ELLIPSOIDS, COMPLETE INTEGRABILITY
AND HYPERBOLIC GEOMETRY

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Dedicated to the memory of J. Moser

ABSTRACT. We describe a new proof of the complete integrability of the two related dynamical systems: the billiard inside the ellipsoid and the geodesic flow on the ellipsoid (in Euclidean, spherical or hyperbolic space). The proof is based on the construction of a metric on the ellipsoid whose nonparameterized geodesics coincide with those of the standard metric. This new metric is induced by the hyperbolic metric inside the ellipsoid (the Caley–Klein model of hyperbolic space).

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1. INTRODUCTION

This paper concerns two closely related dynamical systems: the billiard transformation inside the ellipsoid and the geodesic flow on the ellipsoid in Euclidean space; both provide classical examples of completely integrable systems. We describe a new proof of the complete integrability of these systems given in [27, 31]; an interesting feature of this proof is a rather unexpected connection with hyperbolic geometry. The same proof applies to the ellipsoid in a space of constant curvature, positive or negative. The proof is a byproduct of a study of a new class of dynamical systems, called projective billiards, and related problems of symplectic, Finsler and projective geometry, see [28, 29, 30, 32, 33]; we will discuss relevant part of this theory below.

The geodesic flow on a Riemannian manifold M^n is a flow on the tangent bundle TM: a tangent vector v moves with constant speed along the geodesic tangent to v. From the physical viewpoint, the geodesic flow describes the motion of a free particle on M. Identifying the tangent and cotangent bundles by the metric, the geodesic flow becomes a Hamiltonian vector field on the cotangent bundle T^*M with its canonical symplectic structure "$dp \wedge dq$", the Hamiltonian function being

the energy: $H(q,\, p) = p^2/2$. Complete integrability of the geodesic flow means that the flow has n invariant functions (integrals), independent on an open dense set and Poisson commuting with respect to the canonical symplectic structure on T^*M.

Let M be a compact convex domain with a smooth boundary in $\mathbb{R}^{n+1}$. The billiard system describes the motion of a free particle inside M with elastic reflections off the boundary. One replaces the continuous time system by its discrete time reduction, the billiard transformation. The billiard transformation T is a transformation of the set of oriented lines in $\mathbb{R}^{n+1}$ that intersect M; the map T is defined by the familiar law of geometric optics: *the incoming ray r, the outgoing ray $T(r)$ and the normal to the boundary ∂M at the impact point lie in one 2-plane, and the angles made by r and $T(r)$ with the normal are equal.*

The space $\mathcal{L}$ of oriented lines in $\mathbb{R}^{n+1}$ is a $2n$-dimensional symplectic manifold (we will discuss this symplectic structure ω in Section 2), and the billiard transformation is a symplectomorphism: $T^*(\omega) = \omega$, see e. g. [4, 26]. Complete integrability of T means that there exist n integrals, functionally independent on an open dense subset of $\mathcal{L}$ and Poisson commuting with respect to the symplectic structure ω.

The nondegenerate common level sets of the integrals are Lagrangian submanifolds of the ambient symplectic manifold (T^*M in the case of the geodesic flow and $\mathcal{L}$ in the case of the billiard). One obtains a Lagrangian foliation, leaf-wise invariant under the flow or the transformation. The leaves of a Lagrangian foliation carry a canonical affine structure (see e. g. [5]), that is, an atlas whose transition functions are affine transformations. This affine structure is invariant under the geodesic flow or the billiard transformation. If the leaves of a Lagrangian foliation are compact then they are tori. The geodesic flow is a constant flow and the billiard transformation is a parallel translation in the affine structure on each torus; in short, the dynamics is quasi-periodic. The last statement is, essentially, the Arnold–Liouville theorem, see [4].

Let the ellipsoid under consideration $E \subset \mathbb{R}^{n+1}$ be given by the equation

$$\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2} = 1, \tag{1.1}$$

and assume that all the semiaxes a_i are distinct. The family of confocal quadratic hypersurfaces E_t is given by the equation

$$\sum_{i=1}^{n+1} \frac{x_i^2}{a_i^2 + t} = 1, \qquad t \in \mathbb{R},\ \ t \neq -a_i^2.$$

If E is an ellipse (i. e., $n = 1$) then E_t consists of ellipses and hyperbolas with the same foci as E.

The geometric meaning of complete integrability is as follows. Consider a fixed geodesic γ on E. Then the straight lines, tangent to γ, are also tangent to $n-1$ fixed confocal hypersurfaces $E_{t_1}, \ldots, E_{t_{n-1}}$ where the parameters $t_1, \ldots, t_{n-1}$ depend on γ. Thus one has $n-1$ integrals of the geodesic flow, and one more integral is the energy H. Likewise, consider an oriented line r intersecting E. Then r is tangent to

n confocal hypersurfaces, and all the reflected rays $T(r)$, $T^2(r)$, ... remain tangent to the same n confocal hypersurfaces.

The simplest example is the billiard inside an ellipse E. If a ray r does not pass between the foci of E then r and all the reflected rays are tangent to the same confocal ellipse, and if r passes between the foci then r and all the reflected rays remain tangent to the same confocal hyperbola (the intermediate case is especially well known as the optical property of an ellipse: a ray from a focus reflects to another focus).

The set of rays, tangent to a fixed confocal ellipse, is topologically a circle; it follows from the Arnold–Liouville theorem that the billiard transformation of this circle is conjugated to a rotation. Applying a projective transformation to confocal ellipses, one obtains an arbitrary pair of nested ellipses. One arrives at the famous Poncelet theorem from projective geometry (see [7, 9] for the interesting history and various proofs of the theorem). Given two nested ellipses A, B in the plane one plays the following game: choose a point $x \in B$, draw a tangent line to A through it, find the intersection y with B, and iterate, taking y as a new starting point. The statement is that if x returns back after a number of iterations, then every point of B will return back after the same number of iterations.

The billiard transformation inside the ellipsoid and the geodesic flow on the ellipsoid are closely related. On the one hand, the domain inside the ellipsoid (1.1) can be considered as a degenerate ellipsoid, the limit of the ellipsoids

$$\sum_{i=1}^{n+2} \frac{x_i^2}{a_i^2} = 1$$

as $a_{n+2} \to 0$. Thus the billiard system is obtained from the geodesic flow; in particular, the integrals of the latter are those of the former.

On the other hand, consider a free particle inside the ellipsoid E whose trajectory meets E at a small angle α. In the limit $\alpha \to 0$ one obtains a free particle on E, and the integrals of the billiard transformation inside E provide integrals of the geodesic flow on E.

The complete integrability of the geodesic flow on the triaxial ellipsoid was established by Jacobi in 1838. Jacobi integrated the geodesic flow by separation of variables, see [12]. The appropriate coordinates are called the elliptic coordinates, and this approach works in any dimension. Two other proofs of the complete integrability of the geodesic flow on the ellipsoid, by confocal quadrics and by isospectral deformations, are described in [21, 20, 22]; see also [6, 13, 25].

We mentioned that the complete integrability of the geodesic flow on the ellipsoid implies that of the billiard transformation inside the ellipsoid. G. Birkhoff was, probably, the first to put forward the study of mathematical billiards; the complete integrability of the billiard inside the ellipsoid is discussed in [8]. We refer to [14, 23, 37] for contemporary proofs and to [36] for complete integrability of the billiard inside the ellipsoid in a space of constant curvature.

This is an expository paper, and the proofs, especially the computational ones, are omitted. We use vector notation throughout the paper: if x, $y \in \mathbb{R}^n$ and f is a

function on $\mathbb{R}^n$ then
$$xy = x_1 y_1 + \cdots + x_n y_n, \quad x\,dy = x_1\,dy_1 + \cdots + x_n\,dy_n,$$
$$dx \wedge dy = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n, \quad \operatorname{grad} f = (f_{x_1}, \ldots, f_{x_n}), \quad \text{etc.}$$

2. Bisymplectic maps and projectively equivalent metrics

We use the following naive mechanism to provide integrals of the billiard transformation. Consider a smooth manifold M^{2n} with two symplectic forms ω, Ω and a diffeomorphism that preserves both:
$$T\colon M \to M, \quad T^*(\omega) = \omega, \quad T^*(\Omega) = \Omega.$$
We will call M a *bisymplectic manifold* and T a *bisymplectomorphism*. Then the functions f_i defined by the relation
$$\Omega^i \wedge \omega^{n-i} = f_i\,\omega^n, \qquad i = 1, \ldots, n \tag{2.1}$$
are T-invariant. If the forms ω and Ω are sufficiently generic then these integrals are functionally independent on an open dense subset. The same construction works for a degenerate 2-form Ω.

Alternatively, a bisymplectic structure determines a field E of linear automorphisms of the tangent spaces:
$$\Omega(u, v) = \omega(Eu, v) \qquad \text{for all } u, v \in T_x M. \tag{2.2}$$
The eigenvalues of E all have multiplicity 2; they are invariant functions, and these n integrals functionally depend on the integrals $f_1, \ldots, f_n$.

Remark 1. In general, the integrals $f_1, \ldots, f_n$ are not in involution with respect to either of the symplectic structures. If ω and Ω are Poisson compatible (the sum of the respective Poisson structures is again a Poisson structure) then the integrals $f_1, \ldots, f_n$ Poisson commute with respect to both forms, see e. g. [10, 16] concerning bihamiltonian formalism. However Poisson compatibility is not necessary for the functions $f_1, \ldots, f_n$ to be in involution, see Example 5 below. It would be interesting to give a local classification of bisymplectic structures for which $f_1, \ldots, f_n$ Poisson commute with respect to both symplectic forms.

Now we modify the above integrability mechanism to provide integrals of the geodesic flow. Consider a smooth manifold M^n with two Riemannian metrics g_1, g_2.

Definition 2. The metrics are called *projectively equivalent* if their nonparameterized geodesics coincide.

Projectively equivalent metrics are also called geodesically equivalent. The local study of projectively equivalent metrics is a classical subject, see [3] for a survey.

Example 3. There exists a metric of constant positive curvature in n-dimensional space which is projectively equivalent to the Euclidean one. This metric is obtained from the unit northern hemisphere by the central projection onto the horizontal tangent hyperplane at the north pole.

Another classical example plays the principal role in the present paper. The interior of a unit ball in $\mathbb{R}^n$ provides a model for n-dimensional hyperbolic space

$\mathbb{H}^n$, called the projective or Beltrami–Klein model. The distance between points A and B is $|\ln[A,\, B,\, X,\, Y]|$ where $[\ldots]$ denotes the cross-ratio of four points and X, Y are the points of intersection of the line AB with the boundary unit sphere. The geodesics of this metric are straight segments, and the Beltrami–Klein metric is projectively equivalent to the Euclidean one. The Beltrami–Klein model can be also obtained by a central projection from one sheet of the two-sheeted hyperboloid in Minkowski space to a tangent hyperplane of the hyperboloid.

These two examples exhaust the list of Riemannian metrics projectively equivalent to the Euclidean one: according to a Beltrami theorem, if the geodesics of a Riemannian metric in a domain in $\mathbb{R}^n$ are straight lines then this is a metric of constant curvature, see [11].

If a manifold M^n carries two projectively equivalent metrics then the geodesic flow of each has n integrals. These integrals are constructed as follows (a similar construction was independently discovered by V. Matveev and P. Topalov, see [17, 18, 19, 34, 35]).

Consider a Riemannian manifold $(M^n,\, g_1)$. The cotangent bundle T^*M has a canonical symplectic form, and this form is the differential of the canonical Liouville 1-form. Identifying T^*M with TM by the metrics g_1, one obtains a symplectic form ω_1 and a 1-form λ_1 on the tangent bundle such that $\omega_1 = d\lambda_1$. Let $S_1 \subset TM$ be the unit vector hypersurface. The form λ_1 is a contact form on S_1, that is, $\lambda_1 \wedge \omega_1^{n-1}$ is a nondegenerate volume form. The restriction of ω_1 to S_1 has a 1-dimensional kernel ξ at every point, called the *characteristic direction*, and the curves tangent to ξ are called *characteristic curves*. These curves are the trajectories of the geodesic flow of the metric g_1 (see e.g. [4, 5]). The same considerations apply to the second metric g_2.

Consider the map $\phi\colon TM \to TM$ that rescales the tangent vectors sending S_1 to S_2. This map sends the trajectories of the geodesic flow of g_1 to those of g_2. Consider the two forms $\omega = \omega_1$ and $\Omega = \phi^*(\omega_2)$. These forms have the same characteristic foliation on the hypersurface S_1 and both are holonomy invariant along this foliation. It follows that the functions f_i defined by the equality

$$\lambda_1 \wedge \omega^{n-1-i} \wedge \Omega^i = f_i\, \lambda_1 \wedge \omega^{n-1}, \qquad i = 1,\, \ldots,\, n-1, \tag{2.3}$$

are integrals of the geodesic flow of the metric g_1, the n-th integral being the energy. These integrals are analogous to the integrals (2.1).

Alternatively, the forms ω and Ω determine nondegenerate 2-forms on the quotient $2(n-1)$-dimensional spaces TS_1/ξ. As before, one may consider the field of automorphisms E, analogous to (2.2), given by the formula:

$$\Omega(u,\, v) = \omega(Eu,\, v) \qquad \text{for all } u,\, v \in T_xM/\xi.$$

The eigenvalues of E have multiplicities 2, and they are invariant along the characteristic foliation.

Remark 4. (i) The construction of integrals (2.3) extends to a more general case of projectively equivalent Finsler metrics. Finsler geometry describes the propagation of light in an inhomogeneous anisotropic medium; this means that the velocity of light depends on the point and the direction. Finsler metric is determined by a

field of strictly convex centrally symmetric smooth hypersurfaces centered at the origin in the tangent space at each point; these hypersurfaces consist of the unit tangent vectors and are called *indicatrices* (see [24]). In the Riemannian case the unit hypersurfaces are ellipsoids. The description of Finsler metrics projectively equivalent to the Euclidean one is part of Hilbert's 4-th problem. There is an abundance of such metrics described in terms of integral and symplectic geometry, see [2, 1].

(ii) Similarly to Remark 1, it is an interesting problem to describe the pairs of projectively equivalent Finsler metrics for which the integrals (2.3) are in involution with respect to the symplectic structure on TM obtained from the canonical one on T^*M by the Legendre transform. If both metrics are Riemannian then the integrals (2.3) Poisson commute; this theorem is proved by Matveev and Topalov using local normal forms for projectively equivalent Riemannian metrics that go back to Levi-Civita.

Let $S \subset TM^n$ be the unit vector hypersurface for a Riemannian metric g. Assume that the space of leaves of the characteristic foliation, that is, the space of non-parameterized oriented geodesics of the metric g, is a smooth $(2n-2)$-dimensional manifold $\mathcal{L}$. This manifold has a symplectic structure induced by the symplectic form ω on TM. We say that this symplectic structure on the space of geodesics is associated with the metric g. The same construction applies to Finsler metrics as well.

Example 5. A metric g in a domain in n-dimensional space whose geodesics are straight lines (see Example 3) provides a symplectic structure ω_g on the space of oriented lines $\mathcal{L}^{2n-2}$. Let ω and Ω be two such symplectic forms associated with the Euclidean and hyperbolic metric inside the unit ball. We give explicit formulas for these forms.

Let the ball be centered at the origin. An oriented line $l \in \mathcal{L}$ is characterized by the unit vector q in the direction of l and the vector p, the perpendicular to l from the origin; clearly, $q^2 = 1$, $qp = 0$ and $p^2 < 1$. In terms of these coordinates, one has:

$$\omega = dp \wedge dq, \qquad \Omega = \frac{dp \wedge dq}{(1-p^2)^{1/2}} + \frac{(p\,dp) \wedge (p\,dq)}{(1-p^2)^{3/2}}.$$

The form ω is the invariant symplectic form of the billiard transformation mentioned in Section 1.

Let g be another generic Euclidean metric. Then the forms ω_g and Ω are not Poisson compatible, while the functions (2.1) are in involution with respect to both forms. This is the example promised in Remark 1.

3. PROJECTIVE BILLIARDS AND η-GEODESICS

The billiard dynamical system is defined in metric terms (equal angles). It would be interesting to have a broader class of "billiards" with a projectively-invariant law of reflection; for the lack of a better name, call them *projective billiards*. Informally

speaking, one would like to have the following "proportion":

$$\frac{\text{Conventional Billiards}}{\text{Projective Billiards}} = \frac{\text{Euclidean Geometry}}{\text{Projective Geometry}}.$$

The definition of projective billiards given in [28] is as follows. Let $M \subset \mathbb{R}^{n+1}$ be a smooth closed convex hypersurface. The projective billiard transformation is a transformation of the set of oriented lines intersecting the billiard table M. To define this transformation one needs an extra structure: a smooth field η of transverse directions along M.

Definition 6. The law of the projective billiard reflection reads: the incoming ray, the outgoing ray and the line $\eta(x)$ at the impact point x lie in one 2-plane π, and these three lines, along with the line of intersection of π with the tangent hyperplane $T_x M$, constitute a harmonic quadruple of lines.

Recall that four coplanar and concurrent lines is a harmonic quadruple if the cross-ratio of these lines equals -1. The cross-ratio of four coplanar concurrent lines is the cross-ratio of their intersection points with a fifth line (it does not depend on the choice of this auxiliary line).

If the line field η consists of Euclidean normals to M then the projective billiard coincides with then usual one.

Fix a point $x \in M$. The set of lines through x is the projective space $\mathbf{RP}^n$. The projective billiard reflection is a projective involution of this projective space whose fixed points are the line $\eta(x)$ and every line tangent to M. If ψ is a projective transformation of the ambient space and (M, η) is a "hairy" billiard table then $(\psi(M), d\psi(\eta))$ is a new billiard table. The respective projective billiard transformations T and T_ψ commute with the action of ψ on the space of oriented lines: $T_\psi \circ \psi = \psi \circ T$.

We refer to [28, 30] for some results on projective billiards. One of the most interesting questions is the following

Problem 1. *Describe the pairs (M, η) for which the projective billiard transformation has an invariant symplectic form.*

Roughly speaking, this is equivalent to asking when the projective billiards admit a variational formulation (see [4, 5] for the relation between variational calculus and symplectic geometry) or, in even more general terms, when a projective billiard has a physical meaning. A manifestation of the variational nature of the conventional billiards is that billiard trajectories are extrema of the arclength functional.

Problem 1 is probably rather hard. One example of an invariant symplectic structure is provided by the usual billiards (M, η), where η is the field of normals, and their images under projective maps. Here is another one.

Example 7. Consider a convex domain in space with a metric g of constant curvature projectively equivalent to the Euclidean one, see Example 3. Let M be the boundary hypersurface and η the field of g-normals. Consider the billiard flow in the metric g, that is, the geodesic flow with elastic reflection in M. One obtains the billiard transformation T of the set of oriented lines intersecting M, the geodesics of g. The map T has an invariant symplectic structure ω_g defined in Section 2.

One has the following result: *T coincides with the projective billiard transformation associated with (M, η)*. In particular, this projective billiard transformation has an invariant symplectic form ω_g.

A related example will be discussed in Section 5.

According to Section 1, the geodesics on a hypersurface can be considered as glancing billiard trajectories inside this hypersurface. Similarly one can define an analog of a geodesic curve in the more general setting of projective billiards. Let (M, η) be a smooth hypersurface equipped with a smooth field of transverse lines.

Definition 8. An *η-geodesic* on M is a nonparameterized curve $\gamma \subset M$ such that at every point $x \in \gamma$ the 2-plane generated by the tangent line to γ and the line $\eta(x)$ is second-order tangent to γ.

If M is quadratically-nondegenerate then an η-geodesic is a curve whose osculating 2-plane at every point x contains the respective line $\eta(x)$. If η consists of the Euclidean normals then the η-geodesics are the usual geodesics. An η-geodesic has a preferred parameterization $\gamma(t)$ determined by the condition that the acceleration vector $\gamma''(t)$ is collinear with $\eta(\gamma(t))$ for all t; for the field of normals, this is the arc-length parameterization.

Thus through every point of a hypersurface and in every tangent direction there passes an η-geodesic. These curves are the geodesics of a connection ∇ on M constructed as follows. Let ∇^0 be the Euclidean connection in the ambient space V and let $\pi_x \colon T_x V \to T_x M$ be the projection whose kernel is the line $\eta(x)$. Then the connection ∇ is defined by the formula:

$$\nabla_u(v) = \pi_x \nabla^0_u(v)$$

for every two vectors $u, v \in T_x M$.

One poses the following problem analogous to Problem 1 above.

Problem 2. *Describe the pairs (M, η) for which there exists a (Finsler) metric on M whose nonparameterized geodesics coincide with the η-geodesics.*

The situation with this problem is the same as with Problem 1: in addition to the case when the line field consists of the Euclidean normals and the images of such pairs (M, η) under projective transformations, one has the next example whose modification will be also discussed in Section 5.

Example 9. Let M be a hypersurface in a domain with a metric g of constant curvature projectively equivalent to the Euclidean one, and let η be the field of g-normals. One has the following result: *g-geodesics and η-geodesics coincide*; this is not at all true for a metric g that is not projectively equivalent to the Euclidean one. Moreover, let g be a metric in a domain with the following property: for every smooth hypersurface M there exists a smooth transverse line field η such that g-geodesics on M coincide with η-geodesics. Then g is projectively equivalent to the Euclidean metric, see [31].

4. Digression: exact transverse line fields

In this section we discuss a special class of transverse line fields along smooth hypersurfaces in linear (and projective) space which we call *exact line fields*. The same object is also known as a relative normalization in affine differential geometry, see [15]. Exact line fields enjoy many properties of the Euclidean normals and can be considered as projective analogs of the latter.

Let (M, η) be a smooth hypersurface in affine space equipped with a smooth transverse line field. Choose a nonvanishing vector field v along η so that for every $x \in M$ the line $\eta(x)$ is generated by the vector $v(x)$. Let p be a conormal field, that is, a field of covectors along M such that $\operatorname{Ker} p(x) = T_x M$ for all $x \in M$, normalized by the condition $p(x)v(x) = 1$. One can show that the following definition does not depend on the choice of the vector field v.

Definition 10. The line field η is called exact if the differential 1-form $p\,dv$ is exact on M.

Here v is treated as a vector-valued function on M; accordingly, dv is a vector-valued 1-form and the convolution pdv is a 1-form. If the ambient space has a Euclidean structure, one may identify covectors and vectors; in particular, p can be taken to be the unit normal vector field along M and v chosen accordingly. Mention in passing that exactness can be defined for continuous transverse fields of directions along polyhedral hypersurfaces as well; this is done in [32].

Now we present a number of examples and properties of exact transverse line fields, see [27, 29, 30, 32] for details. We hope to convince the reader that exact transverse line fields is an interesting object of study.

(o) The simplest example of an exact field is the field of the Euclidean normals to a hypersurface. Let n be the unit normal vector field. Then the exactness condition requires the 1-form ndn to be exact. This form actually vanishes since $n^2 = 1$.

Another simple example: if M is star-shaped with respect to a point then the field of lines through this point is exact.

If M is a hyperplane then every transverse line field along M is exact.

(i) Let g be a Riemannian metric of constant positive or negative curvature whose geodesics are straight lines. Then *the field of g-normals to every hypersurface is exact.*

(ii) A Minkowski metric in linear space is a translation-invariant Finsler metric; a Minkowski metric is determined by its unit sphere S, a smooth centrally symmetric quadratically-convex hypersurface centered at the origin. Given a cooriented hypersurface M, the *Minkowski normal* to M at a point $x \in M$ is defined as follows. There is a unique vector $u \in S$ such that the tangent hyperplane $T_u S$ is parallel to $T_x M$; this u is the Minkowski normal to M at x.

One has the following result: *the field of Minkowski normals to every cooriented hypersurface is exact.* More generally, one can classify Finsler metrics with the property that the field of Finsler normals to every hypersurface is exact, see [32].

(iii) The following result puts the previous example (ii) into a more general perspective.

Let (M, η) be a hypersurface with a transverse line field. For every $x \in M$, parallel translate the tangent hyperplane $T_x M$ to the points of the line $\eta(x)$. One obtains a codimension one distribution in a sufficiently small vicinity of M. *The line field η is exact if and only if this distribution is integrable, and the leaves of this foliation are diffeomorphic to M.* As one moves further away from M, the leaves start to develop singularities, and the whole picture resembles propagation of light from the source M. If η is the field of Minkowski normals to M then the leaves of the foliation are hypersurfaces equidistant from M, as follows from the Huygens principle of wave propagation (see [4]). Thus exact transverse line fields satisfy a version of the Huygens principle.

(iv) Given a strictly convex hypersurface M in affine space and a point $x \in M$, consider the $(n-1)$-dimensional sections of M by the hyperplanes parallel to $T_x M$. The centroids of these sections lie on a smooth curve δ starting at x. The *affine normal* to M at x is defined as the tangent line to δ at x.

One has the following result: *the field of affine normals to every convex hypersurface is exact.*

(v) Let γ be a smooth strictly convex closed plane curve. *A smooth transverse line field along γ is exact if and only if this line field is generated by the acceleration vectors $\gamma''(t)$ for some smooth parameterization $\gamma(t)$.*

The following generalization of the classical 4-vertex theorem holds: *given a generic exact transverse line field η along a closed convex smooth plane curve γ, the envelope of the 1-parameter family of lines $\eta(x)$, $x \in \gamma$, has at least 4 cusp singularities*, see [29]. This 4-vertex theorem also has a polygonal analog, see [33].

(vi) Although defined in linear terms, exactness is a projective property. Namely, *if ψ is a projective transformation of the ambient space and η is an exact field along a hypersurface M then $d\psi(\eta)$ is an exact field along $\psi(M)$.* One can give a definition of exactness in purely projective terms, see [32], and one may extend the notion of exactness to hypersurfaces in projective space.

(vii) An easy application of Morse theory shows that, given an immersed closed hypersurface $M \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the number of perpendiculars from x to M is not less than the least number of critical points of a smooth function on the manifold M. In particular, if x is generic then this number is bounded below by the sum of Betti numbers of M.

Another manifestation of analogy between exact fields and Euclidean normals is that one can extend the above result to an immersed *quadratically nondegenerate* closed hypersurface $M \subset \mathbb{RP}^n$ equipped with an exact transverse line field η. *Given a point $x \in \mathbb{RP}^n$, the number of points $y \in M$ such that the line $\eta(y)$ passes through x, admits the same Morse-theoretic lower bound as for the Euclidean normals*, see [30]. Quadratic nondegeneracy is essential as counterexamples in [32] show.

(viii) Let the hypersurface be the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Identify the unit ball with the projective model of hyperbolic space $\mathbb{H}^{n+1}$. Let $\mathcal{L}^{2n}$ be the space of oriented lines in $\mathbb{H}^{n+1}$ with its symplectic structure Ω introduced in Section 2. A field of transverse directions η along S^n determines an embedding $S^n \to \mathcal{L}^{2n}$.

One has the following result: *the field η is exact if and only if the image of this embedding is a Lagrangian submanifold*, see [27].

On the other hand, an n-parameter family of lines is Lagrangian if and only if it consists of the hyperbolic normals to a hypersurface. Thus an exact field η consists of the normals to a hypersurface $M_\eta \subset \mathbb{H}^{n+1}$; in fact, one has a 1-parameter family of equidistant hypersurfaces with a fixed family of normals.

(ix) Finally, we mention another characterization of exact transverse line fields along the unit sphere. Let f be a smooth function on the unit sphere. Then *the line field generated by the vector field $v(x) = x + \operatorname{grad} f(x)$, $x \in S^n$, is exact, and every exact field along the unit sphere is obtained in this way.*

5. Putting it all together: integrability

All pieces of the "jigsaw puzzle" being present, it remains to assemble the picture. First we give a partial answer to Problem 1 from Section 3.

Theorem 11. *Let (S^n, η) be the unit sphere with an exact transverse line field. Then the respective projective billiard transformation has an invariant symplectic structure, namely, the structure Ω associated with (the projective model of) the hyperbolic metric inside the sphere.*

Let us deduce this theorem from Example 7. Consider the 1-parameter family of equidistant hypersurfaces M_η that are perpendicular , in the hyperbolic sense, to the lines from the family η. Choose one of these hypersurfaces, say M_η^0, and denote the hypersurfaces in the family by M_η^t where t is the hyperbolic distance from M_η^0 to M_η^t along the normals. As was pointed out in Example 7, the respective projective billiard transformation inside M_η has an invariant symplectic structure Ω associated with the hyperbolic metric. As $t \to \infty$, the limit of the billiard transformations inside M_η^t is the projective billiard map inside (S^n, η), and this implies Theorem 11.

To obtain integrals of the billiard transformation inside the ellipsoid E one argues as follows. Let n be the field of normals along E; this field is exact. Apply an affine transformation ψ that takes E to the unit sphere S, and let $\eta = \psi(n)$. Then η is an exact field of lines. The map ψ conjugates the original billiard transformation inside E and the projective billiard transformation T associated with the table (S, η). According to Theorem 11, $T^*(\Omega) = \Omega$. On the other hand, the original billiard transformation preserved the canonical symplectic structure $dp \wedge dq$ on the space of oriented lines, therefore T also preserves a symplectic structure ω associated with a Euclidean metric. It remains to feed ω and Ω to the mechanism described in Section 2 to obtain integrals (2.1).

Similarly, we give a partial answer to Problem 2 from Section 3.

Theorem 12. *Let (S^n, η) be the unit sphere with an exact transverse line field. There is a metric g on the sphere such that nonparameterized g-geodesics coincide with η-geodesics. If η is generated by the vector field $v(x) = x + \operatorname{grad} f(x)$, $x \in S^n$, as in Section 4.2 (ix), then the respective metric is conformally equivalent to the Euclidean one with the conformal factor $\exp(-f)$.*

A computation behind this statement is contained in [31]. We explain the relation of the metric g with the hyperbolic geometry inside the sphere. Consider again the 1-parameter family of equidistant hypersurfaces M_η^t orthogonal to the lines η. Let h_t be the metric on M_η^t induced from the ambient hyperbolic space. The intersections with the lines from the family η determine an identification of each M_η^t with S^n, and we consider h_t as metrics on the sphere. The metrics h_t exponentially grow with t but they have a limit after a renormalization. This is the metric g on the sphere from Theorem 12, namely, one has:

$$\lim_{t\to\infty} e^{-t} h_t = g.$$

Return to the original ellipsoid E. Let n be the field of Euclidean normals along E. As before, consider an affine transformation ψ that takes E to the unit sphere S, and let $\eta = \psi(n)$. Then η is an exact transverse line field along S. Let g be the metric on S from Theorem 12. Then the metric $\psi^*(g)$ is a metric on E projectively equivalent to the Euclidean one. An explicit formula for this metric is given in the next theorem, summarizing the above arguments; this theorem was independently proved by Matveev and Topalov.

Theorem 13. *The restrictions of the metrics*

$$\sum_{i=1}^{n+1} dx_i^2 \quad and \quad \frac{\sum_{i=1}^{n+1} a_i\, dx_i^2}{\sum_{i=1}^{n+1} a_i^2\, x_i^2}$$

on the ellipsoid

$$\sum_{i=1}^{n+1} a_i\, x_i^2 = 1$$

are projectively equivalent.

One inputs these projectively equivalent metrics into the mechanism described in Section 2 to obtain integrals (2.3) of the geodesic flow on the ellipsoid. A computation is contained in [31]; remarkably, one obtains the classical integrals as given in [21, 20, 22] (and not just some functions of these integrals).

We conclude with the following remark: due to property (i) in Section 4, one may replace the Euclidean normals in the arguments of Section 5 by the g-normals where g is a metric of constant curvature projectively equivalent to the Euclidean one. As a result, one obtains proofs of the complete integrability of the billiard transformation inside the ellipsoid and the geodesic flow on the ellipsoid in space of constant curvature, positive or negative, cf. [36].

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