

# A Link between Degrees of Freedom and VC-Dimension

By Dan Simon

# Why care?

- VC-dimension can be hard to calculate and non-intuitive
- Number of degrees of freedom is often easy to see and intuitive
- VC-dimension and degrees of freedom are said to be related
- No formal theorems proven about this relationship

# What are degrees of freedom?

- Function from  $\mathbb{R}^n$  to set of hypotheses
- Not just any function (what if function is constant?)
- Require every two values to be different hypotheses
- Allows us to extend distance on  $\mathbb{R}^n$  to distance between hypotheses

# What are degrees of freedom? (continued)

- Define degrees of freedom thus:
- A set of hypotheses has  $n$  degrees of freedom if there is a 1-to-1 map from  $R^n$  to it such that for any two hypotheses  $h_1$  and  $h_2$ , there is a point that all hypotheses close to one of  $h_1$  and  $h_2$  contain and all hypotheses close to the other are missing.

# Benefits of this definition

- Requires any two hypotheses to be different
- Requires everything in a neighborhood of a hypothesis to contain a point
- Only uses topological structure of  $\mathbb{R}^n$

# Conjecture

- If a hypothesis set has  $n$  degrees of freedom, then its VC-dimension is at least  $n$ .

# Why not equality?

- Unit circles have 2 degrees of freedom (location of their center)
- But an equilateral triangle with side length 1 can be shattered by the set of unit circles

# Application to the sphere

- $n$ -dimensional spheres intuitively have  $n + 1$  degrees of freedom
- 1 for the radius,  $n$  for the center
- The radius cannot be negative
- But if we let the first coordinate represent  $\ln(\text{radius})$  and the others represent the coordinates of the center, we get a map  $f$  from  $\mathbb{R}^{n+1}$  to the set of  $n$ -dimensional spheres



## Application to the sphere (continued)

- Does our map  $f$  have the properties required?
- For any two different spheres, there is a point in one (and all spheres close to it) but not in the other (or any spheres close to it).
- So the set of spheres does have  $n + 1$  degrees of freedom

## Application to the sphere (further continued)

- Our conjecture leads us to believe that the VC-dimension of  $n$ -dimensional spheres is at least  $n + 1$  (without us having to come up with explicit points)
- But we know that  $n$ -dimensional spheres have a VC-dimension of exactly  $n + 1$

# References

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