MATH 329

Theory of Interest

Professor Jérome Wahldispühl

Assignment 4

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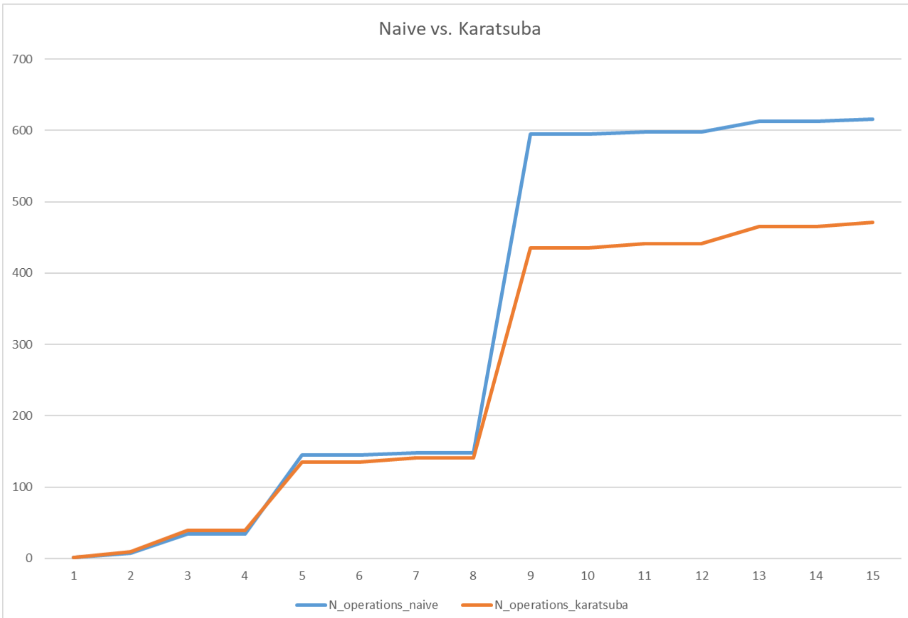
Part 1

: Plot and explanation



You can notice that despite Karatsuba has one more elementary operation in the final calculation compared to the naive method, you can see that one less recursion achieved by the trick showed a huge asymptotic efficiency difference between the two methods. As it was demonstrated in class, Karatsuba's algorithm is more efficient, and the difference gets more pronounced as the size increases. In fact, when the size is small, the naive multiplication algorithm tends to be better (when size <5). This is because Karatsuba's algorithm has a complexity of $\mathbf{\Theta}(n^{1.585})$ whereas the naive one has $\mathbf{\Theta}(n^2)$

Part 2

Theorem Master method: Given

$$a, b \in [1.\infty[$$

$$f : \mathbb{N}^+ -> \mathbb{R}$$

, and

$$\forall n \in \mathbb{N}^+ \;\; T(n) = \; aT(\frac{n}{b}) \; + \; f(n); \;\;\; \frac{n}{b} = \lfloor \frac{n}{b} \rfloor \vee \lceil \frac{n}{b} \rceil$$

Then
1) If $\exists \epsilon > 0$ such that $f(n) \in \mathbf{O}(n^{log_b a - \epsilon})$, $T(n) \in \mathbf{\Theta}(n^{log_b a})$
2) If $\exists p \in \mathbb{N}$ such that $f(n) \in \mathbf{\Theta}(n^{log_b a} log^p(n))$, then $T(n) \in \mathbf{\Theta}(n^{log_b a} log^{p+1}(n))$
3) If
a. $\exists \epsilon > 0$ such that $f(n) \in \mathbf{\Omega}(n^{log_b a + \epsilon})$
b. $\exists c < 1, n_0 \in \mathbb{N}^+$ such that $af(\frac{n}{b}) < cf(n)$ for all $n \geq n_0$
Then $T(n) \in \mathbf{\Theta}(f(n))$

Question 2

$a : T(n) = 25T(\frac{n}{5}) + n$
Solution.
Since we have $log_5(25) = 2$, we do have $1 > 0$ such that

$$f(n) = n \in \mathbf{O}(n^{2-1}) = \mathbf{O}(n)$$

Thus by case 1 $T(n) \in \mathbf{\Theta}(n^2)$ $\square$.

$b : T(n) = 2T(\frac{n}{3}) + nlog(n)$
Solution.

we know:
$log_3 2$ $log_3 2 < 1$ and $nlog(n) \in \mathbf{\Omega}(n)$.
Take $1 - log_3 > 0$ such that

$$nlog(n) \in \mathbf{\Omega}(n^{log_3 2 + (1 - log_3 2)}) = \mathbf{\Omega}(n).$$

We know that $log : \mathbf{R}_0^+ -> \mathbf{R}$ is a monotone increasing function, i.e. $log(\frac{n}{3}) \leq log(n) \;\; \forall n \in \mathbb{N}^+$ Then we would have

$$2(\frac{n}{3} log(\frac{n}{3})) = \frac{2}{3} nlog(\frac{n}{3})) \leq \frac{2}{3} nlog(n)$$

$\forall n \in \mathbb{N}^+$ Thus by case 3 $T(n) \in \mathbf{\Theta}(nlog(n))$ $\square$.

$c : T(n) = T(\frac{3}{4}n) + 1$
Solution.

We have: $log_{3/4}1 = 0$ and we have $1 \in \mathbf{\Theta}(1) = \mathbf{\Theta}(log(n)^{1-1})$
Then by case 2 we would have $T(n) \in \mathbf{\Theta}(log(n))$ $\square$.

$d : T(n) = 7T(\frac{n}{3}) + n^3$
Solution.

We have $log_3 7 > 2$. Then take $\epsilon = 2 - log_3 7 > 0$ such that

$$n^3 \in \mathbf{\Omega}(n^{log_3 7 + (2 - log_3 7)} = \mathbf{\Omega}(n^2)$$

and by Lemma 1 above we have for $n > 1, \frac{4}{5} < 1$

$$7(\frac{n}{3})^3 = \frac{7}{27}n^3 \leq \frac{1}{2}n^3 \quad \forall n$$

Thus by case 3 $T(n) \in \mathbf{\Theta}(n^3)$
$e : T(n) = T(\frac{n}{2}) + n(2 - cos(n))$

Solution.
We know that $-1 \leq cos(x) \leq 1 \; \forall x \in \mathbb{R}$
so naturally it follows that:

$$1 \leq 2 - cos(n) \leq 2 + 1 = 3 \implies n \leq n(2 - cos(n)) \leq 3n$$

and $log_2 1 = 0$ so 1
we have $n(2 - cos(n)) \in \mathbf{\Theta}(n) = \mathbf{\Theta}(n \log^{1-1}(n))$
but $n^0 \in \mathbf{\Omega}(1)$
So we have for 1¿0

$$n(2 - cos(n)) \in \mathbf{\Omega}(n^{0+1})$$

but there is no $c < 1, n_0 \in \mathbb{N}$ where $cf(n)$ can bound $\frac{n}{2}(2 - cos(\frac{n}{2})) \;\; \forall n \geq n_0$
So Master theorem is not applicable.

Question 3

$$T_A(n) = 7T_A(\frac{n}{2}) + n^2$$

$$T_B(n) = \alpha T_B(\frac{n}{4}) + n^2$$

Solution.
$log_2 7 > 2 \implies log_2 7 - 2 > 0$ such that

$$n^2 \in \mathbf{O}(n^{log_2 7 - (log_2 7 - 2)}) = \mathbf{O}(n^2)$$

So $T_A(n) \in \mathbf{\Theta}(n^{log_2 7})$.

For α it needs to hold that

$$log_2 7 = \frac{\ln 7}{\ln 2} > \frac{\ln \alpha}{\ln 4}$$

$\implies$

$$\frac{\ln 4}{\ln 2} = 2 > \frac{\ln \alpha}{\ln 7} = log_7 \alpha$$

so α is at most 48. $\square$