{\mathrm{sn}^2 u \quad \mathrm{cn}^2 \frac{m}{n} K + \mathrm{cn}^2 u \quad \mathrm{sn}^2 \frac{m}{n} K}{\mathrm{sn}^2 u - \mathrm{sn}^2 \frac{m}{n} K},$$

and

$$L = \frac{1}{2^{n-1}} \left(\frac{\theta(0)\theta_1(0)}{\theta(\frac{m}{n}K)\theta_1(\frac{m}{n}K)}\right), \quad k^2 = \frac{2(\beta - \alpha)}{(1-\alpha)(1-\beta)}.$$

Akhiezer proved the following results:

Theorem 12 (Akhiezer). (*a*) *The function $TA_n(x, m, k)$ is a polynomial of degree n in x with the leading coefficient 1 and the second coefficient equal to $-n\tau_1$, where*

$$\tau_1 = -1 + 2\frac{\mathrm{sn}\,\frac{m}{n}K \quad \mathrm{cn}\,\frac{m}{n}K}{\mathrm{dn}\,\frac{m}{n}K}\left(\frac{1}{\mathrm{sn}\,\frac{2m}{n}K} - \frac{\theta'(\frac{m}{n}K)}{\theta(\frac{m}{n}K)}\right).$$

(*b*) *The maximum of the modulus of T_n on the union of the two intervals $[-1, \alpha] \cup [\beta, 1]$ is L.*

(*c*) *The function T_n takes the values $\pm L$ with alternating signs at $\mu = n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $\nu = m + 1$ consecutive points of the interval $[\beta, 1]$. In addition,*

$$T_n(\alpha, m, k) = T_n(\beta, m, k) = (-1)^m L,$$

and for any $x \in (\alpha, \beta)$ the following inequality holds:

$$(-1)^m T_n(x, m, k) > L.$$

(*d*) *Let F be a polynomial of degree n in x with the leading coefficient equal to 1, such that:*

i) $\max |F(x)| = L$ *for* $x \in [-1, \alpha] \cup [\beta, 1]$*;*

ii) *$F(x)$ takes values $\pm L$ with alternating signs at $n - m + 1$ consecutive points of the interval $[-1, \alpha]$ and at $m + 1$ consecutive points of the interval $[\beta, 1]$.*

Then $F(x) = T_n(x, m, k)$.

Let us observe that the polynomials $\hat{p}_n(s)$ satisfy the conditions of item (d) of the Akhiezer theorem, up to an affine change of variables and up to an inessential constant factor. Indeed, let

$$\hat{p}_n(s, 2l)$$

denote a polynomial $\hat{p}_n$ such that it corresponds to the winding numbers $(n, 2l), 2l < n$. Then the corresponding signature is $(n-2l-1, 2l-1)$. The number of alternating points on the interval $[0, c_3]$ of the polynomial $\hat{p}_n(s, 2l)$ is equal to $2l+1$, while the number of its alternating points on the interval $[c_2, c_1]$ is equal to $n-2l+1$. Thus, m from the Akhiezer theorem is

$$m = n - 2l.$$

Now let us determine the affine transformations in two cases: (E) when the caustic is an ellipse and (H) when the caustic is a hyperbola.

Case (E).

For

$$h : [-1, \alpha] \cup [\beta, 1] \to [0, a^{-1}] \cup [b^{-1}, \lambda^{-1}], \quad h(x) = \hat{a}x + \hat{b}$$

we get

$$\hat{a} = \hat{b}, \quad \hat{a} = \frac{1}{a(\alpha+1)}, \quad \frac{\beta+1}{\alpha+1} = \frac{a}{b}.$$

Thus,

$$\lambda = \frac{a(\alpha+1)}{2} = \frac{b(\beta+1)}{2}.$$

We have proved the following theorem.

Theorem 13. *The polynomials $\hat{p}_n$ can be expressed up to an inessential multiplier as a composition of a TA_n polynomial and an affine transformation:*

$$\hat{p}_n(s, 2l) \sim TA_n(a(\alpha+1)s - 1; n-2l, k). \tag{5.24}$$

Case (H).

For n even there is one more option, with the caustic being a hyperbola.

For

$$h_1 : [-1, \alpha] \cup [\beta, 1] \to [0, a^{-1}] \cup [\lambda^{-1}, b^{-1}], \quad h_1(x) = \hat{a}x + \hat{b}$$

we get

$$\hat{a} = \hat{b}, \quad \hat{a} = \frac{1}{2b}, \quad \alpha = \frac{2b}{a} - 1.$$

Thus,

$$\lambda = \frac{2b}{(\beta+1)}.$$

Theorem 14. *The polynomials $\hat{p}_n$ can be expressed up to an inessential multiplier as a composition of a TA_n polynomial and an affine transformation:*

$$\hat{p}_n(s, 2l) \sim TA_n(2bs - 1; n-2l, k). \tag{5.25}$$

The relation between α and β, given by

$$\alpha = 1 - 2\,\mathrm{sn}^2\frac{m}{n}K, \quad \beta = 2\,\mathrm{sn}^2\frac{n-m}{n}K - 1,$$

can be seen more clearly if we introduce

$$Z = \mathrm{sn}\frac{mK}{n}.$$

We have then

$$\operatorname{sn}^2\left(\frac{n-m}{n}K\right) = \frac{1-Z^2}{1-\kappa^2 Z^2}$$

and

$$\alpha = 1 - 2Z^2, \quad \beta = \frac{1+(\kappa^2-2)Z^2}{1-\kappa^2 Z^2}.$$

Example 1. Let us illustrate the last two theorems for $n = 4$. Then $l = 1, \quad m = n - 2l = 2$. It is well known that

$$\operatorname{sn}^2 \frac{K}{2} = \frac{1}{1+\sqrt{1-\kappa^2}}.$$

We have

$$\alpha = -\beta = 1 - 2\operatorname{sn}^2 \frac{K}{2}.$$

Case (E).

Plugging $\beta = -\alpha$ into

$$\frac{\beta+1}{\alpha+1} = \frac{a}{b} = t,$$

we get

$$\alpha = \frac{1-t}{1+t}.$$

From

$$\lambda = \frac{a(\alpha+1)}{2} = \frac{a}{1+t},$$

we get

$$\lambda = \frac{ab}{a+b},$$

which coincides with one of the values obtained from the Cayley-type condition in (4.4). From the relation

$$\alpha = 1 - 2\operatorname{sn}^2 \frac{K}{2} = \frac{1-t}{1+t},$$

we get a relation between t, the ratio of the squares of the semiaxes of the ellipse and the elliptic modulus κ:

$$t = \frac{1}{\sqrt{1-\kappa^2}}.$$

Finally, we get

$$\hat{p}_4(s,2) \sim TA_4\left(2\frac{ab}{a+b}s - 1; 2, \sqrt{\frac{a^2-b^2}{a^2}}\right).$$

Case (H).

From

$$\beta = -\alpha = 1 - \frac{2}{t}$$

we get

$$\lambda = \frac{bt}{t-1},$$

which gives

$$\lambda = \frac{ab}{a-b},$$

which is again one of the values obtained from the Cayley-type condition in (4.4).

From the relation

$$\alpha = 1 - 2\,\mathrm{sn}^2\,\frac{K}{2} = \frac{2}{t} - 1,$$

we get a relation between t, the ratio of the squares of the semiaxes of the ellipse and the elliptic modulus κ:

$$t = \frac{1+\sqrt{1-\kappa^2}}{\sqrt{1-\kappa^2}}.$$

Finally, we get

$$\hat{p}_4(s,2) \sim TA_4\left(2bs-1; 2, \sqrt{\frac{a^2-2ab}{(a-b)^2}}\right).$$

Example 2. Let us consider now the case $n = 5$. Let us denote

$$\mathrm{sn}\quad\frac{K}{5} = Y.$$

Then, from the addition formulae we get

$$\mathrm{sn}^2\quad\frac{2K}{5} = \frac{4Y^2(1-Y^2)(1-\kappa^2Y^2)}{1-\kappa^2Y^4}.$$

Similarly, we get

$$\mathrm{sn}^2\,\frac{4K}{5} = \frac{16(1-\kappa^2Y^4)^8Y^2(1-Y^2)(1-\kappa^2Y^2)(1-\mathrm{sn}^2\quad\frac{2K}{5})(1-\kappa^2\,\mathrm{sn}^2\quad\frac{2K}{5})}{(1-\kappa^2Y^4)^2\big((1-\kappa^2Y^4)^4-16Y^4(1-Y^2)^2(1-\kappa^2Y^2)^2\big)^2}.$$

From the last relation and

$$\mathrm{sn}^2\,\frac{4K}{5} = \frac{1-Y^2}{1-\kappa^2Y^2}$$

we get

$$0 = P(Z,s) = \sum_{p,q} P_{p,q}Z^ps^q = \sum_{p=0}^{16} E_p(s)Z^p = \sum_{q=0}^{8} F_q(Z)s^q,$$

where

$$Z = Y^2, \quad s = \kappa^2$$

and

$$\begin{aligned}
F_0 &= -1 + 16Z - 64Z^2 + 64Z^3,\\
F_1 &= -56Z^2 + 352Z^3 - 416Z^4,\\
F_2 &= 144Z^3 - 1244Z^4 + 2160Z^5 - 1280Z^6 + 896Z^7 - 256Z^8,\\
F_3 &= -160Z^4 + 2144Z^5 - 4744Z^6 + 4160Z^7 - 3264Z^8 + 1024Z^9,\\
F_4 &= 64Z^5 - 1984Z^6 + 5360Z^7 - 5830Z^8 + 5360Z^9 - 1984Z^{10} + 64Z^{11},\\
F_5 &= -160Z^{12} + 2144Z^{11} - 4744Z^{10} + 4160Z^9 - 3264Z^8 + 1024Z^7,\\
F_6 &= 144Z^{13} - 1244Z^{12} + 2160Z^{11} - 1280Z^{10} + 896Z^9 - 256Z^8,\\
F_7 &= -56Z^{14} + 352Z^{13} - 416Z^{12},\\
F_8 &= -Z^{16} + 16Z^{15} - 64Z^{14} + 64Z^{13}.
\end{aligned}$$

Observe the symmetry:

$$P_{p,q} = P_{16-p,8-q}. \tag{5.26}$$

Case $l = 1$. From the Akhiezer formulae we get

$$\alpha = 1 - 2\,\mathrm{sn}^2 \frac{3}{5}K, \quad \beta = 2\,\mathrm{sn}^2 \frac{2}{5}K - 1,$$

and

$$\frac{a}{b} = t = \frac{\mathrm{sn}^2 \frac{2}{5}K}{1 - \mathrm{sn}^2 \frac{3}{5}K}. \tag{5.27}$$

Thus,

$$\lambda_0 = a\left(1 - \mathrm{sn}^2 \frac{3}{5}K\right),$$

and

$$\hat{p}_5(s, 2) \sim TA_4\left(2a\left(1 - \mathrm{sn}^2 \frac{3}{5}K\right)s - 1; 3, \kappa\right).$$

From Eq. (5.27) and one of the additional formulae:

$$\mathrm{sn}^2 \frac{3}{5}K = \frac{1 - \mathrm{sn}^2 \frac{2}{5}K}{1 - \kappa^2\,\mathrm{sn}^2 \frac{2}{5}K}$$

we get

$$\kappa^2 = \frac{t-1}{Wt},$$

where $W = \mathrm{sn}^2 \frac{3}{5}K$.

Case $l = 2$.

Similarly, from

$$\alpha = 1 - 2\,\mathrm{sn}^2 \frac{1}{5}K, \quad \beta = 2\,\mathrm{sn}^2 \frac{4}{5}K - 1$$

and

$$\frac{a}{b} = t = \frac{\mathrm{sn}^2 \frac{4}{5}K}{1 - \mathrm{sn}^2 \frac{1}{5}K}$$

we get

$$\lambda_0 = a\left(1 - \mathrm{sn}^2 \frac{1}{5}K\right)$$

and

$$\hat{p}_5(s, 4) \sim TA_4\left(2a\left(1 - \mathrm{sn}^2 \frac{1}{5}K\right)s - 1; 1, \kappa\right).$$

Here

$$\kappa^2 = \frac{t-1}{W_1},$$

where $W_1 = \mathrm{sn}^2 \frac{4}{5}K$.

6. DISCRIMINANTLY SEPARABLE AND DISCRIMINANTLY FACTORIZABLE POLYNOMIALS

6.1. Definition of Discriminantly Separable Polynomials

We briefly review the basic notions, indicate several relationships and applications to different areas of mathematics and mechanics, and provide a general definition of the discriminantly separable polynomials. By $\mathcal{P}_m^n$ we denote the polynomials of m variables of degree n in each variable.

Definition 2 ([12]). A polynomial $F(x_1, \ldots, x_n)$ is *discriminantly separable* if there exist polynomials $f_1(x_1)$, …, $f_n(x_n)$ such that the discriminant $\mathcal{D}_{x_i}F$ of F with respect to x_i satisfies

$$\mathcal{D}_{x_i}F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j(x_j)$$

for each $i = 1, \ldots, n$. F is *symmetrically discriminantly separable* if $f_2 = f_3 = \ldots = f_n$, while it is *strongly discriminantly separable* if $f_1 = f_2 = f_3 = \ldots = f_n$. It is *weakly discriminantly separable* if there exist polynomials $f_i^j(x_i)$ such that for every $i = 1, \ldots, n$

$$\mathcal{D}_{x_i}F(x_1, \ldots, \hat{x}_i, \ldots, x_n) = \prod_{j \neq i} f_j^i(x_j).$$

6.2. n-valued Groups

The idea of n-valued groups, on a local level, goes back to Buchstaber and Novikov, to their 1971 study of characteristic classes of vector bundles. That concept was significantly developed further by Buchstaber and his collaborators ([8] and references therein). An n-valued group on X can be defined as a map:

$$\begin{aligned} m : &\quad X \times X \to (X)^n \\ m(x, y) &= x * y = [z_1, \ldots, z_n], \end{aligned}$$

where $(X)^n$ denotes the symmetric nth power of X and z_i coordinates therein. Such a map should satisfy the following axioms.

Associativity: The condition of equality of two n^2-sets is

$$\begin{gathered} [x * (y * z)_1, \ldots, x * (y * z)_n] \\ [(x * y)_1 * z, \ldots, (x * y)_n * z] \end{gathered}$$

for all triplets $(x, y, z) \in X^3$.

Unit element: An element $e \in X$ is *a unit* if $e * x = x * e = [x, \ldots, x]$ for all $x \in X$.

Inverse: A map $\mathrm{inv} : X \to X$ is *an inverse* if $e \in \mathrm{inv}(x) * x$ and $e \in x * \mathrm{inv}(x)$ for all $x \in X$.

Buchstaber says that m defines *an n-valued group structure* (X, m, e, inv) if it is associative, with a unit and an inverse.

An n-valued group X acts on a set Y if there is a mapping

$$\phi \ : \ X \times Y \to (Y)^n, \quad \phi(x, y) = x \circ y,$$

such that the two n^2-multisubsets $x_1 \circ (x_2 \circ y)$ and $(x_1 * x_2) \circ y$ of Y are equal for all $x_1, x_2 \in X$, $y \in Y$. It is also assumed $e \circ y = [y, \ldots, y]$ for all $y \in Y$.

A list of elementary n-valued groups has been done in [8]. For a fixed n, the corresponding n-valued group is defined by a symmetric polynomial $p_n \in \mathcal{P}_3^n$.

We recall that the elementary symmetric functions of three variables are denoted as s_1, s_2, s_3:

$$s_1 = x + y + z, \quad s_2 = xy + xz + yz, \quad s_3 = xyz.$$

Let us consider a few simple examples.

Example 3. Two-valued group p_2 is defined by the relation

$$m_2: \quad \mathbf{C} \times \mathbf{C} \to (\mathbf{C})^2,$$
$$x *_2 y = [(\sqrt{x} + \sqrt{y})^2, (\sqrt{x} - \sqrt{y})^2].$$

The product $x *_2 y$ is given by the solutions of the polynomial equation $p_2(z, x, y) = 0$ in z, where

$$p_2(z, x, y) = (x + y + z)^2 - 4(xy + yz + zx).$$

As observed in [12], the polynomial $p_2(z, x, y)$ is strongly discriminantly separable:

$$\mathcal{D}_z(p_2)(x, y) = 2x \cdot 2y, \quad \mathcal{D}_x(p_2)(y, z) = 2y \cdot 2z, \quad \mathcal{D}_y(p_2)(x, z) = 2x \cdot 2z,$$

so it generates a case of the generalized Kowalevski system of differential equations from [12].

Now we can reproduce a small mathematical experiment from [13] with the next cases of elementary n valued groups, with small n.

Example 4 ($n = 3$). $p_3 = s_1^3 - 3^3 s_3$, $\mathcal{D}_z p_3 = y^2 x^2 (x - y)^2$.

Example 5 ($n = 4$).

$$p_4 = s_1^4 - 2^3 s_1^2 s_2 + 2^4 s_2^2 - 2^7 s_1 s_3, \quad \mathcal{D}_z p_4 = y^3 x^3 (x - y)^2 (y + 4x)^2 (4y + x)^2.$$

Example 6 ($n = 5$).

$$p_5 = s_1^5 - 5^4 s_1^2 s_3 + 5^5 s_2 s_3, \quad \mathcal{D}_z p_5 = y^4 x^4 (x - y)^4 (x^2 - y^2 - 11xy)^2 (x^2 - y^2 + 11xy)^2.$$

We see that the polynomials p_3, p_4, p_5 are not any more discriminantly separable. Nevertheless, following [13], we can observe an amazing factorizability property of their discriminants. Unexpectedly, the Hadamard−Hankel determinants which appeared in the study of the Cayley-type conditions in previous sections have the same algebraic properties.

6.3. Cayley-type Conditions and Discriminantly Factorizible Polynomials

Example 7. Denote the numerator in the expression (4.1) as $F_2(\lambda_0, a, b)$:

$$F_2(\lambda_0, a, b) = \lambda_0^2 (a - b)^2 + 2\lambda_0 ab(a + b) - 3a^2 b^2.$$

F_2 is a discriminantly factorizable polynomial. Its discriminant with respect to λ_0 is

$$\mathcal{D}_{\lambda_0} F_2 = 16a^2 b^2 (a^2 - ab + b^2).$$

Example 8. Denote the numerator in (4.3) as $F_3(\lambda_0, a, b)$:

$$\begin{aligned} F_3(\lambda_0, a, b) &= (a\lambda_0 - ab + \lambda_0 b)(a\lambda_0 + ab - \lambda_0 b)(-a\lambda_0 + ab + \lambda_0 b) \\ &= -(a - b)^2 (a + b)\lambda_0^3 + ab(a - b)^2 \lambda_0^2 + a^2 b^2 (a + b)\lambda_0 - a^3 b^3. \end{aligned}$$

Its discriminant with respect to λ_0 is

$$\mathcal{D}_{\lambda_0} F_3 = 64a^8 b^8 (a - b)^2.$$

Example 9. The discriminant with respect to λ_0 of the polynomial numerator of $B_4^2 - B_3 B_5$ equals

$$-3094850098213450687247810560 \cdot a^{74} b^{74} (a - b)^{18} \left(a^2 - ab + b^2\right).$$

Example 10. The discriminant of the numerator of $C_3^2 - C_2 C_4$ with respect to λ_0 is

$$-87960930222080 \cdot a^{38} b^{38} (a - b)^8 \left(27a^6 - 81a^5 b + 322a^4 b^2 - 509a^3 b^3 + 322a^2 b^4 - 81ab^5 + 27b^6\right).$$

Example 11. Denote the numerator in the expression (4.5) as $F_2(\lambda_0, a, b)$:

$$F_2(\lambda_0, a, b) = \lambda_0^2 (a - b)^2 - 2\lambda_0 ab(a + b) + a^2 b^2.$$

F_2 is a strongly discriminantly separable polynomial. Its discriminant with respect to λ_0 is

$$\mathcal{D}_{\lambda_0} F_2 = 16a^3 b^3.$$

Remark 6. We observe that in Examples 3 and 11 we are getting disciminantly separable polynomials. In the remaining examples, 4–10, the polynomials are not discriminantly separable, but discriminantly factorizable. However, it is important to note that the factors in all these latter examples are homogeneous. Thus, by a change of variables in the polynomials p_3, p_4, p_5 $(x, y) \mapsto (x, z)$, $z = x/y$ we are getting discriminantly separable polynomials in the new coordinates (x, z). Similarly, in Examples 7–10, the change of variables $(a, b) \mapsto (a, \hat{b})$, with $\hat{b} = a/b$, transforms the polynomials into discriminantly separable polynomials in new variables $(a, \hat{b})$. It would be very interesting to study further the observed relationship between n-valued groups and Cayley-type conditions with the discriminantly separable polynomials.

6.4. 2-valued Group on $\mathbf{CP}^1$*, Disciminantly Separable Polynomials and the Great Poncelet Theorem for Triangles*

It appears that the general equation of the pencil of conics corresponds to an action of a two-valued group. We used this correspondence to provide a novel interpretation of "the mysterious Kowalevski change of variables". It turned out that the associativity condition for this action is equivalent to the Great Poncelet Theorem for a triangle, as it was observed in [12]. We are going to close the loop in the paper with a brief reminder at the end about the Great Poncelet Theorem for triangles and this relationship to the associativity of the two-valued group, and thus, with the Kowalevski change of variables.

Consider the general pencil equation $\mathcal{F}(s, x_1, x_2) = 0$, with s being the parameter of the pencil, and x_1, x_2 the Darboux coordinates (see [12]). That pencil is related to two elliptic curves: $\tilde{\Gamma}_1 : y^2 = P(x)$ and $\tilde{\Gamma}_2 : t^2 = J(s)$, where the polynomials P, J are of degrees four and three, respectively. These two curves appear to be isomorphic. Rewrite the cubic one $\tilde{\Gamma}_2$ in the canonical form $\tilde{\Gamma}_2 : t^2 = J'(s) = 4s^3 - g_2 s - g_3$. Let $\psi : \tilde{\Gamma}_2 \to \tilde{\Gamma}_1$ be a birational morphism between the curves induced by a fractional-linear transformation $\hat{\psi}$ which maps the three zeros of J' and ∞ to the four zeros of the polynomial P.

The curve $\tilde{\Gamma}_2$ as a cubic has a group structure with the neutral element at infinity. With the subgroup $\mathbf{Z}_2$, it defines the standard two-valued group structure on $\mathbf{CP}^1$ (see [7]):

$$s_1 *_c s_2 = \left[-s_1 - s_2 + \left(\frac{t_1 - t_2}{2(s_1 - s_2)} \right)^2, -s_1 - s_2 + \left(\frac{t_1 + t_2}{2(s_1 - s_2)} \right)^2 \right], \tag{6.1}$$

where $t_i = J'(s_i)$, $i = 1, 2$.

Theorem 15 ([12]). *The general pencil equation after fractional-linear transformations*

$$\mathcal{F}\big(s, \hat{\psi}^{-1}(x_1), \hat{\psi}^{-1}(x_2)\big) = 0$$

induces the two-valued coset group structure $(\tilde{\Gamma}_2, \mathbf{Z}_2)$ *defined by* (6.1).

A proof is given in [12].

The geometric meaning of the pencil equation and the algebraic structure of the two-valued group give together a connection observed in [12] with the Great Poncelet Theorem ([34], see also [6, 19]). We recall the formulation of the Great Poncelet Theorem for triangles in the form we need below.

Theorem 16 (Great Poncelet Theorem for triangles [34]). *Let* $\mathscr{C}_1$*,* $\mathscr{C}_2$*,* $\mathscr{C}_3$*,* $\mathscr{C}$ *be conics from a pencil and* a_1*,* a_2*,* a_3 *tangent lines to* $\mathscr{C}$*, such that* a_1*,* a_2 *intersect on* $\mathscr{C}_1$*,* a_2*,* a_3 *intersect on* $\mathscr{C}_2$ *and* a_1*,* a_3 *intersect on* $\mathscr{C}_3$*. Moreover, we suppose that the tangents to* $\mathscr{C}_1$*,* $\mathscr{C}_2$*,* $\mathscr{C}_3$ *at the intersection points are not concurrent. Suppose that* b_1*,* b_2 *are tangents to* $\mathscr{C}$ *which intersect on* $\mathscr{C}_1$*. Then there exists* b_3*, a tangent to* $\mathscr{C}$ *such that the triplet* (b_1, b_2, b_3) *satisfies all conditions as* (a_1, a_2, a_3)*.*

The associativity condition for the action of the two-valued group $(\Gamma_2, \mathbf{Z}_2)$ is as follows.

Theorem 17 ([12]). *Associativity conditions for the group structure of the two-valued coset group* $(\Gamma_2, \mathbf{Z}_2)$ *and for its action on* $\mathbf{CP}^1$ *are equivalent to the Great Poncelet Theorem for a triangle.*

The proof is given in [12].

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