260677676_Assignment_2_MATH447

#3.5

```
##### Matrix powers ##############################
# matrixpower(mat,k) mat^k
#
matrixpower = function(X, n){

  if(dim(X)[1] != dim(X)[2]){
      throw("Dimensions of the matrix do not match: ", dim(X))
  }
  if (n==0){return(diag(dim(X)[1]))}
  else if(n >0) {return(X %*% matrixpower(X, (n-1)))}
  else {return(matrixpower(solve(X), -n))}
}
```

```
c11 <- c(0,.75,0,0,.25)
c12 <- c(.25,0,0,0,.75)
c13 <- c(0,0,1,0,0)
c14 <- c(0,0,0,1,0)
c15 <- c(.75,.25,0,0,0)
C1 <- cbind(c11, c12, c13, c14, c15)
```

```
#printing p
C1
```

```
##        c11  c12 c13 c14  c15
## [1,] 0.00 0.25   0   0 0.75
## [2,] 0.75 0.00   0   0 0.25
## [3,] 0.00 0.00   1   0 0.00
## [4,] 0.00 0.00   0   1 0.00
## [5,] 0.25 0.75   0   0 0.00
```

##(a) We need to solve equations

$$1 = \frac{3}{4}\pi_2 + \frac{1}{4}\pi_5$$

$$\pi_2 = \frac{1}{4} + \frac{3}{4}\pi_5$$

$$\pi_3 = \pi_3$$

$$\pi_4 = \pi_4$$

$$\pi_5 = \frac{3}{4} + \frac{1}{4}$$

Once we solve the equation we realize that our solution span $\Pi \subset \text{span}\{(1, 1, \pi_3, \pi_4, 1)\}$ such that $\forall a$ that satisfies $(3 + \pi_3 + \pi_4)a = 1$

##(b)

1

```r
library(rlist)
accumulateMtxPower = function(X, n){
  power.list = list(matrixpower(X, n))
  if(n == 1){
    return(power.list)
  } else {
    return(c(accumulateMtxPower(X, n-1), power.list))
  }
}
```

```r
accumulateMtxPower(C1, 20)
```

```
## [[1]]
##      [,1] [,2] [,3] [,4] [,5]
## [1,] 0.00 0.25    0    0 0.75
## [2,] 0.75 0.00    0    0 0.25
## [3,] 0.00 0.00    1    0 0.00
## [4,] 0.00 0.00    0    1 0.00
## [5,] 0.25 0.75    0    0 0.00
##
## [[2]]
##        [,1]   [,2] [,3] [,4]   [,5]
## [1,] 0.3750 0.5625    0    0 0.0625
## [2,] 0.0625 0.3750    0    0 0.5625
## [3,] 0.0000 0.0000    1    0 0.0000
## [4,] 0.0000 0.0000    0    1 0.0000
## [5,] 0.5625 0.0625    0    0 0.3750
##
## [[3]]
##          [,1]     [,2] [,3] [,4]     [,5]
## [1,] 0.437500 0.140625    0    0 0.421875
## [2,] 0.421875 0.437500    0    0 0.140625
## [3,] 0.000000 0.000000    1    0 0.000000
## [4,] 0.000000 0.000000    0    1 0.000000
## [5,] 0.140625 0.421875    0    0 0.437500
##
## [[4]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.2109375 0.4257812    0    0 0.3632812
## [2,] 0.3632812 0.2109375    0    0 0.4257812
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.4257812 0.3632812    0    0 0.2109375
##
## [[5]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.4101562 0.3251953    0    0 0.2646484
## [2,] 0.2646484 0.4101562    0    0 0.3251953
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3251953 0.2646484    0    0 0.4101562
##
## [[6]]
##           [,1]      [,2] [,3] [,4]      [,5]
```

```
## [1,] 0.3100586 0.3010254    0    0 0.3889160
## [2,] 0.3889160 0.3100586    0    0 0.3010254
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3010254 0.3889160    0    0 0.3100586
##
## [[7]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3229980 0.3692017    0    0 0.3078003
## [2,] 0.3078003 0.3229980    0    0 0.3692017
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3692017 0.3078003    0    0 0.3229980
##
## [[8]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3538513 0.3115997    0    0 0.3345490
## [2,] 0.3345490 0.3538513    0    0 0.3115997
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3115997 0.3345490    0    0 0.3538513
##
## [[9]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3173370 0.3393745    0    0 0.3432884
## [2,] 0.3432884 0.3173370    0    0 0.3393745
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3393745 0.3432884    0    0 0.3173370
##
## [[10]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3403530 0.3368006    0    0 0.3228464
## [2,] 0.3228464 0.3403530    0    0 0.3368006
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3368006 0.3228464    0    0 0.3403530
##
## [[11]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3333120 0.3272231    0    0 0.3394649
## [2,] 0.3394649 0.3333120    0    0 0.3272231
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3272231 0.3394649    0    0 0.3333120
##
## [[12]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3302835 0.3379267    0    0 0.3317898
## [2,] 0.3317898 0.3302835    0    0 0.3379267
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3379267 0.3317898    0    0 0.3302835
##
```

```
## [[13]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3363925 0.3314132    0    0 0.3321943
## [2,] 0.3321943 0.3363925    0    0 0.3314132
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3314132 0.3321943    0    0 0.3363925
##
## [[14]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3316085 0.3332439    0    0 0.3351477
## [2,] 0.3351477 0.3316085    0    0 0.3332439
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3332439 0.3351477    0    0 0.3316085
##
## [[15]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3337198 0.3342629    0    0 0.3320173
## [2,] 0.3320173 0.3337198    0    0 0.3342629
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3342629 0.3320173    0    0 0.3337198
##
## [[16]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3337015 0.3324430    0    0 0.3338556
## [2,] 0.3338556 0.3337015    0    0 0.3324430
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3324430 0.3338556    0    0 0.3337015
##
## [[17]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3327961 0.3338170    0    0 0.3333868
## [2,] 0.3333868 0.3327961    0    0 0.3338170
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3338170 0.3333868    0    0 0.3327961
##
## [[18]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3337095 0.3332392    0    0 0.3330513
## [2,] 0.3330513 0.3337095    0    0 0.3332392
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
## [5,] 0.3332392 0.3330513    0    0 0.3337095
##
## [[19]]
##           [,1]      [,2] [,3] [,4]      [,5]
## [1,] 0.3331922 0.3332159    0    0 0.3335919
## [2,] 0.3335919 0.3331922    0    0 0.3332159
## [3,] 0.0000000 0.0000000    1    0 0.0000000
## [4,] 0.0000000 0.0000000    0    1 0.0000000
```

```
## [5,] 0.3332159 0.3335919     0     0 0.3331922
##
## [[20]]
##            [,1]       [,2] [,3] [,4]       [,5]
## [1,] 0.3333099 0.3334920     0     0 0.3331981
## [2,] 0.3331981 0.3333099     0     0 0.3334920
## [3,] 0.0000000 0.0000000     1     0 0.0000000
## [4,] 0.0000000 0.0000000     0     1 0.0000000
## [5,] 0.3334920 0.3331981     0     0 0.3333099
```

$$\lim_{n \to \infty} P^n = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \end{bmatrix}$$

##(c)

First thing to remark that is the transition matrix P is not regular, thus P is not guaranteed to have a limiting distribution in the first place. Secondly, if you remark the rows of the matrix

$$\lim_{n \to \infty} P^n$$

they are not the same.

#3.27

##(a) All vertices communicate via vertex 0, so the chain $X = \{X_t\}_{t \in \mathbb{N}_0}$ is irreducible. Since it is irreducible, all vertices must have the same period. We pick vertex zero to calculate periodicity. The possible return time for vertex 0 is $T = \{n \in \mathbb{N} : n \geq 2\}$ We have $gcd(T) = 1$. Since all vertices would have period 1 the chain is aperiodic.

##(b) Since all vertices besides zero have two choices (either to go forward or to go back to zero), the probability of our chain reaching to vertex n at once is

$$1 * \frac{1}{2} * \frac{2}{3} * ... * \frac{n-2}{n-1} * \frac{n-1}{n} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

Thus our probability of first passage time being finite is

$$P(T_0 < \infty | X_0 = 0) = 1 - \lim_{n \to \infty} \frac{1}{n} = 1$$

Thus vertex 0 is recurrent. Since our chain is irreducible, this implies the entire chain is recurrent.

(Alternative solution lies at part (c))

##(c)

The chain is null recurrent since for all possible integer k and state j

$$P(T_j = k | X_0 = j) = \frac{1}{k(k-1)}$$

and the expected value is an infinite subseries of the harmonic series.

We prove this by induction.

###Base case 1: for state 0

$$P(T_0 = 2|X_0 = 0) = 1 * \frac{1}{2} = \frac{1}{2 * (2 - 1)}$$

And for all others

$$P(T_0 = k|X_0 = 0) = 1 * \frac{1}{2} * ... * \frac{k - 2}{k - 1} * \frac{1}{k} = \frac{(k - 2)!}{k!} = \frac{1}{k(k - 1)}$$

so we have

$$E[T_0|X_0 = 0] = \sum_{j=2}^{\infty} k * \frac{1}{k(k - 1)} = \sum_{j=2}^{\infty} \frac{1}{k - 1}$$

which is harmonic series. Thus state 0, or vertex 0 is null recurrent.

You can also solve part (b) in this way.

$$P(T_0 < \infty|X_0 = 0) = \lim_{n \to \infty} \sum_{i=1}^{n} P(T_0 = i|X_0 = 0) = \sum_{i=2}^{\infty} \frac{1}{i(i - 1)} = \sum_{i=2}^{\infty} (\frac{1}{i - 1} - \frac{1}{i}) = 1 + \sum_{i=3}^{\infty} \frac{1}{i - 1} - \sum_{j=2}^{\infty} \frac{1}{j} = 1$$

Base case 1: for state 1

The argument is analogous to base case 1. In fact, the expected return time given the initial state as 1 is the same as the state 0: the harmonic series.

Inductive step: from state j to state j+1

The first passage time for state j is bounded below by j+1 since you need to visit vertex zero and come back to state j. Suppose our claim holds for 1, ..., j For j+1, the first passage time for state j is j+2. Thus

$$P(T_{j+1} = j + 2|X_0 = j + 1) = \frac{1}{j + 2} * 1 * \frac{1}{2} * ... * \frac{j}{j + 1} = \frac{1}{(j + 2)(j + 1)}$$

Likewise, in general,

$$P(T_{j+1} = j + 2 + k|X_0 = j + 1) = \frac{j + 1}{j + 2} * ... \frac{1}{j + 2 + (k + 1)} * 1 * \frac{1}{2} * ... * \frac{j}{j + 1} = \frac{1}{(j + 2 + k + 1)(j + 2 + k)}$$

It follows that the expected return time would be

$$\sum_{i=j+2}^{\infty} i \frac{1}{i(i - 1)} = \sum_{i=j+2}^{\infty} \frac{1}{(i - 1)}$$

Which is a part of harmonic series, diverging. Thus we have shown that all states are null recurrent.

#3.28

As you can see from the diagrams I attached along with .pdf file,(Appendix 1) you can notice that $\{1, 2, 3\}$ is a recurrent communication class whereas $\{4, 5, 6, 7\}$ is a transient communication class, You can find the stationary distribution for the recurrent class

$$P_{recurrent} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/3 & 0 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

You can the stationary distribution $\pi_{recurrent} = (1/3, 1/3, 1/3)$. For our original TPM P, $\pi = (1/3, 1/3, 1/3, 0, 0, 0, 0)$ Once we take the limit $\lim_{n \to \infty} P^n$, our matrix will be

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \pi \\ \pi \\ \dots \\ \pi \end{bmatrix}$$

Thus the first 3 entries of any row will be 1/3, and the rest of it will stay as zero. Since we determined the numbers, now we can use "technology" to verify our claim:

```
r1 <- c(1/3,1/3,1/3,0,0,0,0)
r2 <- c(2/3,0,1/3,0,0,0,0)
r3 <-c(0,2/3,1/3,0,0,0,0)
r4 <- c(0,1/4,0,1/2,0,1/4,0)
r5 <- c(0,0,1/4,0,1/4,0,1/2)
r6 <- c(0,0,0,0,0,0,1)
r7 <- c(0,1/8,1/8,1/8,1/8,1/4,1/4)
R1 <- rbind(r1,r2,r3,r4,r5,r6,r7)
```

```
matrixpower(R1, 100)
```

```
##          [,1]      [,2]      [,3]         [,4]         [,5]         [,6]
## r1 0.3333333 0.3333333 0.3333333 0.000000e+00 0.000000e+00 0.000000e+00
## r2 0.3333333 0.3333333 0.3333333 0.000000e+00 0.000000e+00 0.000000e+00
## r3 0.3333333 0.3333333 0.3333333 0.000000e+00 0.000000e+00 0.000000e+00
## r4 0.3333333 0.3333333 0.3333333 5.703651e-11 3.124697e-11 6.079466e-11
## r5 0.3333333 0.3333333 0.3333333 5.017660e-11 2.748882e-11 5.348274e-11
## r6 0.3333333 0.3333333 0.3333333 6.910622e-11 3.785926e-11 7.365965e-11
## r7 0.3333333 0.3333333 0.3333333 5.548563e-11 3.039733e-11 5.914158e-11
##            [,7]
## r1 0.000000e+00
## r2 0.000000e+00
## r3 0.000000e+00
## r4 1.382124e-10
## r5 1.215893e-10
## r6 1.674601e-10
## r7 1.344543e-10
```

#3.43

We have a Markov chain with TPM

$$P = \begin{bmatrix} 1/3 & 1/4 & 0 & 1/4 \\ p & 0 & 1-p & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ q & 0 & 1-q & 0 \end{bmatrix}$$

##(a) For what values of p and q is the chain ergodic? If $p, q \notin 0, 1$ it is ergodic since it is irreducible and aperiodic (due to vertex 1 that has a self loop). Otherwise we shall do case analysis (for more details refer to onenote document (or Appendix 2) that will be combined with this r markdown knitted, hopefully), but when $(p, q) = (1, 0)$ or $(p, q) = (0, 1)$ the whole graph stay irreducible and aperiodic, thus it is ergodic.

When $(p, q) = (0, 0), (1, 1)$ however, the chain do not stay irreducible (for case (0,0), vertex 1 becomes transient; for case (1,1), state 3 becomes transient).

##(b) For what values of p and q is the chain reversible?

Recall the detailed balance equations given a stationary distribution $\vec{\pi} = (\pi_1, ..., \pi_n)$ and chain with n*n TPM P

$$\forall i, j \quad \pi_i P_{ij} = \pi_j P_{ji}$$

We find the expression for both hand sides for all possible cases once we find the stationary distribution.

But first, let us find the stationary distribution. If you set $\pi_1 = 1$ (you will standardize it anyway) We have

$$1 = 1/2 + p\pi_2 + q\pi_4 \implies \frac{1}{2} = p\pi_2 + q\pi_4$$

$$\pi_2 = \pi_4 = \frac{1}{4} + \frac{1}{4}\pi_3$$

$$\pi_3 = (1-p)\pi_2 + \frac{1}{2}\pi_3 + (1-q)\pi_4 \implies \frac{1}{2}\pi_3 = (2 - (p+q))\pi_2$$

If we solve the equations we obtain

$$\pi = (\frac{p+q}{3}, \frac{1}{6}, \frac{2-p-q}{3}, \frac{1}{6})$$

Now we dive into the detailed balance equations:

$$\pi_1 P_{12} = \frac{p+q}{3}\frac{1}{4} = \frac{p+q}{12} = \pi_2 P_{21} = \frac{p}{6}$$

$$\pi_1 P_{13} = 0 = \pi_3 P_{31}$$

$$\pi_1 P_{14} = \frac{p+q}{12} = \pi_4 P_{41} = \frac{q}{6}$$

$$\pi_2 P_{23} = \frac{1-p}{6} = \pi_3 P_{32} = \frac{2-p-q}{12}$$

$$\pi_2 P_{24} = 0 = \pi_4 P_{42}$$

$$\pi_3 P_{34} = \frac{2-p-q}{12} = \pi_4 P_{43} = \frac{1-q}{6}$$

All the equations hold once $p = q$. So the chain is reversible when p is equal to q. #3.56,

Let X_n be the most recent subsequence of $HTTHH$.

$$P = \begin{bmatrix} & \emptyset & H & HT & HTT & HTTH & HTTHH \\ \emptyset & 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ H & 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ HT & 0 & 1/3 & 0 & 2/3 & 0 & 0 \\ HTT & 2/3 & 0 & 0 & 0 & 1/3 & 0 \\ HTTH & 0 & 0 & 2/3 & 0 & 0 & 1/3 \\ HTTHH & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In the beginning, we have 2/3 chance of having tails (so we have to start over) and 1/3 chance of having H. Once we have H, we have 2/3 chance of obtaining a T in the following flip and 1/3 chance of getting H (I am deliberately using the alphabet, instead of word to avoid the pun), continuing the sequence. Once we have HT, we have 1/3 chance of getting H (going back to the previous case) and 2/3 chance of getting T in the flip (continuing the sequence). Once we have HTT, we have 2/3 chance of getting T (and we need to start over) and 1/3 chance of getting H in the flip (continuing the sequence). Once we have HTTH (I started copy and pasting a while ago), we have 2/3 chance of getting T, heading to "HT" and 1/3 chance of getting H in the flip, continuing the sequence. And HTTHH is an absorbing state.

Remark that on the fourth row (when we have the subsequence "HTTH"), if the next flip returns T, our most recent subsequence will just return to "HT"; we do not need to start from the beginning. Among classmates there was an argument on this. We will find the fundamental matrix and the expected number of flips through R commands

```
q1 <- c(2/3,1/3,0,0,0)
q2 <- c(0,1/3,2/3,0,0)
q3 <- c(0,1/3,0,2/3,0)
q4 <- c(2/3,0,0,0,1/3)
q5 <- c(0,0,2/3,0,0)
Q = rbind(q1,q2,q3,q4,q5)
finv = diag(5)-Q
f =  solve(finv)

f  %*% c(1,1,1,1,1)
```

```
##          [,1]
## [1,]  63.75
## [2,]  60.75
## [3,]  59.25
## [4,]  57.00
## [5,]  40.50
```

We should expect roughly 64 (63.75) flips till we see the given sequence. Most of the people said 72.75 after messing up the fourth row, since a "trustworthy source" said so!

The pages after this are appendices