A Link between Degrees of Freedom and VC-Dimension

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Why care?

- VC-dimension can be hard to calculate and non-intuitive
- Number of degrees of freedom is often easy to see and intuitive
- VC-dimension and degrees of freedom are said to be related
- No formal theorems proven about this relationship

What are degrees of freedom?

- Function from R^n to set of hypotheses
- Not just any function (what if function is constant?)
- Require every two values to be different hypotheses
- Allows us to extend distance on R^n to distance between hypotheses

What are degrees of freedom? (continued)

- Define degrees of freedom thus:
- A set of hypotheses has n degrees of freedom if there is a 1-to-1 map from R^n to it such that for any two hypotheses h_1 and h_2, there is a point that all hypotheses close to one of h_1 and h_2 contain and all hypotheses close to the other are missing.

Benefits of this definition

- Requires any two hypotheses to be different
- Requires everything in a neighborhood of a hypothesis to contain a point
- Only uses topological structure of R^n

Conjecture

- If a hypothesis set has n degrees of freedom, then its VC-dimension is at least n.

Why not equality?

- Unit circles have 2 degrees of freedom (location of their center)
- But an equilateral triangle with side length 1 can be shattered by the set of unit circles

Application to the sphere

- n-dimensional spheres intuitively have n + 1 degrees of freedom
- 1 for the radius, n for the center
- The radius cannot be negative
- But if we let the first coordinate represent ln(radius) and the others represent the coordinates of the center, we get a map f from R^{n+1} to the set of n-dimensional spheres

Application to the sphere (continued)

- Does our map f have the properties required?
- For any two different spheres, there is a point in one (and all spheres close to it) but not in the other (or any spheres close to it).
- So the set of spheres does have n + 1 degrees of freedom

Application to the sphere (further continued)

- Our conjecture leads us to believe that the VC-dimension of n-dimensional spheres is at least n + 1 (without us having to come up with explicit points)
- But we know that n-dimensional spheres have a VC-dimension of exactly n + 1

References

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