Module 1: Returns and Risk

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Abstract

This module introduces the Capital Asset Pricing Model (CAPM) as a foundational tool for measuring systematic risk and estimating expected returns in portfolio management. Beginning with the theoretical underpinnings of CAPM and its optimization problem—rooted in Markowitz's mean-variance framework—it derives the model's core equation using matrix algebra and Lagrange multipliers. The module explores the interpretation and estimation of beta coefficients, the Security Market Line (SML), and Jensen's Alpha as measures of mispricing and performance. It also derives the portfolio beta both algebraically and through covariance properties, showing its equivalence with regression-based approaches. Finally, it compares the Sharpe and Treynor Ratios, highlighting their respective roles in evaluating performance based on total and systematic risk. This module provides both a mathematical and economic perspective, making it suitable for students and professionals seeking a rigorous understanding of CAPM in applied finance.

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1 Capital Asset Pricing Model

In the discipline of portfolio management, the measurement of risk is fundamental. As we discussed in the previous chapter, diversifying by including negatively correlated assets can help mitigate market risk. However, it is also crucial to understand how sensitive the assets in our portfolio, or the portfolio as a whole, are to fluctuations in the capital markets. To analyze this sensitivity, we use the approach of the **Capital Asset Pricing Model (CAPM)**, which introduces the concept of the beta coefficient. This coefficient measures the sensitivity of an asset or portfolio's return relative to movements in the overall market.

1.1 Fundamentals of the CAPM

The CAPM model, developed by William F. Sharpe, John Lintner, and Jan Mossin, is indeed rooted in an optimization problem, similar to that of Markowitz's Theory. This model introduces several assumptions about markets and the behavior of investors participating in them. The most critical assumptions are quite similar to those in Markowitz's framework: investors are rational, risk-averse, and seek to maximize their utility by selecting optimal portfolios along the Capital Market Line (CML).

This implies that investors will make decisions based on their risk preferences. Some will select the tangency portfolio (the point of tangency between the efficient frontier and the CML) to achieve the highest possible return for a given level of risk. Others may choose to lend money, taking on less risk and earning the risk-free rate. Conversely, more risk-tolerant investors might opt for leverage, borrowing funds to increase their exposure to the market, thereby potentially achieving higher returns. The resulting returns for these investors can be expressed as:

$$\mu_P = \sum_{i=1}^{n} \omega_i \mu_i + \left(1 - \sum_{i=1}^{n} \omega_i\right) r_f \tag{1}$$

This equation shows that some investors prefer to allocate a portion of their wealth to risk-free assets, such as treasury bonds, to reduce the risk in their portfolios. On the other hand, some may choose a 'negative' fraction in risk-free assets (i.e., by taking on debt) to invest a larger portion of their wealth in risky assets. We also assume that investors are always seeking to minimize risk and maximize their utility, consistent with the principles outlined in Markowitz's Theory.

The market equilibrium is achieved when all investors select a portfolio along the Capital Allocation Line, which then transforms into the Capital Market Line (CML). At this point, the new optimization problem becomes minimizing the volatility of a portfolio, subject to the revised form of the portfolio's returns, which now includes the risk-free asset. The key question is: how can we further reduce risk, given that we now have access to securities with zero risk? Then we can propose a Lagrange Equation in the matrix form:

$$\mathcal{L}(\omega, \lambda_1) = \sqrt{\omega^\top \Sigma \omega} - \lambda_1 \left(\omega^\top \mu + \left[1 - \omega^\top \iota\right] r_f - \mu_P\right) \tag{2}$$

Note that, term $\omega^\top \Sigma \omega$ represents the variance of the portfolio, while $\omega^\top \mu$ denotes the portfolio returns generated by the risky assets. Consequently, $\left[1 - \omega^\top \iota\right] r_f$ is to the portion of returns explained by the risk-free asset. To derive the First Order Conditions, we differentiate the objective function with respect to the portfolio weights ω and the Lagrange Multiplier λ_1:

$$\frac{\partial \mathcal{L}(\omega, \lambda_1)}{\partial \omega} = \frac{\Sigma \omega}{\sigma_P} - \lambda_1 \left(\mu - r_f \iota\right) = 0 \tag{3}$$

$$\frac{\partial \mathcal{L}(\omega, \lambda_1)}{\partial \lambda_1} = \omega^\top \mu + \left[1 - \omega^\top \iota\right] r_f - \mu_P = 0 \tag{4}$$

Now, we can take Equation 3 and multiply all the equation by $\omega^\top$. This will help us to transform some of the terms into others that will be easier to handle in this context.

$$\frac{\omega^\top \Sigma \omega}{\sigma_P} = \lambda_1 \left(\omega^\top \mu - r_f \omega^\top \iota\right) \tag{5}$$

Since $\omega^\top \Sigma \omega = \sigma_P^2$, we can simplify the expression on the left side of the equation. We can also recall two key conditions from the Markowitz Theory that apply in this context. First, when the chosen portfolio is the Market Portfolio, the portfolio returns can be written as $\mu_P = \mu_M = \omega^\top \mu$ represents the expected returns of the risky assets. Second, if the market portfolio does not include a risky asset (i.e., it is purely risk-free), then $\omega^\top \iota = 1$. Therefore, we arrive at the following expression:

$$\lambda_1 = \frac{\sigma_M}{\mu_M - r_f} \tag{6}$$

Then, by recalling again Equation 3 and rearranging it, we will obtain the next expression. We can transform it by replacing the form of λ_1:

$$\mu = r_f \iota + \frac{1}{\lambda_1} \cdot \frac{\Sigma \omega}{\sigma_M} \tag{7}$$

$$\mu = r_f \iota + \frac{\mu_M - r_f}{\sigma_M} \cdot \frac{\Sigma \omega}{\sigma_M} \tag{8}$$

Let us review the properties of the term $\Sigma\omega$. The matrix Σ represents the covariance matrix of all the assets in the portfolio, which, in this case, refers to the market portfolio. The vector ω contains the weights of those assets within the portfolio. Therefore, the term $\Sigma\omega$ can be interpreted as the vector of covariances of the portfolio with the individual assets in the market, weighted by the portfolio weights.

$$\Sigma\omega = \begin{bmatrix} \sigma_1^2 & \gamma_{2,1} & \cdots & \gamma_{n,1} \\ \gamma_{1,2} & \sigma_2^2 & \cdots & \gamma_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{1,n} & \gamma_{2,n} & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} \omega_1\sigma_1^2 + \omega_2\gamma_{2,1} + \cdots + \omega_n\gamma_{n,1} \\ \omega_1\gamma_{1,2} + \omega_2\sigma_2^2 + \cdots + \omega_n\gamma_{n,2} \\ \vdots \\ \omega_1\gamma_{1,n} + \omega_2\gamma_{2,n} + \cdots + \omega_n\sigma_n^2 \end{bmatrix}$$

Note that the first element of the vector $\Sigma\omega$ represents the weighted average of the covariances between asset 1 and all the assets in the Market Portfolio (i.e., the entire market). The second element represents the same, but for asset 2, and so on for all the assets in the portfolio. Therefore, we can conclude that the vector $\Sigma\omega$ represents the covariances of each asset with the market portfolio. Specifically, this can be expressed as $\mathrm{Cov}(\mu, \mu_M)$, that we can substitute in the previous equation:

$$\mu = r_f \iota + (\mu_M - r_f) \frac{\mathrm{Cov}(\mu, \mu_M)}{\sigma_P^2} \tag{9}$$

If you remember what we reviewed in the Module 3, the form of a β coefficient is exactly the covariance between the dependent variable and the independent variable, adjusted by the variance of the latter. Then, we can interpret the fraction in Equation 9 as a vector of β coefficients that relate each asset in the market to the market itself. Finally, we can arrive at the next expression, which is the matrix form of the CAPM:

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} r_f \\ r_f \\ \vdots \\ r_f \end{bmatrix} + (\mu_M - r_f) \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

Then, for each asset in the market, we can obtain a linear equation that relates their returns to the risk-free rate and the market's risk premium $(\mu_M - r_f)$ This is the general form of the Capital Asset Pricing Model, proposed by Sharpe, Lintner, and Mossin. Next, we would like to know the form of the β coefficients, which are estimated through the OLS algorithm, as we reviewed in the previous chapter.

$$\mu_i = r_f + \beta_i(\mu_M - r_f) \tag{10}$$

Another way to understand the Capital Asset Pricing Model is by expressing β in its correlation form. In this representation, beta is the product of the Pearson correlation coefficient between the market and an individual asset, and the ratio of their variances. This form is also a widely used representation of the CAPM model, and is mathematically expressed by the following formula:

$$\mu_i = r_f + \rho_{i,M}\frac{\sigma_i}{\sigma_M}(\mu_M - r_f) + \alpha_i \tag{11}$$

Evidently, if the asset is uncorrelated with the market $(\rho_{i,M} = 0)$, the beta will be zero, indicating that the returns of the asset in question are not explained by market movements, and therefore, it has no systematic risk (captured by the α_i). The ratio σ_i/σ_M can help us understand how volatile an asset is compared to the whole market. Highly volatile assets will be more sensitive to the market, even if they are not strongly correlated with it. This form is very useful for understanding how systematic risk affects risky assets.

1.2 Security Market Line and the Jensen's Alpha

The graphical representation of the Capital Asset Pricing Model (CAPM) is the Security Market Line (SML), which illustrates the relationship between an asset's expected return and its systematic risk. As shown in Figure 1, the SML is a straight line with a slope equal to the Market Risk Premium and an intercept at the risk-free rate. This establishes a direct relationship between a stock's return and its exposure to market risk.

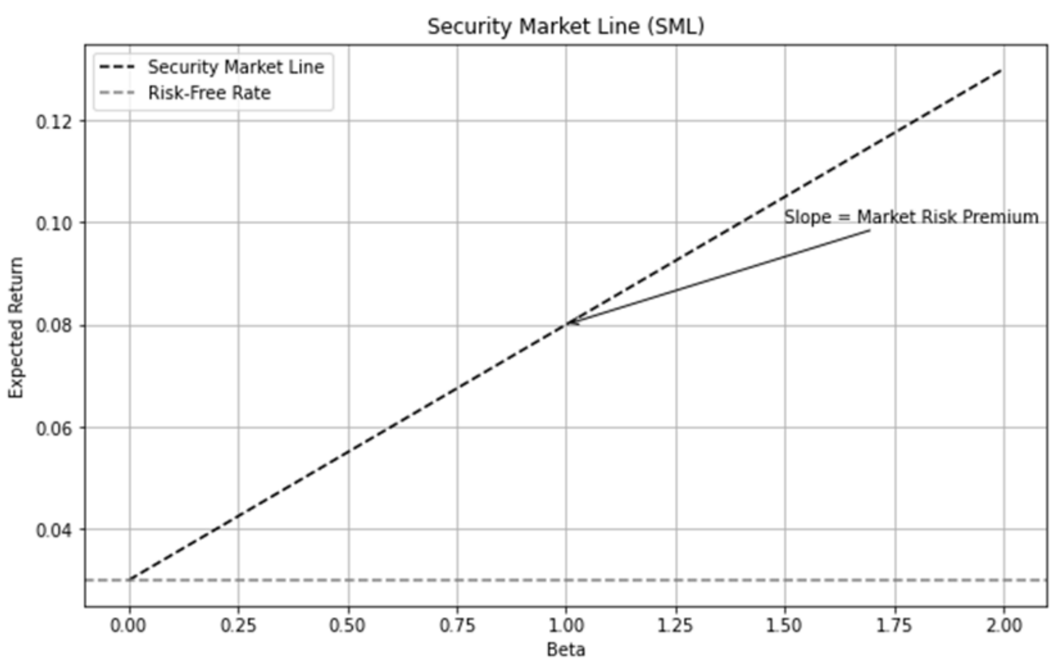

Figure 1: Security Market Line

Recalling the concepts from Markowitz Theory and the construction of portfolios along the efficient frontier, we know that all optimal portfolios will have no diversifiable risk. This is because diversifiable risk is eliminated by selecting the appropriate asset weights to minimize total risk. Consequently, for these portfolios, all remaining risk is market or systematic risk (captured by the beta). Evidently, for the market portfolio, the beta will be exactly one, and that portfolio will be over the SML.

But what does it mean when an asset lies above or below the Security Market Line (SML)? Portfolio managers often use the SML to evaluate assets and stocks. For example, if a particular security is located above the SML, we can conclude that its returns are higher than what the market portfolios offer, after adjusting for risk. In this case, the asset is providing superior returns relative to its risk. Conversely, if an asset lies below the SML, it is considered overvalued, as it offers lower returns than the market portfolios, after adjusting for risk.

What happens with these arbitrage opportunities? For undervalued assets, investors identify investment opportunities that offer better returns than what is predicted by the SML, after adjusting for risk. As a result, many investors are willing to buy the undervalued asset, leading to an excess demand for that specific stock. This increased demand pushes the price of the assets up. Since there is an inverse relationship between price and returns (theoretically speaking), the asset's return decreases as its price increases. Eventually, the asset's expected return will align with the SML, moving it to the appropriate position relative to its risk.

The same dynamic occurs for overvalued assets, but in reverse. Investors recognize that the asset's return is lower than what would be expected given its level of risk, so they sell the overvalued asset. The resulting excess supply drives down the asset's price, which in turn increases its expected return. Over time, this adjustment brings the asset's return back in line with the Security Market Line (SML), thereby correcting the mispricing. Hence, the equation for these abnormal returns is given by:

$$\alpha_i = (\mu_i - r_f) - \beta_i(\mu_M - r_f) \tag{12}$$

This measure is commonly known as Jensen's Alpha (α_i) or Jensen's Measure. It is often used to capture how much an asset outperforms a market portfolio or the market's benchmark. If Jensen's Alpha is zero ($\alpha_i = 0$), we conclude that the stock under study lies exactly on the Security Market Line (SML), and there is no arbitrage opportunities associated with it. Econometrically speaking, this measure might help us understand that other factors could explain the returns of an asset, rather than just market performance.

1.3 The Portfolio's Beta and the Treynor Ratio

We conclude that knowing the beta coefficient for each asset is crucial for understanding the valuation of those stocks. Nevertheless, what about a portfolio? How can we measure the systematic risk of a portfolio composed of risky assets? Mathematically, some propose to measure the portfolio's beta as a weighted average of the betas of all the stocks that compose it, but is this true? Imagine a portfolio of n assets all over the Security Market Line, whose returns will be:

$$\begin{aligned}
\mu_1 &= r_f + \beta_1(\mu_M - r_f) \\
\mu_2 &= r_f + \beta_2(\mu_M - r_f) \\
&\vdots \\
\mu_n &= r_f + \beta_n(\mu_M - r_f)
\end{aligned}$$

Then, recalling the form of the portfolio's return, which is also a weighted average:

$$\mu_P = \omega_1\mu_1 + \omega_2\mu_2 + \cdots + \omega_n\mu_n \tag{13}$$

Replacing the forms of the assets' returns:

$$\mu_P = \omega_1 \left[r_f + \beta_1(\mu_M - r_f)\right] + \omega_2 \left[r_f + \beta_2(\mu_M - r_f)\right] + \cdots + \omega_n \left[r_f + \beta_n(\mu_M - r_f)\right] \tag{14}$$

By rearranging some terms, we will obtain:

$$\mu_P = \omega_1 r_f + \omega_2 r_f + \cdots + \omega_n r_f + \omega_1\beta_1(\mu_M - r_f) + \omega_2\beta_2(\mu_M - r_f) + \cdots + \omega_n\beta_n(\mu_M - r_f) \tag{15}$$

Obtaining common factors:

$$\mu_P = [\omega_1 + \omega_2 + \cdots + \omega_n]\, r_f + [\omega_1\beta_1 + \omega_2\beta_2 + \cdots + \omega_n\beta_n]\,(\mu_M - r_f) \tag{16}$$

And remembering that the sum of weights is one:

$$\mu_P = r_f + \sum_{i=1}^{n} \omega_i\beta_i(\mu_M - r_f) \tag{17}$$

Then we can propose the definition of the portfolio beta:

$$\beta_P = \sum_{i=1}^{n} \omega_i\beta_i \tag{18}$$

Calculating the beta of a portfolio is as straightforward as computing the weighted average of the individual betas. For an equally weighted portfolio, the beta would simply be the arithmetic mean. But why do we use this formula instead of performing a linear regression using ordinary least squares (OLS) on the portfolio returns, as we do with any individual stock? Let's prove mathematically that both methods are theoretically equivalent. Then, the beta would be:

$$\beta_P = \frac{\mathrm{Cov}(\mu_P, \mu_M)}{\sigma_M^2} = \frac{\mathrm{Cov}(\omega_1\mu_1 + \omega_2\mu_2 + \cdots + \omega_n\mu_n, \mu_M)}{\sigma_M^2} \tag{19}$$

One of the covariances properties establishes that the covariance of a random variable with the linear combination of other random variables is $\mathrm{Cov}(X, aY + bZ) = a\mathrm{Cov}(X, Y) + b\mathrm{Cov}(X, Z)$. Then the previous equation can be written as:

$$\beta_P = \frac{1}{\sigma_M^2}\left[\omega_1\mathrm{Cov}(\mu_1, \mu_M) + \omega_2\mathrm{Cov}(\mu_2, \mu_M) + \cdots + \omega_n\mathrm{Cov}(\mu_n, \mu_M)\right] \tag{20}$$

By distributing the market's variance (σ_M^2) and recalling the formula for the individual beta of each asset, we obtain the same form as before as Equation 18:

$$\beta_P = \sum_{i=1}^{n} \omega_i\beta_i$$

Nevertheless, in practice, the two methods might not yield the same results. This discrepancy can be explained by the quality of the data used to build the regression model. In the analysis of the time series econometrics, we will see that failing to properly handle the data can introduce biases in the estimation of the beta coefficients, leading to inconsistencies in the calculations.

Estimating the beta coefficients for an individual stock or an entire portfolio is valuable for various practical applications. For instance, some investors use a portfolio's beta to forecast expected returns over a specific time horizon. Others may use beta to hedge against market risk, especially when speculating about an upcoming crisis. Additionally, some investors might use beta to measure the efficiency of a stock by adjusting the risk premium based on the beta coefficient.

$$T_i = \frac{\mu_i - r_F}{\beta_i} \tag{21}$$

This measure of efficiency is known as the Treynor Ratio, an extension of the Sharpe Ratio, which adjusts an asset's performance by total risk. Unlike the Sharpe Ratio, the Treynor Ratio focuses solely on systematic or market risk, as measured by the beta coefficient. Since the beta coefficient can be negative, it is important to compare Treynor Ratios using their absolute values. A higher Treynor Ratio indicates better performance.

What is the relationship between the Sharpe Ratio and the Treynor Ratio? Since beta only captures systematic risk, which is a component of total risk, it might seem that the Treynor Ratio should always be larger than the Sharpe Ratio. But how accurate is this assertion? To answer this, let's analyze the variance of an asset's returns:

$$\text{Var}(\mu_i) = \text{Var}(r_f + \beta_i(\mu_M - r_f) + \varepsilon_i) \tag{22}$$

The term ε_i represents the residuals of the regression model, which account for all factors, aside from the market risk premium, that explain an asset's returns. Thus, using the properties of variance, we can obtain:

$$\sigma_i^2 = \beta_i^2 \sigma_M^2 + \sigma_\varepsilon^2 \tag{23}$$

The term σ_ε^2 represents the non-systematic, or diversifiable, risk. Recall that one of the assumptions of the Capital Asset Pricing Model (CAPM) states that, by applying Markowitz optimization to construct a portfolio, this risk is either entirely eliminated or minimized as much as possible. Therefore:

$$\sigma_i^2 > \beta_i^2 \sigma_M^2 \tag{24}$$

Rearranging and taking square root:

$$\frac{\sigma_i}{\sigma_M} > \beta_i \tag{25}$$

Recalling the alternative form of the beta in Equation 11:

$$1 \geq |\rho_{i,M}| \tag{26}$$

One of the most interesting conclusions of this inequality is to observe that, unless the assets are perfectly correlated with the market, the non-systematic risk cannot be totally eliminated, then the systematic risk will always be smaller than the total risk. Recalling Equation 24, and inverting it:

$$\frac{1}{\sigma_i} \leq \frac{1}{\beta_i \sigma_M} \tag{27}$$

Then, if and only if $\sigma_M \geq 1$ (which is commonly the case), we can conclude:

$$\frac{1}{\sigma_i} \leq \frac{1}{\beta_i} \tag{28}$$

So we can conclude given the definitions of the Sharpe Ratio and the Treynors Ratio:

$$S = \frac{\mu_i - r_f}{\sigma_i} \leq \frac{\mu_i - r_f}{\beta_i} = T \tag{29}$$

Naturally, there can be anomalies in the relationship between the two ratios. In some cases, the Sharpe Ratio may be higher for assets with low correlation to the market during periods of extreme market volatility. However, such cases are rare exceptions that are rarely observed in practice. The Treynors Ratio, on the other hand, is particularly useful for evaluating the efficiency of optimal portfolios, as non-systematic risk is generally minimized.